Asymptotic Distribution of the Kaplan–Meier *U*-Statistics

Arup Bose

Indian Statistical Institute, Calcutta, India
E-mail: abose@isical.ac.in

and

Arusharka Sen

University of Hyderabad, Hyderabad, India
E-mail: asensm@uohyd.ernet.in

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Consider the Kaplan–Meier estimate of the distribution function for right randomly censored data. We show that a U-statistic defined via this estimate is asymptotically normal. Under a condition of degeneracy, different from the degeneracy condition in uncensored models, it has an asymptotic nonnormal distribution.

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1. INTRODUCTION

Let $X_1, ..., X_n$ be independent random variables with distribution F. Let $h(x_1, ..., x_m)$ be a real valued function symmetric in its arguments. The U-statistic with kernel h is defined as

$$U_n(h) = \binom{n}{m}^{-1} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} h(X_{i_1}, ..., X_{i_m}).$$

$U_n(h)$ is the nonparametric uniformly minimum variance estimator of $\theta = E(h(X_1, ..., X_m))$. It is the minimiser with respect to α of

$$\sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} (h(X_{i_1}, ..., X_{i_m}) - \alpha)^2.$$

Asymptotic properties of these statistics may be found in Lee (1990). For instance:

(i) $U_n(h) \to \theta = E(h(X_1, \ldots, X_m))$ almost surely if $E\,|h(X_1, \ldots, X_m)| < \infty$

(ii) $n^{1/2}(U_n(h) - \theta) \xrightarrow{d} N(0, \sigma_1^2)$ where $\sigma_1^2 = \text{var}(E(h(X_1, \ldots, X_m) \mid X_1)) < \infty$

(iii) If $\sigma_1^2 = 0$, $nU_n(h) \to \sum_{i=1}^{\infty} \lambda_i(\chi_i^2 - 1)$ in distribution where $\{\chi_i^2\}$ is a sequence of iid chi-squared random variables and $\{\lambda_i\}$ are the eigenvalues of an appropriate operator.

Many statistical functionals and estimators are approximately U-statistics and the above results are frequently used to establish their asymptotic properties. For example, M_m estimators, which are minimisers of $\sum_{1 \leqslant i_1 < \cdots < i_m \leqslant n} f(X_{i_1}, \ldots, X_{i_m}, \alpha)$, often have representations with a U-statistics leading term. See Bose (1998) and Lee (1990) for other examples.

Often the original observations are censored from the right. This means that $Y_1, \ldots, Y_n$ is another sequence of iid random variables, independent of $\{X_i\}$, with common distribution G, and one observes $(Z_i = \min[X_i, Y_i]$, $\delta_i = I\{X_i \leqslant Y_i\}$, $1 \leqslant i \leqslant n)$. It is a natural question how the results above get modified under this model. Such results would be potentially useful for studying the asymptotic properties of statistical functionals in the censored case.

Note that the usual U-statistics gives equal weight to all combinations. If the kernel size is one, this amounts to using the empirical distribution which puts mass $1/n$ at all X_i as the estimate of F. This equal weighting is not appropriate in censored data.

There are several estimates of F in the literature for censored data. The most famous is the Kaplan–Meier estimate. To describe this, suppose $Z_{1:n} \leqslant \cdots \leqslant Z_{n:n}$ are the ordered Z-values. Ties within X-values or Y-values may be ordered arbitrarily but ties among X's and Y's are treated as though X precedes Y. Let $\delta_{[i:n]} = \delta_j$ if $Z_{i:n} = Z_j$, $1 \leqslant i, j \leqslant n$. The Kaplan–Meier estimator F_n of F is defined as

$$1 - F_n(x) = \prod_{i=1}^{n} \left(1 - \frac{\delta_{[i:n]}}{n-i+1}\right)^{I\{Z_{i:n} \leqslant x\}}. \tag{1.1}$$

It is easily seen that F_n puts mass only at the Z-values and the mass at $Z_{i:n}$ equals

$$\begin{aligned} W_{in} &= \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1}\right)^{\delta_{[j:n]}} \\ &= \frac{1}{n}\,\delta_{[i:n]} \prod_{j=1}^{i-1} \left(1 + \frac{1-\delta_{[j:n]}}{n-j}\right). \end{aligned} \tag{1.2}$$

Thus, if there is no censoring, then $\delta_{[i:n]} = 1$ and hence $W_{in} = 1/n$ for all $1 \leqslant i \leqslant n$.

Suppose now that $h(\cdot)$ is a real valued function. If we use the Kaplan–Meier weights in the criterion mentioned in the beginning with $m = 1$, we get the estimator $S_{1n}(h)/S_{1n}(1)$, where, for any ϕ, we define

$$S_{1n}(\phi) = \int \phi \, dF_n = \sum_{i=1}^{n} \phi(Z_{i:n}) W_{in}. \tag{1.3}$$

Stute and Wang (1993) and Stute (1995) considered $S_{1n}(\phi)$ to be the analogue of the sample mean of $\{\phi(Z_i)\}$. If the random variables are uncensored it reduces to the usual sample mean. Because of the above discussion, $S_{1n}(h)/S_{1n}(1)$ may also be considered as an analogue of the sample mean. Note that $S_{1n}(1)$ is not equal to 1 in general.

Now consider kernels of size two. By extending the above analogy, Bose and Sen (1999) introduced the following U-statistics of degree two under this model. Let $\phi(\cdot, \cdot)$ be a real function, symmetric in its arguments. Let

$$U_{2n}(\phi) = \frac{\sum_{1 \leqslant i < j \leqslant n} \phi(Z_{i:n}, Z_{j:n}) W_{in} W_{jn}}{\sum_{1 \leqslant i < j \leqslant n} W_{in} W_{jn}} = \frac{S_{2n}(\phi)}{S_{2n}(1)}, \qquad \text{say.} \tag{1.4}$$

Our goal in this paper is to study the asymptotic distributional properties of this statistic. Such results may turn out useful in studying the asymptotic properties of M_2 estimates and other quadratic statistical functionals in the censored case.

Let us introduce some further notations and mention some of the results already known for the above statistics:

Let H be the distribution of Z_1. Then $(1-H) = (1-F)(1-G)$. For any distribution D, let $D\{a\} = D(a) - D(a-)$ be the jump of D at any point a, let $A_D = \{a: D\{a\} > 0\}$ be the set of atoms of D, and let $\tau_D = \inf\{x: D(x) = 1\}$. The following results are known:

(i) (Stute and Wang, 1993) If $\int |\phi(x)| \, F(dx) < \infty$ then as $n \to \infty$,

$$S_{1n}(\phi) \to \int \phi(x) \, \tilde{F}(dx) \qquad \text{almost surely,} \tag{1.5}$$

where

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if} \quad x < \tau_H \\ F(\tau_H-) + I\{\tau_H \in A_H\} \, F\{\tau_H\} & \text{if} \quad x \geqslant \tau_H. \end{cases}$$

(ii) (Stute, 1995, Theorem 1.1) Under assumptions given in (2.12) and (2.13) later, to ensure finite variance and to control the bias, the following representation holds,

$$S_{1n}(\phi) = n^{-1} \sum_{i=1}^{n} \{\phi(Z_i)\, \gamma_0(Z_i)\, \delta_i + \gamma_1(Z_i)(1-\delta_i) - \gamma_2(Z_i)\} + R_n, \quad (1.6)$$

where $R_n = o_P(n^{-1/2})$. The functions $\gamma_j(\cdot)$, $j = 0, 1, 2$, are defined in Section 2. (The representation appears to need a condition stronger than assumed by Stute (1995). See Remark 1 in Section 2.) This implies that $n^{1/2} \int \phi\, d(F_n - \tilde{F}) \to N(0, \sigma^2)$ where

$$\sigma^2 = V\big(\phi(Z)\, \gamma_0(Z)\, \delta + \gamma_1(Z)(1-\delta) - \gamma_2(Z)\big). \quad (1.7)$$

(iii) (Bose and Sen, 1999) If $E_F\, |\phi(X_1, X_2)| < \infty$, then as $n \to \infty$,

$$U_{2n}(\phi) \to S_2(\phi)/S_2(1) \qquad \text{almost surely,} \quad (1.8)$$

where

$$S_2(\phi) = \int \phi(x_1, x_2)\, \tilde{F}(dx_1)\, \tilde{F}(dx_2)$$

$$= \int_{\{x_1 < \tau_H\}} \bar{\phi}(x_1, F)\, F(dx_1) + I\{\tau_H \in A_H\}\, \bar{\phi}(\tau_H, F)\, F\{\tau_H\}$$

and

$$\bar{\phi}(x_1, F) = \int_{\{x_2 < \tau_H\}} \phi(x_1, x_2)\, F(dx_2) + I\{\tau_H \in A_H\}\, \phi(x_1, \tau_H)\, F\{\tau_H\}.$$

Gijbels and Veraverbeke (1991) considered a class of truncated Kaplan–Meier U-statistics, $\int_0^T \cdots \int_0^T h(x_1, ..., x_m) \prod_{i=1}^m F_n(dx_i)$, where $0 < T < \tau_H$ is fixed. They derived the limiting normal distribution for these statistics by using integration-by-parts. They rightly remark that from the point of application, this truncation is undesirable.

Here we study the asymptotic distribution of $U_{2n}(\phi)$ in its full generality. In Section 2 we show that $n^{1/2}(U_{2n}(\phi) - S_2(\phi)/S_2(1))$ is asymptotically normal solely under appropriate moment conditions. Our moment conditions are equivalent to the requirement that $\phi(\cdot, \cdot)$ belongs to the *tensor-product* of appropriate Banach spaces (see Section 3). This extends the CLT of Stute (1995) given above. As already mentioned, it appears that their condition needs to be strengthened slightly to obtain this result.

Our analysis follows Stute (1995). For simplicity, we start with a $\phi(\cdot,\cdot)$ with support $[0,T]\times[0,T]$ for some $T<\tau_H$. For such restricted ϕ, Stute (1995) expressed his statistic as a sum of three leading terms. One of them is a mean of iid random variables and the other two are the final result of projections of several approximating U-statistics of orders two and three. Similarly, we express our statistics as a combination of the projections of *four* U-statistics (in the usual sense) plus a remainder. The asymptotic normality then follows. The support restriction is then removed by an argument similar to that given in Stute (1995).

