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A Combinatorial Approach for Component Importance

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ABSTRACT

A number of structural importance measures like Birnbaum importance, Barlow–Proschan importance etc., are used for studying component importance in a system. A unified approach for calculating component importance in a system was introduced by Seth and Ramamurthy. It is based on the concept of structural matrix which needs to be determined from the simple form of a structure function of a system. In this article, we present a combinatorial expression for

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the elements of the structural matrix. Other related results and applications of this approach to different systems are considered. Results of this article is also applicable to the problem of finding power of a player in game theory.

Key Words: Coherent structure; Structure function; Structural matrix.

1. INTRODUCTION

For a complex system consisting of a number of components, some components have more bearing on system reliability than others. For example, putting a component in series within a system causes it to have more importance for system reliability than the same component would have, if it were placed in parallel within the system. Thus choice of components and their placement may make them more critical than others. To improve system performance we need to identify those components which play a crucial role and put more research and development efforts for improving their performance.

A number of structural importance measures like Birnbaum importance (Birnbaum, 1969), Barlow–Proschan importance (Barlow and Proschan, 1975) and Butler cut importance ranking (Butler, 1979) have been considered in the literature using varying approaches. A structural importance measure requires only the knowledge of the structure function of the system. These measures are more suitable during system design and development phases when the component reliabilities are generally not known. On the other hand reliability importance measures require additional information about component reliabilities apart from the structure function.

A unified approach to determining structural importance was introduced by Seth and Ramamurthy (1991). This approach uses the concept of structural matrix which needs to be determined from the simple form of the structure function. Different measures of structural importance of components can then be obtained using the structural matrix. In this article, we give a combinatorial expression for the elements of a structural matrix in terms of the number of path sets. This approach provides a new look for better understanding of the structural importance problem. It also makes us possible to study the properties of the structural matrix. Results of this article is also applicable to the problem of finding power of a player in game theory.

2. NOTATIONS AND DEFINITIONS

Let $N = \{1, 2, \ldots, n\}$ be the set of components of the system. The state of component $i \in N$ will be described by a binary variable $x_i : x_i = 1$ if component i is working and $x_i = 0$ if component i is failed. The vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in B^N$ where $B = \{0, 1\}$, is a state vector. The state of the system is completely and uniquely determined by the state of its components only and is expressed in terms of the so called structure function, $\phi(\mathbf{x})$ defined as $\phi : B^n \to B$ where $\phi(\mathbf{x}) = 1$ (0) means the system is working (failed). A monotone non-decreasing structure function, $\phi(\mathbf{x})$ with $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$ where $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$ is called a monotone structure. We denote such a system by (N, ϕ). A monotone structure (N, ϕ) is said to be a coherent structure if all its components are relevant. A component $i \in N$ is said to be relevant to (N, ϕ) if $\phi(1_i, \mathbf{x}) \neq \phi(0_i, \mathbf{x})$ for some $\mathbf{x} \in \mathbf{B}^n$ where $(1_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ and $(0_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$.

A path (cut) set is a subset of components whose functioning (failing) ensures system functions (fails).

A structure function ϕ, can be expressed as

$$\phi(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j \quad \text{for} \quad \mathbf{x} \in B^n \tag{2.1}$$

which is called the *simple form* of $\phi(\mathbf{x})$, by Ramamurthy (1990). For $S = \emptyset$, we take $\prod_{j \in S} x_j = 1$ and the a_S's are some integers. The structural matrix $M(\phi) = ((m(\phi))_{ij}$ is a square matrix of order n with elements given by

$$m(\phi)_{ij} = \sum_{S \in A_{ij}} a_S \tag{2.2}$$

where

$$A_{ij} = \{S : S \subseteq N,\ i \in S,\ \text{and}\ |S| = j\}$$

for all $i, j \in N$.

Different measures of importance of component have been proposed in the literature using different approaches like critical path vectors, etc. The concept of structural matrix was introduced by Seth and Ramamurthy (1991) to develop a unified approach for calculating the structural importance of components. They showed that $M(\phi)\mu$ gives the vector of Birnbaum structural importance of components where $\mu \in R^n$ is a column vector with $\mu_j = (1/2)^{j-1} (j = 1, 2, \ldots, n)$ and $M(\phi)\mu$ gives the vector of Barlow–Proschan structural importance if we take $\mu_j = 1/j$, $(j = 1, 2, \ldots, n)$. The determination of $M(\phi)$ as per their methods requires the availability of $\phi(\mathbf{x})$ in its simple form. However, a problem arises if we know only the path sets or cut sets of the system and $\phi(\mathbf{x})$ is not known in the required form.

