isid/ms/2002/14
May 10, 2002
http://www.isid.ac.in/ ~ statmath/eprints

On the exponential metric increasing property

Rajendra Bhatia

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

On the exponential metric increasing property

Rajendra Bhatia
Indian Statistical Institute
7, S. J. S. Sansanwal Marg
New Delhi - 110 016 (INDIA)
Email:rbh@isid.ac.in

Abstract

The metric increasing property of the exponential map is known to be equivalent to the fact that the set of positive definite matrices is a Riemannian manifold of nonpositive curvature. We show that this property is an easy consequence of the logarithmic-geometric mean inequality for positive numbers. Operator versions of this inequality lead to a generalisation of the exponential metric increasing property to all Schatten-von Neumann norms.

1 Introduction

For a fixed n, let $\mathbb{M}$ be the space of $n \times n$ complex matrices, $\mathbb{S}$ the (real vector) subspace of $\mathbb{M}$ consisting of Hermitian matrices, and $\mathbb{P}$ the subset consisting of positive (definite) matrices.

Let $s_1(A) \geq \cdots \geq s_n(A)$ be the singular values of A (these are the positve square roots of the eigenvalues of $A^\star A$). For each $A \in \mathbb{M}$ let $\|A\|_2 := \left[\sum s_j^2(A)\right]^{\frac{1}{2}} = (\text{tr } A^\star A)^{\frac{1}{2}}$. This is the norm associated with the inner product $\langle A, B\rangle =$ tr $A^\star B$ on $\mathbb{M}$. We denote by $\lambda_i(A), 1 \leq i \leq n,$ the eigenvalues of $A,$ and by $\text{Eig}A$ a diagonal matrix with diagonal entries $\lambda_i(A)$.

If A, B are positive, then the product AB has positive eigenvalues. For $A, B \in \mathbb{P}$ let

$$\delta_2(A, B) := \|\log \text{Eig}(AB^{-1})\|_2 = \left[\sum_{i=1}^{n} \log^2 \lambda_i(AB^{-1})\right]^{\frac{1}{2}}. \quad (1)$$

This defines a metric on the manifold $\mathbb{P}$. The tangent space to $\mathbb{P}$ at any of its points A is the space $T_A\mathbb{P} = \{A\} \times \mathbb{S}$. The metric (1) is the distance associated with the arc length with respect to the Riemannian metric $ds^2 =$ tr $(A^{-1}dA)^2$.

The exponential map is a bijection of $\mathbb{S}$ onto $\mathbb{P}$. The *exponential metric increasing property* (EMI) says that for all $A, B \in \mathbb{S}$

$$\|A - B\|_2 \leq \delta_2(\exp A,\ \exp B) \quad (2)$$

and if A, B are on the same line through the origin, then (2) is an equality.

This property is equivalent to another important property of $\mathbb{P}$: it is a Riemannian manifold with nonpositive curvature; see Chapters XI and XII of S. Lang [10].(The discussion there is confined to real matrices.) The standard general reference for the subject is [1].

In this note we present a proof of (2) that is much shorter than that of Lang [10] which, in turn, is based on the exposition in Mostow [13]. Our proof may provide a little more insight into this inequality as it reduces it to the classical logarithmic-geometric mean inequality for positive numbers. After that we show that the EMI remains true when the norm $\|.\|_2$ is replaced by any of the Schatten-von Neumann norms (also called unitarily invariant norms).

Every Schatten-von Neumann norm on $\mathbb{M}$ arises as a *symmetric gauge function* of the singular values [2]. We use the notation $\|.\|_\Phi$ for the norm on $\mathbb{M}$ corresponding to the symmetric gauge function Φ on $\mathbb{R}^n$. To show that each such function when $\|.\|_2$, in the definition (1) is replaced by $\|.\|_\Phi$, leads to a metric δ_Φ on $\mathbb{P}$ we use a theorem of Lidskii [12].This is one from the circle of ideas that have recently come into prominence because of the solution of Horn's problem [3,8]. To prove a general version of (2) we use an operator analogue of the logarithmic-geometric mean inequality proved in [9], and later in [5]. The inequalities obtained in this note have an independent interest as operator inequalities. For example, the generalized EMI is a strengthening in several ways of the famous Golden-Thompson inequality of mathematical physics.

2 A simple proof of the EMI

Let $\exp'(A)$ denote the derivative, at a point A, of the exponential map $\exp$ from $\mathbb{S}$ onto $\mathbb{P}$. By standard calculus arguments the EMI (2) is a consequence of the inequality

$$\|B\|_2 \leq \| \exp(-A) \exp'(A)(B)\|_2 \tag{3}$$

valid for all $A,\ B \in \mathbb{S}$.

