The measurement of multidimensional poverty

FRANÇOIS BOURGUIGNON[1] and SATYA R. CHAKRAVARTY[2]
[1] *Delta, ENS, 48 Bd Jourdan, 75014 Paris, France*
[2] *Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700 035, India*

Abstract. Many authors have insisted on the necessity of defining poverty as a multidimensional concept rather than relying on income or consumption expenditures per capita. Yet, not much has actually been done to include the various dimensions of deprivation into the practical definition and measurement of poverty. Existing attempts along that direction consist of aggregating various attributes into a single index through some arbitrary function and defining a poverty line and associated poverty measures on the basis of that index. This is merely redefining more generally the concept of poverty, which then essentially remains a one dimensional concept. The present paper suggests that an alternative way to take into account the multi-dimensionality of poverty is to specify a poverty line for *each* dimension of poverty and to consider that a person is poor if he/she falls below *at least one* of these various lines. The paper then explores how to combine these various poverty lines and associated one-dimensional gaps into multidimensional poverty measures. An application of these measures to the rural population in Brazil is also given with poverty defined on income and education.

Key words: multidimensional, poverty measure.

1. Introduction

Poverty has been in existence for many years and continues to exist in a large number of countries. Therefore, targeting of poverty alleviation remains an important issue in many countries. In order to understand the threat that the problem of poverty poses, it is necessary to know its dimension and the process through which it seems to be deepened. A natural question that arises here is how to quantify the extent of poverty. In an important contribution, Sen [20] viewed the poverty measurement problem as involving two exercises: (i) the identification of the poor, and (ii) aggregation of the characteristics of the poor into an overall indicator. In the literature, the first problem has been solved mostly by the income (or consumption) method, which requires the specification of a subsistence income level, referred to as the poverty line. A person is said to be poor if his/her income falls below the poverty line. On the aggregation issue, Sen [20] criticised two crude poverty measures, the head count ratio (proportion of persons with incomes less than the poverty line) and the income gap ratio (the gap between the poverty line and average income of the poor, expressed as a proportion of the poverty line), because they are insensitive to the redistribution of income among the poor and the former also remains unaltered if the position of a poor worsens. He also suggested a more sophisticated index of poverty using an axiomatic approach.[1]

However, the well-being of a population and, hence its poverty, which is a manifestation of insufficient well-being, depend on both monetary and non-monetary variables. It is certainly true that with a higher income or consumption budget a person may be able to improve the position of some of his/her monetary and non-monetary attributes. But at the same time it may be the case that markets for some non-monetary attributes do not exist, for example, with some public good. It may also happen that markets are highly imperfect, for instance, in the case of rationing. Therefore, income as the sole indicator of well-being is inappropriate and should be supplemented by other attributes or variables, e.g., housing, literacy, life expectancy, provision of public goods and so on. The need for such a multidimensional approach to the measurement of inequality in well-being was already emphasised, among others, by Kolm [15], Atkinson and Bourguignon [1], Maasoumi [17] and Tsui [25]. Concerning poverty, Ravallion [19] argued in a recent paper that four sets of indicators can be defended as ingredients for a sensible approach to poverty measurement. These are: (i) real expenditure per single adult on market goods, (ii) non-income indicators as access to non-market goods, (iii) indicators of intra-household distribution such as child nutritional status and (iv) indicators of personal characteristics which impose constraints on the ability of an individual, such as physical handicap. In other words, a genuine measure of poverty should depend on income indicators as well as non-income indicators that may help in identifying aspects of welfare not captured by incomes.

We can cite further rationales for viewing the problem of measurement of well-being of a population from a multidimensional structure. For instance, the basic needs approach advocated by development economists regards development as an improvement in an array of human needs and not just as growth of income – see Streeten [23]. There exists a debate about the importance of low incomes as a determinant of under-nutrition – see Lipton and Ravallion [16]. Finally, well-being is intrinsically multidimensional from the view point of 'capabilities' and 'functionings', where functionings deal with what a person can ultimately do and capabilities indicate the freedom that a person enjoys in terms of functionings – Sen [21, 22]. In the capability approach functionings are closely approximated by attributes such as literacy, life expectancy, etc. and not by income per se. An example of multidimensional measure of well-being in terms of functioning achievements is the Human Development Index suggested by UNDP [23]. It aggregates at the country level functioning achievements in terms of the attributes life expectancy, per capita real GDP and educational attainment rate.

For reasons stated above we deviate in the present paper from the single dimensional income approach to the measurement of poverty and adopt an alternative approach which is of multidimensional nature. In our multidimensional framework instead of visualising poverty or deprivation using income or consumption as the sole indicator of well-being, we formalise it in terms of functioning failures, or, more precisely, in terms of shortfalls from threshold levels of attributes themselves. We then examine various aggregation rules which permit to quantify the overall magnitude of poverty using these shortfalls. It may be important to note that the threshold levels are determined independently of the attribute distributions. In this sense the concept of poverty measurement we explore here is of 'absolute' type.

We begin the paper by discussing the problem of identifying the poor in Section 2. Section 3 then suggests reasonable properties for a multidimensional poverty index. Since we view poverty measurement from a multidimensional perspective, a very important issue that needs to be discussed is the trade off among attributes. It is shown that the possibility/impossibility of such trade-offs drops out as an implication of different postulates for a multidimensional measure of poverty. This is presented in Section 4 of the paper. Section 5 introduces some functional forms for a multidimensional poverty measure whereas Section 6 shows how they may be practically implemented by considering the evolution of 'income/education poverty' in rural Brazil. Section 7 concludes.

2. Identification of the poor

The purpose of this section is to determine the set of poor persons. We begin with notational definitions. With a population of size n, person i possesses an m-row vector of attributes, $x_i \in R^m_+$, where R^m_+ is the non-negative orthant of the Euclidean m-space R^m. The vector x_i is the ith row of a $n \times m$ matrix $X \in M^n$, where M^n is the set of all $n \times m$ matrices whose entries are non-negative reals. The (i, j)th entry of X gives the quantity of attribute j possessed by person i. Therefore the jth column of X gives a distribution of attribute j among n persons. Let $M = \bigcup_{n \in N} M^n$, where N is the set of positive integers. For any $X \in M$, we write $n(X)$ – or, n – for the corresponding population size. It should be noted that quantitative specifications of different attributes preclude the possibility that a variable can be of qualitative type – for instance, of the type whether a person is ill or not.

A simple way of dealing with the multidimensionality of poverty is to assume that the various attributes of an individual may be aggregated into a single cardinal index of 'well-being' and that poverty may be defined in terms of that index. In other words, an individual can be said poor if his/her index of aggregate well-being falls below some poverty line. However, such an approach would be severely restrictive and would mostly amount to considering multidimensional poverty as single dimensional income poverty, with some appropriate generalisation of the concept of 'income'. Although there sometimes may be a good justification for such an approach,[2] this is the case that we do not want to consider here because it is conceptually strictly equivalent to the case of income poverty. The fundamental point in all what follows is that a multidimensional approach to poverty defines poverty as a shortfall from a threshold on each dimension of an individual's well being. In other words, *the issue of the multidimensionality of poverty arises because individuals, social observers or policy makers want to define a poverty limit* *on* each *individual attribute: income, health, education, etc*.... All the arguments presented in this paper are based on this idea.[3]

In agreement with this basic principle, a direct method to check whether a person is poor in the multidimensional framework where he/she is characterised by m attributes is to see whether he/she has the subsistence or threshold level of each attribute. Let $z \in Z$ be a vector of thresholds, or 'minimally acceptable levels' – Sen [22], p. 139 – for different attributes,[4] where Z is a subset of R^m_+. The problem is now to determine whether a person, i, is poor or not on the basis of his/her, x_i and the vector z.

One unambiguous way of counting the number of poor in this context is to identify those for whom the levels of *all* attributes fall below the corresponding thresholds. But this definition does not exhaust the entire set of poor persons. For example, an old beggar certainly cannot be regarded as rich because of his longevity, though the above notion excludes him from the set of poor. Therefore this definition seems to be inappropriate.

