

# On the Symmetries of the Differential Equation Governing Damped Harmonic Vibration

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*Dedicated to Prof.Dr. Constantin UDRIȘTE  
on the occasion of his sixtieth birthday*

## Abstract

The application of group theoretic method of Lie to ODE is discussed as it provides a unified view of the diverse and ad-hoc integration procedures of such equations. Following consequences of one-parameter point transformations, the symmetries of the differential equation describing damped harmonic vibration are determined. A particular case is considered to illustrate the generality of the approach.

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**Key words:** harmonic vibration, infinitesimal symmetries, group-invariant solutions

## 1 Introduction

The symmetries of differential equations are often found useful in finding their solutions. The knowledge about the underlying transformations and their generators are essential for this purpose. Let us consider the invertible point transformation of the  $x$ - $y$  plane

$$(1.1) \quad x = \phi(x, y, \gamma), \quad y = \psi(x, y, \gamma)$$

with the conditions

$$(1.2) \quad \phi|_{\gamma=0} = x \quad \text{and} \quad \psi|_{\gamma=0} = y$$

The point transformations are said to form a one-parameter group if successive action of two transformations is equivalent to the action of another transformation of the form (1.1) (Ibragimov [1]). Taylor series expansions with respect to  $\gamma$  in the neighbourhood of  $\gamma = 0$  of (1.1) yield the infinitesimal transformations

$$(1.3) \quad x \rightarrow x + \xi(x, y)\gamma, \quad y \rightarrow y + \eta(x, y)\gamma$$

where

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$$(1.4) \quad \xi(x, y) = \left. \frac{\partial \phi}{\partial \gamma} \right|_{\gamma=0}, \quad \eta(x, y) = \left. \frac{\partial \psi}{\partial \gamma} \right|_{\gamma=0}$$

A tangent vector field in terms of the first-order differential operator, is written as

$$(1.5) \quad X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

Lie [2] termed  $X$  as the infinitesimal generator of the transformation. The unknown functions  $\xi(x, y)$  and  $\eta(x, y)$  may be found out from the determining equation for Lie point symmetries.

For the second-order differential equation

$$(1.6) \quad y'' = f(x, y, y'); \quad y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dx^2}$$

the determining equation for the Lie point symmetries derived as (Ibragimov [1])

$$(1.7) \quad \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} - (\eta_y - 2\xi_x - 3y'\xi_y)f \\ - [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] f_{y'} - \xi f_x - \eta f_y = 0$$

where  $f(x, y, y')$  is a known function and

$$(1.8) \quad \xi_x = \frac{\partial \xi}{\partial x}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_x = \frac{\partial \eta}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y}, \quad \eta_{xx} = \frac{\partial^2 \eta}{\partial x^2}, \quad \xi_{xx} = \frac{\partial^2 \xi}{\partial x^2}, \quad \xi_{yy} = \frac{\partial^2 \xi}{\partial y^2}, \\ \xi_{xy} = \frac{\partial^2 \xi}{\partial x \partial y}, \quad \eta_{yy} = \frac{\partial^2 \eta}{\partial y^2}, \quad \eta_{xy} = \frac{\partial^2 \eta}{\partial x \partial y}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_x = \frac{\partial f}{\partial x}, \quad f_{y'} = \frac{\partial f}{\partial y'}$$

Treating the left-hand side of equation (1.7) as a polynomial of third degree in  $y'$ , we can split it into an overdetermined system of equations. The symmetries of the differential equation (1.6) are determined generally, by integrating the system. The principle discussed above is to be applied here to the dynamical equation governing the damped harmonic vibration.

## 2 Symmetries of the differential equation for damped harmonic vibration

If a particle be oscillating in a fluid (air or liquid) the equation of motion for its small displacement can be written as

$$(2.1) \quad \frac{d^2 y}{dt^2} - k \frac{dy}{dt} + \omega^2 y = 0$$

where  $k$  is defined as the fluid resistance or the damping effect to the motion of a particle of unit mass moving with unit velocity,  $y$  is the co-ordinate and  $\omega$  is the angular velocity.

