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A NOTE ON MIXING PROCESSES

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SUMMARY. In this paper it is shown that in the space of discrete stationary stochastic processes under the weak topology finite Markov Chains are dense and the set of weakly mixing processes is a dense G_δ.

The purpose of this note is to show that any real valued discrete stationary process can be approximated by means of strongly mixing Markov Chains and deduce that the set of weakly mixing processes is a set of the second category under the weak topology. This answers a question raised by Kolmogorov (1962).

DEFINITIONS AND NOTATIONS

Let R denote the real and R^I the countable product of R over all the integers. T denotes the shift transformation. $\mathfrak{M}$ is the space of all distributions on R^I which are invariant under T. $\mathfrak{M}$ is assigned the weak topology which makes it a complete separable metric space.

THEOREMS

Theorem 1 : *The set of strongly mixing Markov Chains is everywhere dense in* $\mathfrak{M}$.

Proof : Consider points of the type x such that $T^k x = x$ for some k. The smallest k for which this is valid is called the period of x. The measure which assigns mass $1/k$ to the points $x, Tx, \dots, T^{k-1}x$ is a periodic ergodic measure [as described by the author (Parthasarathy, 1961)]. Such measures have been proved to be dense in $\mathfrak{M}$ (cf. Parthasarathy, 1961). From this we deduce the following.

Consider sequences of the following type

$$x = (\dots x_{-1}, x_0, x_1, x_2, \dots)$$

where the numbers $x_0, x_1, \dots x_{k-1}$ are all distinct and $x_r = x_{k+r}$ and measures with mass $1/k$ at $x, Tx, \dots, T^{k-1}x$. Such measures are dense in $\mathfrak{M}$.

We shall now approximate measures of this type by strongly mixing Markov Chains. To this end, we consider the Markov Chains with states $x_0, x_1, \ldots, x_{k-1}$, transition matrix

$$\begin{pmatrix} \epsilon/k-1 & 1-\epsilon & \epsilon/k-1 \ldots \epsilon/k-1 & & \\ \epsilon/k-1 & \epsilon/k-1 & 1-\epsilon & \epsilon/k-1 \ldots & \epsilon/k-1 \\ \ldots & \ldots & \ldots \quad \ldots & \ldots \quad \ldots & \\ 1-\epsilon & \epsilon/k-1 & \epsilon/k-1 \ldots & \epsilon/k-1 \ldots & \epsilon/k-1 \end{pmatrix}$$

and initial distribution $(1/k, 1/k, \ldots, 1/k)$ for $x_0, x_1, \ldots, x_{k-1}$. These are strongly mixing Markov Chains. As $\epsilon \to 0$ these Markov Chains converge weakly to the distribution with mass $1/k$ at the points $x, Tx, \ldots, T^{k-1}x$. This proves Theorem 1.

Corollary : *In particular the set of weakly mixing distributions is everywhere dense.*

Theorem 2 : *The set of weakly mixing distributions is a set of the second category in $\mathfrak{M}$.*

Proof : Let $\mathfrak{M}_w$ denote the set of weakly mixing distributions on R^I. It is enough to show that $\mathfrak{M}_w$ is a G_δ. We shall just indicate the proof since the arguments go exactly on the same lines as in the case of ergodic measures (Parthasarathy, 1961). We choose a metric in R^I such that the space of bounded uniformly continuous functions is separable under the uniform topology. Such a possibility is shown by Varadarajan (1958). A distribution μ is weakly mixing if and only if, for every (real valued) bounded uniformly continuous function f on R^I,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} \left| \int f(T^r x) f(x) d\mu - \left(\int f \, d\mu\right)^2 \right| = 0.$$

This condition can be replaced by

$$\lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} \left| \int f(T^r x) \, f(x) d\mu - \left(\int f \, d \, \mu\right)^2 \right|^2 = 0. \qquad \ldots \quad (1)$$

Since $\mu \times \mu$ is invariant under $T \times T$ and

$$\frac{1}{n} \sum_{1}^{n} \left| \int f(T^r x) \, f(x) \, d\mu - \left(\int f \, d \, \mu\right)^2 \right|^2$$

$$= \frac{1}{n} \sum_{1}^{n} \iint (f(T^r x) - Ef)(f(T^r y) - Ef) \cdot (f(x) - Ef)(f(y) - Ef) d\mu \times d\mu \qquad \ldots \quad (2)$$

the limit as $n \to \infty$ of the expression (2) exists for every stationary distribution. Therefore in condition (1) we can replace $\lim\limits_{n \to \infty}$ by $\liminf\limits_{n \to \infty}$. Hereafter the proof is exactly the same as in the case of ergodic distributions.

REFERENCES

KOLMOGOROV, A. N. (1962) : Lecture delivered at the Indian Statistical Institute. April 1962.

PARTHASARATHY, K. R. (1961) : On the category of ergodic measures. *Illinois. J. Math.*, 5, No. 4, 648-656.

VARADARAJAN, V. S. (1958) : Weak convergence of measures on separable metric spaces. *Sankhyā*, 19, 15-13.

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