Exact solutions of cylindrical and spherical dust ion acoustic waves

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Cylindrical and spherical modified Korteweg–de Vries (KdV) equations are derived for dust ion acoustic waves. It is shown that a suitable coordinate transformation reduces the cylindrical KdV equation into the ordinary KdV equation which can be solved analytically. A completely different analytical solution is obtained using the group analysis method. However, for cylindrical and spherical modified KdV equations group analysis method yields trivial analytical solutions. Numerically, solutions to these modified KdV equations are obtained assuming initial profiles similar to those in one-dimensional soliton solutions.

Nonlinear waves in dusty plasma have attracted a great deal of interest in recent years.[1–7] Dusty plasma is a fully or partially ionized electron–ion plasma containing charged micrometer sized dust particles. Dusty plasma can be found in many parts of our cosmic environment (for a detailed survey, see Ref. 8). Dusty plasma supports two types of acoustic modes: high frequency ion acoustic mode involving mobile ions and static grains, and a low frequency dust acoustic wave involving mobile dust grains. Both of these modes have been observed in experiments.[9,10] One particular field of study which has received a lot of attention is that of solitary waves and shock waves. Usually, the reduction perturbation technique (RPT) is applied to study solitary waves (both ion acoustic and dust acoustic solitary waves) and RPT (Refs. 11–14) gives rise to the famous Korteweg–de Vries (KdV) or modified KdV (MKdV) equation, which, have been studied extensively. However, most of the studies are limited to one-dimensional geometry. If the geometry is extended to two and three dimensions and one assumes radial symmetry then RPT would derive, respectively, cylindrical and spherical KdV equations. Recently Mamun and Shukla[15,16] derived the cylindrical and spherical KdV equations for dust acoustic waves and dust ion acoustic waves. However, in the case of, say, dust ion acoustic waves, the KdV equation fails to describe solitary waves, when μ, the ratio of electron and ion densities (initial) takes a particular value. For this one has to consider cylindrical and spherical MKdV equations. The motivation for writing the present note is twofold. First we would derive the cylindrical and spherical MKdV equations. Second we would show that contrary to the observations made in Ref. 16 exact solutions exist for both cylindrical and spherical KdV equations. In these cases analytical solutions can be obtained by the group analysis of the generalized KdV equation, first discussed by Zakharov and Korobeinikov.[17] Earlier Hirota[18] showed how cylindrical KdV equation can be transformed into the usual KdV equation by coordinate transformations. However, Hirota's method cannot be applied for spherical KdV and cylindrical or spherical MKdV equations. But the group analysis method gives analytical solutions for the spherical KdV equation though it does not give any nontrivial solutions for the MKdV equations. For the MKdV equations we use a numerical method to find solitary wave solutions.

To derive the cylindrical and spherical KdV equations we consider a nonplanar cylindrical or spherical geometry and study the nonlinear propagation of dust ion acoustic solitary (DIAS) waves in an unmagnetized dusty plasma whose constituents are inertial ions, Boltzmann electrons and stationary dust particles.

The nonlinear dynamics of the DIAS waves whose phase speed is much smaller (larger) than the electron (ion) thermal speed (in nonplanar cylindrical and spherical geometries) is governed by

$$\frac{\partial n_i}{\partial t}+\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu n_i u_i)=0, \tag{1}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial r}=-\frac{\partial \phi}{\partial r}, \tag{2}$$

$$\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\phi}{\partial r}\right)=\mu\exp(\phi)-n_i+(1-\mu), \tag{3}$$

where $\nu=0$, for one-dimensional geometry and $\nu=1, 2$ for cylindrical and spherical geometry, respectively, n_i is the ion number density normalized to its equilibrium value n_{i0}, u_i is the ion fluid speed normalized to $C_i=(K_BT_e/m_i)^{1/2}$, and ϕ

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is the electrostatic wave potential normalized by $K_B T_e/e$. The time and space variables are in units of the ion plasma period $\omega_{Pi}^{-1}=(m_i/4\pi n_{i0}e^2)^{1/2}$ and the Debye radius $\lambda_{Dm}=(K_B T_e/4\pi n_{i0}e^2)^{1/2}$, respectively. We have denoted $\mu=n_{e0}/n_{i0}$.