The normal limit for usual U-statistics can be degenerate under the well-known condition of degeneracy. Similarly, the above limit in the censored case can also be degenerate. We address this issue in Section 3. There we show that under the appropriate condition of degeneracy in the censored case, and with an appropriate centering θ, $n(U_{2n}(\phi)-\theta)$ has an asymptotic distribution given by a *double Wiener integral*. The condition for degeneracy is different from the uncensored case. Thus, situations can arise where the uncensored U-statistic is not degenerate but its censored version is and vice-versa. We demonstrate all these by a few examples.

In the degenerate case also, we start with a restricted ϕ. But the analysis is more delicate and cumbersome. First, extracting the relevant leading (degenerate) U-statistics requires significant additional work. Second, the remainder term needs to be of a much smaller order. Third, removal of the compact support assumption is more involved. In particular, all these need higher order Taylor's expansion, leading to several additional U-statistics. Once the representation is established, the limit distribution is then given by the double Wiener integrals of the degenerate U-statistic kernels which appear as the leading terms. See Theorem 3(a) and Remark 4 for a discussion of this. Our method may be extended to establish limit theorems for kernels of degree greater than two. But the algebra will become quite formidable. We wonder if there is some simpler approach to this problem.

Interestingly, U-statistic leading terms have also been encountered by earlier authors in the context of the Kaplan–Meier process. See, for example, Susarla and Van Ryzin (1980), who consider $n^{1/2}\int_0^{M_n}(\hat{F}_n-F)(x)\,dx$, and Koul and Susarla (1982), who study $n^{1/2}\sigma_n^{-1}\int_0^{t_n}(\hat{F}_n-F_n^0)(x)\,h_n(x)\,dx$. (Here $M_n>0$, $t_n>0$, $\sigma_n>0$ are appropriate constants, $h_n(\cdot)$ deterministic, and F_n^0 survival functions, $n\geqslant 1$.) However, they work with a slightly modified estimator $\hat{F}_n$.

2. THE NORMAL CONVERGENCE

In this section we tackle the regular case and prove a central limit theorem for $U_{2n}(\phi)$. First we define the functions involved in Eq. (1.6) to establish notation and to facilitate a comparison with the result of Stute (1995).

Define

$$H_0(z) = P(Z \leqslant z, \delta = 0) = \int_{-\infty}^{z} (1-F(y))\, G(dy)$$

$$H_1(z) = P(Z \leqslant z, \delta = 1) = \int_{-\infty}^{z} (1-G(y-))\, F(dy)$$

$$\gamma_0(x) = \exp\left(\int_{-\infty}^{x-} \frac{H_0(dz)}{1-H(z)}\right)$$

$$\gamma_1(x) = \frac{1}{1-H(x)} \int I\{x < w\}\, \phi(w)\, \gamma_0(w)\, H_1(dw)$$

$$\gamma_2(x) = \int\int I\{v < x,\, v < w\}\, \frac{\phi(w)\, \gamma_0(w)}{[1-H(v)]^2}\, H_0(dv)\, H_1(dw)$$

$$C(x) = \int I\{y < x\}(1-H(y))^{-2}\, H_0(dy)$$

$$= \int_{-\infty}^{x-} \frac{G(dy)}{(1-H(y))(1-G(y))}.$$

Define the empirical (sub)distribution function estimators of H, H_0, and H_1 as

$$H_n(z) = n^{-1} \sum_{i=1}^{n} I\{Z_i \leqslant z\}$$

$$H_{n0}(z) = n^{-1} \sum_{i=1}^{n} I\{Z_i \leqslant z, \delta_i = 0\}$$

$$H_{n1}(z) = n^{-1} \sum_{i=1}^{n} I\{Z_i \leqslant z, \delta_i = 1\}.$$

Lemma 1 expresses $S_{2n}(\phi)$ as a function of $\{(Z_i, \delta_i), 1 \leqslant i \leqslant n\}$ and H_n, H_{n0}, H_{n1} when H is continuous. The proof is along the proof of Lemma 2.1 of Stute (1995) and hence is omitted. It may be found in Bose and Sen (1996).

Lemma 1. *If H is continuous, then*

$$S_{2n}(\phi)=\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\phi(Z_i,Z_j)\,\delta_i\,\delta_j\exp\left\{n\int_{-\infty}^{Z_i-}\log\left(1+\frac{1}{n(1-H_n(s))}\right)H_{n0}(ds)\right.$$
$$\left.+n\int_{-\infty}^{Z_j-}\log\left(1+\frac{1}{n(1-H_n(t))}\right)H_{n0}(dt)\right\}.$$

We now extract the leading terms of $S_{2n}(\phi)$ by expanding the exponent above. A higher order expansion will be used in the next section. However, we decided to present the simpler expansion of this section separately for the sake of transparency and also as an insight into the more detailed analysis of the next section.

Define

$$\begin{aligned}A_{in}&=n\int_{-\infty}^{Z_i-}\log\left(1+\frac{1}{n(1-H_n(z))}\right)H_{n0}(dz)-\int_{-\infty}^{Z_i-}\frac{H_0(dz)}{1-H(z)}\\&=a_{in}-a_{i0}\\&=B_{in}+C_{in},\end{aligned}$$

where

$$B_{in}=n\int_{-\infty}^{Z_i-}\log\left(1+\frac{1}{n(1-H_n(z))}\right)H_{n0}(dz)-\int_{-\infty}^{Z_i-}\frac{H_{n0}(dz)}{1-H_n(z)}$$
$$C_{in}=\int_{-\infty}^{Z_i-}\frac{H_{n0}(dz)}{1-H_n(z)}-\int_{-\infty}^{Z_i-}\frac{H_0(dz)}{1-H(z)}.$$

Observe the two terms in the exponent for the expression of $S_{2n}(\phi)$ given in Lemma 1. Using the bivariate Taylor's expansion for $\exp(x+y)$ around (a_{i0}, a_{j0}) (note that $\exp(a_{i0})=\gamma_0(Z_i)$),

$$S_{2n}(\phi)=\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n}\phi(Z_i,Z_j)\,\delta_i\,\delta_j\left[\gamma_0(Z_i)\,\gamma_0(Z_j)\{1+B_{in}+C_{in}+B_{jn}+C_{jn}\}\right.$$
$$\left.+\frac{1}{2}e^{\Delta_i+\Delta_j}\{B_{in}+C_{in}+B_{jn}+C_{jn}\}^2\right],$$

where Δ_i and Δ_j lie between (a_{in}, a_{i0}) and (a_{jn}, a_{j0}), respectively.

The contributions to the leading term come from the C_{in} terms. The C_{in} term called T_1 below will be seen to contribute *three* (approximately) V statistics called T_{n1}, T_{n2}, and T_{n3}.

Lemma 2(a) claims the negligibilty of the remaining contribution from T_1. Lemma 2(b) and 2(c) claim the neglibility of the terms involving B_{in} and the terms involving the squares. The lemma is proved under a support condition on ϕ and the square integrability of the kernel. The omitted proof is availale in Bose and Sen (1996) but we stress that the main benefit of the support condition (2.8) is that all denominators are bounded away from 0 and thus integrability problems are avoided.

To isolate the leading terms, first note that for $z < Z_{n:n}$,

$$\frac{1}{1-H_n(z)} = -\frac{1-H_n(z)}{[1-H(z)]^2} + \frac{2}{1-H(z)} + \frac{[H_n(z)-H(z)]^2}{[1-H(z)]^2\,[1-H_n(z)]}.$$

Hence

$$\begin{aligned} C_{in} = &-\int_{-\infty}^{\tau_H}\int_{-\infty}^{\tau_H} I\{t<Z_i\}\, I\{t<s\}(1-H(t))^{-2}\, H_n(ds)\, H_{n0}(dt) \\ &+2\int_{-\infty}^{\tau_H} I\{t<Z_i\}(1-H(t))^{-1}\, H_{n0}(dt) \\ &-\int_{-\infty}^{\tau_H} I\{t<Z_i\}(1-H(t))^{-1}\, H_0(dt) \\ &+\int_{-\infty}^{\tau_H} I\{t<Z_i\}\, \frac{[H_n(t)-H(t)]^2}{[1-H(t)]^2\,[1-H_n(t)]}\, H_{n0}(dt). \end{aligned}$$

Let

$$g_\phi(u, v) = \phi(u, v)\, \gamma_0(u)\, \gamma_0(v). \tag{2.1}$$

Then

$$\begin{aligned} T_1 := &\frac{1}{n^2}\sum_{1\leqslant i<j\leqslant n} \phi(Z_i, Z_j)\, \delta_i\, \delta_j\, \gamma_0(Z_i)\, \gamma_0(Z_j)\{C_{in}+C_{jn}\} \\ = &\frac{1}{n^2}\sum_{1\leqslant i\neq j\leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j\, C_{in} \\ = &-\int_{\{u\neq v\}} g_\phi(u, v)\, I\{t<u, t<s\}(1-H(t))^{-2} \\ &\times H_n(ds)\, H_{n0}(dt)\, H_{n1}(du)\, H_{n1}(dv) \\ &+2\int_{\{u\neq v\}} g_\phi(u, v)\, I\{t<u\}(1-H(t))^{-1}\, H_{n0}(dt)\, H_{n1}(du)\, H_{n1}(dv) \end{aligned}$$

$$-\int_{\{u\neq v\}} g_\phi(u,v)\, I\{t<u\}(1-H(t))^{-1}\, H_0(dt)\, H_{n1}(du)\, H_{n1}(dv)$$

$$+\int_{\{u\neq v\}} g_\phi(u,v)\, I\{t<u\}\, \frac{[H_n(t)-H(t)]^2}{[1-H(t)]^2\,[1-H_n(t)]}$$

$$\times H_{n0}(dt)\, H_{n1}(du)\, H_{n1}(dv)$$

$$= -T_{n1}+2T_{n2}-T_{n3}+R_{n0}, \qquad \text{say.} \tag{2.2}$$

Now

$$-T_{n1} = -\int_{\{u\neq v\}} g_\phi(u,v)\, I\{t<u,\, t<s\}(1-H(t))^{-2}$$
$$\times [H_n(ds)\, H_0(dt)\, H_1(du)\, H_1(dv)$$
$$+H(ds)\, H_{n0}(dt)\, H_1(du)\, H_1(dv)+H(ds)\, H_0(dt)\, H_{n1}(du)\, H_1(dv)$$
$$+H(ds)\, H_0(dt)\, H_1(du)\, H_{n1}(dv)-3H(ds)\, H_0(dt)\, H_1(du)\, H_1(dv)]-R_{n1} \tag{2.3}$$

$$2T_{n2} = 2\int_{u\neq v} g_\phi(u,v)\, I\{t<u\}(1-H(t))^{-1}\, [H_{n0}(dt)\, H_1(du)\, H_1(dv)$$
$$+H_0(dt)\, H_{n1}(du)\, H_1(dv)+H_0(dt)\, H_1(du)\, H_{n1}(dv)$$
$$-2H_0(dt)\, H_1(du)\, H_1(dv)]+R_{n2} \tag{2.4}$$

$$-T_{n3} = -\int_{u\neq v} g_\phi(u,v)\, I\{t<u\}(1-H(t))^{-1}\, H_0(dt)[H_{n1}(du)\, H_1(dv)$$
$$+H_1(du)\, H_{n1}(dv)-H_1(du)\, H_1(dv)]+R_{n3}. \tag{2.5}$$

