Thermocapillarity in a liquid film on an unsteady stretching surface

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Abstract

The influence of thermocapillarity on the flow and heat transfer in a thin liquid film on a horizontal stretching sheet is analysed. The time-dependent governing boundary layer equations for momentum and thermal energy are reduced to a set of coupled ordinary differential equations by means of an exact similarity transformation. The resulting three-parameter problem is solved numerically for some representative values of an unsteadiness parameter S and a thermocapillarity number M for Prandtl numbers from 0.001 to 100. The thermocapillary surface forces drag the liquid film in the same direction as the stretching sheet and a local velocity minimum occurs inside the film. The surface velocity, the film thickness, and the Nusselt number at the sheet increase with M for $Pr \lesssim 10$. For higher Prandtl numbers, the thermal boundary layer is confined to the lower part of the liquid film and the temperature at the free surface remains equal to the slit temperature and the thermocapillary forces vanish.

Keywords: Boundary layer; Thermocapillary; Thin films

1. Introduction

A class of flow problems with obvious relevance to polymer extrusion is the flow induced by the stretching motion of a flat elastic sheet. In a melt-spinning process, the extrudate from the die is generally drawn and simultaneously stretched into a filament or sheet, which is thereafter solidified through rapid quenching or gradual cooling by direct contact with water or chilled metal rolls. In fact, stretching imports a unidirectional orientation to the extrudate, thereby improving its mechanical properties and the quality of the final product greatly depends on the rate of cooling. Crane [1] was the first who studied the motion set up in the ambient fluid due to a linearly stretching surface. Several authors, see e.g. the references cited in [2], have subsequently explored various aspects of the accompanying heat transfer occurring in the infinite fluid medium surrounding the stretching sheet. The hydrodynamics of a finite fluid medium, i.e. a thin liquid film, on a stretching sheet was first considered by Wang [3] who by means of a similarity transformation reduced the unsteady Navier–Stokes equations to a non-linear ordinary differential equation. The accompanying heat transfer problem was solved more recently by Andersson et al. [2]. In these studies the film surface was planar and free of any stresses.

The purpose of the present paper is to explore how the hydrodynamics and heat transfer in a liquid film on an unsteady stretching surface are affected by thermocapillarity, i.e. thermally induced surface-tension gradients along the horizontal interface between the passive gas and the liquid film. These surface-tension gradients generate an interfacial flow that, through viscous drag, either oppose or support the shear-driven motion due to the stretching sheet. The presence of thermocapillarity

Nomenclature

b	stretching rate [s^{-1}]
C_f	local skin friction coefficient, Eq. (26)
c_p	specific heat [$J\,kg^{-1}\,K^{-1}$]
f	dimensionless stream function, Eq. (13)
h	film thickness [m]
M	thermocapillarity number, Eq. (24)
Ma	Marangoni number, Eq. (25)
Nu_x	local Nusselt number, Eq. (27)
Pr	Prandtl number, ν/κ
q	heat flux, $-\rho c_p \kappa \partial T/\partial y$ [$J\,s^{-1}\,m^{-2}$]
Re_x	local Reynolds number, Ux/ν
S	unsteadiness parameter, α/b
t	time [s]
T	temperature [K]
U	sheet velocity [$m\,s^{-1}$]
u	horizontal velocity component [$m\,s^{-1}$]
v	vertical velocity component [$m\,s^{-1}$]
x	horizontal coordinate [m]
y	vertical coordinate [m]

Greek symbols

α	constant [s^{-1}]
β	dimensionless film thickness
γ	constant [K^{-1}]
η	similarity variable, Eq. (15)
θ	dimensionless temperature, Eq. (14)
κ	thermal diffusivity [$m^2\,s^{-1}$]
μ	dynamic viscosity [$kg\,m^{-1}\,s^{-1}$]
ν	kinematic viscosity [$m^2\,s^{-1}$]
ρ	density [$kg\,m^{-3}$]
σ	surface tension [$kg\,s^{-2}$]
τ	shear stress, $\mu\partial u/\partial y$ [$kg\,m^{-1}\,s^{-2}$]
ψ	stream function [$m^2\,s^{-1}$]

Subscripts

i	isothermal sheet
o	origin
ref	reference value
s	sheet
x	local value

couples the hydrodynamic and the thermal boundary layer problems. It will, nevertheless, be demonstrated that exact similarity can be achieved also in the presence of thermocapillarity. Accurate numerical solutions will be provided for the resulting three-parameter problem covering the range of Prandtl numbers from 0.001 to 100. Typical values of the dimensionless unsteadiness parameter introduced in Ref. [3] and a new thermocapillarity parameter will be considered.