More generally, person i may be called poor with respect to attribute j if $x_{ij} < z_j$. Person i is regarded as rich if $x_{ij} \geq z_j$ for all j. Analogously, attribute j for person i is said to be meagre or non-meagre according as $x_{ij} < z_j$ or $x_{ij} \geq z_j$. For any $X \in M$, let $S_j(X)$ (or S_j) be the set of persons who are poor with respect to attribute j. One may argue that the total number of poor persons can be obtained by adding the number of people in S_j over j. But this procedure may lead to *double counting*. To see this, suppose that there are two attributes, 1 and 2. The subsistence levels z_1 and z_2 are represented by the lines CD and AB respectively in Figure 1. U_1 and U_2 are upper bounds on the quantities of the attributes. Clearly the total number of poor in this two-attribute case becomes the number of persons for whom

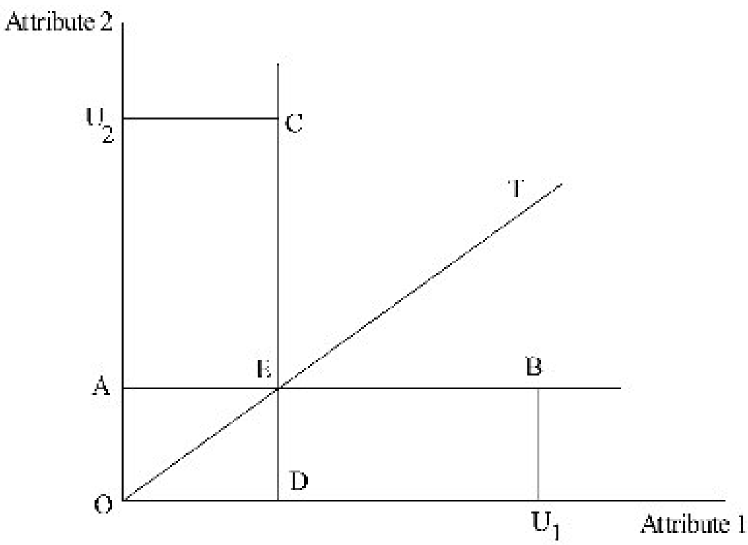

Figure 1.

the attribute quantities lie inside the space ($OABU_1 + ODCU_2$). This shows that the number of persons in OAED is counted twice in this calculation. The double counting may be avoided if we subtract OAED from ($OABU_1 + ODCU_2$). But with an increase in the number of attributes the number of sets on which double counting occurs will increase. Consequently, given that we should avoid double counting, determination of the total number of poor using S_j's will be very intricate.

A simpler way of defining poverty and counting the number of poor is to explicitly account for the possibility of being poor in any poverty dimension. A straightforward way of doing so is to define the poverty indicator variable:

$$\begin{aligned} \rho(x_i; z) &= 1 \quad \text{if } \exists j \in (1, 2, \ldots, m) : x_{ij} < z_j \quad \text{and} \\ \rho(x_i; z) &= 0, \quad \text{otherwise.} \end{aligned} \tag{1}$$

Then the number of poor is simply given by:

$$H = \sum_i \rho(x_i; z). \tag{2}$$

For further reference and in line with the preceding arguments, it will be convenient to adopt the following definitions. The region OAED in Figure 1, where person i is poor with respect to both attributes, will be called the '*two dimensional poverty*' region (PR2). In contrast, the spaces $AECU_2$ and $DEBU_1$ can be called the 'one-dimensional poverty regions' (PR1) because the quantity of only one of the attributes is above the subsistence level in these spaces.

3. Properties for a multidimensional poverty index

In this section, we lay down the postulates for a measure of multidimensional poverty. A formal statement of all these postulates is given in the Appendix to this paper. The following discussion is essentially verbal.

A multidimensional poverty index is a non-constant function $P: M \times Z \to R'$. For any $X \in M$, $z \in Z$, $P(X; z)$ gives the extent of poverty associated with the attribute matrix X and thresholds z. Thus, though we view the poverty measurement problem from a multidimensional perspective, we indicate the magnitude of overall poverty by a real number. The index P may be assumed to satisfy certain postulates. A first set of postulates includes the following: STRONG FOCUS (SF), WEAK FOCUS (WF), SYMMETRY (SM), MONOTONICITY (MN), CONTINUITY (CN), PRINCIPLE OF POPULATION (PP), SCALE INVARIANCE (SI), and SUBGROUP DECOMPOSABILITY (SD).

These postulates are straight generalisations of the desiderata suggested for a single dimensional poverty index.[5] As such, most of them are little debatable. SF demands that for any two attribute matrices X and Y, if Y is obtained from X by changing some non-poor attainment quantities so that the set of poor persons as well as their attribute levels below the relevant threshold remain the same, then the poverty levels associated with X and Y must be equal. In other words, we say that the poverty index is independent of the non-poor attribute quantities. Therefore SF does not allow the possibility that a person can give up some amount of a non-meagre attribute to improve the position of a meagre attribute. If one views poverty in terms of deprivation from thresholds, then SF is quite reasonable. In contrast to SF, WF, the weak version of the focus axiom, says that the poverty index is independent of the attribute levels of the non-poor persons only. SM states that any characteristic of persons other than the quantities of attributes used to define poverty is unimportant for measuring poverty. According to MN if the position of person i who is poor with respect to attribute j improves then overall poverty should not increase. It may be noted that the improvement may make the beneficiary non-poor with respect to the attribute under consideration. Continuity (CN) requires P to vary continuously with x_{ij}'s and is essentially a technical requirement. Continuity ensures in particular that the poverty index will not be oversensitive to minor observational errors on quantities of attributes. PP is necessary for cross population comparisons of poverty. SI says that the poverty index should be invariant under scale transformation of attributes and thresholds. In other words, what matters for poverty measurement is only the relative distance at which the quantities of *all* attributes are from their poverty thresholds. SD shows that if the population is partitioned into several subgroups with respect to some homogeneous characteristic, say age, sex, race, region, etc., then the overall poverty is the population share weighted average of subgroup poverty levels. Therefore SD enables us to calculate percentage contributions of different subgroups to total poverty and hence to identify the subgroups that are most afflicted by poverty.[6]

We now consider postulates which may less easily be generalized to a multidimensional framework or are specific to it. We first focus on redistribution criteria that involve a transfer of a fixed amount of some attribute from one person to another. We say that matrix X is obtained from Y by a *Pigou–Dalton progressive transfer of attribute* j from one poor person to another if the two matrices X and Y are exactly the same except that the richer poor i – with respect to attribute j – has θ units less of attribute j in Y than in X whereas poorer poor t has θ units more. Equivalently, we say that Y results from X through a regressive Pigou–Dalton transfer in attribute j. It is quite reasonable to argue that under such a progressive (regressive) transfer poverty should not increase (decrease). This is what is demanded by the ONE DIMENSIONAL TRANSFER PRINCIPLE (OTP).

A straightforward extension of that principle that generalises in a simple manner the Pigou–Dalton transfer principle used in income poverty measurement, is a variant of the following multidimensional transfers principle introduced by Kolm [15]. The Kolm property says that the distribution of a set of attributes summarised by some matrix X is more equal than another matrix Y (whose rows are not identical) if and only if $X = BY$, where B is some bistochastic matrix[7] and X cannot be derived from Y by permutation of the rows of Y. Intuitively, multiplication of Y by **B** makes the resulting distribution less concentrated. In effect, this transformation is equivalent to replacing the original bundles of attributes of any pair of individuals by some convex combination of them. Following Tsui [26], the analogous property applied to the set of poor is the MULTIDIMENSIONAL TRANSFER PRINCIPLE (MTP). There is no more poverty with X than with Y if X is obtained from Y simply by redistributing the attributes of the poor according to the bistochastic transformation.[8]

Instead of the single dimensional and multidimensional transfer principles OTP and MTP, we now consider a redistributive criterion involving two attributes, but without tying down the proportions in which they are exchanged as in MTP. For this, suppose two persons, i and t, are in the two-dimensional poverty space associated with attributes j and k, and i has more of k but less of j. Let us interchange the amounts of attribute j between the two persons. As person i who had more of k has now more of j too, there is an increase in the correlation of the attributes within the population. It is reasonable to expect that such a switch will not decrease or increase poverty according as the two attributes correspond to similar or different aspects of poverty. The NON-DECREASING POVERTY UNDER CORRELATION INCREASING SWITCH (NDCIS) postulate says that poverty cannot decrease with such correlation increasing switches. The converse property will be denoted by NICIS. The exact meaning of both postulates will be discussed more explicitly in the next section.