The stretched coordinates

$$\xi=-\epsilon^{1/2}(r+v_0 t) \quad \text{and} \quad \tau=\epsilon^{3/2}t \tag{4}$$

are used to derive cylindrical and spherical KdV equations. The KdV type equation derived by using the stretched coordinates (4) is given by[16]

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{\nu}{2\tau}\phi^{(1)}+A\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+B\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \tag{5}$$

where

$$A=\frac{\sqrt{\mu}}{2}\left(3-\frac{1}{\mu}\right) \quad \text{and} \quad B=\frac{1}{2\mu^{3/2}}. \tag{6}$$

We see that for $\mu=\frac{1}{3}$ cylindrical and spherical KdV solutions collapse. One then considers cylindrical and spherical MKdV solutions.

For this we introduce the stretched coordinates, $\xi=\epsilon(r-v_0 t), \tau=\epsilon^3 t$ and expand n_i, u_i, and ϕ in a power series of ϵ as

$$n_i=1+\epsilon n_i^{(1)}+\epsilon^2 n_i^{(2)}+\cdots, \tag{7}$$

$$u_i=\epsilon u_i^{(1)}+\epsilon^2 u_i^{(2)}+\cdots, \tag{8}$$

$$\phi=\epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\cdots, \tag{9}$$

and develop equations in various powers of ϵ.

To lowest order in ϵ, Eqs. (1)–(3) gives $n_i^{(1)}=u_i^{(1)}/v_0$, $u_i^{(1)}=\phi^{(1)}/v_0$, and $v_0=1/\sqrt{\mu}$.

To next higher order in ϵ, we obtain a set of equations,

$$v_0 n_i^{(2)}=n_i^{(1)}u_i^{(1)}+u_i^{(2)}, \tag{10}$$

$$v_0 u_i^{(2)}-\tfrac{1}{2}(u_i^{(1)})^2=\phi^{(2)}, \tag{11}$$

$$\mu\phi^{(2)}+\frac{\mu}{2}(\phi^{(1)})^2=n_i^{(2)}. \tag{12}$$

From (10)–(12), we get

$$\phi^{(2)}\left(v_0\mu-\frac{1}{v_0}\right)+(\phi^{(1)})^2\left(\frac{\mu v_0}{2}-\frac{3}{2v_0^3}\right)=0. \tag{13}$$

If $v_0=1/\sqrt{\mu}$ and $\mu=\frac{1}{3}$, then Eq. (13) is an identity.

To next higher order in ϵ, we obtain a set of equations

$$\frac{\partial n_i^{(1)}}{\partial\tau}-v_0\frac{\partial n_i^{(3)}}{\partial\xi}+\frac{\partial}{\partial\xi}(u_i^{(3)}+n_i^{(1)}u_i^{(2)}+n_i^{(2)}u_i^{(1)})+\frac{\nu u_i^{(1)}}{v_0\tau}=0, \tag{14}$$

$$\frac{\partial u_i^{(1)}}{\partial\tau}-v_0\frac{\partial u_i^{(3)}}{\partial\xi}+u_i^{(1)}\frac{\partial u_i^{(2)}}{\partial\xi}+u_i^{(2)}\frac{\partial u_i^{(1)}}{\partial\xi}=-\frac{\partial\phi^{(3)}}{\partial\xi}, \tag{15}$$

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2}-\mu\phi^{(3)}-\mu\phi^{(1)}\phi^{(2)}-\frac{\mu}{6}(\phi^{(1)})^3+n_i^{(3)}=0. \tag{16}$$

