MAXWELL-CHERN-SIMONS THEORY IS FREE FOR MARGINALLY NONCOMMUTATIVE SPACETIMES

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Abstract:

We have conclusively established the duality between noncommutative Maxwell-Chern-Simons theory and Self-Dual model, the latter in ordinary spacetime, to the first non-trivial order in the noncommutativity parameter $\theta^{\mu\nu}$, with $\theta^{0i} = 0$. This shows that the former theory is free for marginally noncommutative spacetimes. A θ-generalized covariant mapping between the variables of the two models in question has been derived explicitly, that converts one model to the other, including the symplectic structure and action.

Key Words: Noncommutative gauge theory, Seiberg-Witten map, Duality, Maxwell-Chern-Simons theory, Self-Dual model.

In this Letter we provide an example of a Non-Commutative (NC) *free* field theory in 2+1-dimensions - the NC abelian Maxwell-Chern-Simons (MCS) theory. Our analysis is perturbative in $\theta^{\mu\nu}$ - the noncommutativity parameter - to the first nontrivial order. Hence the result is valid for spacetimes with small noncommutativity. This is a non-trivial result since the noncommutativity generates non-linear derivative type of interaction terms in the action. We show in a conclusive way that NC MCS theory is dual to the abelian Self-Dual (SD) model (in *ordinary* spacetime). The latter model was shown to represent a free massive spin one excitation by Deser and Jackiw [1]. They also proved that SD model was dual to the well-known MCS (topologically massive gauge) theory [2][1]. The importance of the SD model was further enhanced when it was shown to appear in the bosonization [3] of the fermionic massive Thirring model in the large fermion mass limit. In a generic way, the planar gauge theories have played important roles in the context of physically interesting phenomena (that are effectively planar), such as quantum Hall effect, high-T_C superconductivity, to name a few and in anyon physics, where excitations having arbitrary spin and statistics appear.

We restrict ourselves to only spatial noncommutativity ($\theta^{0i} = 0$) and the results are valid to the first non-trivial order in $\theta^{\mu\nu}$ - the noncommutativity parameter, as defined below,

$$[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}. \tag{1}$$

The $*$-product is given by the Moyal-Weyl formula,

$$p(x) * q(x) = pq + \frac{i}{2}\theta^{\rho\sigma}\partial_\rho p \partial_\sigma q + \ O(\theta^2). \tag{2}$$

The reason for invoking $\theta^{0i} = 0$ is that space-time noncommutativity can induce higher order time derivatives leading to a loss of causality. Also, even to $O(\theta)$, it can alter the symplectic structure in a significant way, that might result in a non-perturbative change in the dynamics, which we want to avoid.

The study of NC quantum field theory has acquired a prominent place after the seminal work of Seiberg and Witten [4], who showed that NC manifolds emerge naturally in the context of D-branes on which an open string can terminate, (in the presence of a two-form background field). Field theories living on D-branes are essentially NC. The effects of noncommutativity can be systematically studied in a perturbative way by exploiting the Seiberg-Witten map [4], which converts an NC theory to a conventional theory in ordinary spacetime. The NC effects appear as interaction terms. Even though the basic field theoretic framework remains unaltered (for $\theta^{0i} = 0$), the noncommutativity induces a plethora of distinctive features, such as UV/IR mixing [5], presence of solitons in higher dimensional scalar theory [6], excitations of dipolar nature [7], to name a few. Thus one is interested to know how the noncommutativity affects established properties of conventional field theories and one such area is the duality (or equivalence) between field theories - in particular the MCS-SD duality [1], the case of present interest. It has been shown [8] that the NC Chern-Simons theory is free. However, in our approach of exploiting the Seiberg-Witten map, this result is expected since under the above mapping, the NC Chern-Simons theory reduces to comutative Chern-Simons theory to all orders of θ and hence the results corresponding to commutative Chern-Simons theory should hold.