4. Implications of properties

This section discusses some implications of the properties suggested in the previous section.

In the rest of this paper we will consider mostly subgroup decomposable measures. A trivial implication of SD is that a poverty index defined on M^n can be written as:

$$P(X; z) = \frac{1}{n} \cdot \sum_{i=1}^{n} p(x_i; z).$$

In this expression $p(x_i; z)$ may clearly be interpreted as the level of poverty associated with a single person i possessing attribute vector x_i. Most of our arguments in this section are presented in terms of this ‘individual poverty function’.

Our first proposition, whose proof is easy, makes a simple but extremely important observation about the shape of an isopoverty contour in a single dimensional poverty region.

PROPOSITION 1. *Under SF, the isopoverty contours of an individual in a one-dimensional poverty space are parallel to the axis that shows the quantities of the attribute with respect to which he/she is poor.*

This proposition is extremely important because it conveys the essence of multidimensional poverty measurement. If one insists on defining a poverty threshold *independently for each attribute*, then at the same time one cannot suppose that the poverty shortfall in a given attribute may be compensated and possibly eliminated by increasing the quantity of another attribute indefinitely above its threshold level. If I am poor because my income is below the poverty limit, a very long life expectancy cannot make my poverty disappear. More precisely, Proposition 1 does not allow trade off between meagre and non-meagre attribute quantities of a person.

Things are slightly different when using WF rather than SF. Since WF assumes that the poverty index is independent of attribute levels of *non-poor persons only*, it does not rule out the possibilities of trade offs. WF ignores information on attributes of nonpoor persons but, unlike SF, takes into account the non-poor attributes of a poor person, that is, of a person who has at least one poor attribute. Therefore, we can no longer have straight line isopoverty contours in one-dimensional poverty spaces if we assume WF.

In fact, if we assume convexity of isopoverty contours in single dimensional poverty regions,[9] then the following variant of Proposition 1 emerges.

PROPOSITION 1*. *Under WF, the convex isopoverty contours in single dimensional poverty regions have vertical and horizontal asymptotes.*

The reasoning behind this proposition is as follows. Although trade off is allowed under WF in one-dimensional poverty spaces, poverty is never eliminated. That is, there is a positive lower bound of the poverty index along any vertical or horizontal axis in the poverty space. This means that the contour becomes a horizontal or a vertical line asymptotically. However, this property leads to analytically difficult problems and we shall be working mostly with SF in what follows.

Propositions 1 and 1* do not give any idea about the existence or nonexistence of trade-offs in the two-dimensional poverty space. The following proposition describes the nature of trade offs in that space.

PROPOSITION 2 (Convexity of isopoverty contours). *Suppose that* $m = 2$ *and that the poverty index satisfies MN, CN, SD and OTP or MTP. Then the isopoverty contours in the two-dimensional poverty region are decreasing convex to the origin.*

Proof. That the isopoverty contour is decreasing is guaranteed by MN. The convexity makes use of OTP or MTP. Denote the two attributes for which contours are to be examined by 1 and 2. Since we will restrict our attention to the two-dimensional space only, let us suppose that $x_{ij} < z_j$ for $j = 1, 2$ and for two persons 1 and 2. Let their attributes (x_{11}, x_{12}) and (x_{21}, x_{22}) be represented by points A_1 and A_2 in Figure 2. Consider a transfer of attributes between these two persons which makes their bundles identical. Under SD, the change in the overall poverty index is given by:

$$\Delta P = \frac{1}{n}[2 \cdot p((x_{11} + x_{21})/2, (x_{12} + x_{22})/2; z) - \\ - p(x_{11}, x_{12}; z) - p(x_{21}, x_{22}; z)]. \quad (3)$$

Both OTP and MTP imply that this expression is non-positive. If I is the midpoint of the segment A_1A_2 in Figure 2, CN and MN then imply that I lies above the

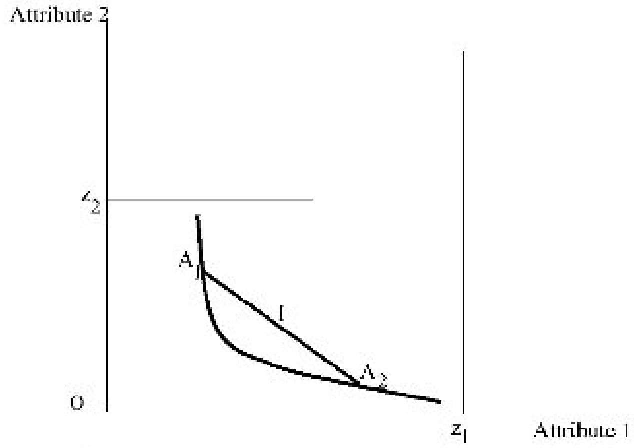

Figure 2.

isopoverty contour going through the bundle A_1 or A_2, where individual poverty is maximum. If A_1 and A_2 are on the same isopoverty contour it follows that all bundles on the segment A_1A_2 lie on isopoverty contours with lower poverty. □

This proposition shows that non-increasingness of the marginal rate of substitution between two attributes for a person in the two-dimensional poverty region is an implication of OTP or MTP. The notion of substitutability between attributes in something different and will be taken up below.

It should be clear that under SF, the poverty indifference curves in the one-dimensional poverty regions will be either horizontal or vertical straight lines depending on which axis of the graph represents quantities of which attribute. Given the shapes of the curves in the respective poverty spaces, we can combine them to generate isopoverty contours for the entire domain. Continuity enables us to connect the curves over the intervals $[z_1 - \varepsilon, z_1]$ and $[z_2 - \varepsilon, z_2]$, by continuous curves where $\varepsilon > 0$ is infinitesimally small. We show the combined graphs in Figure 3. Q_1, Q_2, and Q_3 are three overall isopoverty curves. The poverty levels associated with Q_1 is higher than that corresponds to Q_2, and Q_2 represents more poverty than Q_3.

In the preceding proposition, OTP and MTP have an identical role. It is clear, however, that requiring validity of the transfers principle for all attributes is more demanding than that for one attribute only. Therefore the set of poverty indices satisfying OTP must be more restrictive than those satisfying MTP. Our next proposition shows that indeed the former includes only those individual poverty functions that are additive across components.

PROPOSITION 3 (Additivity). *Suppose that a subgroup decomposable poverty index satisfying OTP possesses first-order partial derivatives. Then it is additive*

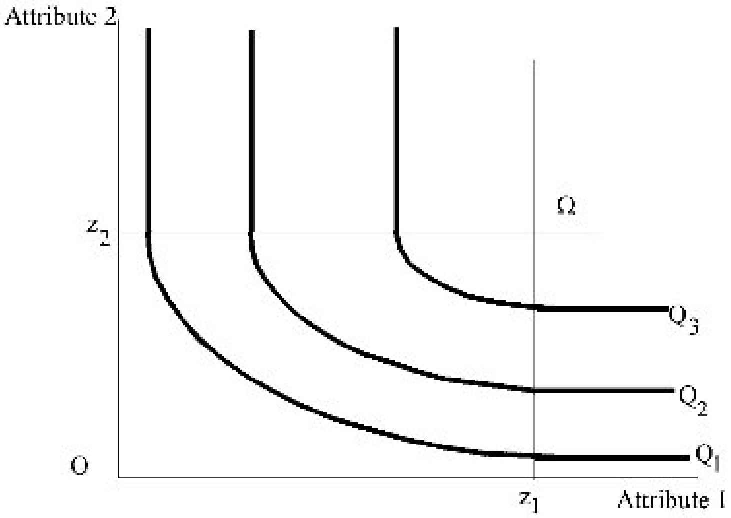

Figure 3.

across attributes, that is,

$$P(X; z) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} p^j(x_{ij}, z_j), \tag{4}$$

where $p^j()$ is the individual poverty function associated with attribute j.