2. Mathematical formulation

2.1. Governing equations and boundary conditions

Let us first consider a thin elastic sheet that emerges from a narrow slit at the origin of a Cartesian coordinate system, as shown schematically in Fig. 1. The

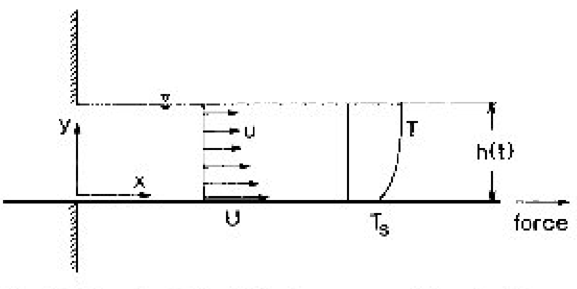

Fig. 1. Schematic of a liquid film flow on a stretching elastic sheet.

continuous sheet at $y = 0$ is parallel with the x-axis and moves in its own plane with the velocity

$$U = bx/(1 - \alpha t) \tag{1}$$

where b and α are both positive constants with dimension time^{-1}. Similarly, the surface temperature T_s of the stretching sheet varies with the distance x form the slit as

$$T_s = T_o - \frac{1}{2} T_{ref} \cdot Re_x (1 - \alpha t)^{-1/2} = T_o - T_{ref}[bx^2/2\nu](1 - \alpha t)^{-3/2} \tag{2}$$

where

$$Re_x = Ux/\nu = bx^2/\nu(1 - \alpha t) \tag{3}$$

is a local Reynolds number based on the sheet velocity U. Here, T_o denotes the temperature at the slit and T_{ref} can be taken as a constant reference temperature such that $0 \leqslant T_{ref} \leqslant T_o$. The expression (1) for the sheet velocity $U(x,t)$ reflects that the elastic sheet, which is fixed at the origin, is stretched by applying a force in the positive x-direction. The effective stretching rate $b/(1 - \alpha t)$ increases with time since $\alpha > 0$. Similarly, the expression (2) for the temperature $T_s(x,t)$ of the sheet represents a situation in which the sheet temperature decreases from T_o at the slit in proportion to x^2 and such that the amount of temperature reduction along the sheet increases with time. The particular forms of the above expressions for $U(x,t)$ and $T_s(x,t)$ are the same as in [2,3] and were chosen in order to be able to devise a new similarity transformation which transforms the governing partial differential equations for heat and momentum into a set of ordinary differential equations, thereby facilitating the exploration of the effects of the controlling parameters.

A thin liquid film of uniform thickness $h(t)$ lies on the horizontal sheet (cf. Fig. 1). The fluid motion within the film is primarily caused by the stretching of the elastic sheet. Buoyancy is neglected since the liquid layer is relatively thin, but not so thin that intermolecular forces come into play. The volatility of the Newtonian liquid is low and evaporation from the surface can be neglected. The fluid properties are assumed to be constant, except the surface tension which varies linearly with temperature:

$$\sigma = \sigma_o[1 - \gamma(T - T_o)] \tag{4}$$

This is a commonly made assumption, e.g. [4,5]. For most liquids the surface tension decreases with temperature, i.e. γ is a positive fluid property. The ambient gas does not exert any interfacial shear on the liquid film and there is no direct effect of the surface tension since the interface remains planar. However, the variation of σ along the interface, i.e.

$$\frac{\partial \sigma}{\partial x} = \frac{\partial \sigma}{\partial T} \cdot \frac{\partial T}{\partial x} \tag{5}$$

may generate interfacial motion, which is of primary concern in the present study. The velocity and temperature fields in the thin liquid layer are governed by the two-dimensional boundary layer equations for mass, momentum and thermal energy:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} \tag{8}$$

in which viscous dissipation of energy has been assumed negligible. The pressure is constant in the surrounding gas phase and the gravity force gives rise to a hydrostatic pressure variation in the liquid film.