Equation (2.1) is re-written in the form

$$(2.2) \quad y'' = f(t, y, y') = -ky' - \omega^2 y; \quad y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dx^2}$$

Let the infinitesimal generator in this case is

$$(2.3) \quad X = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}$$

$\xi = \xi(t, y)$ ,  $\eta = \eta(t, y)$ , and the determining equation for the Lie symmetry, is given by

$$(2.4) \quad \begin{aligned} & \eta_{tt} + (2\eta_{ty} - \xi_{tt})y' + (\eta_{yy} - 2\xi_{ty})y'^2 - y'^3 \xi_{yy} + (\eta_y - 2\xi_t - 3y'\xi_y)f \\ & - [\eta_t + (\eta_y - \xi_t)y' - y'^2 \xi_y] f_{y'} - \xi f_t - \eta f_y = 0 \end{aligned}$$

where

$$\begin{aligned} \xi_t &= \frac{\partial \xi}{\partial t}, \quad \xi_y = \frac{\partial \xi}{\partial y}, \quad \eta_t = \frac{\partial \eta}{\partial t}, \quad \eta_y = \frac{\partial \eta}{\partial y}, \quad \eta_{tt} = \frac{\partial^2 \eta}{\partial t^2}, \quad \xi_{tt} = \frac{\partial^2 \xi}{\partial t^2}, \quad \xi_{yy} = \frac{\partial^2 \xi}{\partial y^2}, \\ \xi_{ty} &= \frac{\partial^2 \xi}{\partial t \partial y}, \quad \eta_{yy} = \frac{\partial^2 \eta}{\partial y^2}, \quad \eta_{ty} = \frac{\partial^2 \eta}{\partial t \partial y}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_t = \frac{\partial f}{\partial t}, \quad f_{y'} = \frac{\partial f}{\partial y'} \end{aligned}$$

Treating the left-hand side of equation (2.4) as a polynomial in  $y'$ , we can split it up into the following system of equations :

$$(2.5) \quad y'^3 : \xi_{yy} = 0$$

$$(2.6) \quad y'^2 : \eta_{yy} - 2\xi_{ty} + 2k\xi_y = 0$$

$$(2.7) \quad y' : 2\eta_{ty} - \xi_{tt} + k\xi_t + 3\omega^2 y \xi_y = 0$$

$$(2.8) \quad y'^0 : \eta_{tt} - \omega^2 y(\eta_y - 2\xi_t) - \eta_t - k\eta_t = 0$$

Solution of (2.5) can be written as

$$(2.9) \quad \xi = p(t)y + a(t)$$

In view of (2.9), equation (2.6) is transformed to

$$(2.10) \quad \eta_{yy} - p_t + 2kp = 0; \quad p_t = \frac{dp}{dt}$$

Solving (2.10), we obtain  $\eta$  as

$$(2.11) \quad \eta = (p_t - kp)y^2 + b(t)y + c(t)$$

Taking (2.9) and (2.11) into account in equation (2.7), we obtain

$$(2.12) \quad 3y (p_{tt} - kp_t + \omega^2 p) - (a_{tt} - ka_t - 2b_t) = 0$$

Since left-hand side of (2.12) can be treated as a simple polynomial in  $y$ , we can write

$$(2.13) \quad p_{tt} - kp_t + \omega^2 p = 0$$

$$(2.14) \quad a_{tt} - ka_t - 2b_t = 0$$

where

$$a_t = \frac{\partial a}{\partial t}, \quad p_{tt} = \frac{\partial^2 p}{\partial t^2}, \quad a_{tt} = \frac{\partial^2 a}{\partial t^2}, \quad b_t = \frac{\partial b}{\partial t}$$

Two independent solutions of equation (2.13), are given by

$$(2.15) \quad p_1 = A_1 e^{\frac{\omega + \sqrt{\omega^2 - k^2}}{2} t}$$

$$(2.16) \quad p_2 = A_2 e^{\frac{\omega - \sqrt{\omega^2 - k^2}}{2} t}$$

where  $A_1, A_2$  are constants, and the general solution can be written as

$$(2.17) \quad p = A_1 e^{\frac{\omega + \sqrt{\omega^2 - k^2}}{2} t} + A_2 e^{\frac{\omega - \sqrt{\omega^2 - k^2}}{2} t}$$

Integrating (2.14), we obtain

$$(2.18) \quad a(t) = e^{kt} \int e^{-kt} [2b(t) + a_1] dt - a_2$$

where  $a_1$  and  $a_2$  are constants of integration.