Proof. For simplicity let us consider the two-person, two-attribute case. But one may check that the result remains valid in the general case too.

Consider two individuals 1 and 2 with attribute levels (x_{11}, x_{12}) and (x_{21}, x_{22}) respectively. Then for $x_{11} < x_{21}$ OTP implies the following:

$$p(x_{11} - \varepsilon, x_{12}) + p(x_{21} + \varepsilon, x_{22}) - p(x_{11}, x_{12}) - p(x_{21}, x_{22}) \geq 0$$
$$\text{for all } (x_{12}, x_{22}, \varepsilon > 0).$$

Letting ε tend toward 0 and taking limits leads to:

$$p_1(x_{21}, x_{22}) - p_1(x_{11}, x_{12}) \geq 0 \quad \text{for all } (x_{12}, x_{22}, \text{ and } x_{11} < x_{21}), \tag{5}$$

where $p_1()$ is the partial derivative of $p()$ with respect to its first argument. Define now:

$$\begin{aligned} g(t) &= \text{Max}\, p_1(t, s) \quad \text{for } s \in [0, \infty[\quad \text{and} \\ h(t) &= \text{Min}\, p_1(t, s) \quad \text{for } s \in [0, \infty[. \end{aligned} \tag{6}$$

Then (5) implies:

$$h(x_{21}) - g(x_{11}) \geq 0 \quad \text{for all } x_{11} < x_{21}. \tag{7}$$

But, by definition of $g()$ and $h()$ in (6), we have:

$$\begin{aligned} h(x_{11}) - g(x_{11}) &\leq 0 \quad \text{for all } x_{11} \quad \text{and} \\ h(x_{21}) - g(x_{21}) &\leq 0 \quad \text{for all } x_{21}. \end{aligned} \tag{8}$$

Allowing x_{11} to tend toward x_{21} from below shows a contradiction between (7) and (8), unless $h(t) = g(t)$ for all t. $h(t) = g(t)$ implies that $p_1(t, s)$ is independent of s, which in turn shows that $p(t, s)$ can be written as $p_1(t) + p_2(s)$. □

Using (4) we can determine the shares of different attributes to total poverty. If a poverty index exhibits additivity in conjunction with SD, then we have a two-way poverty breakdown and can calculate the contributions of alternative subgroups to aggregate poverty with respect to different attributes. Consequently, identification of the subgroup-attribute combinations that are more susceptible to poverty can be made. Isolation of such subgroup-attribute combinations becomes important in designing antipoverty policies when a society's limited resource does not enable it to eliminate poverty for an entire subgroup or for a specific attribute.[10] We shall study later the practical implications of this additivity property and see that they may not always be convenient.

We finally consider the last transfer properties introduced in the preceding section, non-decrasing (non-increasing) poverty under correlation increasing switch. To understand this issue, define substitutability as proximity in the nature of attributes. A correlation increasing switch means that a person who has higher amount of one attribute gets higher amount of the other through a rank reversing transfer. If attributes are close to each other – i.e. they are substitutes – such a transfer should not decrease poverty. The poorer person cannot compensate the lower quantity of one attribute by a higher quantity of the other. A similar argument can be provided for the complementarity case. Atkinson and Bourguignon [1] argued rigorously that welfare should not increase under a correlation increasing switch if the attributes involved in the switch are substitutes, where substitute attributes are such that the marginal utility of one attribute decreases when the quantity of the other increases. The equivalent definition in terms of the individual poverty function $p(x; z)$ – assuming that this function is twice differentiable – is that two attributes j and k are substitutes *whenever* $p_{jk}(x; z) > 0$ for all x. In other words, poverty decreases less with an increase in attribute j for persons with larger quantities of k. For instance, the drop in poverty due to a unit increase in income is less important for people who have an educational level close to the education poverty threshold than for persons with very low education, if income and education are considered as substitutes. On the contrary, the drop in poverty is larger for persons with higher education if these two attributes are supposed to be complements. Thus, the equivalent of the Atkinson and Bourguignon property in the case of poverty is:

PROPOSITION 4. *Under SD, non-decreasing (non-increasing) poverty under increasing correlation switch holds for attributes which are substitutes (complements) in the individual poverty function.*

Of course, we observe that with $P()$ in (4), attributes are neither substitutes nor complements. As expected OTP makes the properties NDCIS or NICIS irrelevant. However, this is not the case with MTP. There will be indices satisfying MTP and NICIS and others satisfying MTP and NDCIS. Tsui [26] argued that a poverty index should be unambiguously nondecreasing under a correlation increasing switch. But there is no a priori reason for a person to regard attributes as substitutes only. Some of the attributes can as well be complements.

5. Some functional forms for multidimensional poverty indices

Assuming that we may require multidimensional poverty indices to satisfy MN, FC, CN and SD, the preceding section led us to distinguish poverty indices satisfying OTP from those satisfying MTP. Further, among the latter, there are indices that meet NDCIS (NICIS) but not NICIS (NDCIS). In this section, we consider simple functional forms for poverty indices from these three sets, imposing in addition scale invariance. We will start from the two-dimensional case and try to generalise whenever this is possible.

The Set of Additive Multidimensional Poverty Indices

As seen above, poverty indices satisfying OTP are additive so that the general form of the individual poverty function in the two-dimensional case is simply:

$$p(x_{i1}, x_{i2}; z_1, z_2) = \begin{cases} f_1\left(\dfrac{x_{i1}}{z_1}\right) & \text{if } x_{i1} < z_1 \text{ and } x_{i2} \geq z_2, \\ f_1\left(\dfrac{x_{i1}}{z_1}\right) + f_2\left(\dfrac{x_{i2}}{z_2}\right) & \text{if } x_{i1} < z_1 \text{ and } x_{i2} < z_2, \\ f_2\left(\dfrac{x_{i2}}{z_2}\right) & \text{if } x_{i1} \geq z_1 \text{ and } x_{i2} < z_2, \end{cases} \tag{9}$$

where $f_j()$ are continuous, decreasing and convex function such that $f_j(u) = 0$ for $u \geq 1$. Note that homogeneity with respect to x and z results from the SI property. (9) may also written under a more compact form as:

$$p(x_{i1}, x_{i2}; z_1, z_2) = f_1\left(\frac{x_{i1}}{z_1}\right) \cdot S_1^i + f_2\left(\frac{x_{i2}}{z_2}\right) \cdot S_2^i, \tag{10}$$

where S_j^i is the indicator function such that $S_j^i = 1$ if $i \in S_j$ and $S_j^i = 0$, otherwise.

In the general case of m attributes and n individuals, the expression for the poverty index P corresponding to (10) becomes:

$$P(X; z) = \frac{1}{n} \sum_{j=1}^{m} \sum_{i \in S_j} f_j\left(\frac{x_{ij}}{z_j}\right), \tag{11}$$

where $X \in M^n$, $n \in N$, $z \in Z$ are arbitrary, f_j: $[0, \infty[\rightarrow R^1$ is continuous, non-increasing, convex and $f_j(t) = 0$ for all $t \geq 1$.

To illustrate the preceding formula let us choose:

$$f_j(t) = a_j(1-t)^{\theta_j}, \quad 0 \le t < 1, \tag{12}$$

where $\theta_j > 1$ and a_j (> 0) may be interpreted as the 'weight' given to attribute j in the overall poverty index. Then the resulting measure is:

$$P_\theta(X; z) = \frac{1}{n} \sum_{j=1}^{m} \sum_{i \in S_j} a_j \cdot \left(1 - \frac{x_{ij}}{z_j}\right)^{\theta_j}. \tag{13}$$

This is a simple multidimensional extension of the Foster–Greer–Thorbecke [12] index. If $\theta_j = 1$ for all j, then P_θ becomes a weighted sum of poverty gaps in all dimensions. On the other hand, if $\theta_j = 2$ for all j, then

$$P_2(X; z) = \frac{1}{n} \sum_{j=1}^{m} a_j \cdot F_j \cdot [A_j^2 + (1 - A_j^2) \cdot V_j^2], \tag{14}$$

where F_j is the population size in S_j as a fraction of n, A_j is the average relative poverty shortfall of persons in S_j and V_j is the coefficient of variation of the distribution of attribute j among those in S_j.