[1] It is important to note that the effect of the Chern-Simons term is not perturbative in nature. In its absence, the 2+1-dimensional pure Maxwell theory describes a free massless spin zero excitation [2].

It is worthwhile to point out the rationale of restricting the analysis to $O(\theta)$ only in the present context. Ever since the advent of noncommutativity feature in spacetime, $O(\theta)$ results in noncommutative field theory have played a significant role, primarily because in many cases, $O(\theta)$ modifications tend to depart from commutative spacetime results in a nontrivial way. As we are going to study the effects of NC spacetime on an already established result in ordinary spacetime, (*i.e.* the MCS and Self-Dual model equivalence [2]), it is indeed logical to look for modifications in the leading order in θ. This is quite in keeping with the spirit of our work [9, 10] where effects of noncommutativity are looked at in the context of $CP(1)$ solitons [11] and in solitons in the Chern-Simons-Higgs system [12] respectivly. On the other hand, as a contrasting example, one should consider the case of NC solitons in a scalar theory in [6], where a *large* θ limit has been pursued. This is natural since these latter NC solitons are completely new entities of NC spacetime, having no counterpart in the corresponding ordinary spacetime theory. Furthermore, our methodology relies heavily on the Seiberg-Witten map [4], which is free from any ambiguity up to $O(\theta)$.

Recently several papers have appeared in this context [13] [14] [15] [16] and the results have not always agreed. The primary reason for the ambiguity is that all the works have tried to establish the duality by way of comparing the actions and showing that they are related in some way [14] [15], such as via a "Master" Lagrangian [1]. In [16], two actions are termed as dual when one of them becomes equal to the other on the surface of the equations of motion. But in both of the above instances, it is not clear whether the two actions, satisfying *only* the above criteria of duality, share the same symplectic structure and subsequent dynamics.

We point out that no discussions on the symplectic structure of the variables (that dictates the dynamics) or an explicit mapping between the degrees of freedom of the two purported dual theories, (NC MCS and NC SD), have been attempted so far. By itself, relating the actions can not prove duality conclusively and should only be considered as a confirmatory test of duality, obtained in a more fundamental way, concerning the basic fields. The latter scheme obviously suggests the former relation at the level of actions. In fact it should be remembered that in the original work [1], the duality was first proved at the level of symplectic structures of MCS and SD models. Subsequently a mapping was provided which can bodily convert one model in to the other totally and only then a Master action was provided to corroborate the previous findings. We will precisely follow this route but will not attempt the last one - construction of the Master action.

Our metric is $g^{\mu\nu} = diag(1,-1,-1)$. The NC MCS model is defined in the following way [14],

$$\hat{\mathcal{L}}_{MCS} = \int d^3x[-\frac{1}{4}\hat{F}^{\mu\nu} * \hat{F}_{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}(\hat{A}_\mu * \partial_\nu \hat{A}_\lambda + \frac{2}{3}i\hat{A}_\mu * \hat{A}_\nu * \hat{A}_\lambda)], \tag{3}$$

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu.$$

We use the NC extension of the CS action derived in [17]. Utilizing the Seiberg-Witten map, to the lowest non-trivial order in θ,

$$\hat{A}_\mu = A_\mu + \theta^{\sigma\rho}A_\rho(\partial_\sigma A_\mu - \frac{1}{2}\partial_\mu A_\sigma)\ ;\ \hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma}(F_{\mu\rho}F_{\nu\sigma} - A_\rho\partial_\sigma F_{\mu\nu}), \tag{4}$$

we arrive at the following $O(\theta)$ modified form of the NC MCS theory, expressed in terms of ordinary spacetime variables

