ESTIMATION OF DIMENSION FUNCTIONS OF BAND-LIMITED WAVELETS

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ABSTRACT. The dimension function D_ψ of a band-limited wavelet ψ is bounded by n if $\hat{\psi}$ is supported in $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$. For each $n \in \mathbb{N}$ and for each ϵ, $0 < \epsilon < \delta = \delta(n)$, we construct a wavelet ψ with supp $\hat{\psi} \subseteq [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ such that $D_\psi > n$ on a set of positive measure, which proves that $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$ is the largest symmetric interval for estimating the dimension function by n. This construction also provides a family of (uncountably many) wavelet sets each consisting of infinite number of intervals.

1. Introduction

A wavelet is a function $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k) : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Given a wavelet ψ of $L^2(\mathbb{R})$, there is an associated function D_ψ, called the dimension function of ψ, defined by

$$D_\psi(\xi) = \sum_{j\geq 1}\sum_{k\in\mathbb{Z}} |\hat{\psi}(2^j(\xi + 2k\pi))|^2. \tag{1}$$

A simple periodization argument shows that $\int_0^{2\pi} D_\psi(\xi)d\xi = 2\pi\|\psi\|_2^2$, if $\psi \in L^2(\mathbb{R})$. So the function D_ψ is well defined and is finite a.e. Observe that D_ψ is 2π-periodic. P. G. Lemarié [6, 7] used this function to show that certain wavelets are associated with a multiresolution analysis (MRA) of $L^2(\mathbb{R})$. P. Auscher [1] proved that if ψ is a wavelet, then the function D_ψ is the dimension of certain closed subspaces of the sequence space $l^2(\mathbb{Z})$ (hence the name dimension function, a term coined by Guido Weiss). This result in particular proves that D_ψ is integer valued a.e. G. Gripenberg [4] and X. Wang [8], independently, characterized all wavelets of $L^2(\mathbb{R})$ associated with an MRA. This well known characterization states that a wavelet ψ of $L^2(\mathbb{R})$ is associated with an MRA if and only if $D_\psi = 1$ a.e. The article [2] contains a characterization of all dimension functions.

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A function is said to be band-limited if its Fourier transform is compactly supported. It is easy to see that the dimension function of a band-limited wavelet is bounded.

Proposition 1. *Let $n \in \mathbb{N}$. If ψ is a wavelet such that supp $\hat{\psi} \subset [-2n\pi, 2n\pi]$, then $D_\psi \leq n$ a.e.*

The above proposition is not optimal. For example, D_ψ is still bounded by 1 for wavelets ψ such that supp $\hat{\psi} \subseteq [-\frac{8}{3}\pi, \frac{8}{3}\pi]$, which is proved in [5] (see section 3.4). The authors of [3] constructed an example of a wavelet ψ with supp $\hat{\psi} \subseteq [-\frac{8}{3}\pi, \frac{8}{3}\pi + \epsilon], 0 < \epsilon < \frac{2}{3}\pi$, such that $D_\psi \geq 2$ a.e. on a set of positive measure, which shows that $[-\frac{8}{3}\pi, \frac{8}{3}\pi]$ is the largest symmetric interval for estimating the dimension function by 1. A natural question to ask is whether there are optimal symmetric intervals to estimate the dimension function by n, $n \geq 2$. The following theorem sheds light to the above question.

Theorem 1. *Let $n \in \mathbb{N}$. If ψ is a wavelet such that $\hat{\psi}$ is supported in $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$, then $D_\psi \leq n$ a.e.*

This result was also proved by Z. Rzeszotnik and D. Speegle in an unpublished article. They also proved that for every positive integer n and every ϵ, $0 < \epsilon < \delta(n)$, there exists an MSF wavelet ψ such that supp $\hat{\psi} \subset [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ and $\|D_\psi\|_\infty > n$. This shows that $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi]$ is the optimal symmetric interval for estimating the dimension function by n. We thank Professor Guido Weiss for kindly providing the above information to us. The purpose of this article is to construct such a wavelet explicitly. We shall prove the following theorem in a constructive manner.