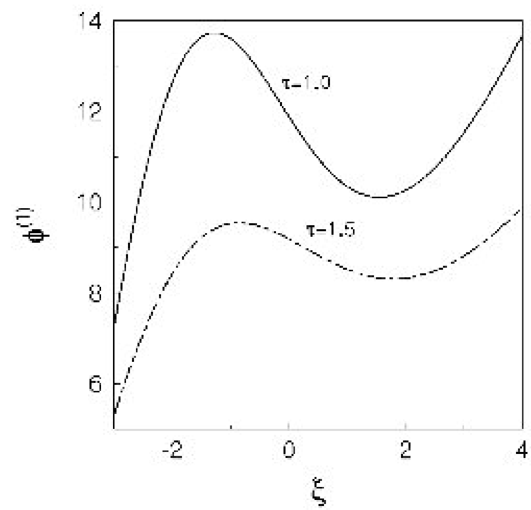

FIG. 1. Plot of $\phi^{(1)}$ vs ξ for the solution (21), where $\mu=0.4$ and $V=1$.

Combining Eqs. (14)–(16), we deduce a modified KdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{\nu}{2\tau}\phi^{(1)}+A_1(\phi^{(1)})^2\frac{\partial\phi^{(1)}}{\partial\xi}+B_1\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \tag{17}$$

where

$$A_1=\frac{1}{4\sqrt{\mu}}(15\mu^2-1), \quad B_1=\frac{1}{2\mu^{3/2}}. \tag{18}$$

To obtain an analytical solution of cylindrical KdV equation [Eq. (5) for $\nu=1$], let us first use Hirota's transformation[18] given by

$$\phi^{(1)}=\frac{v}{\tau}+\frac{\xi}{2A\tau}.$$

Under this transformation Eq. (5) transforms to

$$\frac{\partial v}{\partial\tau}+\frac{Av}{\tau}\frac{\partial v}{\partial\xi}+\frac{\xi}{2\tau}\frac{\partial v}{\partial\xi}+B\frac{\partial^3 v}{\partial\xi^3}=0. \tag{19}$$

Again we use a transformation given by

$$\tau'=-2\tau^{-1/2}, \quad \xi'=\xi\tau^{-1/2}.$$

Then Eq. (19) reduces to

$$\frac{\partial v}{\partial\tau'}+Av\frac{\partial v}{\partial\xi'}+B\frac{\partial^3 v}{\partial\xi'^3}=0. \tag{20}$$

Equation (20) is the usual KdV equation. So the solution of Eq. (20) is given by

$$v=\frac{3V}{A}\operatorname{sech}^2\left[\sqrt{\frac{V}{4B}}(\xi'-V\tau')\right],$$

assuming appropriate boundary conditions where V is the solitary wave velocity.

The exact solution of Eq. (5) is given by

$$\phi^{(1)}=\frac{1}{\tau}\left[\frac{\xi}{2A}+\frac{3V}{A}\operatorname{sech}^2\left(\sqrt{\frac{V}{4B\tau}}(\xi+2V)\right)\right]. \tag{21}$$

This solution is valid for $\tau>0$. In Fig. 1 solution (21) is plotted for two values of τ viz., $\tau=1.0$ and $\tau=1.5$ (the other parameters are $\nu=1$, $\mu=0.4$, $V=1$).

Another solution of the cylindrical KdV equation can be obtained by the group analysis of the generalized KdV equation, first discussed by Zakharov and Korobeinikov.[17] The details of Lie point symmetry, which gives rise to the invariants of any differential equations, are given in Ref. 19. For the sake of brevity we just quote the results essential to obtain the analytical solutions of the cylindrical KdV equation.