Note that after cancellations,

$$T_1 = -T_{n1}+2T_{n2}-T_{n3}+R_{n0}$$
$$= -\int_{u\neq v}\int g_\phi(u,v)\, I\{t<u,\, t<s\}(1-H(t))^{-2}\, H_n(ds)\, H_0(dt)\, H_1(du)\, H_1(dv)$$
$$+\int_{u\neq v} g_\phi(u,v)\, I\{t<u\}(1-H(t))^{-1}\, H_{n0}(dt)\, H_1(du)\, H_1(dv)$$
$$+R_{n1}+R_{n2}+R_{n3}+R_{n0}. \tag{2.6}$$

LEMMA 2. *Suppose the following two conditions hold:*

$$E\phi^2(X_1, X_2) < \infty. \tag{2.7}$$

$$\text{There exists } 0 < T < \tau_H \text{ such that } \phi(x, y) = 0 \quad \text{for } T < x, y < \tau_H. \tag{2.8}$$

Then

(a) *for* $j = 1, 2, 3,$

$$R_{nj} = \begin{cases} O_P(n^{-1}) \\ O(n^{-1} \log n) \qquad \text{almost surely.} \end{cases}$$

$$R_{n0} = O(n^{-1} \log \log n) \qquad \text{almost surely.}$$

(b)

$$T_2 := \frac{1}{n^2} \sum_{1 \leqslant i \neq j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j\, B_{in} = O(n^{-1}) \qquad \text{almost surely}$$

(c)

$$T_3 := \frac{1}{2n^2} \sum_{i \leqslant i < j \leqslant n} \phi(Z_i, Z_j)\, \delta_i\, \delta_j \exp(\Delta_i + \Delta_j)(B_{in} + C_{in} + B_{jn} + C_{jn})^2$$
$$= O(n^{-1} \log \log n) \qquad \text{almost surely.}$$

Having identified the leading terms, it is now a matter of expressing them in a more compact form. From (2.6), and Lemma 2(a)–2(c), it follows that if (2.7) and (2.8) hold,

$$S_{2n}(\phi) - \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j = T_1 + T_2 + T_3$$
$$= (-T_{n1} + 2T_{n2} - T_{n3}) + R_n, \tag{2.9}$$

where $R_n = O(n^{-1} \log n)$ almost surely. Expression for $(-T_{n1} + 2T_{n2} - T_{n3})$ is given in (2.6).

Using the fact that for $x < \tau_H$,

$$\gamma_0(x) = (1 - G(x-))^{-1}$$
$$H_1(dv) = (1 - G(v-))\, F(dv),$$
$$H_0(dt) = (1 - F(t))\, G(dt),$$

the *first term* in (2.6) may be written as $n^{-1} \sum_{i=1}^{n} \gamma_4(Z_i)$, where

$$\gamma_4(x) = \int g_\phi(u, v)\, C(u \wedge x)\, H_1(du)\, H_1(dv), \tag{2.10}$$

and the *second term* in (2.6) may be written as $n^{-1} \sum_{i=1}^{n} \gamma_3(Z_i)(1-\delta_i)$, where

$$\gamma_3(x) = \int \frac{1}{1-H(x)}\, g_\phi(u, v)\, I(x < u)\, H_1(du)\, H_1(dv). \tag{2.11}$$

Define

$$P_\phi(x) = E[\phi(x, Z)\, \gamma_0(Z)\, \delta],$$
$$L(Z, \delta; \phi) = P_\phi(Z)\, \gamma_0(Z)\, \delta + (1-\delta)\, \gamma_3(Z) - \gamma_4(Z).$$

Note here that

$$\gamma_3(Z) = S_\phi^{(1)}(Z)/(1-H(Z)),$$
$$\gamma_4(Z) = \int I\{w < Z\}\, S_\phi^{(1)}(w)(1-H(w))^{-2}\, H_0(dw)$$
$$= \int I\{w < Z\}\, \gamma_3(w)(1-H(w))^{-1}\, H_0(dw),$$

where

$$S_\phi^{(1)}(w) := \int g_\phi(u, v)\, I\{u > w\}\, H_1(du)\, H_1(dv)$$
$$= \int P_\phi(u)\, I\{u > w\}\, H_1(du).$$

We are now ready to give the central limit theorem. The proof is given in the Appendix.

THEOREM 1. *Suppose that the following conditions hold:*

$$E\, \delta_1\, \delta_2\, g_\phi^2(Z_1, Z_2) < \infty \tag{2.12}$$
$$E\, \delta_1\, \delta_2\, |g_\phi(Z_1, Z_2)|\, C(Z_1)\, C(Z_2) < \infty$$
$$E\, \delta_1 \{E(\delta_2\, g_\phi^2(Z_1, Z_2) \mid Z_1)\}^{1/2}\, C(Z_1) < \infty$$
$$E\, \delta_1 \{E(\delta_2\, g_\phi(Z_1, Z_2)\, C(Z_2) \mid Z_1)\}^2 < \infty. \tag{2.13}$$

Then

$$n^{-2} \sum_{1 \leqslant i < j \leqslant n} \phi(Z_{i:n}, Z_{j:n}) W_{in} W_{jn} = n^{-1} \sum_{i=1}^{n} L(Z_i, \delta_i; \phi) + o_P(n^{-1/2}). \quad (2.14)$$

Remark 1. Let us consider now the conditions under which Stute (1995) obtained his representation (Eq. (1.6)):

$$E[\phi(Z_1)\, \gamma_0(Z_1)\, \delta_1]^2 < \infty, \quad (S1)$$

$$\int |\phi(x)|\, C^{1/2}(x)\, \tilde{F}(dx) < \infty. \quad (S2)$$

Condition (S1) (Condition (1.5) in Stute (1995)) is the appropriate second moment condition and Condition (S2) (Condition (1.6) in Stute (1995)) controls the bias $n^{1/2}(E\int \phi\, dF_n - \int \phi\, d\tilde{F})$ which may not converge to zero. See Stute (1994) for a discussion of this issue.

Notice that Condition (2.12) is the exact analogue of Condition (S1), and the *first* condition in (2.13) is the analogue of Condition (S2). The other two conditions in (2.13) arise because we are now dealing with $\phi(\cdot,\cdot)$ belonging to a certain *tensor-product* space. This space is defined in Section 3 (Eqs. (3.6)–(3.7)) where we also include a discussion on these four conditions. Note also that in all the three conditions in (2.13), we have used $C(\cdot)$ rather than $C^{1/2}(\cdot)$ as in (S2). Replacing $C^{1/2}(\cdot)$ by $C(\cdot)$ in (S2) strengthens the assumption but this seems to be an essential requirement. We were unable to establish the square-integrability of $\gamma_2(\cdot)$ in Theorem 1.1 of Stute (1995) otherwise. The analogues of $\gamma_2(\cdot)$ in our case are $\gamma_4(\cdot)$ above and $\gamma_{3j}(\phi)(\cdot,\cdot)$, $j = 1, 2, 3$, in (3.8)–(3.9) (Theorem 2, Section 3) which are actually the tensor-products of $\gamma_2(\cdot)$ with itself and the other two functions of Eq. (1.5) (i.e., $\gamma_{3j}(\phi)(\cdot,\cdot) = \gamma_2(\cdot) \otimes \gamma_{j-1}(\cdot)$, $j = 1, 2, 3$).

Remark 2. Consider the denominator $S_{2n}(1)$ of $U_{2n}(\phi)$. For the sake of simplicity, assume that $F(\cdot)$ and $G(\cdot)$ are continuous. Then the leading summand $L(Z_1, \delta_1; 1)$ of (2.14) takes the form

$$\begin{aligned} L(Z_1, \delta_1; 1) &= \delta_1 F(\tau_H)(1 - G(Z_1))^{-1} \\ &\quad + (1 - \delta_1)[F(\tau_H)(1 - G(Z_1))^{-1} - F(\tau_H)(1 - F(\tau_H))(1 - H(Z_1))^{-1}] \\ &\quad - \Big[F(\tau_H) \int I\{y < Z_1\}(1 - G(y))^{-2}\, G(dy) \\ &\quad - F(\tau_H)(1 - F(\tau_H)) \int I\{y < Z_1\}(1 - G(y))^{-2}\, (1 - F(y))^{-1}\, G(dy) \Big]. \end{aligned}$$

Suppose $F(\tau_H)=1$; then $L(Z_1, \delta_1; 1) \equiv 1 = S_2(1)$. Thus if conditions (2.12) and (2.13) hold for $\phi \equiv 1$, then

$$n^{1/2}(S_{2n}(1)-1) = o_P(1). \tag{2.15}$$

Now,

$$n^{1/2}(U_{2n}(\phi) - S_2(\phi)/S_2(1)) = n^{1/2}(S_{2n}(\phi) - S_2(\phi))/S_{2n}(1) \\ - n^{1/2} S_2(\phi)(S_{2n}(1) - S_2(1))/(S_{2n}(1)\, S_2(1)).$$

Thus if ϕ and the constant function 1 satisfy (2.12) and (2.13), we get a representation for $c_1 S_{2n}(\phi) + c_2 S_{2n}(1)$ by Theorem 1 and consequently the asymptotic normality of $U_{2n}(\phi)$ (note that $S_{2n}(1) \to \tilde{F}^2(\tau_H)$ almost surely under our conditions by the SLLN of Bose and Sen (1999)). In the special case of continuous $F(\cdot)$ and $G(\cdot)$ and $F(\tau_H)=1$, $\lim_{n\to\infty} n^{1/2}(U_{2n}(\phi) - S_2(\phi)/S_2(1)) = \lim_{n\to\infty} n^{1/2}(S_{2n}(\phi) - S_2(\phi))$ in distribution by (2.15) above.