Theorem 2. *For each $n \in \mathbb{N}$ and $0 < \epsilon < \delta = \delta(n)$, there exists a wavelet ψ such that supp $\hat{\psi} \subseteq [-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ and $\|D_\psi\|_\infty > n$.*

A wavelet ψ of $L^2(\mathbb{R})$ is said to be a *minimally supported frequency* (MSF) wavelet if $|\hat{\psi}|$ is the characteristic function of some measurable subset K of $\mathbb{R}$. The associated set K is called a *wavelet set*. A simple characterization of such sets is the following (see [5] for a proof):

A set $K \subset \mathbb{R}$ is a wavelet set if and only if both the collections $\{K + 2k\pi : k \in \mathbb{Z}\}$ and $\{2^j K : j \in \mathbb{Z}\}$ are partitions of $\mathbb{R}$.

It is not always easy to construct wavelet sets satisfying desired properties. The concepts of translation and dilation equivalence of subsets of $\mathbb{R}$ are useful for this purpose. A set $A \subset \mathbb{R}$ is said to be 2π-*translation equivalent* to a set $B \subset \mathbb{R}$ if there exists a partition $\{A_n : n \in \mathbb{Z}\}$ of A such that $\{B_n \equiv A_n + 2n\pi : n \in \mathbb{Z}\}$ is a partition of B. Similarly, A

is said to be 2-*dilation equivalent* to B if there exists another partition $\{A'_n : n \in \mathbb{Z}\}$ of A such that $\{B'_n \equiv 2^n A'_n : n \in \mathbb{Z}\}$ is a partition of B.

In view of the characterization of wavelet sets stated above, it is now clear that a subset K of $\mathbb{R}$ is a wavelet set if and only if K is 2π-translation equivalent to some interval of length 2π, $K \cap (0, \infty)$ is 2-dilation equivalent to $[a, 2a]$ for some $a > 0$, and $K \cap (-\infty, 0)$ is 2-dilation equivalent to $[-2b, -b]$ for some $b > 0$.

2. Proofs of the theorems

Proof of Proposition 1. Let $F(\xi) = \sum_{j\geq 1} |\hat{\psi}(2^j\xi)|^2$. The condition on the support of $\hat{\psi}$ implies that supp $F \subset [-n\pi, n\pi]$. Since the equality,

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R},$$

is satisfied by every wavelet ψ, we have $F \leq 1$. Therefore, we get $F \leq \chi_{[-n\pi,n\pi]}$. This implies that

$$D_\psi(\xi) = \sum_{k\in\mathbb{Z}} F(\xi + 2k\pi) \leq \sum_{k\in\mathbb{Z}} \chi_{[-n\pi,n\pi]}(\xi + 2k\pi) = n,$$

which proves the proposition. □

Proof of Theorem 1. Since the function D_ψ is 2π-periodic, it is enough to prove that if ψ satisfies the hypothesis, then $D_\psi(\xi) \leq n$ for $\xi \in [-\pi, \pi]$. For $\xi \in [-\pi, \pi]$, we have $(2k-1)\pi \leq \xi + 2k\pi \leq (2k+1)\pi$ for all $k \in \mathbb{Z}$.

(i) $j = n$. If $k \geq 2$, then $2^j(\xi + 2k\pi) \geq 2^j(2k-1)\pi = 2^n(2k-1)\pi \geq 3 \cdot 2^n\pi \geq \frac{2^{n+2}}{3}\pi$. Similarly, if $k \leq -2$, then $2^n(\xi + 2k\pi) \leq -\frac{2^{n+2}}{3}\pi$. Hence, for $j = n$, the only non-zero terms contributing to D_ψ are for $k = -1, 0, 1$.

(ii) $j \geq n+1$. If $k \geq 1$, then $2^j(\xi + 2k\pi) \geq 2^j(2k-1)\pi \geq 2^{n+1}(2k-1)\pi \geq 2^{n+1}\pi \geq \frac{2^{n+2}}{3}\pi$. Similarly, if $k \leq -1$, then $2^j(\xi+2k\pi) \leq -\frac{2^{n+2}}{3}\pi$. Hence, for $j \geq n+1$, only contributing k to D_ψ is $k = 0$.