For the cylindrical KdV equation

$$\frac{\partial u}{\partial t}+\frac{u}{2t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3\phi}{\partial x^3}=0,$$

the generators or transform operators are given by

$$X_1=\frac{\partial}{\partial x},\quad X_2=3t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u},$$

$$X_3=2\sqrt{t}\frac{\partial}{\partial x}+\frac{1}{\sqrt{t}}\frac{\partial}{\partial u},$$

$$X_4=2x\sqrt{t}\frac{\partial}{\partial x}+4t\sqrt{t}\frac{\partial}{\partial t}+\left(\frac{x}{\sqrt{t}}-4u\sqrt{t}\right)\frac{\partial}{\partial u}.$$

$F(x,t)$ is an invariant of the equation iff $X_iF=0$ ($i=1,2,3,4$). For example, $X_4F=0$ gives the invariant $\xi=xt^{-1/2}$ and suggests the transformation

$$u(x,t)=\frac{\frac{x}{2}+U(\lambda)}{t},\quad \lambda=xt^{-1/2}.$$

So, for the cylindrical KdV Eq. (5) we make the following transformation:

$$\phi^{(1)}=\frac{\frac{\xi}{2A}+v(\eta)}{\tau},\quad \eta=\xi\tau^{-1/2}. \tag{22}$$

Then Eq. (5) reduces to

$$B\frac{d^3v}{d\eta^3}+Av\frac{dv}{d\eta}=0. \tag{23}$$

Integrating (23),

$$B\frac{d^2v}{d\eta^2}=c-\frac{Av^2}{2}, \tag{24}$$

where c is an integration constant. Equation (24) is of the Sagdeev-type and can be solved with the help of the so-called tanh method.[20] The solution turns out to be

$$v(\eta)=a_0+a_2\tanh^2\eta, \tag{25}$$

where

$$a_0=\frac{8B}{A}\ \text{and}\ a_2=-\frac{12B}{A}. \tag{26}$$

Hence,

$$\phi^{(1)}=\frac{\xi}{2A\tau}+\frac{1}{\tau}(a_0+a_2\tanh^2(\xi\tau^{-1/2})). \tag{27}$$

Another transformation $\phi^{(1)}=(\xi/2A\tau)+U(\tau)$, gives the solution as

$$\phi^{(1)}=\frac{\xi}{2A\tau}+\frac{c}{\tau}, \tag{28}$$

where c is an arbitrary constant.

For the spherical KdV equation,

$$\frac{\partial u}{\partial t}+\frac{u}{t}+u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}=0,$$

the generators are

$$X_1=\frac{\partial}{\partial x},\quad X_2=\ln t\frac{\partial}{\partial x}+\frac{1}{t}\frac{\partial}{\partial u},$$

$$X_3=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}.$$

Some of the nontrivial invariant solutions are

$$u_1=\frac{x}{t\ln t}+U(t),\quad u_2=\frac{x}{t}U(\lambda),\quad \lambda=xt^{-1/3},$$

$$u_3=t^{-2/3}U(\lambda)-\frac{1}{t},\quad \lambda=(x+\ln t+3)t^{-1/3}.$$

The solution u_1, when applied to the equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{\phi^{(1)}}{\tau}+A\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+B\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \tag{29}$$

gives the following solution:

$$\phi^{(1)}=\frac{\xi}{A\tau\ln\tau}+\frac{c}{\tau\ln\tau}. \tag{30}$$

The solution u_2, when applied to Eq. (29), gives the following equation:

$$B\lambda\frac{d^3U}{d\lambda^3}+3B\frac{d^2U}{d\lambda^2}+\lambda^2U\frac{dU}{d\lambda}-\frac{1}{3}\lambda^2\frac{dU}{d\lambda}+\lambda U^2=0. \tag{31}$$

Again the solution u_3, when applied to Eq. (29), gives the following equation:

$$B\frac{d^3U}{d\lambda^3}+U\frac{dU}{d\lambda}-\frac{\lambda}{3}\frac{dU}{d\lambda}+\frac{1}{3}U=0. \tag{32}$$

Analytical solutions of (31) and (32) are difficult to obtain.