In this article, we consider a different route for calculating the structural matrix. We basically develop a combinatorial approach using path sets instead of the simple form of a structure for determining $M(\phi)$.

3. COMBINATORIAL APPROACH

For any $S \subseteq N$, we associate a binary vector $e^S \in \{0, 1\}^n$ where $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. Let p_{ij} denote the number of path sets of ϕ that contain component i and are of cardinality j. A path set is a subset of components whose functioning ensure system functioning. Further let q_{ij} represent the number of path sets of ϕ that do not contain component i but are of cardinality j. Obviously, we have

$$p_{ij} = \sum_{S \in A_{ij}} \phi(e^S) \quad \text{and} \quad q_{ij} = \sum_{S \in \bar{A}_{ij}} \phi(e^S)$$

where

$$A_{ij} = \{S : S \subseteq N,\ i \in S \text{ and } |S| = j\} \quad \text{and}$$
$$\bar{A}_{ij} = \{S : S \subseteq N,\ i \notin S \text{ and } |S| = j\} \quad \text{for} \quad i,\ j \in N.$$

Also we have $p(j) = p_{ij} + q_{ij}$, where $p(j)$ is the number of path sets of ϕ which are of size j.

Proposition 1. *Let ϕ be a structure function defined on N. For any $S \subseteq N$ and in view of formula* (2.1), *we have*

$$a_S = \sum_{T \subseteq S} (-1)^{|S-T|} \phi(e^T) \tag{3.1}$$

Proof. We have $\phi(\mathbf{x}) = \sum_{S \subseteq N} a_S \prod_{j \in S} x_j$ for $\mathbf{x} \in B^n$. This gives us for any $S \subseteq N$

$$\phi(e^S) = \sum_{T \subseteq S} a_T$$

and the required result follows from the well known Mobius Inversion Theorem (see Berge (1971), pp. 83–85 or Ramamurthy (1990), p. 31).

Proposition 2. *For a monotone structure* ϕ, *the element* $m(\phi)_{ij}$ *of the structural matrix*, $M(\phi)$ *as stated by formula* (2.2), *is given by*

$$m(\phi)_{ij} = p_{ij} + \sum_{r=1}^{j-1} (-1)^{j-r} \left\{ \binom{n-r}{j-r} p_{ir} + \binom{n-r-1}{j-r-1} q_{ir} \right\} \tag{3.2}$$

Proof. For $T \subseteq N, 1 \leq i, j \leq n$, define $h_{ij}(T) = |\{S : S \in A_{ij}, S \supseteq T\}|$, we then have

$$\begin{aligned} h_{ij}(T) &= 0 && \text{if} \quad |T| > j \\ & && \text{or} \quad |T| = j \quad \text{and} \quad i \notin T \\ &= 1 && \text{if} \quad |T| = j \quad \text{and} \quad i \in T \\ &= \binom{n-r}{j-r} && \text{for} \quad i \in T, \quad \text{and} \quad |T| = r < j\,(1 \leq r < j) \\ &= \binom{n-r-1}{j-r-1} && \text{for} \quad i \notin T \quad \text{and} \quad |T| = r < j\,(0 \leq r < j). \end{aligned}$$

This gives us

$$\begin{aligned} m(\phi)_{ij} &= \sum_{S \in A_{ij}} a_S \\ &= \sum_{S \in A_{ij}} \sum_{T \subseteq S} (-1)^{|S-T|} \phi(e^T) \\ &= \sum_{T \subseteq N} h_{ij}(T)\, \phi(e^T)(-1)^{j-|T|} \\ &= \sum_{i \in T \subseteq N} (-1)^{j-|T|} h_{ij}(T) \phi(e^T) + \sum_{i \notin T \subseteq N} (-1)^{j-|T|} h_{ij}(T) \phi(e^T) \\ &= p_{ij} + \sum_{r=1}^{j-1} (-1)^{j-r} \left[\binom{n-r}{j-r} p_{ir} + \binom{n-r-1}{j-r-1} q_{ir} \right] \end{aligned}$$

Remark 1. $m(\phi)_{ij} = 0$ for $j < r_0$ and $i = 1, 2, \ldots, n$ where r_0 is the smallest integer for which $p(r_0) > 0$.