Thus, we have

$$\begin{aligned} D_\psi(\xi) &= \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2 \\ &\quad + \sum_{k=-1}^{1} |\hat{\psi}\big(2^n(\xi + 2k\pi)\big)|^2 + \sum_{j\geq n+1} |\hat{\psi}(2^j\xi)|^2 \qquad (2) \\ &= \sum_{j=1}^{n-1}\sum_{k\in\mathbb{Z}} |\hat{\psi}\left(2^j(\xi + 2k\pi)\right)|^2 \end{aligned}$$

$$+\left\{|\hat{\psi}\left(2^n(\xi-2\pi)\right)|^2+|\hat{\psi}\left(2^n(\xi+2\pi)\right)|^2\right\}+\sum_{j\geq n}|\hat{\psi}(2^j\xi)|^2. \tag{3}$$

If $\xi\in[-\frac{2}{3}\pi,\frac{2}{3}\pi]$, then $2^n(\xi+2\pi)\geq 2^n\cdot\frac{4}{3}\pi=\frac{2^{n+2}}{3}\pi$. Similarly, $2^n(\xi-2\pi)\leq-\frac{2^{n+2}}{3}\pi$. So both the terms inside the curly bracket in (3) are zero, and we get $D_\psi\leq(n-1)+1=n$. Now, if $\xi\in[\frac{2}{3}\pi,\pi]$, then for all $j\geq n+1$, we have $2^j\xi\geq 2^{n+1}\cdot\frac{2}{3}\pi=\frac{2^{n+2}}{3}\pi$. So the last sum in (2) is zero and again $D_\psi\leq n$. In a similar manner, it can be shown that $D_\psi\leq n$ if $\xi\in[-\pi,-\frac{2}{3}\pi]$. This finishes the proof. □

Proof of Theorem 2. The wavelets we construct to prove Theorem 2 are MSF wavelets so that it suffices to construct the associated wavelet sets. In addition to proving the theorem, this construction also provides an example of a family of wavelet sets which are union of infinite number of intervals. We will treat the even and odd cases separately.

Case I. n is even

For a real number ϵ such that $0<\epsilon<\delta=\frac{2^{n+2}}{3(2^{n+2}-1)}\pi$, let $S_i,\ 1\leq i\leq 6$ be the following sets.

$$\begin{aligned}
S_1 &= \left[-\tfrac{2^{n+2}}{3}\pi,-\tfrac{2^{n+2}}{3}\pi+\epsilon\right],\\
S_2 &= \left[-\tfrac{1}{3}\pi+\tfrac{\epsilon}{2^{n+2}},-\tfrac{1}{6}\pi\right],\\
S_3 &= \left[\tfrac{2^{n+2}}{3}\pi+\epsilon-2\pi,\tfrac{2^{n+2}}{3}\pi-\tfrac{5}{3}\pi+\tfrac{\epsilon}{2^{n+2}}\right],\\
S_4 &= \left[\tfrac{2^{n+2}}{3}\pi-\tfrac{3}{2}\pi,\tfrac{2^{n+2}}{3}\pi-\tfrac{7}{6}\pi+\tfrac{\epsilon}{2^{n+3}}\right],\\
S_5 &= \left[\tfrac{2^{n+2}}{3}\pi-\pi,\tfrac{2^{n+2}}{3}\pi-\tfrac{2}{3}\pi\right],\\
S_6 &= \left[\tfrac{2^{n+2}}{3}\pi-\tfrac{2}{3}\pi+\epsilon,\tfrac{2^{n+2}}{3}\pi+\epsilon\right].
\end{aligned}$$

Define the sets X_0, Y_0 and Z_0 as

$$\begin{aligned}
X_0 &= \left[\tfrac{1}{6}\pi+\tfrac{\epsilon}{2^{n+3}},\tfrac{1}{3}\pi-\tfrac{1}{2^{n+1}}\pi+\tfrac{\epsilon}{2^{n+2}}\right],\\
Y_0 &= \tfrac{1}{2^{n+2}}\left(S_2+2\cdot\tfrac{2^{n+1}-2}{3}\pi\right),\\
Z_0 &= \left[\tfrac{1}{3}\pi-\tfrac{1}{3\cdot 2^{n+1}}\pi,\tfrac{1}{3}\pi-\tfrac{1}{3\cdot 2^{n+1}}\pi+\tfrac{\epsilon}{2^{n+2}}\right].
\end{aligned}$$