It may be important to note that though the use of S_j sets for determining the number of poor leads to double counting, their use in the construction of a poverty index of the form (11) (excluding the headcount ratio) does not involve this problem. The reason behind this is that we are not counting the number of poor but aggregating their poverty shortfalls in the various dimensions. However, as mentioned earlier, these measures are not sensitive to a correlation increasing switch.

Non-additive Poverty Indices Satisfying MTP

As seen above, a more general family of poverty indices is that satisfying MTP rather than OTP. It may be obtained in the two-dimensional case from isopoverty contours which are convex to the origin. These poverty contours may be generated by taking non-decreasing and quasi-concave transformations of the relative shortfalls of the two attributes. The following functional form for the individual poverty function $p(x; z)$ is a compact way of representing the iso-poverty contours shown in Figure 3:

$$p(x; z) = I\left[\mathrm{Max}\left(1 - \frac{x_1}{z_1}, 0\right), \mathrm{Max}\left(1 - \frac{x_2}{z_2}, 0\right)\right], \tag{15}$$

where $I(u_1, u_2)$ is an increasing, continuous, quasi-concave function with $I(0, 0) = 0$. The corresponding poverty index becomes:

$$P(X; z) = \frac{1}{n} \sum_{i=1}^{n} I\left[\mathrm{Max}\left(1 - \frac{x_{i1}}{z_1}, 0\right), \mathrm{Max}\left(1 - \frac{x_{i2}}{z_2}, 0\right)\right]. \tag{16}$$

Clearly, the additive case analysed above is a particular case of (16) where $I(u_1, u_2) = f_1(u_1) + f_2(u_2)$. Different forms of the poverty index may now be generated from alternative specifications of $I()$. An appealing specification may be derived from the CES form:

$$I(u_1, u_2) = f[(a_1 \cdot u_1^{\theta} + a_2 \cdot u_2^{\theta})^{1/\theta}], \tag{17}$$

where $f()$ is an increasing and convex function such that $f(0) = 0$, a_1 and a_2 are positive weights attached to the two attributes and θ permits to parameterise the elasticity of substitution between the shortfalls of the various attributes. Note, however, that in order to generate isopoverty contours convex to the origin in the two-dimensional region of the space of attributes, (17) must lead to isopoverty contours that are concave to the origin in the space of shortfalls. This is what is shown in Figure 3 when isopoverty contours are looked at from the origin, O, or from the no-poverty point, Ω. This concavity requirement imposes that $\theta > 1$ in (17).

The full specification of poverty indices based on the individual poverty function (17) is obtained by combining (16) and (17).

$$P(X; z) = \frac{1}{n} \cdot \sum_{i=1}^{n} f\left[\left\{a_1 \cdot \left[\mathrm{Max}\left(1 - \frac{x_{i1}}{z_1}, 0\right)\right]^{\theta} + a_2 \cdot \left[\mathrm{Max}\left(1 - \frac{x_{i2}}{z_2}, 0\right)\right]^{\theta}\right\}^{1/\theta}\right]. \tag{18}$$

This index seems a rather flexible functional form consistent with MTP. Note, however, that it is not clear a priori whether it satisfies NDCIS or NICIS. It is easy to see that MTP implies that $\theta > 1$, which in turn implies that the cross second derivative of $I()$ is negative. However, the two shortfalls may still be complement in determining poverty depending on the shape of the function $f()$.[11]

Three particular cases of (18) are worth stressing. The first case is when θ tends toward infinity so that the substitutability between the two shortfalls or equivalently the two attributes in the definition of poverty tends toward zero. In that case, the isopoverty contours become rectangular curves even within the two-dimensional poverty space. This is the shape shown in Figure 4. It is interesting to note that in this case the two attributes must necessarily combine within the two-dimensional poverty space in the same proportions as the threshold levels z_1 and z_2.[12] The expression for the poverty index then becomes:

$$\begin{aligned} P(X; z) &= \frac{1}{n} \cdot \sum_{i=1}^{n} f\left[\mathrm{Max}\left\{\mathrm{Max}\left(1 - \frac{x_{i1}}{z_1}, 0\right), \mathrm{Max}\left(1 - \frac{x_{i2}}{z_2}, 0\right)\right\}\right] \\ &= \frac{1}{n} \cdot \sum_{j=1}^{2} \sum_{i \in I_j} f\left(1 - \frac{x_{ij}}{z_j}\right), \end{aligned} \tag{19}$$

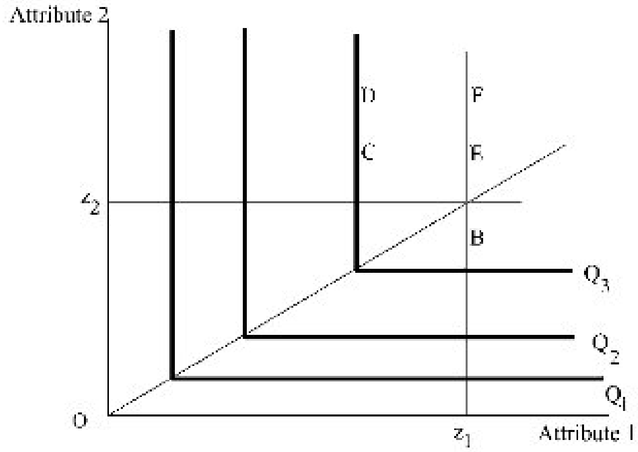

Figure 4.

where

$$I_1 = \left\{ i : \frac{x_{i1}}{z_1} \leq \text{Min}\left[\frac{x_{i2}}{z_2}, 1\right]\right\}, \qquad I_2 = \left\{ i : \frac{x_{i2}}{z_2} < \text{Min}\left[\frac{x_{i1}}{z_1}, 1\right]\right\}.$$

These two sets may be called ‘exclusive poverty sets’ where two-dimensional poverty is transformed into one-dimensional poverty with respect to the attribute that is the farthest away from its poverty line. Expression (19) is analogous to that for additive poverty indices except that the poverty sets S_j are replaced by the sets I_j, and the poverty functions are the same for the various attributes. The extreme parsimony of this family of poverty indices is to be noted. It actually requires no more than the knowledge of the threshold levels and a conventional one-dimensional poverty index $f()$, for instance, the well-known Foster–Greer–Thorbecke P_α index. Of course, these poverty indices satisfy MTP and NICIS.

The second particular case is at the other extreme when the two-attributes are perfect substitutes in the two-dimensional poverty space. The isopoverty contour is then a straight line in that space which connects the horizontal and vertical straight lines in one-dimensional poverty spaces, as in Figure 5. The general expression of the corresponding poverty indices is:

$$P(X; z) = \frac{1}{n} \cdot \sum_{i=1}^{n} f\left[a_1 \cdot \text{Max}\left(1 - \frac{x_{i1}}{z_1}, 0\right) + a_2 \cdot \text{Max}\left(1 - \frac{x_{i2}}{z_2}, 0\right)\right], \quad (20)$$

where, again, $f()$ may be any one-dimensional poverty index, like the Foster–Greer–Thorbecke P_α index, and, as before, the positive coefficients a_j represent the weight given to the attributes and determine the slope of the iso-poverty contour in the two-dimensional poverty space. Poverty indices of type (20) satisfy MTP and NDCIS or NICIS depending on whether the one-dimensional poverty function $f()$ is concave or convex.