Remark 2. If the cut sets of the system are known we can use the same approach to get $M(\phi^D)$ where ϕ^D is a dual structure of ϕ given by $\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$ for $\mathbf{x} \in B^n$.

Example. Consider the system of components given in Fig. 1.

The minimal path sets of this system are: $\{1,3\}, \{2,3\}$, and $\{4\}$. Also we have

$$((p_{ij})) = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \quad ((q_{ij})) = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$p(1) = 1,\ p(2) = 5,\ p(3) = 4,\ p(4) = 1$. This gives us

$$\begin{aligned} m(\phi)_{i1} &= p_{i1} \\ m(\phi)_{i2} &= p_{i2} - 3p_{i1} - q_{i1} \\ m(\phi)_{i3} &= p_{i3} + 3p_{i1} + 2q_{i1} - 2p_{i2} - q_{i2} \\ m(\phi)_{i4} &= p_{i4} - p(1) + p(2) - p(3). \end{aligned}$$

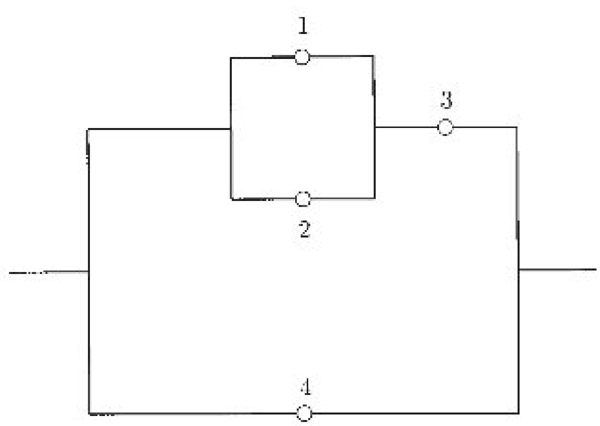

Figure 1. Block diagram.

for $i = 1, 2, 3, 4$. Hence we obtain

$$M(\phi) = \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & -2 & 1 \end{bmatrix}$$

In a monotone structure (N, ϕ) components i and $k \in N$ are said to be in series (parallel) if whenever component i occurs in a minimal path (cut) set of (N, ϕ), component k also occurs in the same minimal path (cut) set of (N, ϕ) and conversely. We say that component $i \in N$ is in series (parallel) with the rest of the system if every path (cut) set contains i.

Proposition 3. *If components i and k are either in both series or both parallel in a coherent system* (N, ϕ), *we then have*

$$m(\phi)_{ij} = m(\phi)_{kj} \quad \textit{for} \quad j = 1, 2, \ldots, n.$$

Proof.

(i) If components i and $k \in N$ are in series in (N, ϕ), it follows that $p_{ij} = p_{kj}$ and $q_{ij} = q_{kj}$ for $j = 1, 2, \ldots, n$. Hence we have from formula (3.2) that $m(\phi)_{ij} = m(\phi)_{kj}$ for $j = 1, 2, \ldots, n$ whenever i and j are in series.

(ii) Suppose that components i and k are in parallel in (N, ϕ), it implies that $m(\phi^D)_{ij} = m(\phi^D)_{kj}$ for $j = 1, 2, \ldots, n$. We also note from Seth and Ramamurthy (1991), that $M(\phi) = M(\phi^D)T_n$ where $T_n = ((t_{ij}))$ is a square matrix with

$$t_{ij} = (-1)^{j-1}\binom{i-1}{j-1} \quad \text{for} \quad i \geq j$$
$$= 0 \qquad \text{for} \quad i < j$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Hence we have $m(\phi)_{ij} = m(\phi)_{kj}$ for $j = 1, 2, \ldots, n$ whenever i and j are in parallel in (N, ϕ).