Lastly, taking the relations (2.9) and (2.11) into account in equation (2.8), we obtain after some calculations

$$(2.19) \quad y^2 [p_{ttt} - p_t (\omega^2 - k^2) + kp\omega^2] - y (b_{tt} + 2\omega^2 a_t + kb_t) + (c_{tt} + c\omega^2 + kc_t) = 0$$

Considering (2.19) as a polynomial in  $y$ , we can write

$$(2.20) \quad p_{ttt} - p_t (\omega^2 - k^2) + kp\omega^2 = 0$$

$$(2.21) \quad b_{tt} + 2\omega^2 a_t + kb_t = 0$$

$$(2.22) \quad c_{tt} + c\omega^2 + kc_t = 0$$

where

$$p_{ttt} = \frac{d^3 p}{dt^3}$$

Equation (2.20) is automatically satisfied since it can be rearranged as

$$(2.23) \quad \frac{d}{dt} (p\dot{u} - kp\dot{t} + p\omega^2) + k (p\dot{u} - kp\dot{t} - p\omega^2) = 0$$

and  $p\dot{u} - kp\dot{t} + p\omega^2 = 0$  by virtue of equation (2.13).

Solution of (2.21), is given by

$$(2.24) \quad b(t) = e^{-kt} \int [-2\omega^2 a(t) + b_1] e^{kt} dt + b_2$$

where  $b_1$  and  $b_2$  are constants of integration.

Two independent solutions of equation (2.22), are given

$$(2.25) \quad c_1 = C_1 e^{\frac{-k + \sqrt{k^2 - 4\omega^2}}{2} t}$$

$$(2.26) \quad c_2 = C_2 e^{\frac{-k - \sqrt{k^2 - 4\omega^2}}{2} t}$$

where  $C_1, C_2$  are constants. The general solution of equation (2.22) can therefore be written as

$$(2.27) \quad c = C_1 e^{\frac{-k + \sqrt{k^2 - 4\omega^2}}{2} t} + C_2 e^{\frac{-k - \sqrt{k^2 - 4\omega^2}}{2} t}$$

Thus with the symmetry conditions prescribed through the relations (2.5) – (2.8), the generator  $X$  is given by

$$(2.28) \quad X = [p(t)y + a(t)] \frac{\partial}{\partial t} + [(p\dot{t} - kp)y^2 + b(t)y + c(t)] \frac{\partial}{\partial y}$$

where  $p(t)$  and  $c(t)$  are respectively, given by the relations (2.17) and (2.27).  $a(t)$  and  $b(t)$  are determined by the coupled relations (2.18) and (2.24).

It is to be noticed that the generator (2.28) exhibits a case of eight-parameter symmetry of the differential equation (2.1) as it involves exactly eight parametric constants  $A_1, A_2, C_1, C_2, a_1, a_2, b_1$  and  $b_2$ . Stephani [3] has shown that the maximum number of symmetries admitted by a second-order ODE is also eight.

### 3 A particular case

We attempt now for some simple solutions of the problem by taking  $a = b = 0$  and considering the independent solutions for  $p$  and  $c$  as given, respectively by (2.15), (2.16) and (2.25), (2.26). Equations (2.9) and (2.11) reduce simply to

$$(3.1) \quad \xi = p(t)y \quad \text{and} \quad \eta = (p\dot{t} - kp)y^2 - c(t)$$

For the sake of brevity, we proceed by writing  $p = Ae^{\alpha t}$  and  $c = Ce^{\beta t}$ . Thus,  $\xi$  and  $\eta$  can be written as

$$(3.2) \quad \xi = Ae^{\alpha t} \quad \text{and} \quad \eta = Ae^{\alpha t}(\alpha - k)y^2 + Ce^{\beta t}$$

Substituting (3.2) for  $\xi$  and  $\eta$  in the characteristic equation (Olver [4])  $b_1$

$$(3.3) \quad \frac{dt}{\xi} = \frac{dy}{\eta}$$

we obtain

$$(3.4) \quad \frac{dt}{y} = \frac{dy}{(\alpha - k)y^2 - Be^{(\beta - \alpha)t}}; \quad B = \frac{C}{A}$$