In order to justify the boundary layer approximation, the length scale in the primary flow direction must be significantly larger than the length scale in the cross-stream direction. Now, if $(\nu/b)^{1/2}$ is a representative measure of the film thickness, the scale ratio $x/(\nu/b)^{1/2} \gg 1$. It is readily seen that the local Reynolds number in Eq. (3) initially equals the square of this scale ratio. Thus, just as in aerodynamic boundary layer theory, cross-stream diffusion of momentum and thermal energy can only be neglected at high Reynolds numbers.

The associated boundary conditions are

$$u = U; \quad v = 0; \quad T = T_s \quad \text{at } y = 0 \tag{9}$$

$$\mu\partial u/\partial y = \partial\sigma/\partial x \quad \text{at } y = h \tag{10}$$

$$\partial T/\partial y = 0 \quad \text{at } y = h \tag{11}$$

$$v = \mathrm{d}h/\mathrm{d}t \quad \text{at } y = h \tag{12}$$

Here, it is implicitly assumed that the mathematical problem is defined only for $x \geqslant 0$. It is moreover assumed that the surface of the planar liquid film is smooth and free of surface waves. The influence of interfacial shear due to the quiescent atmosphere is negligible and Eq. (10) states a balance between the viscous shear stress $\tau = \mu\partial u/\partial y$ and the net surface tension. The heat flux $q = -\rho c_p \kappa \partial T/\partial y$ vanishes at the adiabatic free surface, cf. Eq. (11), whereas Eq. (12) imposes a kinematic constraint on the fluid motion.

2.2. Similarity transformation

The special case of an isothermal sheet with $T_s = T_o$, i.e. $T_{ref} = 0$, was treated separately in the appendix of Ref. [2]. In that case the trivial solution $T(x, y, t) = T_o$ applies for all Prandtl numbers. Although surface tension effects were not taken into consideration in Ref. [2], it is intuitively clear that this uniform-temperature solution excludes any thermocapillary fluid motion. Let us now exclude the case $T_{ref} = 0$ and introduce new dimensionless variables f and θ and the similarity variable η:

$$\psi = \{\nu b(1 - \alpha t)^{-1}\}^{1/2} \cdot x \cdot f(\eta) \tag{13}$$

$$T = T_o - T_{ref}[bx^2/2\nu](1 - \alpha t)^{-3/2}\theta(\eta) \tag{14}$$

$$\eta = (b/\nu)^{1/2}(1 - \alpha t)^{-1/2} y \tag{15}$$

in which $\psi(x, y, t)$ is the physical stream function which automatically assures mass conservation (6). The velocity components are readily obtained as:

$$u = \partial\psi/\partial y = bx(1 - \alpha t)^{-1} f'(\eta) \tag{16}$$

$$v = -\partial\psi/\partial x = -\{\nu b(1 - \alpha t)^{-1}\}^{1/2} f(\eta) \tag{17}$$

The mathematical problem defined in Eqs. (6)–(12) transforms exactly into a set of ordinary differential equations and their associated boundary conditions:

$$S\left(f' + \frac{\eta}{2}f''\right) + (f')^2 - f''f = f''' \tag{18}$$

$$Pr[(S/2)(3\theta + \eta\theta') + 2\theta f' - \theta' f] = \theta'' \tag{19}$$

$$f'(0) = 1; \quad f(0) = 0; \quad \theta(0) = 1 \tag{20}$$

$$f''(\beta) = M \cdot \theta(\beta) \tag{21}$$

$$f(\beta) = S\beta/2 \tag{22}$$

$$\theta'(\beta) = 0 \tag{23}$$

where a prime denotes differentiation with respect to η. Here Eq. (21) represents the only and yet crucial distinction from the earlier analysis of the same problem in the absence of thermocapillarity [2].