The parameter ϵ is chosen in a suitable manner to make the above sets non-empty. For $j\geq 1$, let the sets X_j,Y_j,Z_j be defined recursively as follows:

$$P_j=\tfrac{1}{2^{n+2}}\left(P_{j-1}+2\cdot\tfrac{2^{n+1}-2}{3}\pi\right),\ j\geq 1,\quad P\in\{X,Y,Z\}.$$

By a routine calculation, we can easily verify the following facts:

(i) $P_j\subset\left[\frac{1}{6}\pi+\frac{\epsilon}{2^{n+3}},\frac{1}{3}\pi\right], j\geq 0,\quad P\in\{X,Y,Z\}$.

(ii) $2^{n+2}P_j\subset\left[\frac{2^{n+2}}{3}\pi-\frac{7}{6}\pi+\frac{\epsilon}{2^{n+3}},\frac{2^{n+2}}{3}\pi-\pi\right]$ for $j\geq 1,\quad P\in\{X,Y,Z\}$.

(iii) $\{X_j,Y_j,Z_j: j\geq 0\}$ is a disjoint collection.

(iv) X_j lies to the left of Y_j, and Y_j lies to the left of Z_j for $j \geq 0$.
(v) $X_{j+1}, Y_{j+1}, Z_{j+1}$ lie between Y_j and $Z_j, j \geq 0$.

Let

$$X = \bigcup_{j \geq 0} X_j,\ Y = \bigcup_{j \geq 0} Y_j,\ Z = \bigcup_{j \geq 0} Z_j,$$

and

$$V = [\tfrac{2^{n+2}}{3}\pi - \tfrac{7}{6}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{2^{n+2}}{3}\pi - \pi] \setminus \Big\{ \bigcup_{j \geq 1} 2^{n+2}(X_j \cup Y_j \cup Z_j) \Big\}.$$

Now define

$$W = \Big(\bigcup_{i=1}^{6} S_i \Big) \cup (X \cup Y \cup Z) \cup V. \tag{4}$$

Claim. W is a wavelet set.

Translation equivalence: We will show that W is translation equivalent to the interval $[\frac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ of length 2π. Note that

$$S_3 \cup \big(S_2 + 2 \cdot \tfrac{2^{n+1}-2}{3}\pi\big) \cup S_4 = \big[\tfrac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \tfrac{2^{n+2}}{3}\pi - \tfrac{7}{6}\pi + \tfrac{\epsilon}{2^{n+3}}\big],$$

and

$$S_5 \cup \big(S_1 + 2 \cdot \tfrac{2^{n+2}-1}{3}\pi\big) \cup S_6 = \big[\tfrac{2^{n+2}}{3}\pi - \pi, \tfrac{2^{n+2}}{3}\pi + \epsilon\big].$$

Now,

$$\begin{aligned}
&(X \cup Y \cup Z) + 2 \cdot \tfrac{2^{n+2}-1}{3}\pi \\
&= \bigcup_{j \geq 0} \Big\{ (X_j + 2 \cdot \tfrac{2^{n+1}-2}{3}\pi) \cup (Y_j + 2 \cdot \tfrac{2^{n+1}-2}{3}\pi) \cup (Z_j + 2 \cdot \tfrac{2^{n+1}-2}{3}\pi) \Big\} \\
&= \bigcup_{j \geq 0} 2^{n+2}(X_{j+1} \cup Y_{j+1} \cup Z_{j+1}) = \bigcup_{j \geq 1} 2^{n+2}(X_j \cup Y_j \cup Z_j).
\end{aligned}$$

Therefore, by the definition of V

$$V \cup \big\{ (X \cup Y \cup Z) + 2 \cdot \tfrac{2^{n+1}-2}{3}\pi \big\} = \big[\tfrac{2^{n+2}}{3}\pi - \tfrac{7}{6}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{2^{n+2}}{3}\pi - \pi\big].$$

Observe that $\frac{2^{n+1}-2}{3}\pi$ and $\frac{2^{n+2}-1}{3}\pi$ are integers, since n is even. We have proved that appropriate translations of the partition of W in (4) form a partition of the interval $[\frac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$. So W is translation equivalent to this interval.