We have the well-known formula [2, p.311]

$$\exp'(A)(B) = \int_0^1 e^{tA} B e^{(1-t)A} dt. \tag{4}$$

The *logarithmic mean* of two positive numbers a, b is the quantity

$$L(a,b) := \frac{a-b}{\log a - \log b} = \int_0^1 a^t b^{1-t} dt. \tag{5}$$

It is easy to see that this number lies in between the geometric and the arithmetic means of a and b. Here is a simple proof of the part that we need:

$$\sqrt{ab} \leq \frac{a-b}{\log a - \log b}. \tag{6}$$

To see this assume $b < a$, divide both sides by b, and then replace a/b by x^2. The inequality (6) then reduces to

$$2\log x \leq \frac{x^2-1}{x} \text{ for } x \geq 1.$$

When $x = 1$ the two sides are zero, and for $x > 1$ the derivative of the left-hand side is smaller than that of the right-hand side.

Now let A be any positive matrix. Then for any matrix X we have

$$\|A^{\frac{1}{2}}XA^{\frac{1}{2}}\|_2 \leq \|\int_0^1 A^t X A^{1-t} dt\|_2 \tag{7}$$

To see this choose an orthonormal basis in which A is diagonal with diagonal entries λ_i. Then the matrix on the left-hand side of (7) has entries $\sqrt{\lambda_i\lambda_j}x_{ij}$, that on the right has entries $\left[\int_0^1 \lambda_i^t\lambda_j^{1-t}dt\right] x_{ij}$. So, the inequality (7) follows from the logarithmic-geometric mean inequality.

Now to prove (3) let A, B be any two Hermitian matrices. Write $B = e^{A/2}\left(e^{-\frac{A}{2}}Be^{-\frac{A}{2}}\right)e^{\frac{A}{2}}$. Then using (7) we have

$$\begin{aligned}\|B\|_2 &\leq \|\int_0^1 e^{tA}\left(e^{-A/2}Be^{-A/2}\right)e^{(1-t)A}dt\|_2 \\ &= \|e^{-A/2}\left[\int_0^1 e^{tA}Be^{(1-t)A}dt\right]e^{-A/2}\|_2.\end{aligned}$$

Using (4) this gives

$$\|B\|_2 \leq \|e^{-\frac{A}{2}}\left[\exp'(A)(B)\right]e^{-\frac{A}{2}}\|_2. \tag{8}$$

To get the inequality (3) from this, we use the fact that if a matrix product XY is Hermitian, then

$$\|XY\|_2 \leq \|YX\|_2. \tag{9}$$

To see this note that the singular values of a Hermitian matrix are the absolute values of its eigenvalues. So,

$$\begin{aligned}\|XY\|_2^2 &= \sum s_j^2(XY) = \sum \lambda_j^2(XY) = \sum \lambda_j^2(YX) \\ &\leq \sum s_j^2(YX) = \|YX\|_2^2.\end{aligned}$$

(The inequality used in the above chain can be derived from Schur's theorem that says every operator can be reduced to an upper triangular form in some basis.)

3 The generalized EMI

In this section we replace the norm $\|.\|_2$ by a more general class of norms. A norm Φ on $\mathbb{R}^n$ is called a *symmetric gauge function* if it is invariant under permutations and sign changes of coordinates. It is customary to assume a normalisation condition $\Phi(1,0,\ldots,0) = 1$. The norms $\|x\|_p = \left(\sum |x_j|^p\right)^{\frac{1}{p}}, 1 \leq p \leq \infty$, are examples of such norms. For $A \in \mathbb{M}$, let

$\|A\|_\Phi = \Phi(\{s_j(A)\})$ where $\{s_j(A)\}$ are the singular values of A. Then $\|A\|_\Phi$ is a norm on $\mathbb{M}$ that is unitarily invariant, i.e. $\|UAV\|_\Phi = \|A\|_\Phi$ for all unitary U,V. By a theorem of von Neumann all unitarily invariant norms on $\mathbb{M}$ arise in this way. The ones corresponding to $\|.\|_p$ are called the Schatten p-norms, $1 \leq p \leq \infty$.

For any vector $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ we write $x^\downarrow = (x_1^\downarrow, \ldots, x_n^\downarrow)$ for the vector whose coordinates are obtained by arranging the x_j in decreasing order and $x^\uparrow = (x_1^\uparrow, \ldots, x_n^\uparrow)$ for the vector obtained by arranging them in increasing order. We say $x \prec_w y$ if $\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow$ for all $1 \leq k \leq n$; and $x \prec y$ if, in addition, we have $\sum_{j=1}^n x_j^\downarrow = \sum_{j=1}^n y_j^\downarrow$.