The three dimensionless parameters, which appear explicitly in the transformed problem, are the unsteadiness parameter $S \equiv \alpha/b$, the Prandtl number $Pr = \nu/\kappa$, and the thermocapillarity number M defined as:

$$M \equiv \frac{\gamma\sigma_o T_{ref}}{\mu(b\nu)^{1/2}} \tag{24}$$

The latter parameter, which emerges naturally from the similarity analysis, is closely related to the Marangoni number. The Marangoni number is a frequently used parameter in the analysis of thermocapillarity-driven flows and involves a characteristic length scale. In the present context the thickness of the liquid layer is of the order $(\nu/b)^{1/2}$ and the Marangoni number based on this scale becomes:

$$Ma \equiv \frac{\gamma\sigma_o T_{ref}\sqrt{\nu/b}}{\mu\kappa} = Pr \cdot M \tag{25}$$

In the transformed problem, boundary conditions are imposed at $\eta = 0$ and $\eta = \beta$, where β denotes the value of the similarity variable η at the free surface. Thus Eq. (15) gives $\beta = (b/\nu)^{1/2}(1-\alpha t)^{-1/2}h$ for $y = h$ where β is a yet unknown constant which should be determined as an integral part of the boundary-value problem. The kinematic constraint (12) at $y = h(t)$ thus transforms into the free-surface condition (22) and the interfacial stress balance (10) leads to (21) which serves to couple the momentum boundary layer problem to the thermal boundary layer problem. Of particular practical relevance is the local skin friction coefficient

$$C_f \equiv \frac{2\tau_s}{\rho U^2} = -2f''(0) \cdot Re_x^{-1/2} \tag{26}$$

and the local Nusselt number

$$Nu_x \equiv -\frac{x}{T_{ref}}\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{1}{2}(1-\alpha t)^{-1/2} \cdot \theta'(0) \cdot Re_x^{3/2} \tag{27}$$

where Re_x is the local Reynolds number defined in Eq. (3). Thus, C_f decreases linearly with the distance from the slit, whereas Nu_x increases as x^3.

3. Numerical procedure

The non-linear differential equations (18) and (19) subject to the boundary conditions (20)–(23) constitute a two-point boundary-value problem, which was solved by the method of adjoints [6]. The two ODEs (18) and (19) were first formulated as a set of five first-order equations. For a tentative value of β, this set subjected to the three explicit initial conditions (20), the explicit terminal condition (23) and the implicit terminal condition (21) was solved by the method of adjoints. The numerical solution did generally not satisfy the auxiliary terminal condition (22), and the estimated value of β was therefore systematically adjusted until Eq. (22) was satisfied to within 10^{-4}. For non-linear two-point boundary-value problems, the method of adjoints involves forward integration of the five ODEs and multiple backward integrations of the five corresponding adjoint equations, i.e. equations which are adjunct to analytically determined variational equations. The iterative process, as described in more detail in chapter 3 of Roberts and Shipman [6], was terminated when Eqs. (21) and (23) were satisfied to within 10^{-8}.

4. Results and discussions

4.1. Absence of thermocapillarity ($M = 0$)

The thickness $h(t)$ of the thin liquid film has been assumed to be uniform, although decreasing monotonically with time. Therefore the only dynamic effect of surface tension in the present problem stems from thermocapillarity, i.e. thermally-induced variation of the surface tension along the film surface. In the particular case of temperature-independent surface tension, i.e. $\gamma = 0$ in Eq. (4), thermocapillarity is absent ($M = 0$) and the boundary condition (21) at the film surface simplifies to $f''(\beta) = 0$. In this case the momentum boundary layer problem defined by the ODE (18) and its associated boundary conditions (20)–(22) decouples from the thermal problem. The hydrodynamic part of this problem was solved by Wang [3], whereas Andersson et al. [2] solved the accompanying thermal problem. Let us briefly recall their major findings.