$$\hat{\mathcal{A}}_{MCS} = \int d^3x[-\frac{1}{4}(F^{\mu\nu}F_{\mu\nu} + 2\theta^{\rho\sigma}(F^{\mu}{}_{\rho}F^{\nu}{}_{\sigma}F_{\mu\nu} - \frac{1}{4}F_{\rho\sigma}F^{\mu\nu}F_{\mu\nu})) + \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda], \quad (5)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It should be remembered that under the Seiberg-Witten map, the NC CS term exactly reduces to the CS term in ordinary spacetime. In 2+1-dimensions, the action (5) is further simplified to,

$$\hat{\mathcal{A}}_{MCS} = \int d^3x[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{1}{8}\theta^{\rho\sigma}F_{\rho\sigma}F^{\mu\nu}F_{\mu\nu}], \quad (6)$$

and leads to the equation of motion,

$$\partial_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} + \frac{1}{4}\partial_\mu(\theta^{\mu\nu}F^2 + 2(\theta.F)F^{\mu\nu}) = 0. \quad (7)$$

As we are interested in the symplectic structure, we now move on to the Hamiltonian formulation of the above model and introduce the canonical momenta and Poisson brackets,

$$\Pi_\mu \equiv \frac{\delta\hat{\mathcal{L}}_{MCS}}{\delta\dot{A}^\mu} \; ; \; \{A^\mu(x,t), \Pi^\nu(y,t)\} = -g^{\mu\nu}\delta(x-y). \quad (8)$$

Here the non-trivial momenta are

$$\Pi_i = (1-\theta B)F^{0i} + \frac{m}{2}\epsilon^{ij}A^j. \quad (9)$$

Reverting to a non-covariant notation [1],

$$E^i = F^{i0}, \; B = -\epsilon^{ij}\partial^i A^j = -F^{12}$$

the momenta and Hamiltonian are obtained as,

$$\Pi_i = (1-\theta B)E_i - \frac{m}{2}\epsilon_{ij}A_j, \quad (10)$$

$$\begin{aligned}\hat{\mathcal{H}}_{MCS} \equiv \Pi_i\dot{A}^i - \hat{\mathcal{L}}_{MCS} &= \frac{1}{2}(1-\theta B)[(1-\theta B)^{-2}(\Pi^i + \frac{m}{2}\epsilon^{ij}A^j)^2 + B^2] \\ &+ A^0[\partial^i(\Pi^i + \frac{m}{2}\epsilon^{ij}A^j) + mB] \\ &= \frac{1}{2}(1-\theta B)(E^iE^i + B^2) + A^0\hat{\mathcal{G}}, \quad (11)\end{aligned}$$

where the Gauss law constraint appears as

$$\hat{\mathcal{G}} \equiv \partial^i[(1-\theta B)E^i] + mB \approx 0. \quad (12)$$

The relation (13) has been used to derive (11). Inverting the relation (9) to express the electric field in terms of phase space variables,

$$E^i = (1-\theta B)^{-1}(\Pi^i + \frac{m}{2}\epsilon^{ij}A^j) \approx (1+\theta B)(\Pi^i + \frac{m}{2}\epsilon^{ij}A^j), \quad (13)$$

it is straightforward to compute the following algebra among the electric and magnetic fields,

$$\{E^i(x), E^j(y)\} = m\epsilon^{ij}(1 + 2\theta B)\delta(x - y) - \theta[\epsilon^{kj}E^i(x) + \epsilon^{ki}E^j(y)]\partial^k_{(x)}\delta(x - y),$$

$$\{E^i(x), B(y)\} = \epsilon^{ij}[1 + \theta B(x)]\partial^j_{(x)}\delta(x - y) \; ; \;\; \{B(x), B(y)\} = 0. \tag{14}$$

Interestingly, similar to the $\theta = 0$ theory [1], there exists a free field representation of E^i and B in terms of (φ, π) obeying $\{\varphi(x), \pi(y)\} = \delta(x - y)$,

$$B \equiv \sqrt{-\nabla^2}\varphi \; , \;\; E^i \equiv (1 + \theta B)(\epsilon^{ij}\hat{\partial}_j\pi - m\hat{\partial}_i\varphi), \tag{15}$$

that satisfies the algebra (14). The notations used are $\nabla^2 \equiv \partial^i\partial^i$, $\hat{\partial}^i \equiv \frac{\partial^i}{\sqrt{-\nabla^2}}$.