Example 1. Consider the variance kernel, namely,

$$\phi(x_1, x_2) = (x_1 - x_2)^2/2.$$

To facilitate comparison with the uncensored case, as well as to keep the discussion technically simple, assume that F and G are both continuous and $F(\tau_H)=1$. Let

$$\mu = \int_{-\infty}^{\infty} x F(dx) \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 F(dx).$$

Then,

$$\begin{aligned} P_\phi(x) &= \int \phi(x, v)\, \gamma_0(v)\, H_1(dv) \\ &= \int_{-\infty}^{\tau_H} \phi(x, v)\, F(dv) \\ &= (x-\mu)^2/2 + \sigma^2/2, \\ S_\phi^{(1)}(x) &= \int \phi(u, v)\, \gamma_0(u)\, \gamma_0(v)\, I\{u > x\}\, H_1(du)\, H_1(dv) \\ &= \int P_\phi(u)\, \gamma_0(u)\, I\{u > x\}\, H_1(du) \\ &= \int_{-\infty}^{\tau_H} [(u-\mu)^2/2 + \sigma^2/2]\, I\{u > x\}\, F(du) \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\tau_H} (u-\mu)^2 I\{u > x\}\, F(du) + \sigma^2(1 - F(x))\right], \end{aligned}$$

so that

$$\gamma_3(x) = \frac{S_\phi^{(1)}(x)}{1-H(x)}$$

$$= \frac{1}{2}\left[(1-H(x))^{-1} \int_{-\infty}^{\tau_H} (u-\mu)^2 I\{u>x\}\, F(du) + \sigma^2 (1-G(x))^{-1} \right]$$

$$\gamma_4(x) = \int I\{y<x\}(1-H(y))^{-1}\, \gamma_3(y)\, H_0(dy)$$

$$= \frac{1}{2}\Bigg[\int_{-\infty}^{\tau_H} I\{y<x\}((1-H(y))(1-G(y)))^{-1}$$

$$\times \left\{ \int_{-\infty}^{\tau_H} (u-\mu)^2 I\{u>y\}\, F(du) \right\} G(dy)$$

$$+\sigma^2 \int_{-\infty}^{\tau_H} I\{y<x\}(1-G(y))^{-2}\, G(dy) \Bigg].$$

Finally we get,

$$L(Z_1, \delta_1; \phi) - \sigma^2 = P_\phi(Z_1)\, \delta_1 \gamma_0(Z_1) + (1-\delta_1)\, \gamma_3(Z_1) - \gamma_4(Z_1) - \sigma^2$$

$$= \tfrac{1}{2}\, [\delta_1 \gamma_0(Z_1)(Z_1-\mu)^2 - \sigma^2]$$

$$+ \tfrac{1}{2}\, (1-\delta_1)(1-H(Z_1))^{-1} \int_{-\infty}^{\tau_H} (u-\mu)^2\, I\{u>Z_1\}\; F(du)$$

$$-\tfrac{1}{2} \int_{-\infty}^{\tau_H} I\{y<Z_1\}((1-H(y))(1-G(y)))^{-1}$$

$$\times \left\{ \int_{-\infty}^{\tau_H} (u-\mu)^2 I\{u>y\}\, F(du) \right\} G(dy).$$

Note that the first term on the right side above is exactly the Hajek projection of the variance kernel in the uncensored case (that is, when $\delta_1 \equiv 1$ and G is the point mass at ∞).

Example 2. Consider the problem of testing H_0: $F = F_0$ vs $H_1: F \neq F_0$. Koziol and Green (1976) studied the Cramer–von Mises test-statistic $\int_{-\infty}^{\infty} (F_n(x) - F_0(x))^2\, F_0(dx)$ in the random censoring framework. Let us now modify this statistic slightly as follows, in conformity with the structure of our U-statistic $U_{2n}(\phi)$. Define, with $S_{1n}(1)=\sum_{i=1}^{n} W_{in}$,

$$
\begin{aligned}
CV_n &= \int \left(\sum_{i=1}^{n} W_{in} I\{Z_{i:n} \leqslant x\}/S_{1n}(1) - F_0(x) \right)^2 F_0(dx) \\
&= (S_{1n}(1))^{-2} \sum_{i=1}^{n} W_{in}^2 \int (I\{Z_{i:n} \leqslant x\} - F_0(x))^2 F_0(dx) \\
&\quad + (S_{1n}(1))^{-2} \sum_{i \neq j} W_{in} W_{jn} \int (I\{Z_{i:n} \leqslant x\} - F_0(x)) \\
&\quad \times (I\{Z_{j:n} \leqslant x\} - F_0(x))\, F_0(dx) \\
&= (S_{1n}(1))^{-2} \left[\sum_{i=1}^{n} W_{in}^2 \phi(Z_{i:n}, Z_{i:n}) + 2 \sum_{i<j} W_{in} W_{jn} \phi(Z_{i:n}, Z_{j:n}) \right],
\end{aligned}
$$

where

$$
\phi(x_1, x_2) = \int (I\{x_1 \leqslant x\} - F_0(x))(I\{x_2 \leqslant x\} - F_0(x))\, F_0(dx).
$$

Assume, again, that F_0, G are continuous. We shall now show that, if $F_0(\tau_H) < 1$, then under H_0, $n^{1/2}(CV_n - c)$ is asymptotically normal for an appropriate centering constant c—a scenario quite different from the uncensored case where $\phi(x_1, x_2)$ is a *degenerate* kernel under H_0 and hence nCV_n is asymptotically a weighted sum of i.i.d χ_1^2 random variables.

In Section 3 (Example 5), we show that if $F_0(\tau_H) = 1$, $\phi(\cdot, \cdot)$ is a *C-degenerate* kernel (Definition 1, Section 3), and the limit of nCV_n is similar to that in the uncensored case.

Now under continuity and with $F_0(\tau_H) < 1$, we have under H_0,

$$
\begin{aligned}
P_\phi(Z_1) &= \int_{-\infty}^{\tau_H} \phi(Z_1, u)\, F_0(du) \\
&= \int_{-\infty}^{\infty} (I\{Z_1 \leqslant x\} - F_0(x))(F_0(x \wedge \tau_H) - F_0(x)\, F_0(\tau_H))\, F_0(dx)
\end{aligned}
$$

and

$$
\begin{aligned}
E[\delta_1\, \delta_2 \gamma_0(Z_1)\, \gamma_0(Z_2)\, \phi(Z_1, Z_2)] &= \int_{-\infty}^{\infty} (F_0(x \wedge \tau_H) - F_0(x)\, F_0(\tau_H))^2\, F_0(dx) \\
&= F_0^2(\tau_H)(1 - F_0(\tau_H))^2/3.
\end{aligned}
$$

Further, by reworking the arguments of Lemma 2.1 through Theorem 1.1 of Stute (1995), with W_{in} replaced by W_{in}^2, we get

$$\begin{aligned} n^{1/2} \sum_{i=1}^{n} W_{in}^2 \phi(Z_{i:n}, Z_{i:n}) &= n^{-1/2} \left[\frac{1}{n} \sum_{i=1}^{n} \phi(Z_i, Z_i)\, \delta_i \exp\left\{ 2n \int_{-\infty}^{Z_i-} \log\left(1 + \frac{1}{n(1-H_n(s))}\right) H_{n0}(ds) \right\} \right] \\ &= n^{-1/2} \left[\frac{1}{n} \sum_{i=1}^{n} \left(\phi(Z_i, Z_i)\, \delta_i \gamma_0^2(Z_i) + 2(1-\delta_i)\, \gamma_1(Z_i) - 2\gamma_2(Z_i) \right) + o_P(n^{-1/2}) \right] \\ &= n^{-1/2} O_P(1) \to 0 \qquad \text{as} \quad n \to \infty, \end{aligned}$$

under the stronger conditions that (see (S1), (S2) in Remark 1)

$$E[\phi(Z_1, Z_1)\, \gamma_0^2(Z_1)\, \delta_1]^2 < \infty, \tag{SS1}$$

$$\int |\phi(x, x)|\, \gamma_0(x)\, C^{1/2}(x)\, \tilde{F}(dx) < \infty. \tag{SS2}$$

(We remark here that the negligibility of $n^{1/2} \sum_{i=1}^{n} W_{in}^2 \phi(Z_{i:n}, Z_{i:n})$ perhaps holds under weaker conditions, by analogy with the uncensored case. This can probably be established by the martingale arguments of Stute and Wang (1993), which, however, is beyond the scope of the present work. Hence we take the weak representation approach of Stute (1995)).

By these observations, and by Theorem 1 and Remark 2, it follows that under H_0, continuity, and for $F_0(\tau_H) < 1$,

$$n^{1/2}(CV_n - (1 - F_0(\tau_H))^2/3) \to N(0, v(F_0, G)) \qquad \text{as} \quad n \to \infty,$$

for an appropriate $v(F_0, G) > 0$, provided the conditions of Theorem 1 hold for $\phi(\cdot, \cdot)$ and the constant function 1, and (SS1)–(SS2) hold for the function $\phi(x, x)$. (Note here that $\lim_{n \to \infty} (S_{1n}(1))^2 = F_0^2(\tau_H)$).