For the cylindrical and spherical modified KdV equation

$$\frac{\partial u}{\partial t}+\frac{\nu}{2t}u+u^2\frac{\partial u}{\partial x}+\beta\frac{\partial^3u}{\partial x^3}=0, \tag{33}$$

the only generators are

$$X_1=\frac{\partial}{\partial x}\ \text{and}\ X_2=\frac{\partial}{\partial t}.$$

Only trivial analytical solutions like $u=ct^{-\nu/2}$ and $u=u(x)$ can be obtained. Next we discuss the numerical solutions of cylindrical and spherical modified KdV equations.

It may be noted that Eq. (17) has the following solitary wave solution for $\nu=0$:

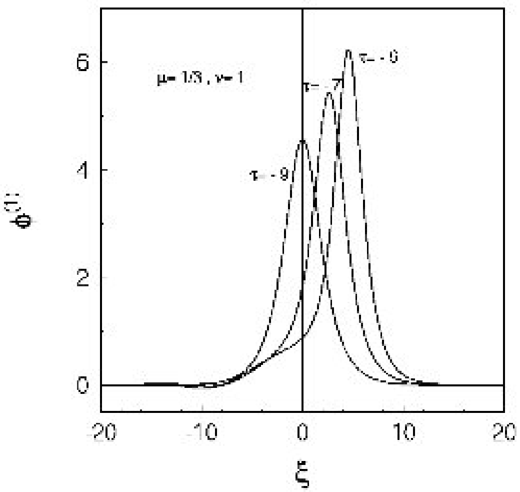

FIG. 2. Numerical solutions for cylindrical MKdV equation for different values of τ, where $\mu=1/3$.

$$\phi^{(1)}=\sqrt{\frac{6U}{A_1}}\,\mathrm{sech}\left[\sqrt{\frac{U}{B_1}}(\xi-U\tau)\right], \tag{34}$$

where U is the soliton velocity. With this initial profile at $\tau=-9$ we solve the cylindrical and spherical modified KdV equations. Figures 2 and 3 show, respectively, the solutions for $\nu=1$ and $\nu=2$ for several values of τ ranging from $\tau=-6$ to $\tau=-9$. It is seen that as magnitude of τ increases the solutions look like those for one-dimensional MKdV solitons. This is because the extra term $(\nu/2\tau)\phi^{(1)}$ becomes small for large values of τ and we get back the old solution. However, the singularities of the solution are not reflected in the initial profile. This can be guessed from the exact analytical solutions of cylindrical and spherical KdV equations. All the solutions have singularity at $\tau=0$.

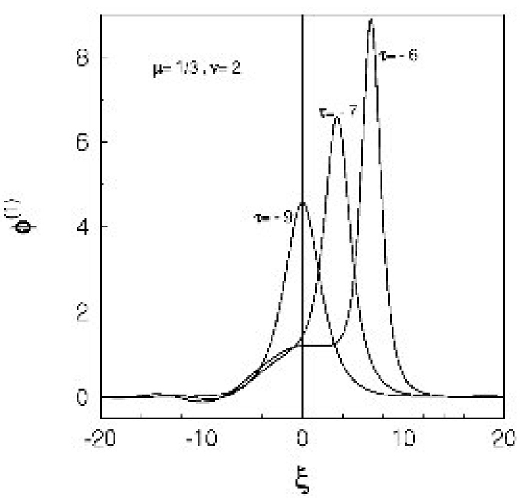

FIG. 3. Numerical solutions for spherical MKdV equation for different values of τ, where $\mu=1/3$.

To summarize, we have derived cylindrical and spherical MKdV equations for dust ion acoustic waves using the reductive perturbation technique. We have also found exact analytical solutions for cylindrical and spherical KdV equations for dust ion acoustic waves (previously derived by Mamun and Shukla[16]) using the group analysis method. Our solutions are also valid for spherical and cylindrical KdV equations for dust acoustic solitary waves derived by Mamun and Shukla.[15] For the cylindrical and spherical MKdV equations, however, the group analysis yields trivial results. Here a numerical method has been applied assuming a initial profile similar to the one-dimensional soliton solution. It is found that, as expected, for large values of τ the solution is similar to that of the one-dimensional MKdV equation.

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