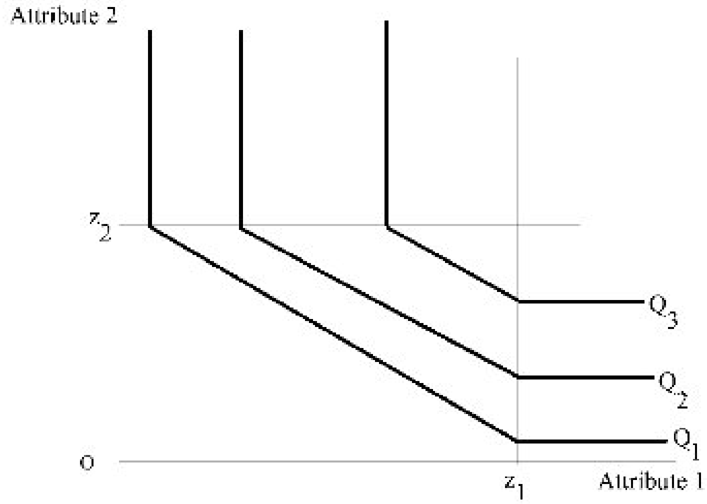

Figure 5.

A third particular case of (18) is obtained by using the Foster–Greer–Thorbecke P_α index for the function $f()$. One then obtains:

$$P_\alpha^\theta(X;z) = \frac{1}{n}\cdot\sum_{i=1}^{n}\left[a_1\cdot\left[\text{Max}\left(1-\frac{x_{i1}}{z_1},0\right)\right]^\theta + \right.$$
$$\left. + a_2\cdot\left[\text{Max}\left(1-\frac{x_{i2}}{z_2},0\right)\right]^\theta\right]^{\alpha/\theta}, \qquad (21)$$

where α is a positive parameter. The interpretation of that measure is straightforward. The poverty shortfalls in the two dimensions are first aggregated into some 'average' shortfall through function $I()$ with a particular value of θ and the coefficients a_j. Multidimensional poverty is then defined as the average of that aggregate shortfall, raised to the power α, over the whole population. This seems to be the measure the closest to one-dimensional poverty measurement concepts and the simplest generalisation of these concepts. With $\alpha = 0$, (21) yields the multidimensional headcount. With $\alpha = 1$, P_α^θ becomes a multidimensional poverty gap obtained by some particular averaging of the poverty gaps in the two dimensions. Higher values for α may be interpreted, as in the one-dimensional case, as higher aversion towards extreme poverty. An interesting property of that P_α^θ measure is that it satisfies NDCIS or NICIS depending on whether α is greater or less than θ.

These three families of poverty indices may easily be generalised to any number of attributes. However, doing so implies assuming the same elasticity of substitution between attributes, and therefore the resulting poverty indices are NDCIS or NICIS for all pairs of attributes. This may not be very satisfactory and other more complex specifications have to be designed to avoid this.

Another interesting generalisation of the preceding measures consists of assuming that the substitutability between the poverty shortfalls in the two attributes

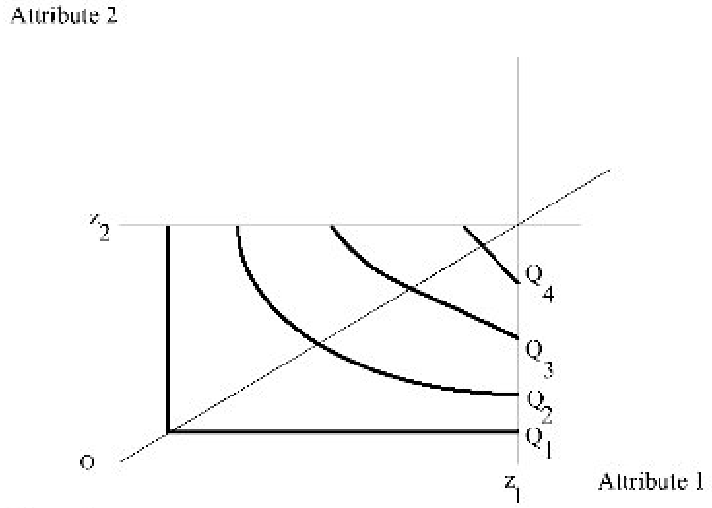

Figure 6.

changes with the extent of poverty. When someone is very poor in one of the two dimensions, one may be willing to assume that the elasticity of substitution between the two dimensions of poverty is of minor importance. For instance, if a person is 50 per cent below the poverty line in terms of food, it is probably immaterial whether he/she is 10 or 20 per cent below the poverty line for educational attainment for evaluating his/her overall poverty. On the contrary, if the food poverty gap is only 10 per cent, then the extent of the poverty gap in education becomes a more important determinant of overall poverty. The corresponding shape of the iso-poverty contours is shown in Figure 6. But one may also be willing to assume the opposite, namely that the substitutability between the two attributes decreases with the extent of poverty. Analytically, a simple way of allowing for this dependency between the substitutability of attributes and the extent of poverty consists of making the θ parameter in (18) a function of the level of poverty. Within a P_α framework, individual poverty is then defined implicitly by the following equation:

$$\left[\left[\mathrm{Max}\left(1-\frac{x_{i1}}{z_1},0\right)\right]^{a(p)}+\left[\mathrm{Max}\left(1-\frac{x_{i2}}{z_2},0\right)\right]^{a(p)}\right]^{\alpha/a(p)} = p(x_{i1},x_{i2},z_1,z_2), \tag{22}$$

where $a(p)$ is a function that describes how attribute substitutability changes with the extent of poverty. Obvious candidates for this function are $a(p) = 1/p$ and $a(p) = 1/(1-p)$, assuming p is normalized so as to lie between 0 and 1. With these functions, solving numerically Equation (22) is not difficult. It leads to poverty functions with the same properties as (21), except for the fact that correlation increasing switches may now increase or decrease overall poverty depending on whether they are performed among very poor or moderately poor persons. We shall refer to these indices respectively as $P_\alpha^{1/p}$ and $P_\alpha^{1/(1-p)}$.

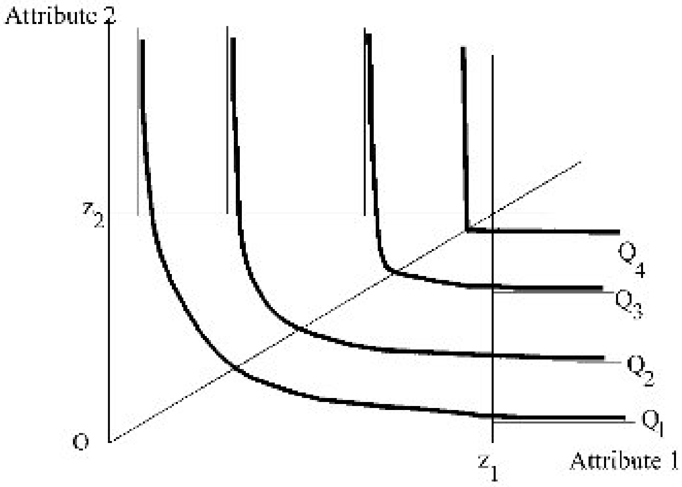

Figure 7.

It is worth stressing that all preceding multi-dimensional poverty indices actually rely on the SF postulate. In effect, it may be shown that the weak focus postulate (WF) rules out functional forms of poverty indices that are additive as well as the CES-like P_α^θ measures, or even their varying substitutability generalisations, $P_\alpha^{1/p}$ and $P_\alpha^{1/(1-p)}$. As a matter of fact we have not been able to find relatively simple functions leading to iso-poverty contours consistent with WF as shown in Figure 7, and the other properties of the individual poverty function $p(\,)$.

6. An example of application

To illustrate the use of the preceding measures as well as the concepts behind them, we analyze here the evolution of multidimensional poverty in rural Brazil during the 1980s. Poverty includes two dimensions: income on the one hand and educational attainment on the other. The analysis is performed on the rural population only, because this is where Brazilian poverty tends to concentrate. It is also restricted to the adult population, so as to avoid the problem of imputing some final educational level to children who are still going to school. Samples from the PNAD household surveys for the years 1981 and 1987 are being used.[13] The reason for choosing these years is that they happen to correspond to an increase in income poverty in the rural population. So, we felt it could be interesting to use the measures presented in the previous section to see whether this increase in income poverty had possibly been compensated by a drop in educational poverty. But, of course, this issue of the trade-off between these two particular dimensions of poverty would also arise in very different contexts. For instance, designing anti-poverty policies may require deciding whether it is better to reduce more income or education poverty.