Dilation equivalence: It is enough to show that $W \cap (-\infty, 0)$ and $W \cap (0, \infty)$ are respectively dilation equivalent to the intervals $[-\frac{2^{n+2}}{3}\pi, -\frac{2^{n+1}}{3}\pi]$ and $[\frac{2^{n+1}}{3}\pi + \frac{\epsilon}{2}, \frac{2^{n+2}}{3}\pi + \epsilon]$.

$$S_1 \cup (2^{n+2} S_2) \quad = \quad \big[-\tfrac{2^{n+2}}{3}\pi, -\tfrac{2^{n+1}}{3}\pi\big],$$

$$
\begin{aligned}
(2^{n+2}X_0) \cup S_3 \cup (2^{n+2}Y_0) \cup S_4 &= \left[\tfrac{2^{n+1}}{3}\pi + \tfrac{\epsilon}{2}, \tfrac{2^{n+2}}{3}\pi - \tfrac{7}{6}\pi + \tfrac{\epsilon}{2^{n+3}}\right], \\
S_5 \cup (2^{n+2}Z_0) \cup S_6 &= \left[\tfrac{2^{n+2}}{3}\pi - \pi, \tfrac{2^{n+2}}{3}\pi + \epsilon\right], \\
\left\{2^{n+2}\Big(\bigcup_{j\geq 1}(X_j \cup Y_j \cup Z_j)\Big)\right\} \cup V &= \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{7}{6}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{2^{n+2}}{3}\pi - \pi\right].
\end{aligned}
$$

Hence, W is a wavelet set.

Case II. n is odd

This case is dealt in a similar manner, but we have to start with different sets. For $0 < \epsilon < \frac{2^{n+2}}{3(2^{n+2}-1)}\pi$, let the sets S_1, S_2 be as above. Define

$$
\begin{aligned}
S_3 &= \left[\tfrac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \tfrac{2^{n+2}}{3}\pi - \tfrac{4}{3}\pi\right], \\
S_4 &= \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{4}{3}\pi + \epsilon, \tfrac{2^{n+2}}{3}\pi - \pi + \tfrac{\epsilon}{2^{n+2}}\right], \\
S_5 &= \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{5}{6}\pi, \tfrac{2^{n+2}}{3}\pi - \tfrac{1}{2}\pi + \tfrac{\epsilon}{2^{n+3}}\right], \\
S_6 &= \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi, \tfrac{2^{n+2}}{3}\pi + \epsilon\right].
\end{aligned}
$$

Let

$$
\begin{aligned}
X_0 &= \left[\tfrac{1}{6}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{1}{3}\pi - \tfrac{1}{2^{n+1}}\pi + \tfrac{\epsilon}{2^{n+2}}\right], \\
Y_0 &= \left[\tfrac{1}{3}\pi - \tfrac{1}{3\cdot 2^n}\pi, \tfrac{1}{3}\pi - \tfrac{1}{3\cdot 2^n}\pi + \tfrac{\epsilon}{2^{n+2}}\right], \\
Z_0 &= \tfrac{1}{2^{n+2}}\left(S_2 + 2\cdot\tfrac{2^{n+1}-1}{3}\pi\right).
\end{aligned}
$$

In this case also, the choice of ϵ ensures that the above sets are non-empty. Define the sets X, Y and Z as in Case I. Let

$$
V = \left[\tfrac{2^{n+2}}{3}\pi + \tfrac{\epsilon}{2^{n+3}} - \tfrac{1}{2}\pi, \tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi\right] \setminus \left\{\bigcup_{j\geq 1} 2^{n+2}\big(X_j \cup Y_j \cup Z_j\big)\right\},
$$

and let W be defined by (4).

As in the case when n is even, to show that W is a wavelet set, we show the translation equivalence of W with the interval $[\frac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$; and the dilation equivalence of $W \cap (-\infty, 0)$ and $W \cap (0, \infty)$ with the intervals $[-\frac{2^{n+2}}{3}\pi, -\frac{2^{n+1}}{3}\pi]$ and $[\frac{2^{n+1}}{3}\pi + \frac{\epsilon}{2}, \frac{2^{n+2}}{3}\pi + \epsilon]$ respectively.