For positive values of the unsteadiness parameter S, Wang observed that solutions exist only for $0 \leqslant S \leqslant 2$. Moreover, when S tended to zero the solution approached the analytical solution due to Crane [1] for an infinitely thick layer of fluid, i.e. $\beta \to \infty$. On the other hand, the limiting solution as $S \to 2.0$ represents a liquid film with infinitesimal thickness, i.e. $\beta \to 0$. Andersson et al. [2] found that when the temperature T_s of the stretching sheet decreases with the distance from the slit, the temperature inside the film increased monotonically from the elastic sheet to the free surface. At sufficiently high Prandtl numbers, i.e. low thermal diffusivity, the surface temperature remained practically equal to the slit temperature T_o throughout the entire liquid film.

4.2. Effects of thermocapillarity ($M > 0$)

The velocity and temperature profiles for the parameter combination $S = 0.8$ and $Pr = 0.1$, as presented

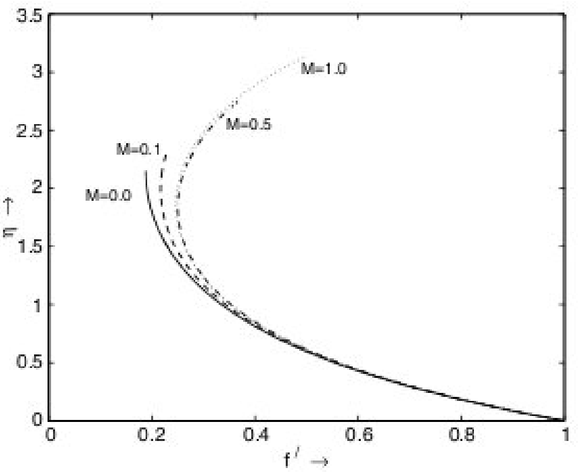

Fig. 2. Similarity velocity profiles $f'(\eta)$ for $S=0.8$ and $Pr=0.1$ for different values of the thermocapillarity number M.

in Figs. 2 and 3, respectively, show that the thermocapillarity number M has a significant influence on the variation of the velocity and the temperature in the film. For all values of M, the dimensionless temperature θ decreases monotonically with the distance η from the elastic sheet. This implies that the temperature T gradually increases from T_s ($\leqslant T_o$) at $\eta=0$ and towards the free surface. Since $\theta(\beta)>0$ the temperature at the free surface is, however, below the slit temperature T_o. The reduction of the surface temperature with x implies that the surface tension σ increases with the distance from the slit, i.e. the net surface tension on an interfacial fluid element is thus in the positive x-direction. Accordingly, the fluid layer just below the free surface is dragged along by the top layer due to viscous shear. This explains the characteristic velocity profiles in Fig. 2, which exhibit a local minimum inside the liquid film. In the mathematical formulation of the problem, this phenomenon is brought about by the interfacial boundary condition (21). The dimensionless shear stress $f''(\beta)$ increases gradually with M until the reduction of $\theta(\beta)$ tends to partially outweigh further increase in M. For M-values above 0.5, the interfacial shear stress is only modestly affected by M, whereas the film is substantially thickened.

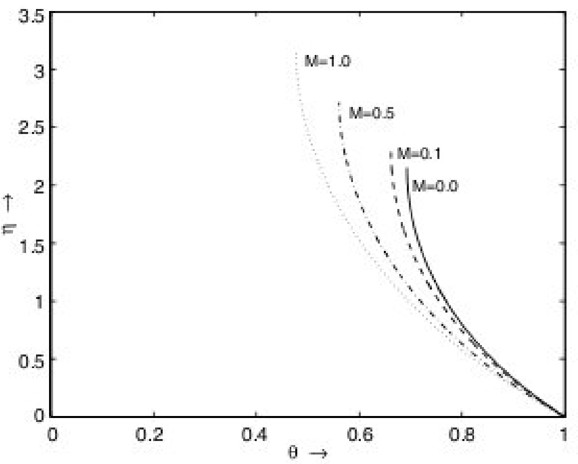

Fig. 3. Similarity temperature profiles $\theta(\eta)$ for $S=0.8$ and $Pr=0.1$ for different values of the thermocapillarity number M.