Poverty is measured at the individual level. Each individual is given the income per capita within the household he/she belongs to. The income poverty threshold is 2$ a day, at 1985 ppp corrected prices. The educational poverty threshold is defined as the end of primary school, that is, 4 years of schooling. The educational poverty shortfall is defined as the number of years of schooling short of that level. It may thus take only 4 values. Yet, we treat it as a continuous variable.

The first two columns of Table I show the level of poverty as measured by the conventional P_α measures separately for income and education. It may be seen that income poverty increased from 1981 to 1987, whereas education poverty fell. The "alpha $= 0$" rows show that there were 40.5 per cent of rural adults below the poverty line in 1981 whereas 74.4 per cent had not completed primary school. Six years later these proportions were 42.1 and 68 per cent respectively, indicating an increase in income poverty and a fall in education poverty. The poverty gap ("alpha $= 1$") and higher levels of the P_α measures show the same evolution.

We now consider multidimensional poverty measures of the P_α^θ type, with α taking the same values as for the one-dimensional poverty measures that we just reviewed and θ taking the values 1, which corresponds to perfect substitutability as in (20) above, 2 and 5. We also use the varying substitutability measures, $P_\alpha^{1/p}$ and $P_\alpha^{1/(1-p)}$. The evaluation of multidimensional poverty for 1981 and 1987 according to these various measures are reported in Table I for two sets of weights for the income and education attributes. The first set gives equal weights to the two dimensions whereas the second gives more weight to income.

Consider first the first two rows which correspond to headcount poverty measures. In the multidimensional case, the headcount corresponds to individuals who are poor either in terms of income or in terms of education. Accordingly there were 79.7 per cent poor in 1981 versus 75.6 per cent in 1987. From these figures and the headcounts in one dimension, it is easy to derive the proportion of people who were poor in *both* dimensions. They were 35.2 per cent in 1981 and 34.4 per cent in 1987.

Reading down the other rows, one may check that the multidimensional P_α^θ measures – as well as the measures with variable substitutability – are commensurate with the one-dimensional P_α measures for income and education. There is nothing surprising here. As noted above, the multidimensional P_α^θ measures are designed in such a way that they may be interpreted as some particular mean of one-dimensional measures. This mean depends on the weighing coefficients, a_1 and a_2, but also on the substitutability parameter, θ. So, multidimensional measures is higher when more weight is given to education because one-dimensional poverty is higher for education, as shown in the first two columns of Table I. But multidimensional poverty also tends to increase when the substitutability of the two attributes falls, or equivalently the θ parameter increases. As suggested by the argument leading to (19) above, this is because low substitutability between the two attributes gives more weight for each observation to the attribute with the largest shortfall.

Table I. Multidimensional measurement of poverty with P_α^θ, $P_\alpha^{1/p}$ and $P_\alpha^{1/(1-p)}$ measures. Income/education poverty in rural Brazil, 1981–1987

(Per cents)

Aversion for poverty	One-dimensional poverty		Multidimensional income/education poverty: Income weight = 50%, education weight = 50%					Income weight = 80%, education weight = 20%				
	Income	Education	Theta = 1	2	5	$1/p$	$1/(1-p)$	Theta = 1	2	5	$1/p$	$1/(1-p)$
Alpha = 0 (headcount)												
1981	**40.52**	74.41	79.7	79.7	79.7	79.7	79.7	79.7	79.7	79.7	79.7	79.7
1987	**42.07**	67.99	75.65	75.65	75.65	75.65	75.65	75.65	75.65	75.65	75.65	75.65
Alpha = 1 (poverty gap)												
1981	**15.93**	57.82	36.87	45.63	53.28	43.8	55.18	**24.31**	33.7	46	35.67	30.02
1987	**17.64**	52.93	35.28	42.83	49.97	41.57	51.4	**24.7**	32.82	43.56	34.52	29.8
Alpha = 2												
1981	**8.48**	51.23	*20.67*	29.86	40.92	31.07	28.97	***11.15***	**17.03**	30.24	24.25	**11.84**
1987	**9.66**	46.87	*20.24*	28.27	38.12	29.45	27.53	***11.97***	**17.11**	28.65	22.46	**12.69**
Alpha = 3												
1981	**5.21**	47.97	***12.87***	*20.94*	33.29	24.39	15.38	***6.43***	***9.59***	21.02	17.22	**6.66**
1987	**6.02**	43.86	***12.96***	*19.98*	30.9	23.07	15.26	***7.12***	***9.94***	19.96	16.56	**7.38**
Alpha = 5												
1981	**2.47**	44.68	***6.12***	*11.27*	23.57	17.4	6.58	***2.94***	***3.86***	10.91	11.4	**3**
1987	**2.89**	40.84	***6.49***	*11*	21.86	16.37	6.96	***3.34***	***4.22***	10.48	10.86	**3.41**

Cells in bold refer to cases where there is more poverty in 1987 than in 1981. Cells in italics refer to cases where the NICIS property holds ($\alpha > \theta$).

Bold figures in Table I correspond to situations where poverty measures indicate more poverty in 1987 than in 1981. We see that this occurrence is more frequent when the weight given to the income dimension is higher. There is nothing really surprising here since we have seen that there was more poverty, in a one dimension sense, with income than with education. It is more interesting to notice that poverty appears to be higher in 1987 than in 1981 when the poverty aversion parameter, α, is high enough, although the value of that parameter for which this happens is not systematically shown in the table. This is true for each value of the substitutability parameter, θ, as well as for both systems of weights. This is true also with the variable substitutability measure, $P_\alpha^{1/(1-p)}$. A possible explanation for this pattern would be that the worsening of the bi-dimensional income/education distribution in rural Brazil may have its roots at the very bottom of the distribution, where poverty is more severe. In other words, income losses may have been more serious predominantly for people with low income and low education.

Regarding the correlation between the two dimensions of poverty, still a more interesting feature in Table I is the fact that poverty tends to be higher in 1987 in cases where the NICIS property holds. It was seen in the previous section that the P_α^θ measure was such that poverty would increase with increasing correlation switches when $\alpha < \theta$. It happens in Table I that cases where poverty is higher in 1987 than in 1981 occur only when the opposite is true. This suggests that the increase in one-dimensional income poverty was accompanied by a drop in the correlation with educational levels.

The varying substitutability measures give still another information. First, it may be seen that 1987 never exhibits more poverty than 1981 with the $P_\alpha^{1/p}$ measure. It does however with the $P_\alpha^{1/(1-p)}$ measure for high values of α when both dimensions have equal weight and much sooner when more weight is put on the income dimension. This evolution is consistent with the idea that income losses were more pronounced for poorer people with a larger income than education shortfall. With the $P_\alpha^{1/(1-p)}$ there is limited substitutability for them and the drop in income could not be compensated by a possible increase in the educational level.

This interpretation of the figures reported in Table I would need to be confirmed by a more careful analysis of the bi-dimensional distribution of education and income in rural Brazil. Within the present framework, what matters is that measures directly inspired from the familiar one-dimensional P_α poverty indices and enlarged through a reduced set of parameters – 2 parameters in the case of P_α^θ and a single one in the case of $P_\alpha^{1/p}$ or $P_\alpha^{1/(1-p)}$ – permit to describe adequately the extent of poverty in a multidimensional perspective.

7. Conclusion

We have explicitly argued in this paper why poverty should be regarded as the failure to reach 'minimally acceptable' levels of different monetary and non-monetary attributes necessary for a subsistence standard of living. That is, poverty is essentially a multidimensional phenomenon. The problems of counting the number of poor in this framework and then combining the information available on them into a statistic that summarises the extent of overall poverty have been discussed rigorously. Using different postulates for a measure of poverty, shapes of isopoverty contours of a person have been derived in alternative dimensions. This in turn establishes a person's nature of trade off between attributes in different poverty spaces.