Lemma 1. [2,p.45] If $x, y \in \mathbb{R}_+^n$ and $x \prec_w y$, then $\Phi(x) \leq \Phi(y)$ for every symmetric gauge function Φ.

The following theorem is a corollary of a theorem of Gel'fand and Naimark, and is sometimes called Lidskii's theorem.

Theorem 2.[2, p.73] Let $A, B \in \mathbb{P}$. Then

$$\{\log \lambda_i^\downarrow(A) + \log \lambda_i^\uparrow(B)\} \prec \{\log \lambda_i(AB)\} \prec \{\log \lambda_i^\downarrow(A) + \log \lambda_i^\downarrow(B)\}. \tag{10}$$

Let $A, B \in \mathbb{P}$ and let Φ be any symmetric gauge function. Define

$$\delta_\Phi(A,B) = \Phi\left(\left\{\log \lambda_i(AB^{-1})\right\}\right). \tag{11}$$

Proposition 3. δ_Φ is a metric on $\mathbb{P}$.

Proof Only the triangle inequality is nontrivial. Let A, B, C be three positive matrices. Then

$$\begin{aligned} \{\log \lambda_i(AC^{-1})\} &= \{\log \lambda_i(B^{-\frac{1}{2}}AC^{-1}B^{\frac{1}{2}})\} \\ &= \{\log \lambda_i(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}.B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})\}. \end{aligned}$$

So, by Theorem 2

$$\begin{aligned} \{\log \lambda_i(AC^{-1})\} &\prec \{\log \lambda_i^\downarrow(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) + \log \lambda_i^\downarrow(B^{\frac{1}{2}}C^{-1}B^{\frac{1}{2}})\} \\ &= \{\log \lambda_i^\downarrow(AB^{-1}) + \log \lambda_i^\downarrow(BC^{-1})\}. \end{aligned}$$

Using Lemma 1, we get from this

$$\delta_\Phi(A,C) \leq \delta_\Phi(A,B) + \delta_\Phi(B,C). \qquad \blacksquare$$

For any $X \in GL_n$, the transformation $\Gamma_X(A) := X^\star AX$ is called a congruence on $\mathbb{M}$. Any element of $\mathbb{P}$ is congruent to the identity I. So the group of congruences acts transitively on $\mathbb{P}$. It is easy to see that every congruence is an isometry of δ_Φ and so is the inversion map; i.e.

$$\delta_\Phi(X^\star AX, X^\star BX) = \delta_\Phi(A,B) = \delta_\Phi\left(A^{-1}, B^{-1}\right). \tag{12}$$

Next we show that the inequality (3) remains true for all norms $\|.\|_\Phi$. The proof uses the same ideas as in Section 2; the tools needed are harder. Instead of (7) we need the inequality

$$\|A^{\frac{1}{2}}XA^{\frac{1}{2}}\|_\Phi \leq \|\int_0^1 A^tXA^{1-t}dt\|_\Phi \tag{13}$$

for positive A and for all Φ. This is true [5,9], though harder to prove. The more general version of (9)

$$\|XY\|_\Phi \leq \|YX\|_\Phi \tag{14}$$

whenever XY is Hermitian is also known to be true [2,p.253]. Combining these we can prove the general version of (3), and as a corollary the following.

Theorem 4. For any $A, B \in \mathbb{S}$ and for any symmetric gauge function Φ we have

$$\|A-B\|_\Phi \leq \delta_\Phi(\exp A, \exp B). \tag{15}$$

The two sides of (15) are equal if A, B lie on the same line through the origin.

(The second statement of the theorem is easy to prove.) So the EMI is true for all Schatten-von Neumann norms.

One special case of this, that of the norm $\|A\|_\infty = \max s_j(A)$ corresponding to the symmetric gauge function $\|x\|_\infty = \max|x_j|$, has been studied earlier by Corach, Porta, and Recht [7].

Let us discuss the inequality (15) in the context of known matrix inequalities. We can write it as

$$\|A-B\|_\Phi \leq \|\log(e^{-B/2}e^Ae^{-B/2})\|_\Phi \tag{16}$$

for Hermitian A, B and for all unitarily invariant norms. Using the properties of the exponential function with respect to the order $\prec$ [2, Chapter II] one gets the weaker inequality

$$\|e^{A-B}\|_\Phi \leq \|e^{-B/2}e^Ae^{-B/2}\|_\Phi. \tag{17}$$

The special case $\|.\|_\Phi = \|.\|_\infty$ is known as Segal's inequality [15, p.260]. Changing signs and using (14) we get from (17) another known result [2, p.261]. We have

$$\|e^{A+B}\|_\Phi \leq \|e^Ae^B\|_\Phi. \tag{18}$$

Choosing the special norm $\|A\|_\Phi = \|A\|_{\mathrm{tr}} = \sum_{j=1}^n s_j(A)$, reduces (18) to the famous Golden-Thompson inequality

$$\mathrm{tr}(e^{A+B}) \leq \mathrm{tr}\, e^Ae^B.$$

Generalisations of this inequality have been sought and proved by several mathematicians and physicists [2,p.285], [15,p.333]. The general version of the EMI we have obtained continues this tradition.