Solution of (3.4), is given by

$$(3.5) \quad y^2 = \frac{2B}{(\beta - 3\alpha + 2k)} e^{(\beta - \alpha)t} + \text{constant} \cdot e^{-2(k - \alpha)t}$$

Employing (3.5), independent solutions for  $y$  corresponding to paired values of  $\alpha$  and  $\beta$  possible are written as

(i)

$$(3.6) \quad \left[ \begin{array}{l} \alpha = e^{\frac{k - \sqrt{k^2 - 4\omega^2}}{2}}, \beta = e^{\frac{-k - \sqrt{k^2 - 4\omega^2}}{2}}, B = \frac{C_1}{A_1}; \\ y = \left\{ \frac{2B}{-\sqrt{k^2 - 4\omega^2}} e^{-kt} - \text{constant} \cdot e^{-kt} \cdot e^{\sqrt{k^2 - 4\omega^2} t} \right\}^{\frac{1}{2}} \end{array} \right.$$

(ii)

$$(3.7) \quad \left[ \begin{array}{l} \alpha = e^{\frac{k - \sqrt{k^2 - 4\omega^2}}{2}}, \beta = e^{\frac{-k - \sqrt{k^2 - 4\omega^2}}{2}}, B = \frac{C_2}{A_1}; \\ y = \left\{ \frac{2B}{-2\sqrt{k^2 - 4\omega^2}} e^{(k + \sqrt{k^2 - 4\omega^2}) t} + \text{constant} \cdot e^{-kt} \cdot e^{\sqrt{k^2 - 4\omega^2} t} \right\}^{\frac{1}{2}} \end{array} \right.$$

(iii)

$$(3.8) \quad \left[ \begin{array}{l} \alpha = e^{\frac{k - \sqrt{k^2 - 4\omega^2}}{2}}, \beta = e^{\frac{-k + \sqrt{k^2 - 4\omega^2}}{2}}, B = \frac{C_1}{A_2}; \\ y = \left\{ \frac{2B}{2\sqrt{k^2 - 4\omega^2}} e^{(k + \sqrt{k^2 - 4\omega^2}) t} + \text{constant} \cdot e^{-kt} \cdot e^{\sqrt{k^2 - 4\omega^2} t} \right\}^{\frac{1}{2}} \end{array} \right.$$

(iv)

$$(3.9) \quad \left[ \begin{array}{l} \alpha = e^{\frac{k - \sqrt{k^2 - 4\omega^2}}{2}}, \beta = e^{\frac{-k - \sqrt{k^2 - 4\omega^2}}{2}}, B = \frac{C_2}{A_2}; \\ y = \left\{ \frac{2B}{\sqrt{k^2 - 4\omega^2}} e^{-kt} + \text{constant} \cdot e^{-kt} \cdot e^{-\sqrt{k^2 - 4\omega^2} t} \right\}^{\frac{1}{2}} \end{array} \right.$$

By the process of linear combination more solutions can be constructed from (3.6) – (3.9). Now, in addition to  $a = b = 0$  if we take  $c = 0$  then all the first terms from the right hand sides of equations (3.6) – (3.9) will vanish. This is obvious since  $C_1 = C_2 = 0$  will make  $B = 0$  in each cases.

From the reduced versions (i.e.,  $C_1 = C_2 = 0$ ) we can combine (3.6) linearly with (3.8) or (3.9), and (3.7) linearly with (3.8) or (3.9) to obtain an exact solution of the form

$$(3.10) \quad y = e^{-\frac{\gamma}{2} t} \left[ R_1 e^{\frac{\sqrt{\kappa^2 - \omega^2}}{2} t} - R_2 e^{-\frac{\sqrt{\kappa^2 - \omega^2}}{2} t} \right]$$

where  $R_1$  and  $R_2$  are constants.

(3.10) is well referred solution of equation (2.1) for the simple harmonic vibration (Ahsan [5]).

**Remarks:** From the above analysis, we may conclude the following :

- I. Lie's group theoretic approach is very useful in finding the symmetries, if they exist of the differential equations and integrate them in turn.
- II. It may not always be simple to find the Lie point symmetry, even of the ordinary second-order linear differential equation.
- III. Whenever an exact solution of a differential equation can be obtained, the underlying property is the symmetry of that equation.

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