Remark 3. Theorem 1 in principle may be extended to statistics of the form below, involving higher order kernels by using arguments similar to those used above. However, the algebra may be quite formidable.

$$U_{kn}(\phi) = \frac{\sum_{1 \le i_1 < i_2 \cdots < i_k \le n} \phi(Z_{i_1:n}, Z_{i_2:n}, \ldots, Z_{i_k:n})\, W_{i_1 n} W_{i_2 n} \cdots W_{i_k n}}{\sum_{1 \le i_1 < i_2 \cdots < i_k \le n} W_{i_1 n} W_{i_2 n} \cdots W_{i_k n}}.$$

3. THE DEGENERATE CASE

In this section, we obtain the limit distribution of $S_{2n}(\phi)$, and hence of $U_{2n}(\phi)$, when the kernel $\phi(\cdot,\cdot)$ satisfies a degeneracy condition. Note that the two terms in the representation (2.14) are all first order projections of several U-statistics of order two or more. The first term is the projection $P_\phi(Z)\,\delta\gamma_0(Z) = E[\delta\delta_1\, g_\phi(Z, Z_1) \mid \delta, Z]$ of the second order U statistic $\bar{U}_{2n}(\phi)$ given by

$$\begin{aligned}\bar{U}_{2n}(\phi) &= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} g_\phi(Z_i, Z_j)\,\delta_i\,\delta_j \\ &= \int_{\{u<v\}} g_\phi(u, v)\, H_{n1}(du)\, H_{n1}(dv). \qquad (3.1)\end{aligned}$$

The other terms are the projections of the U-statistic in (2.2).

If $P_\phi(Z) = 0$, it is easy to see that the leading term in (2.14), $L(Z_1, \delta_1; \phi)$, equals zero.

Alternatively, if $P_\phi(Z)$ is a constant and $E[\delta\gamma_0(Z)] = 1$, then it is easy to see that $L(Z_1, \delta_1; \phi)$ is a constant. For example, if $\phi \equiv 1$, and $F(\tau_H) = 1$, then under continuity, $L(Z_1, \delta_1; 1) = 1$. See Remark 2. This motivates the following definition of degeneracy under censoring.

Definition 1. Call $\phi(\cdot,\cdot)$ *C-degenerate* if (2.12) holds and *at least one* of the following holds:

(A)

$$P_\phi(x) = E[\phi(x, Z)\,\delta\gamma_0(Z)] = \int \phi(x, u)\,\gamma_0(u)\,H_1(du) = 0 \qquad \forall x$$

(B)

$$P_\phi(x) \equiv c \neq 0 \qquad \text{and} \qquad E\,\delta\gamma_0(Z) = \tilde{F}(\tau_H) = 1.$$

Under continuity of $F(\cdot)$ and $G(\cdot)$, $P_\phi(x) = \int_{-\infty}^{\tau_H} \phi(x, u)\, F(du)$ and $E\,\delta\gamma_0(Z) = F(\tau_H)$.

Recall that the condition of degeneracy for uncensored U-statistics is $\int_{-\infty}^{\infty} \phi(x, u)\, F(du) = c\ \forall x \in \mathbb{R}$. Thus random censoring may enforce C-degeneracy—when $\tilde{F}(\tau_H) < 1$—on an otherwise nondegenerate $\phi(\cdot,\cdot)$. Conversely, a $\phi(\cdot,\cdot)$ degenerate in the standard sense may become non-C-degenerate under censoring, as we saw in the case of the Cramer–von Mises statistic in Example 2, Section 2.

EXAMPLE 3. Let F be continuous and ξ_p be the pth quantile of F for some $p<\frac{1}{2}$. In other words, $F(\xi_p)=p$. Now suppose $\tau_G=\xi_{2p}<\tau_F$ so that $\tau_H=\xi_{2p}$; that is, $F(\tau_H)=2p$. Let $\tilde{\phi}(x)=sign(x-\xi_p)$ and $\phi(x_1,x_2):=\tilde{\phi}(x_1)\,\tilde{\phi}(x_2)$. Then we have

$$\begin{aligned}P_\phi(x)&=\tilde{\phi}(x)\,E(\tilde{\phi}(Z)\,\delta\gamma_0(Z))\\&=\tilde{\phi}(x)\int_0^{\tau_H}\tilde{\phi}(u)\,F(du)\\&=\tilde{\phi}(x)(-F(\xi_p)+(F(\tau_H)-F(\xi_p)))\\&=\tilde{\phi}(x)(-p+(2p-p))\equiv 0.\end{aligned}$$

Hence $\phi(x_1,x_2)$ is a C-degenerate kernel, but not degenerate in the usual sense.

EXAMPLE 4. Let, again, F be continuous and define $\mu(\tau_H)=\int_{-\infty}^{\tau_H}xF(dx)$. Let $\tilde{\phi}(x)=x-\mu(\tau_H)$ and $\phi(x_1,x_2)=\tilde{\phi}(x_1)\,\tilde{\phi}(x_2)$. Then

$$P_\phi(x)=\tilde{\phi}(x)\int_{-\infty}^{\tau_H}(u-\mu(\tau_H))\,F(du)=\tilde{\phi}(x)\,\mu(\tau_H)(1-F(\tau_H)).$$

Thus if either $F(\tau_H)=1$ or $\mu(\tau_H)=0$, $\phi(\cdot,\cdot)$ becomes a degenerate kernel.

EXAMPLE 5. Consider the Cramer–von Mises statistic CV_n of Example 2, Section 2. Assume now that $F_0(\tau_H)=1$, in addition to continuity of F_0, G. The $\phi(\cdot,\cdot)$ defined in that example now becomes C-degenerate, since

$$\begin{aligned}P_\phi(v)&=\int_{-\infty}^{\tau_H}\phi(v,u)\,F_0(du)\\&=\int_{-\infty}^{\tau_H}(I\{v\leqslant x\}-F_0(x))(F_0(x\wedge\tau_H)-F_0(x)\,F_0(\tau_H))\,F_0(dx),\end{aligned}$$

and $F_0(x\wedge\tau_H)-F_0(x)\,F_0(\tau_H)\equiv 0$.

In Definition 1, note that if Case (B) holds, we may reduce it to Case (A) by considering the kernel $(\phi(\cdot,\cdot)-c)$. Hence in the following, by C-degeneracy we shall mean Case (A) WOLG. To derive the limit distribution, note that by C-degeneracy, Eqs. (2.9)–(2.11) and (2.14), since $P_\phi(z)=\gamma_3(z)=\gamma_4(z)=0$, we get $n^{1/2}S_{2n}(\phi)=o_P(1)$ as $n\to\infty$. Since $n^{-2}\sum_{1\leqslant i<j\leqslant n}g_\phi(Z_i,Z_j)\,\delta_i\,\delta_j$ is now a (first-order) degenerate U-statistic, we seek the limit distribution of $nS_{2n}(\phi)$, as $n\to\infty$. Thus we need an $o_P(n^{-1})$ expansion of $S_{2n}(\phi)$. It turns out that there are additional U-statistics which now contribute. Since these

U-statistics are degenerate in any case, they had not appeared in the discussion of the normal limit. Below, we express $S_{2n}(\phi)$ as a sum of $\bar{U}_{2n}(\phi)$ and *five* other degenerate U-statistics, plus an $o_p(n^{-1})$ remainder. In obtaining these expansions, we again first impose the restriction (2.8), and continuity of $H(\cdot)$, *in addition to* C-degeneracy. Subsequently, the restriction is removed as in the proof of Theorem 1 of Section 2.

Refer to the representation for $S_{2n}(\phi)$ in Lemma 1. We now use a *degree 3* expansion for $\exp(x+y)$ around (a_{i0}, a_{j0}) (recall that $\exp(a_{i0}) = \gamma_0(Z_i)$, see definition of a_{i0} after the statement of Lemma 1) and get

$$\begin{aligned}
S_{2n}(\phi) &= \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} \phi(Z_i, Z_j)\, \delta_i\, \delta_j \bigg[\gamma_0(Z_i)\, \gamma_0(Z_j) \bigg\{ 1 + (B_{in} + C_{in} + B_{jn} + C_{jn}) \\
&\qquad + \frac{1}{2}(B_{in} + C_{in} + B_{jn} + C_{jn})^2 \bigg\} + \frac{1}{6} e^{\Delta_i + \Delta_j}(A_{in}^3 + A_{in}^2 A_{jn} + A_{in} A_{jn}^2 + A_{jn}^3) \bigg] \\
&= \bar{U}_{2n}(\phi) + \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (B_{in} + B_{jn}) \\
&\qquad + \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (C_{in} + C_{jn}) \\
&\qquad + \frac{1}{2n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (B_{in}^2 + B_{jn}^2) \\
&\qquad + \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j B_{in} B_{jn} \\
&\qquad + \frac{1}{2n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (C_{in}^2 + C_{jn}^2) \\
&\qquad + \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j C_{in} C_{jn} \\
&\qquad + \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (B_{in} + B_{jn})(C_{in} + C_{jn}) \\
&\qquad + \frac{1}{6n^2} \sum_{1 \leqslant i < j \leqslant n} \phi(Z_i, Z_j)\, \delta_i\, \delta_j e^{\Delta_i + \Delta_j}(A_{in}^3 + A_{in}^2 A_{jn} + A_{in} A_{jn}^2 + A_{jn}^3) \\
&= \bar{U}_{2n}(\phi) + \sum_{k=1}^{8} \bar{T}_k, \qquad \text{say}, \qquad (3.2)
\end{aligned}$$

where Δ_i, A_{in}, B_{in}, C_{in}, $1 \leqslant i \leqslant n$, are defined as before, and the definition of $\bar{T}_k$, $1 \leqslant k \leqslant 8$, is clear.

The terms $\bar{T}_2$ and $\bar{T}_6$, in addition to $\bar{U}_{2n}(\phi)$, determine the limit distribution of $S_{2n}(\phi)$. All other terms will be shown to be $o(n^{-1})$ almost surely. Further, $\bar{T}_2$ and $\bar{T}_6$ admit representations as a sum of degenerate U-statistics plus a remainder. For this purpose, we need a higher-order expansion for C_{in} than that in Section 2.

Recall that $C_{in} = \int I(s < Z_i)\, H_{n0}(ds)/(1 - H_n(s)) - \int I(s < Z_i)\, H_0(ds)/(1 - H(s))$.