The backbone of our subsequent analysis is the crucial observation that a new set of variables $(\tilde{E}^i, \; \tilde{B})$ can be introduced,

$$\tilde{E}^i \equiv (1 - \theta B)E^i \; , \;\; \tilde{B} \equiv B, \tag{16}$$

that obeys the $\theta = 0$ algebra,

$$\{\tilde{E}^i(x), \tilde{E}^j(y)\} = m\epsilon^{ij}\delta(x - y),$$

$$\{\tilde{E}^i(x), \tilde{B}(y)\} = \epsilon^{ij}\partial^j_{(x)}\delta(x - y) \; ; \;\; \{\tilde{B}(x), \tilde{B}(y)\} = 0. \tag{17}$$

Exploiting the inverse relations,

$$E^i = (1 - \theta B)^{-1}\tilde{E}^i \approx (1 + \theta\tilde{B})\tilde{E}^i \; ; \;\; B = \tilde{B}, \tag{18}$$

the Gauss law constraint and the Hamiltonian is rewritten below,

$$\partial^i\tilde{E}^i + m\tilde{B} \approx 0, \tag{19}$$

$$\mathcal{H}_{MCS} = \frac{1}{2}[\tilde{E}^i\tilde{E}^i + \tilde{B}^2 + \theta\tilde{B}(\tilde{E}^i\tilde{E}^i - \tilde{B}^2)]. \tag{20}$$

Since the $(\tilde{E}^i, \; \tilde{B})$ algebra has become θ-independent, we can use the ordinary spacetime free field representation,

$$\tilde{B} \equiv \sqrt{-\nabla^2}\varphi \; , \;\; \tilde{E}^i \equiv \epsilon^{ij}\hat{\partial}_j\pi - m\hat{\partial}_i\varphi. \tag{21}$$

Indeed, the theory appears to be far from being free since the θ-contribution in the Hamiltonian (20) has apparently turned the theory in to a non-local one. This is because unlike the $\theta = 0$ part, which is quadratic, the θ-term is of higher order and the non-local operators involved in the free field representation (21) can not be shifted around, even under the integral. However, we now show that to $O(\theta)$, this theory can be identified to the abelian Self Dual theory *in ordinary spacetime* by means of a Lorentz covariant mapping of the degrees of freedom. This constitutes our main result. The SD theory was solved long time ago [1]. It represents a single free massive spin one mode.

The all important mapping between NC MCS variables $(\tilde{E}^i, \tilde{B})$ and SD variables (f^μ) is

$$\tilde{E}^i \equiv \epsilon^{ij}f^j \; , \;\; \tilde{B} \equiv -(f^0 + \theta X), \tag{22}$$

which has the covariant structure,

$$\frac{1}{2}\epsilon^{\mu\nu\lambda}\tilde{F}_{\nu\lambda} \equiv f^\mu + \frac{1}{2}\epsilon^{\mu\nu\lambda}\theta_{\nu\lambda}X. \tag{23}$$

X is an as yet unknown scalar variable. Note that the $\theta = 0$ mapping was first given in [1]. The NC extension of the map is such that *only* the identification between the non-dynamical time-components of the respective vector fields is affected.