From the first Lidskii inequality in (10) one gets, using standard arguments [2],

$$\|\mathrm{Eig}^{\downarrow} A - \mathrm{Eig}^{\downarrow} B\|_{\Phi} \leq \delta_{\Phi}(\exp A, \exp B) \tag{19}$$

for any two Hermitian matrices A, B. The inequality (15) is stronger than this. Here it may be appropriate to remark that non-Riemannian differential geometry involving explicit computation of geodesic length has been used earlier [4] in obtaining tight bounds for spectral variation of unitary matrices.

Let us put the generalized EMI in the context of geodesics in metric spaces [6]. A geodesic segment in a metric space X is a distance-preserving map from a compact interval into X. The space X is said to be a *geodesic space* if the distance between any two points is equal to the length of a geodesic segment joining them. We have shown that the space $\mathbb{P}$ with the metric δ_{Φ} is a geodesic space. The curve e^{tA} is a geodesic segment joining I and e^{A}.

Some symmetric gauge functions Φ on $\mathbb{R}^n$ have the property that all geodesic segments (in the metric induced by Φ) are straight lines. The Schatten p -norms have this property for $1 < p < \infty$. In these cases, for any two points x, y in $\mathbb{R}^n$ their algebraic mid-point $z = (x+y)/2$ is the only *metric mid-point* (a point equidistant from x and y). The norms $\|.\|_{\Phi}$ on $\mathbb{M}$ inherit this property from Φ. For all such norms any two points in $\mathbb{P}$ have a unique δ_{Φ} geodesic segment joining them. (Compare with Theorem 3.6 of Lang [10,p.313].)

There is a very interesting description of the mid-point on this segment.

The *geometric mean* of two elements A, B of $\mathbb{P}$ is the matrix

$$A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}, \tag{20}$$

a definition introduced by Pusz and Woronowicz [14]. This object has many interesting properties, including symmetry not so obvious from the above definition. It is easy to see that

$$\{\lambda_i(AB^{-1})\} = \{\lambda_i^2(A(A\#B)^{-1})\}$$

Thus

$$\delta_{\Phi}(A, B) = 2\delta_{\Phi}(A, A\#B).$$

This shows that $A\#B$ is a metric midpoint between A and B for every metric δ_{Φ}. For certain metrics, such as δ_p induced by the Schatten norms for $1 < p < \infty$, the geometric mean is the unique midpoint between A and B.

For an interesting recent discussion of the geometric mean see [11].

References

1. W. Ballmann, M. Gromov and V. Schroeder, Manifolds of Nonpositive Curvature, Birkhäuser, 1995.

2. R. Bhatia, Matrix Analysis, Springer, 1997.

3. R. Bhatia, Linear algebra to quantum cohomology: the story of Alfred Horn's inequalities, Amer. Math. Monthly, 108 (2001) 289-318.

4. R. Bhatia and J.A.R. Holbrook, Short normal paths and spectral variation, Proc. Amer. Math. Soc., 94 (1985) 377-382.

5. R. Bhatia and K. R. Parthasarathy, Positive definite functions and operator inequalities, Bull. London Math. Soc., 32 (2000) 214-228.

6. M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Springer, 1999.

7. G. Corach, H. Porta, and L. Recht, Geodesics and operator means in the space of positive operators, International J. Math., 4(1993)193-202.

8. W. Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc., 37 (2000) 209-249.

9. F. Hiai and H. Kosaki, Comparison of various means for operators, J. Funct. Anal., 163 (1999) 300-323.

10. S. Lang, Fundamentals of Differential Geometry, Springer, 1999.

11. J. D. Lawson and Y. Lim, The geometric mean, matrices, metrics, and more, Amer. Math. Monthly, 108 (2001) 797-812.

12. V. B. Lidskii, On the proper values of a sum and product of symmetric matrices, Dokl. Akad. Nauk SSSR, 75 (1950) 769-772.

13. G. D. Mostow, Some new decomposition theorems for semisimple groups, Memoirs Amer. Math. Soc., 14 (1955) 31-54.

14. W. Pusz and S. L. Woronowicz, Functional calculus for sesquilinear forms and the purification map, Rep. Math. Phys., 8 (1975) 159-170.

15. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol II, Academic Press, 1975.