Write, for $s < Z_{n:n}$,

$$\frac{1}{1-H_n(s)} = \frac{(1-H_n(s))^2}{(1-H(s))^3} - 3\,\frac{1-H_n(s)}{(1-H(s))^2} + 3\,\frac{1}{(1-H(s))} + \frac{(H_n(s)-H(s))^3}{(1-H(s))^3\,(1-H_n(s))}.$$

Thus

$$\begin{aligned} C_{in} = {} & \int I(s < Z_i)\, I(s < t_1)\, I(s < t_2)(1 - H(s))^{-3}\, H_n(dt_1)\, H_n(dt_2)\, H_{n0}(ds) \\ & - 3 \int I(s < Z_i)\, I(s < t)(1 - H(s))^{-2}\, H_n(dt)\, H_{n0}(ds) \\ & + 3 \int I(s < Z_i)(1 - H(s))^{-1}\, H_{n0}(ds) \\ & - \int I(s < Z_i)(1 - H(s))^{-1}\, H_0(ds) \\ & + \int I(s < Z_i)\, R_{n0}(ds), \end{aligned} \tag{3.3}$$

where $R_{n0}(ds) = [H_n(s) - H(s)]^3\,(1 - H(s))^{-3}\,(1 - H_n(s))^{-1}\, H_{n0}(ds)$.

For real variables z, z_i, $i \geqslant 1$, define the functions

$$a_n(z_1, \ldots, z_n) = I(z_1 < z_2) \cdots I(z_1 < z_n)(1 - H(z_1))^{n-1}, \qquad n \geqslant 2,$$

and

$$a_1(z) = \int_{\mathbb{R}} a_2(x, z)\, H_0(dx).$$

The following lemma gives the representation for $\bar{T}_2$. Its proof, given in the Appendix, uses an argument similar to that used in the proof of Lemma 2(a) along with properties of degenerate U-statistics.

LEMMA 3. *Under* (2.7) *and* (2.8), *we have*

$$\bar{T}_2 := \frac{1}{n^2} \sum_{1 \leq i < j \leq n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (C_{in} + C_{jn})$$

$$= -\int_{\{v \neq t\}} \left[\int g_\phi(u, v)\, a_3(s, t, u)\, H_0(ds)\, H_1(du) \right] (H_n - H)(dt)\, H_{n1}(dv)$$

$$+ \int_{\{v \neq s\}} \left[\int g_\phi(u, v)\, a_2(s, u)\, H_1(du) \right] (H_{n0} - H_0)(ds)\, H_{n1}(dv)$$

$$-n^{-1} \alpha_1(\phi) + o_P(n^{-1}) \qquad as \quad n \to \infty, \tag{3.4}$$

where $\alpha_1(\phi) = \int_{\mathbb{R}^3} g_\phi(u, v)\, a_3(s, u, v)\, H_0(ds)\, H_1(du)\, H_1(dv)$.

Let

$$\alpha_2(\phi) = \int_{\mathbb{R}^5} g_\phi(u, v)\, a_3(s, u, t)\, a_3(\bar{s}, v, t)\, H_0(ds)\, H_1(du)\, H_0(d\bar{s})\, H_1(dv)\, H(dt),$$

$$\alpha_{12}(\phi) = \int_{\mathbb{R}^4} g_\phi(u, v)\, a_3(s, u, \bar{s})\, a_2(\bar{s}, v)\, H_0(ds)\, H_0(d\bar{s})\, H_1(du)\, H_1(dv),$$

i.e., $\alpha_2(\phi)$, $\alpha_{12}(\phi)$, $\alpha_1(\phi)$ are the expectations of the three U-kernels given in (3.5) below with $\{t = \bar{t}\}$, $\{t = \bar{s}\}$, and $\{s = \bar{s}\}$, respectively.

The next lemma provides the expansion for $\bar{T}_6$. Its proof is given in the Appendix.

LEMMA 4. *Under* (2.7) *and* (2.8)

$$\bar{T}_6 \equiv \frac{1}{2n^2} \sum_{1 \leq i \neq j \leq n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j C_{ij} C_{jn}$$

$$= \frac{1}{2} \int_{\{t \neq \bar{t}\}} \left[\int_{\mathbb{R}^4} g_\phi(u, v)\, a_3(s, u, t)\, a_3(\bar{s}, v, \bar{t})\, H_0(ds) \right.$$

$$\left. \times H_1(du)\, H_0(d\bar{s})\, H_1(dv) \right] (H_n - H)(dt)(H_n - H)(d\bar{t})$$

$$- \int_{\{t \neq \bar{s}\}} \left[\int_{\mathbb{R}^3} g_\phi(u, v)\, a_3(s, u, t)\, a_2(\bar{s}, v)\, H_0(ds) \right.$$

$$\left. \times H_1(du)\, H_1(dv) \right] (H_n - H)(dt)(H_{n0} - H_0)(d\bar{s})$$

$$+ \frac{1}{2} \int_{\{s \neq \bar{s}\}} \left[\int_{\mathbb{R}^2} g_\phi(u, v)\, a_2(s, u)\, a_2(\bar{s}, v)\, H_1(du)\, H_1(dv) \right]$$

$$\times (H_{n0} - H_0)(ds)(H_{n0} - H_0)(d\bar{s})$$

$$+ n^{-1} \left(\frac{1}{2} \alpha_2(\phi) - \alpha_{12}(\phi) + \frac{1}{2} \alpha_1(\phi) \right) + o_P(n^{-1}). \tag{3.5}$$

Lemma 5 provides the negligibility of the terms $\bar{T}_1, \bar{T}_3 - \bar{T}_5, \bar{T}_7, \bar{T}_8$. The term $\bar{T}_1$, involving B_{in} and B_{jn}, is similar to that treated in Lemma 2(b) and a refinement of that proof is now needed. Similarly, the proof for $\bar{T}_2$ is a refinement of the proof of Lemma 2(c) along with properties of cumulative hazard functions and degenerate U-statistics. The proofs for the other terms are easier and very similar to those of the proofs of Lemmas 2(b) and 2(c). We refer the reader to Bose and Sen (1997) for the proofs.

LEMMA 5. *Under* (2.7) *and* (2.8), *we have a.s., as* $n \to \infty$,

$$
\begin{aligned}
\bar{T}_1 &= \frac{1}{n^2} \sum_{1 \leq i < j \leq n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (B_{in} + B_{jn}) = o(n^{-1}) \\
\bar{T}_3 &= \frac{1}{2n^2} \sum \sum \{\ldots\} (B_{in}^2 + B_{jn}^2) = O(n^{-2}) \\
\bar{T}_4 &= \frac{1}{n^2} \sum \{\ldots\}\, B_{in} B_{jn} = O(n^{-2}) \\
\bar{T}_5 &= \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j (C_{in}^2 + C_{jn}^2) = o(n^{-1}) \\
\bar{T}_7 &= \frac{1}{n^2} \sum \{\ldots\} (B_{in} + B_{jn})(C_{in} + C_{jn}) = O(n^{-3/2} (\log n)^{1/2}) \\
\bar{T}_8 &= \frac{1}{6n^2} \sum \{\ldots\}\, e^{A_i + A_j} (A_{in}^3 + A_{in}^2 A_{jn} + A_{in} A_{jn}^2 + A_{jn}^3) \\
&= O(n^{-3/2} (\log n)^{3/2}).
\end{aligned}
$$

Theorems 2 and 3 are the main results of this section. Theorem 2 provides a representation for $S_{2n}(\phi)$ as a sum of *six* degnerate U-statistics with an appropriate remainder for a general ϕ, which *does not* necessarily satisfy the truncation condition (2.8). However, we impose the integrability restrictions on ϕ given in Theorem 1. These restrictions mean that $\phi(\cdot, \cdot)$ belongs to a tensor-product space which seems to be the proper anologue, under random censoring, of the usual *symmetric* L_2-*tensor product* space

$$\{\phi: \mathbb{R}^2 \to \mathbb{R} \mid \phi(\cdot, \cdot) \text{ symmetric}, E\phi(X_1, x_2) = 0\ \forall x_2 \in \mathbb{R}, E\phi^2(X_1, X_2) < \infty\}$$

in the case of classical U-statistics (see, for example, Dynkin and Mandelbaum (1983)).

Theorem 3 describes the limit distribution of $nS_{2n}(\phi)$ in terms of appropriate *double Wiener integrals* which are limits of linear combinations of products of mean-zero normal random variables. It is based on the representation given in Theorem 2. For the limit distribution of $nU_{2n}(\phi)$, see Remark 5 after Theorem 3.

To state Theorem 2, first recall that $g_\phi(z_1, z_2) = \phi(z_1, z_2)\, \gamma_0(z_1)\, \gamma_0(z_2)$. Then C-degeneracy is, of course, equivalent to $E\{\delta_2 g_\phi(z_1, Z_2)\} = 0\ \forall z_1 \in \mathbb{R}$.

The following are the four conditions on ϕ, given in Eqs. (2.12) and (2.13), which are really the finiteness of the following four vector space norms. Note that (N_1) is just condition (2.12).

$$(N_1)\quad \|\phi(\cdot,\cdot)\|_1^2 := E\,\delta_1\,\delta_2 g_\phi^2(Z_1, Z_2) < \infty$$
$$(N_2)\quad \|\phi(\cdot,\cdot)\|_2 := E\,\delta_1\,\delta_2\,|g_\phi(Z_1, Z_2)|\,C(Z_1)\,C(Z_2) < \infty$$
$$(N_{12})\quad \|\phi(\cdot,\cdot)\|_{12} := E\,\delta_1\{E(\delta_2 g_\phi^2(Z_1, Z_2)\,|\,Z_1)\}^{1/2}\,C(Z_1) < \infty$$
$$(N_{21})\quad \|\phi(\cdot,\cdot)\|_{21}^2 := E\,\delta_1\{E(\delta_2 g_\phi(Z_1, Z_2)\,C(Z_2)\,|\,Z_1)\}^2 < \infty$$

Denoting the Banach spaces

$$\mathscr{V}_2 = \{\phi(\cdot,\cdot)\,|\,\phi \text{ symmetric, } C\text{-degenerate and } \|\phi\|_1 + \|\phi\|_2 + \|\phi\|_{12} + \|\phi\|_{21} < \infty\},$$
$$\mathscr{V}_1 = \{\phi\colon \mathbb{R}\to\mathbb{R}\,|\,E\,\delta_1\phi(Z_1)\,\gamma_0(Z_1) = 0,\ E\,\delta_1\,\phi^2(Z_1)\,\gamma_0^2(Z_1) < \infty,$$
$$E\,\delta_1\,|\phi(Z_1)|\,\gamma_0(Z_1)\,C(Z_1) < \infty\}, \tag{3.6}$$

it is clear that $N_1 - N_{21}$ are equivalent to $\phi \in \mathscr{V}_2$. It can be shown that the class of functions $\{\phi(\cdot)\,\phi(\cdot)\,|\,\phi(\cdot)\in\mathscr{V}_1\}$ is *total* in $\mathscr{V}_2$. In other words,

$$\mathscr{V}_2 = \mathscr{V}_1 \otimes \mathscr{V}_1 \equiv \text{symmetric tensor-product of } \mathscr{V}_1 \text{ with itself.} \tag{3.7}$$

Note that for $\phi \in \mathscr{V}_1$, the norm is

$$\|\phi\| = (E\,\delta_1\phi^2(Z_1)\,\gamma_0^2(Z_1))^{1/2} + E\,\delta_1\,|\phi(Z_1)|\,\gamma_0(Z_1)\,C(Z_1).$$

The norm on $\mathscr{V}_2$ is thus a *cross-norm* generated by that on $\mathscr{V}_1$. (See Light and Cheney (1985, Chap. 1).)