Examples. In case of a series structure, we have for $i = 1, 2, \ldots, n$

$$p_{ij} = \begin{cases} 0 & \text{for} \quad j < n \\ 1 & \text{for} \quad j = n \end{cases} \quad \text{and}$$
$$q_{ij} = 0, \quad \text{for} \quad j = 1, 2, \ldots, n.$$

It follows that

$$m(\phi)_{ij} = \begin{cases} 0 & \text{for} \quad j < n \\ 1 & \text{for} \quad j = n \end{cases}$$

For parallel structure. A parallel structure (N, ϕ) is the dual of the series structure. Recall the relationship $M(\phi) = M(\phi^D)T_n$ and note that $T_n^2 = I_n$. We then have for the parallel structure $m(\phi)_{ij} = \binom{n-1}{j-1}(-1)^{j-1}$

Proposition 4. *For a monotone structure (N, ϕ), we have $\sum_{j=1}^{n} m(\phi)_{ij} =$ 0 or 1 for every $i \in N$. Furthermore $\sum_{j=1}^{n} m(\phi)_{ij} = 1$ if and only if component i is in series with the rest of the system.*

Proof. We have for any $i \in N$

$$\begin{aligned}\sum_{j=1}^{n} m(\phi)_{ij} &= \sum_{j=1}^{n}\sum_{r=1}^{j}(-1)^{j-r}\binom{n-r}{j-r}p_{ir} + \sum_{j=1}^{n}\sum_{r=1}^{j-1}(-1)^{j-r}\binom{n-r-1}{j-r-1}q_{ir} \\ &= \sum_{r=1}^{n} p_{ir}\sum_{j=r}^{n}(-1)^{j-r}\binom{n-r}{j-r} \\ &\quad - \sum_{r=1}^{n-1} q_{ir}\sum_{j=r+1}^{n}(-1)^{j-r-1}\binom{n-r-1}{j-r-1} \\ &= p_{in} - q_{i,n-1}\end{aligned}$$

We note that $p_{in} = 1$ for any monotone structure. Since there is only one set of size $n-1$ which does not contain i, it follows that $q_{i,n-1} = 0$ or 1. It is easy to verify that for a monotone structure, $q_{i,n-1} = 0$ if and only if i is in series with the rest of the system. This completes the proof.

Proposition 5. *Let (N, ϕ) be a monotone structure with structural matrix $M(\phi)$. Consider the monotone structure $(\overline{N}, \overline{\phi})$ where $\overline{N} = N \cup \{n+1\}$ and component $(n+1)$ is in series with the original system (N, ϕ). We then have*

$$\begin{aligned} m(\overline{\phi})_{i,j+1} &= m(\phi)_{ij} \quad \textit{for} \quad j = 1, 2, \ldots, n \quad \textit{and} \quad i = 1, 2, \ldots, n \\ m(\overline{\phi})_{i1} &= 0 \quad \textit{for} \quad i = 1, 2, \ldots, n+1 \\ m(\overline{\phi})_{ij} &= \sum_{r=1}^{j-1}(-1)^{j-1-r}\left[\binom{n-r}{j-1-r}p(r)\right], \\ &\qquad i = n+1 \quad \textit{and} \quad j = 2, \ldots, n+1 \end{aligned}$$

Proof. Let

$$\phi(x_1, x_2, \ldots, x_n) = \sum_{S\subseteq N} b_S \prod_{j\in S} x_j.$$

This gives us

$$\bar{\phi}(x_1, x_2, \ldots, x_n, x_{n+1}) = \sum_{S\subseteq N} b_S \left(\prod_{j\in S} x_j\right) x_{n+1}.$$

The first two results easily follow. Making use of formula (3.1), we have for $j > 2$

$$m(\bar{\phi})_{n+1,\, j} = \sum_{|S|=j-1} b_S = \sum_{|S|=j-1} \sum_{T\subseteq S} (-1)^{|S-T|} \phi(e^T).$$