The primary characteristics of the hydrodynamic part of the problem, i.e. the dimensionless film thickness β, the surface velocity $f'(\beta)$ and the sheet shear stress $f''(0)$, are presented in Figs. 4 and 5. These results are obtained for $Pr=0.1$ and it should be recalled that the influence of the Prandtl number on the hydrodynamics is only indirect via the free-surface boundary condition (21). It is readily observed from Figs. 4 and 5 that the film thickness and the surface velocity increase with M for all values of the unsteadiness parameter S. The particular value S_o of S, above which no solutions could be obtained, corresponds to an infinitely thin film ($\beta\to 0$). In the absence of thermocapillarity effects (i.e. $M=0$), Wang [3] found $S_o=2.0$ and the accompanying surface velocity $f'(\beta)=1$. The present results show a modest increase in S_o with M and, moreover, that the surface velocity $f'(\beta)$ exceeds unity. This excess velocity can obviously be ascribed to the action of thermocapillary forces. The variation with S of the velocity gradient $f''(0)$ at the stretching sheet in Fig. 5 shows essentially the same trend in the presence of thermocapillarity as reported by Wang [3] for $M=0$. For small values of S, i.e. thick films, thermocapillarity exhibits a negligible effect on the friction between the liquid and the elastic sheet. Even for $S=0.8$ the slopes of the different velocity profiles in Fig. 2 at $\eta=0$ are scarcely discernible. When S is about unity, the magnitude of $f''(0)$ appears to decrease slightly in the presence of thermocapillarity, obviously due to the thickening of the liquid layer with M. However, the different curves in Fig. 5 inevitably intersect at somewhat higher S-values since the upper bound S_o, beyond which no solution exists, increases with M.

The primary thermal characteristics are presented in Fig. 6 over a wide range of Prandtl numbers and for the same two representative values of the unsteadiness parameter S as considered in [2]. In the absence of thermocapillarity effects, Andersson et al. [2] argued that at sufficiently high Prandtl numbers, i.e. $Pr \gtrsim 10$, the thermal boundary layer is confined within the liquid film. Thus, θ decreases monotonically from 1 to 0 across this boundary layer and remains zero in the upper part of the film. In the isothermal surface layer the temperature equals the slit temperature T_o. In this high-Prandtl-number regime, the local heat transfer rate $-\theta'(0)$ at the

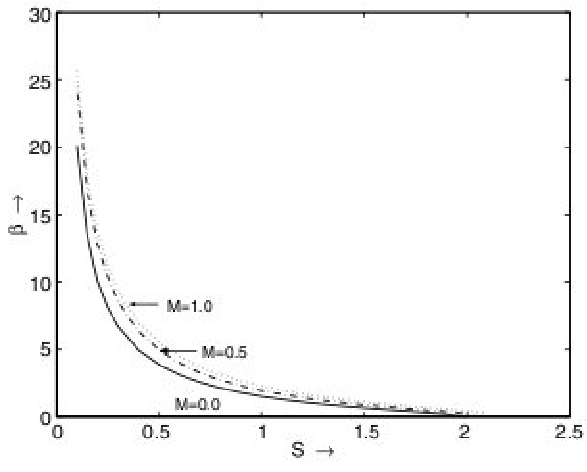

Fig. 4. Film thickness β versus unsteadiness parameter S for $Pr = 0.1$.

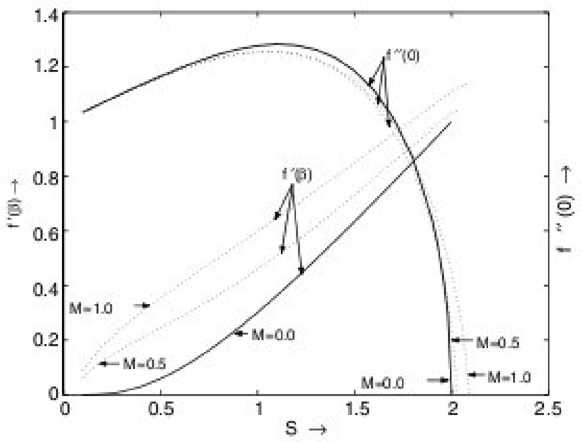

Fig. 5. Free-surface velocity $f'(\beta)$ and sheet shear stress $-f''(0)$ versus unsteadiness parameter S for $Pr = 0.1$.