Next for the ease of writing, we introduce the following short-hand for the (degenerate) U-statistics obtained in Lemmas 3 and 4. Let

$$\gamma_{21}(\phi)(s, v) := \int \phi(u, v)\,\gamma_0(u)\,\gamma_0(v)\,a_2(s, u)\,H_1(du)$$

$$\gamma_{31}(\phi)(t, v) := \int\int \phi(u, v)\,\gamma_0(u)\,\gamma_0(v)\,a_3(s, u, t)\,H_0(ds)\,H_1(du)$$

$$\gamma_{22}(\phi)(s, \bar{s}) := \int\int \phi(u, v)\,\gamma_0(u)\,\gamma_0(v)\,a_2(s, u)\,a_2(\bar{s}, v)\,H_1(du)\,H_1(dv)$$

$$\gamma_{32}(\phi)(t, \bar{s}) := \int\int\int \phi(u, v)\,\gamma_0(u)\,\gamma_0(v)\,a_3(s, u, t)\,a_2(\bar{s}, v) \times H_0(ds)\,H_1(du)\,H_1(dv)$$

$$\gamma_{33}(\phi)(t, \bar{t}) := \int\int\int\int \phi(u, v)\,\gamma_0(u)\,\gamma_0(v)\,a_3(s, u, t)\,a_3(\bar{s}, v, \bar{t}) \times H_0(ds)\,H_1(du)\,H_0(d\bar{s})\,H_1(dv). \tag{3.8}$$

Further, write the corresponding U-statistics in terms of the respective empirical measures,

$$\int\int_{\{t \neq \bar{t}\}} \gamma_{33}(\phi)(t, \bar{t})(H_n - H)(dt)(H_n - H)(d\bar{t}) := \gamma_{33}(\phi)(H_n - H, H_n - H),$$

and so on. Then we have the following theorem. The proof is given in the Appendix.

THEOREM 2. *Assume* (N_1), (N_2), (N_{12}), *and* (N_{21}) *hold. Then* $S_{2n}(\phi)$ *admits the following representation:*

$$\begin{aligned} S_{2n}(\phi) = \bar{U}_{2n}(\phi) &+ \gamma_{21}(\phi)(H_{n0} - H_0, H_{n1}) - \gamma_{31}(\phi)(H_n - H, H_{n1}) - n^{-1}\alpha_1(\phi) \\ &+ \tfrac{1}{2}\gamma_{22}(\phi)(H_{n0} - H_0, H_{n0} - H_0) - \gamma_{32}(\phi)(H_n - H_1, H_{n0} - H_0) \\ &+ \tfrac{1}{2}\gamma_{33}(\phi)(H_n - H, H_n - H) \\ &+ n^{-1}\left(\tfrac{1}{2}\alpha_2(\phi) - \alpha_{12}(\phi) + \tfrac{1}{2}\alpha_1(\phi)\right) + o_P(n^{-1}). \end{aligned} \tag{3.9}$$

(Note that $\bar{U}_{2n}(\phi)$ *could be written, using the above convention, as* $g_\phi(H_{n1}, H_{n1})$.*)*

Before describing the limiting distribution, let us take a look at the representation (3.9). Note that, of the U-kernels appearing in (3.9), $g_\phi(\cdot,\cdot)$, $\gamma_{22}(\phi)(\cdot,\cdot)$, and $\gamma_{33}(\phi)(\cdot,\cdot)$ are *symmetric* while $\gamma_{21}(\phi)(\cdot,\cdot)$, $\gamma_{31}(\phi)(\cdot,\cdot)$, and $\gamma_{32}(\phi)(\cdot,\cdot)$ are *asymmetric* (see (3.8)). Further, denoting $V_i := (Z_i, \delta_i)$, $i \geqslant 1$, let us write

$$\bar{U}_{2n}(\phi) = \frac{1}{n^2}\sum_{i<j} \delta_i\, \delta_j\, g_\phi(Z_i, Z_j) := \frac{1}{n^2}\sum_{i<j} \bar{g}_\phi(V_i, V_j),$$

$$\begin{aligned} &\frac{1}{2}\gamma_{22}(\phi)(H_{n0} - H_0, H_{n0} - H_0) \\ &\quad = \frac{1}{n^2}\sum_{i<j} [(1-\delta_i)(1-\delta_j)\,\gamma_{22}(Z_i, Z_j) - (1-\delta_i)\, E(1-\delta)\,\gamma_{22}(\phi)(Z_i, Z) \\ &\qquad - (1-\delta_j)\, E(1-\delta)\,\gamma_{22}(\phi)(Z, Z_j) + E\{(1-\delta_1)(1-\delta_2)\,\gamma_{22}(\phi)(Z_1, Z_2)\}] \\ &\quad := \frac{1}{n^2}\sum_{i<j} \bar{\gamma}_{22}(\phi)(V_i, V_j) \end{aligned}$$

$$\frac{1}{2}\gamma_{33}(\phi)(H_n - H, H_n - H)$$

$$= \frac{1}{n^2}\sum_{i<j}[\gamma_{33}(\phi)(Z_i, Z_j) - E\gamma_{33}(\phi)(Z_i, Z)$$

$$-E\gamma_{33}(\phi)(Z, Z_j) + E\gamma_{33}(\phi)(Z_1, Z_2)]$$

$$:= \frac{1}{n^2}\sum_{i<j}\bar{\gamma}_{33}(V_i, V_j)$$

$$\gamma_{21}(\phi)(H_{n0} - H_0, H_{n1})$$

$$= \frac{1}{n^2}\sum_{i\neq j}[(1-\delta_i)\,\delta_j\gamma_{21}(\phi)(Z_i, Z_j) - \delta_j E(1-\delta)\,\gamma_{21}(\phi)(Z, Z_j)]$$

$$:= \frac{1}{n^2}\sum\sum_{i\neq j}\bar{\gamma}_{21}(\phi)(V_i, V_j), \tag{3.10}$$

and define $\bar{\gamma}_{31}(\cdot,\cdot)$, $\bar{\gamma}_{32}(\cdot,\cdot)$ analogously.

Note also that

$$\alpha_1(\phi) = E\delta\gamma_{31}(\phi)(Z, Z) = E(1-\delta)\,\gamma_{22}(\phi)(Z, Z)$$

$$\alpha_2(\phi) = E\gamma_{33}(\phi)(Z, Z)$$

$$\alpha_{12}(\phi) = E(1-\delta)\,\gamma_{32}(\phi)(Z, Z).$$

Since $\bar{g}_\phi$, $\bar{\gamma}_{ij}$ are degenerate kernels (of (V_1, V_2)), the limit distribution of $nS_{2n}(\phi)$ can now be expressed as a constant plus *double Wiener integral* of the kernel $(\bar{g}_\phi + \sum_{(i,j)} d_{ij}\bar{\gamma}_{ij})$, where $d_{ij} = -1$ if $(i, j) \in \{(3, 1), (3, 2)\}$ and $d_{ij} = 1$ otherwise.

For details on multiple Wiener integrals, see Parthasarathy (1992, pp. 105–111) and Dynkin and Mandelbaum (1983). For a generalization, see Prakasa Rao and Sen (1995, Section 2).

Consider the closed vector spaces (writing $E \equiv E_{F,G}$)

$$L_0 := \{\psi(v) \mid E\psi(V) = 0, E\psi^2(V) < \infty\}$$

and

$$L_0 \otimes L_0 := \{\psi(V_1, V_2) \mid \psi(\cdot,\cdot) \text{ symmetric}, E\psi(v_1, V_2) = 0$$
$$\forall v_1, E\psi^2(V_1, V_2) < \infty\}.$$

Let $W_i(\xi_i)$ be the Wiener integrals of $\xi_1 \in L_0$ and $\xi_2 \in L_0 \otimes L_0$ (of first and second order, respectively). $W_1(\xi_1)$ is normal with mean 0 and variance $E\xi_1^2(V)$. Note that for $\psi_1, \psi_2 \in L_0$, considering $W_2((\psi_1 + \psi_2)(\cdot)(\psi_1 + \psi_2)(\cdot))$ and by linearity of $W_2(\cdot)$,

$$W_2(\psi_1(v_1)\,\psi_2(v_1) + \psi_2(v_1)\,\psi_1(v_2)) = (W_1(\psi_1)\,W_1(\psi_2) - E\psi_1\psi_2)\sqrt{2}\,.$$

Now recall the spaces $\mathscr{V}_2$ and $\mathscr{V}_1$ defined before Theorem 2 and the functions $\gamma_1(z) \equiv \gamma_1(\tilde{\phi})(z)$, $\gamma_2(z) \equiv \gamma_2(\tilde{\phi})(z)$ defined at the beginning of Section 2 (cf. Theorem 1.1 of Stute (1995)), where $\tilde{\phi}(z) \in \mathscr{V}_1$. Note that for $\phi(z_1, z_2) = \tilde{\phi}(z_1)\,\tilde{\phi}(z_2)$, $\tilde{\phi} \in \mathscr{V}_1$, we have

$$\begin{aligned}\alpha_1(\phi) &= E\gamma_2(\tilde{\phi})(Z)\,\tilde{\phi}(Z)\,\gamma_0(Z) = E(1-\delta)\,\gamma_2^2(\tilde{\phi})(Z)\\ \alpha_2(\phi) &= E\gamma_2^2(\tilde{\phi})(Z)\\ \alpha_{12}(\phi) &= E\gamma_2(\tilde{\phi})(Z)(1-\delta)\,\gamma_1(\tilde{\phi})(Z).\end{aligned}$$

Now we are ready to present Theorem 3.