We now directly exploit (22) to express $\mathcal{H}_{MCS}$ in (20) in terms of f^μ variables,

$$\mathcal{H}_f = \frac{1}{2}[f^i f^i + f^0 f^0 + 2\theta f^0 X + \theta f^0 f^\sigma f_\sigma]. \tag{24}$$

This is trivially obtained from the first order Lagrangian,

$$\mathcal{L}_f = \frac{1}{2} f^\mu f_\mu - \frac{1}{2m}\epsilon^{\mu\nu\lambda} f_\mu \partial_\nu f_\lambda - \frac{1}{4}\epsilon^{\mu\nu\lambda}\theta_{\nu\lambda} f_\mu f^\sigma f_\sigma - \frac{1}{2}\epsilon^{\mu\nu\lambda}\theta_{\nu\lambda} f_\mu X. \tag{25}$$

The θ-term in (25) can be removed by putting $X = -\frac{1}{2} f^\sigma f_\sigma$ and we are left with the abelian Self Dual Model in ordinary spacetime,

$$\mathcal{L}_{SD} = \frac{1}{2} f^\mu f_\mu - \frac{1}{2m}\epsilon^{\mu\nu\lambda} f_\mu \partial_\nu f_\lambda. \tag{26}$$

This is our cherished result.

Obviously this is the most economical form of the dual theory. Some amount of non-uniqueness creeps in through a non-vanishing X. However, as we have emphasized, X has to be such that it does not vitiate the symplectic structure between f^i, the independent dynamical degrees of freedom.

Some comments of the related recent works in the perspective of the present analysis is in order. The bone of contention happens to be the NC generalization of the SD model. We have argued before (Ghosh in [13], [18], [19]) that since the SD theory is a quadratic theory with no gauge invariance, its natural extension in the NC regime should be the same as the original theory. This has also been demonstrated in the context of NC Soldering phenomena [19] (see [20] for reviews on Soldering formalism). This idea is echoed in the present work as well.

Next we come to the work in [15] where the duality between NC MCS and NC SD model was studied from the Master action point of view, where the NC SD model contains the NC Chern-Simons term. At first sight it seems that this observation can be accommodated in our analysis, since for $\theta^{0i} = 0$, the extra θ-dependant three f^μ term $\epsilon^{\mu\nu\lambda}\theta^{\alpha\beta} f_\mu \partial_\alpha f_\nu \partial_\beta f_\lambda$ in the NC SD action of [15] reduces to $\epsilon^{ij}\theta^{kl} f_0 \partial_k f_i \partial_l f_j$ modulo total derivatives. Note that this term will not affect the symplectic structure of the SD model but will modify the constraint connecting f^0 to f^i. But the problem is that this action can not be generated from NC MCS theory by any covariant mapping between F^μ variables (of NC MCS) and f^μ variables (of NC SD) as in (23), without modifying the $\{f^i, f^j\}$ symplectic structure, that governs the dynamics. The speciality of the mapping (23) is that it keeps the $F^i \Leftrightarrow f^i$ identification unaltered. In this sense the duality derived in [15] is weaker because the covariance in the mapping will be lost.

On the other hand, [16] has started from the NC SD model (with the NC Chern-Simons term) and has obtained a dual theory which differs from the NC MCS theory. It will be interesting to redo the analysis along the lines demonstrated here taking the particular version of NC SD theory in [16] as the starting point.

As a final remark, we mention that our analysis of the duality relation is classical in nature and so we were able to exploit the classical result that the NC CS theory is mapped to the ordinary CS theory via the Seiberg-Witten map. Obviously, the non-perturbative effects of θ

leading to the quantization of the level of the NC CS model [21] in the quantum theory can not be addressed in the perturbative framework of the Seiberg-Witten map. In fact, even in a perturbative computational scheme, recently it has been shown [22] that mapping between NC and ordinary CS theories as quantum theories, requires a modification in the Seiberg-Witten map itself, where quantum corrections are to be incorporated. As an interesting future problem, one might study the quantum equivalence between the NC and ordinary Self-Dual models along the lines of [22] since in this case also, both ordinary and NC forms of the actions are known exactly. This is essential for the perturbative analysis [23]. This will also establish the duality between NC MCS and NC Self-Dual models in their quantized version, thus generalizing the $O(\theta)$ classical result presented here.

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