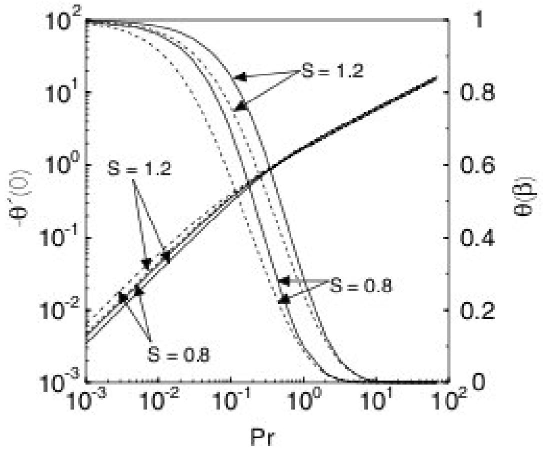

Fig. 6. Dimensionless heat flux $-\theta'(0)$ at the sheet and dimensionless surface temperature $\theta(\beta)$ versus Prandtl number for $M = 0$ (solid lines) and for $M = 0.5$ (broken lines) for two different S-values.

stretching sheet is controlled by the velocity in the immediate vicinity of the sheet, the latter which is practically unaffected by thermocapillarity.

For Prandtl numbers of the order of unity and below, on the other hand, the temperature gradients extend all the way to the free surface and the surface temperature $\theta(\beta)$ attains a finite value below 1; see Fig. 6 and the sample temperature profiles in Fig. 3 for $Pr = 0.1$. It is readily seen from Fig. 6 that thermocapillarity tends to reduce $\theta(\beta)$ for a given Prandtl number, obviously due to the thickening of the liquid film caused by the thermocapillarity forces. Andersson et al. [2] argued that the observed reduction of the heat flux $-\theta'(0)$ with S in the high-diffusivity (low Pr) limit is a result of the film thickness β being a rapidly decaying function of S. Now, since a striking effect of increasing thermocapillarity is the thickening of the liquid film, cf. Fig. 4, the enhanced heat flux for $M = 0.5$ in Fig. 6 is also ascribed to the thickening of the film.

5. Concluding remarks

So far it has been assumed that the reference temperature T_{ref} in Eq. (2) is positive, or more specifically that $0 < T_{ref} \leqslant T_o$. Likewise, γ in Eq. (4) was taken as a positive constant. However, the similarity analysis is equally valid if either T_{ref} or γ or both are negative. $T_{ref} < 0$ implies that the sheet temperature T_s increases with the distance from the slit, whereas $\gamma < 0$ represents a liquid in which the surface tension σ increases with temperature. If either T_{ref} or γ is negative, the thermocapillarity number M defined in Eq. (24) becomes negative, too, and further computations are needed. On the other hand, if both $T_{ref} < 0$ and $\gamma < 0$, $M > 0$ and the numerical results presented in §4.2 still apply, although the physical interpretations are different.

In this paper it has been demonstrated that the similarity transformation devised by Andersson et al. [2] for heat transfer in a liquid film on an unsteady stretching surface in the absence of thermocapillary effects also applies when thermally-induced variations of the surface tension are taken into consideration. Numerical solutions of the resulting ODEs reveal a number of important phenomena. First of all, the thermally-induced variation of the surface tension gives rise to a net force in the direction of the stretching sheet. Thus, the shear-driven motion of the fluid adjacent to the free surface and in the vicinity of the stretching sheet results in a local velocity minimum in the interior of the film. The surface velocity may even exceed the sheet velocity in some cases. The thermocapillarity influence also tends to thicken the film and increase the rate of heat transfer between the sheet and the film. At high Prandtl numbers, i.e. $Pr \gtrsim 10$, the thermal boundary layer is confined to the lower part of the liquid film whereas the temperature in the isothermal surface layer remains equal to the slit temperature T_o. The surface tension is therefore constant and no thermocapillary effects are present at high Pr.

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