To see the translation equivalence, observe that

$$
S_3 \cup \left(S_1 + 2\cdot\tfrac{2^{n+2}-2}{3}\pi\right) \cup S_4 \cup \left(S_2 + 2\cdot\tfrac{2^{n+1}-1}{3}\pi\right) \cup S_5 =
$$

$$
\left[\tfrac{2^{n+2}}{3}\pi + \epsilon - 2\pi, \tfrac{2^{n+2}}{3}\pi + \epsilon\right],
$$

$$
S_6 = [\tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi, \tfrac{2^{n+2}}{3}\pi + \epsilon].
$$

It can be shown, in a manner similar to Case I, that

$$V \cup \left\{(X \cup Y \cup Z) + 2 \cdot \tfrac{2^{n+1}-1}{3}\pi\right\} = \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{1}{2}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi\right].$$

For dilation equivalence, we observe

$$S_1 \cup (2^{n+2}S_2) = \left[-\tfrac{2^{n+2}}{3}\pi, -\tfrac{2^{n+1}}{3}\pi\right],$$

$$(2^{n+2}X_0) \cup S_3 \cup (2^{n+2}Y_0) \cup S_4 \cup (2^{n+2}Z_0) \cup S_5 =$$

$$\left[\tfrac{2^{n+1}}{3}\pi + \tfrac{\epsilon}{2}, \tfrac{2^{n+2}}{3}\pi - \tfrac{1}{2}\pi + \tfrac{\epsilon}{2^{n+3}}\right],$$

$$S_6 = \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi, \tfrac{2^{n+2}}{3}\pi + \epsilon\right],$$

$$\left\{2^{n+2}\Big(\bigcup_{j\geq 1}(X_j \cup Y_j \cup Z_j)\Big)\right\} \cup V = \left[\tfrac{2^{n+2}}{3}\pi - \tfrac{1}{2}\pi + \tfrac{\epsilon}{2^{n+3}}, \tfrac{2^{n+2}}{3}\pi - \tfrac{1}{3}\pi\right].$$

Hence, in this case also we have proved that W is a wavelet set.

By definining $\hat{\psi} = \chi_W$, we get a wavelet ψ such that $\hat{\psi}$ is supported in $[-\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$, since W is a subset of this interval.

Finally, to complete the proof of Theorem 2, we have to show that $\|D_\psi\|_\infty > n$, where $\hat{\psi} = \chi_W$. We prove $D_\psi(\xi) \geq n+1$ for a.e. $\xi \in [\frac{2}{3}\pi, \frac{2}{3}\pi + \frac{\epsilon}{2^{n+1}}]$.

For $1 \leq j \leq n+1$, let $k_j = \frac{2^{(n+1-j)}-1}{3}$ and $l_j = -\frac{2^{(n+1-j)}+1}{3}$. Observe that k_j is an integer if $n-j$ is odd, and l_j is an integer when $n-j$ is even.

Let $\xi \in [\frac{2}{3}\pi, \frac{2}{3}\pi + \frac{\epsilon}{2^{n+1}}]$. If n is even, then for $j = 1, 3, \ldots, n+1$, we have $2^j(\xi + 2k_j\pi) = 2^j(\xi - \frac{2}{3}\pi) + \frac{2^{n+2}}{3}\pi \in [\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon] \subset S_6$. Also, for $j = 2, 4, 6, \ldots n$, $2^j(\xi + 2l_j\pi) \in [-\frac{2^{n+2}}{3}\pi, -\frac{2^{n+2}}{3}\pi + \epsilon] = S_1$. Similarly, if n is odd, then $2^j(\xi + 2k_j\pi) \in [\frac{2^{n+2}}{3}\pi, \frac{2^{n+2}}{3}\pi + \epsilon]$ if $j = 2, 4, 6, \ldots n+1$, and $2^j(\xi + 2l_j\pi) \in [-\frac{2^{n+2}}{3}\pi, -\frac{2^{n+2}}{3}\pi + \epsilon]$ if $j = 1, 3, 5, \ldots, n$.

In each case, there are $n+1$ different pairs of (j,k), with $j \geq 1$ and $k \in \mathbb{Z}$, such that $2^j(\xi + 2k\pi)$ is contained in W which is the support of $\hat{\psi}$. Each such pair will contribute 1 to the sum $D_\psi(\xi)$ defined in (1). Therefore, $\|D_\psi\|_\infty \geq n+1$. □

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