We make a distinction between additive and non-additive poverty measures satisfying the strong version of the 'Focus Axiom', which demands independence from non-poor attribute quantities in poverty measurement. One problem with additive measures is that they are insensitive to a correlation increasing switch. A correlation increasing switch requires giving more of one attribute to a person who has already more of another. A finer subdivision among nonadditive measures is possible depending on whether a measure decreases or increases under such a switch. Specific functional forms have been proposed that fit these various properties depending on the value of a small number of key parameters and generalizing in an easy way the familiar P_α family. As an illustration, the resulting measures have been used to evaluate the evolution of income/education poverty in rural Brazil in the 1980s.

Appendix. Formal statement of the axioms used in the paper

STRONG FOCUS (SF). For any $n \in N$, $(X, Y) \in M^n$, $z \in Z$, $j \in \{1, 2, \ldots, m\}$, if (i) for any i such that $x_{ij} \geq z_j$, $y_{ij} = x_{ij} + \delta$, where $\delta > 0$, (ii) $y_{tj} = x_{tj}$ for all $t \neq i$, and (iii) $y_{is} = x_{is}$ for all $s \neq j$ and for all i, then $P(Y; z) = P(X; z)$.

WEAK FOCUS (WF). For any $n \in N$, $(X, Y) \in M^n$, $z \in Z$, if for some i $x_{ik} \geq z_k$ for all k and (i) for any $j \in \{1, 2, \ldots, m\}$, $y_{ij} = x_{ij} + \delta$, where $\delta > 0$, (ii) $y_{it} = x_{it}$ for all $t \neq j$, and (iii) $y_{rs} = x_{rs}$ for all $r \neq i$ and all s, then $P(Y; z) = P(X; z)$.

SYMMETRY (SM). For any $(X; z) \in M \times Z$, $P(X; z) = P(\Pi X; z)$, where Π is any permutation matrix of appropriate order.

MONOTONICITY (MN). For any $n \in N$, $X \in M^n$, $z \in Z$, $j \in \{1, 2, \ldots, m\}$, if: (i) for any i, $y_{ij} = x_{ij} + \delta$, where $x_{ij} < z_j$, $\delta > 0$, (ii) $y_{tj} = x_{tj}$ for all $t \neq i$, and (iii) $y_{is} = x_{is}$ for all $s \neq j$ and for all i, then $P(Y; z) \leq P(X; z)$.

CONTINUITY (CN). For any $z \in Z$, $P()$ is continuous on M.

PRINCIPLE OF POPULATION (PP). For any $(X; z) \in M \times Z$, $k \in N$, $P(X^k; z) = P(X; z)$ where X^k is the k-fold replication of X.

SCALE INVARIANCE (SI). For any $(X; z) \in M \times Z$, $P(X; z) = P(X'; z')$ where $X' = \Lambda X$, $z' = \Lambda z$, Λ being the diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, $\lambda_i > 0$ for all i.

SUBGROUP DECOMPOSABILITY (SD). For any $X^1, X^2, \ldots, X^K \in M$ and $z \in Z$:

$$P(X^1, X^2, \ldots, X^K; z) = \sum_{i=1}^{K} \frac{n_i}{n} P(X^i; z),$$

where n_i is the population size corresponding to X^i and $n = \Sigma n_i$.

DEFINITION OF A PIGOU–DALTON PROGRESSIVE TRANSFER. Matrix X is said to be obtained from $Y \in M^n$ by a *Pigou–Dalton progressive transfer of attribute* j from one poor person to another if for some *persons* i, t: (i) $y_{tj} < y_{ij} < z_j$, (ii) $x_{tj} - y_{tj} = y_{ij} - x_{ij} > 0$, $x_{ij} \geq x_{tj}$, (iii) $x_{rj} = y_{rj}$ for all $r \neq i, t$, and (iv) $x_{rk} = y_{rk}$ for all $k \neq j$ and all r.

ONE DIMENSIONAL TRANSFER PRINCIPLE (OTP). For all $n \in N$ and $Y \in M^n$, if X is obtained from Y by a Pigou–Dalton progressive transfer of some attribute between two poor, then $P(X; z) \leq P(Y; z)$, where $z \in Z$ is arbitrary.

MULTIDIMENSIONAL TRANSFER PRINCIPLE (MTP). For any $(Y; z) \in M \times Z$, if X is obtained form Y by multiplying Y_p by a bistochastic matrix B and $B.Y_p$ is not a permutation of the rows of Y_p, then $P(X; z) \leq P(Y; z)$, given that the attributes of the non-poor remain unchanged, where Y_p is the bundle of attributes possessed by the poor as defined with the attribute matrix Y.

DEFINITION OF A CORRELATION IN CREASING SWITCH. For any $X \in M^n$, $n \geq 2$, $(j, k) \in \{1, 2, \ldots, m\}$, suppose that for some i, t, $x_{ij} < x_{tj} < z_j$ and $x_{tk} < x_{ik} < z_k$. Y is then said to be obtained from X by a 'correlation increasing switch' between two poor if: (i) $y_{ij} = x_{tj}$, (ii) $y_{tj} = x_{ij}$; (iii) $y_{rj} = x_{rj}$ for all $r \neq i, t$, and (iv) $y_{rs} = x_{rs}$ for all $s \neq j$ and for all r.

NON-DECREASING POVERTY UNDER CORRELATION INCREASING SWITCH (NDCIS). For any $n \in N$ and $n \geq 2$, $X \in M^n$, $z \in Z$, if Y is obtained from X by a correlation increasing switch, then $P(Y; z) \geq P(X; z)$.

The converse property is denoted by NICIS.

Notes

[1] Alternatives and variations of the Sen index have been suggested, among others, by Takayama [24], Blackorby and Donaldson [2], Kakwani [14], Clark, Hemming and Ulph [6], Foster, Greer and Thorbecke [12], Chakravarty [4], and Bourguignon and Fields [3].

[2] Tsui [26] provides an axiomatic justification of such an approach. Note also that this approach may go quite beyond aggregating a few goods or functionings through using appropriate prices or weights. For instance Pradhan and Ravallion [18] tried to integrate into the analysis unobserved welfare determinants summarized by reported subjective perception of poverty.

[3] Note that poverty limits in all dimensions are defined independently of the quantity of other attributes an individual may enjoy. For a more general statement see Duclos *et al.* [9].

[4] Using the same attributes as UNDP, empirical examples of these threshold quantities could be an income of 1$ (ppp corrected) a day, primary education, and 50 year life expectancy.

[5] For discussion of properties for a single dimensional poverty index, see among others, Foster [11], Donaldson and Weymark [8], Cowell [7], Chakravarty [4], Foster and Shorrocks [13] and Zheng [28].

[6] For further discussion, see Tsui [26] and Chakravarty, Mukherjee and Ranade [5]. Also it may be noted that that SD is not the same as subgroup consistency discussed in Foster and Shorrocks [13].

[7] A square matrix is called a bistochastic matrix if each of its entries is non-negative and each of its rows and columns sums to one. Evidently, a permutation matrix is a bistochastic matrix but the converse is not necessarily true.

[8] It is well known that the one-dimensional Pigou–Dalton transfer principle is connected to Lorenz dominance through the Hardy–Littlewood–Polya theorem. No such theorem is available in the multi-attribute case.

[9] Convexity of the contours implicitly assumes that MTP holds throughout the entire poverty space.

[10] For a numerical illustration of this two-way decomposability formula, see Chakravarty, Mukherjee and Ranade [5].

[11] To see this, note that the cross second derivative of the individual poverty function $p(x_1, x_2; z_1, z_2)$ writes with obvious notations: $p_{12} = f' \cdot I_{12} + f'' \cdot I_1 \cdot I_2$. The condition $\theta > 1$ implies that I_{12} is negative, but p_{12} may still be positive because of the second term on the RHS.

[12] If this were note the case, a point like B in Figure 4 could be the summit of a rectangular isopoverty contour, which is obviously contradictory since poverty is zero for high values of an attribute on the vertical branch and non-zero on the horizontal branch.

[13] Irrespectively of the fact that rural incomes are known to be imperfectly observed in PNAD – see for instance Elbers *et al.* [10]. The calculations below must therefore be taken as mostly illustrative.

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