By changing the order of summation, we get

$$\sum_{|S|=j-1} b_S = \sum_{r=1}^{j-1} (-1)^{j-1-r} \binom{n-r}{j-1-r} p(r).$$

Proposition 6. *Let $(\bar{N}, \phi^*)$ represent a monotone structure in which component $(n+1)$ is in parallel with a given monotone structure (N, ϕ). Then the structural matrix $M(\phi^*)$ of $(\bar{N}, \phi^*)$ is given by*

$$\begin{aligned} m(\phi^*)_{ij} &= m(\phi)_{ij} - m(\phi)_{i,j-1},\ i = 1, 2, \ldots, n \quad \textit{and} \quad j = 2, \ldots, n \\ &= m(\phi)_{i1} \ \textit{for}\ i = 1, 2, \ldots, n \quad \textit{and} \quad j = 1 \\ &= 1 \quad \textit{for} \quad i = n+1,\ j = 1 \\ &= \sum_{r=1}^{j-1} (-1)^{j-r} \left[\binom{n-r}{j-r-1} p(r) \right] \quad \textit{for} \quad i = n+1 \end{aligned}$$

Proof. We have

$$\begin{aligned} &\phi^*(x_1, x_2, \ldots, x_n, x_{n+1}) \\ &\quad = \phi(x_1, x_2, \ldots, x_n) + x_{n+1} - \phi(x_1, x_2, \ldots, x_n) x_{n+1} \end{aligned}$$

if

$$\phi(x_1, x_2, \ldots, x_n) = \sum_{S\subseteq N} b_S \prod_{j\in S} x_j$$

we get

$$\phi^*(x_1, x_2, \ldots, x_n, x_{n+1}) = \sum_{S\subseteq N} b_S \prod_{j\in S} x_j + x_{n+1} - x_{n+1} \sum_{S\subseteq N} b_S \prod_{j\in S} x_j.$$

The first three required results follows trivially from the above. To prove the remaining part, note that for $j \geq 2$

$$m(\phi^*)_{n+1,\, j} = -\sum_{\substack{S \subseteq N \\ |S| = j-1}} b_S$$
$$= -\sum_{r=1}^{j-1} (-1)^{j-1-r} \binom{n-r}{j-1-r} p(r)$$
$$= \sum_{r=1}^{j-1} (-1)^{j-r} \binom{n-r}{j-1-r} p(r).$$

Hence the required result follows.

4. CONCLUSIONS

The problem of determining component importance in a system is of a great practical importance to reliability and design engineers. Different approaches followed are based on critical path vectors, structure function and so on. In case of game theory, a similar problem is faced while determining the importance of players in a simple game. In simple games a set of individuals or players, $N = \{1, 2, \ldots, n\}$ must collectively decide to accept or reject a given proposal or bill. Each player either votes yes or no. A set of individuals is called a coalition. A coalition is called a winning(blocking) coalition if it ensures acceptance(rejection) of the bill by voting yes(no). An important problem often faced is of finding numerical indices to represent the amount of influence a player has on the outcome of the game. Conceptually coherent systems of the reliability theory and simple games have the same underlying monotone set system. For example path sets (cut sets) in reliability theory are called winning coalitions (blocking coalitions) in game theory, etc. For detailed correspondence between the terminology of reliability and game theory see Ramamurthy (1990). In fact two popular measures used for quantifying the importance of a player in the game theory were rediscovered in the reliabilty theory (see Ramamurthy (1990)).

A unified approach for determining component importance was developed by Seth and Ramamurthy (1991). This approach uses the concept of structural matrix of the system. The structural matrix is determined from the simple form of the structure function. In this article we have considered the determination of elements of the structural matrix from the path sets or cut sets of the system using a combinatorial approach. Because of the conceptual similarities between coherent systems and simple games, the results of this article are also applicable to the problems in simple games.

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REFERENCES

Barlow, R. E., Proschan, F. (1975). Importance of system components and fault tree events. *Stochastic Processes and Their Applications* 3:153–173.

Berge, C. (1971). *Principles of Combinatorics*. New York: Academic Press. 83–85.

Birnbaum, Z. W. (1969). On the importance of different components in multicomponent system. In: Krishnaiah, P. R., ed. *Multivariate Analysis—II*. New York: Academic Press, pp. 581–592.

Butler, D. A. (1979). A complete importance ranking for components of binary coherent system with extensions to multi-state system. *Naval Research Logistics Quarterly* 26:565–578.

Ramamurthy, K. G. (1990). *Coherent Structures and Simple Games*. The Netherlands: Kluwer Academic Publishers, pp. 28–35.

Seth, A., Ramamurthy, K. G. (1991). Structural importance of components. *Opsearch* 28:88–101.