THEOREM 3.

(a) *For* $\phi \in \mathscr{V}_2$,

$$nS_{2n}(\phi) - \left(\frac{1}{2}(\alpha_2(\phi) - 2\alpha_{12}(\phi) + \alpha_1(\phi)) - \alpha_1(\phi)\right) \xrightarrow{d} \frac{1}{\sqrt{2}} W_2\left(\bar{g}_\phi + \sum_{(i,j)} d_{ij}\bar{\gamma}_{ij}\right)$$
$$= \frac{1}{\sqrt{2}}\left(W_2(\bar{g}_\phi) + \sum_{(i,j)} d_{ij} W_2(\bar{\gamma}_{ij}(\phi)\right), \qquad as \quad n \to \infty.$$

(b) *For* $\phi = \tilde{\phi}(\cdot)\,\tilde{\phi}(\cdot)$, $\tilde{\phi} \in \mathscr{V}_1$,

$$\begin{aligned}&nS_{2n}(\phi) - (\tfrac{1}{2} E(\gamma_2(\tilde{\phi})(Z) - (1-\delta)\,\gamma_1(\tilde{\phi})(Z))^2 - E\gamma_2(\tilde{\phi})(z)\,\tilde{\phi}(Z)\,\delta\gamma_0(Z))\\ &\quad \xrightarrow{d} \{\tfrac{1}{2}(W_1^2(\tilde{\phi}\delta\gamma_0) - E\tilde{\phi}^2\delta\gamma_0^2) + W_1(\tilde{\phi}\delta\gamma_0)\,W_1((1-\delta)\,\gamma_1(\tilde{\phi}))\\ &\quad\quad - (W_1(\tilde{\phi}\delta\gamma_0)\,W_1(\gamma_2(\tilde{\phi})) - \alpha_1(\phi))\\ &\quad\quad + \tfrac{1}{2}(W_1^2(\gamma_2(\tilde{\phi})) - \alpha_2(\phi)) - (W_1(\gamma_2(\tilde{\phi}))\,W_1((1-\delta)\,\gamma_1(\tilde{\phi}))\\ &\quad\quad - \alpha_{12}(\phi)) + \tfrac{1}{2}(W_1^2((1-\delta)\,\gamma_1(\tilde{\phi})) - \alpha_1(\phi))\}.\end{aligned}$$

The proof of Theorem 3 is given in the Appendix.

Remark 4. Unlike in the case of classical degenerate U-statistics, the limit distribution of $nS_{2n}(\phi)$ is not centered in general, as the quantity (see Theorem 3(a))

$$b(\phi) := \tfrac{1}{2}(\alpha_2(\phi) - 2\alpha_{12}(\phi) + \alpha_1(\phi)) - \alpha_1(\phi)$$

need not be zero. However, in case $\phi(\cdot,\cdot) = \tilde{\phi}(\cdot)\,\tilde{\phi}(\cdot)$, $\tilde{\phi} \in \mathscr{V}_1$, we have (see Theorem 3(b))

$$\begin{aligned}b(\phi) = 0 &\Leftrightarrow E((1-\delta)\,\gamma_1(\tilde{\phi})(Z) - \gamma_2(\tilde{\phi})(Z))^2 = 2E\{\gamma_2(\tilde{\phi})(Z)\,\tilde{\phi}(Z)\,\delta\gamma_0(Z)\}\\ &\Leftrightarrow E(\tilde{\phi}(Z)\,\delta\gamma_0(Z) + (1-\delta)\,\gamma_1(\tilde{\phi})(Z) - \gamma_2(\tilde{\phi})(Z))^2 = E\tilde{\phi}^2(Z)\,\delta\gamma_0^2(Z).\end{aligned} \tag{3.11}$$

Note that the left side of the last step above is precisely the limiting variance for $\int \tilde{\phi}\, dF_n$ in Theorem 1.1 of Stute (1995), as $E\tilde{\phi}(Z)\,\delta\gamma_0(Z)=0$. Thus for $\phi(\cdot,\cdot)=\tilde{\phi}(\cdot)\,\tilde{\phi}(\cdot)$, $\tilde{\phi}\in\mathscr{V}_1$, the limiting distribution of $nS_{2n}(\phi)$ is centered if and only if the mean-zero random variable $((1-\delta)\,\gamma_1(\tilde{\phi})(Z)-\gamma_2(\tilde{\phi})(Z))$ *does not contribute* to the limiting variance of $\int \tilde{\phi}\, dF_n$.

Remark 5. The limit distribution of $nU_{2n}(\phi)$ can now be written down easily from Theorem 3(a), via Slutsky's theorem, since

$$nU_{2n}(\phi)=(S_{2n}(1))^{-1}\,(nS_{2n}(\phi)-b(\phi))+(S_{2n}(1))^{-1}\,b(\phi),$$

and $S_{2n}(1)\to S_2(1)$ almost surely.

Remark 6. The U-statistics corresponding to the C-degenerate product kernels presented in Examples 3 and 4 above may now be treated using Theorem 3(b) and Remark 5. However, it may be noted that the expressions for the limiting random variables in Theorem 3(b) will not become any simpler even in these special cases.

For the Cramer–von Mises statistic of Example 5, we will have to apply Theorem 3(a). However, this example could also be treated by the *functional* CLT that follows from Theorem 1.1 of Stute (1995) (see Remark on p. 438 of Stute (1995)). By that theorem we have, in the setup of Example 5, for $x\leqslant\tau_H$,

$$\begin{aligned}\sum_{i=1}^{n} W_{in} I\{Z_{i:n}\leqslant x\}-F_0(x)\\ =n^{-1}\sum_{i=1}^{n}\Big[\,&\delta_i(1-G(Z_i))^{-1}\,I\{Z_i\leqslant x\}-F_0(x)\\ &+(1-\delta_i)(1-H(Z_i))^{-1}\,(F_0(x)-F_0(Z_i))\,I\{Z_i\leqslant x\}\\ &-\int_{-\infty}^{\tau_H} I\{w\leqslant x\}\,C(Z_i\wedge w)\,F_0(dw)\Big]+o_P(n^{-1/2}),\end{aligned} \tag{3.12}$$

and by Remark 2, $S_{1n}(1)=1+o_P(n^{-1/2})$, so that

$$\begin{aligned}nCV_n &=\int_{-\infty}^{\tau_H}\left(n^{1/2}\left[\sum_{i=1}^{n} W_{in} I\{Z_{i:n}\leqslant x\}/S_{1n}(1)-F_0(x)\right]\right)^2 F_0(dx)\\ &\xrightarrow{d}\int_{-\infty}^{\tau_H} W^2(x)\,F_0(dx),\end{aligned}$$

where $W(x)$, $-\infty < x \leqslant \tau_H$, is a mean-zero Gaussian process whose covariance is the same as that of the leading empirical process on the right side of (3.12).

APPENDIX

Here we collect the proofs of the results for the leading terms. The proofs for the terms which are negligible are omitted and the reader is referred to Bose and Sen (1996, 1997) for these.

Proof of Theorem 1. First assume that in addition to (2.12) and (2.13), (2.8) holds and H is continuous. This implies (2.7) holds. From Lemma 2 and relations (2.9), (2.10), and (2.11),

$$S_{2n}(\phi) = n^{-2} \sum_{1 \leqslant i < j \leqslant n} g_\phi(Z_i, Z_j)\, \delta_i\, \delta_j + n^{-1} \sum_{i=1}^{n} \{\gamma_3(Z_i)(1-\delta_i) - \gamma_4(Z_i)\} + R_n, \tag{A1}$$

where $R_n = O(n^{-1} \log n)$ almost surely. The theorem follows in this special case by taking another projection of the first term. We now remove all the restrictions imposed so far.

First, drop condition (2.8), but retain the continuity of H. Observe that $H\{\tau_H\} = 0$. Fix $\varepsilon > 0$. Since (2.12) and (2.13) are valid, choose a function ϕ_0 satisfying (2.12) and (2.13) and vanishing outside of $(-\infty, T] \times (-\infty, T]$ for some $T < \tau_H$ such that, with $\phi_\varepsilon := \phi - \phi_0$,

$$\begin{aligned} E\, \delta_1\, \delta_2\, g^2_{\phi_\varepsilon}(Z_1, Z_2) &< \varepsilon^2 \\ E\, \delta_1\, \delta_2\, |g_{\phi_\varepsilon}(Z_1, Z_2)|\, C(Z_1)\, C(Z_2) &< \varepsilon \\ E\, \delta_1 \{E(\delta_2\, g^2_{\phi_\varepsilon}(Z_1, Z_2) \mid Z_1)\}^{1/2}\, C(Z_1) &< \varepsilon \\ E\, \delta_1 \{E(\delta_2\, g_{\phi_\varepsilon}(Z_1, Z_2)\, C(Z_2) \mid Z_1)\}^2 &< \varepsilon^2. \end{aligned} \tag{A2}$$

It is then enough to show that $n^{1/2}(S_{2n}(\phi_\varepsilon) - E_F \phi_\varepsilon(X_1, X_2)) = O_P(\varepsilon)$. This follws exactly as in Stute (1995) except that we are now dealing with a sum over two indices instead of one. We omit the details (however, see the proof of Theorem 2) but mention the crucial facts needed:

(i) $|B_{in}| \leqslant 1$
(ii) $\sup_{1 \leqslant i \leqslant n} |C_{in}| = O_P(1)$
(iii) $(1 - H_n(t))^{-1}\,(1 - H(t))$ is bounded from above on $t < Z_{n:n}$
(iv) Expansion for variance of U-statistics.

Now consider the situation when F and G may have separate discontinuities but $A_F \cap A_G = \phi$. A quantile transformation traces everything back to uniformly distributed Z's. See Bose and Sen (1996) and Stute and Wang (1993).

Finally, if there are common jumps, say $\{x_i\}$, of F and G, replace each x_i by $[x_i, x_i+\varepsilon_i]$ where $\sum \varepsilon_i < \infty$ and move the G mass at x_i to $x_i+\varepsilon_i$. Extend the time scale for F and ϕ, by putting for example $F(x)=F(x_i)$ if $x_i \leqslant x \leqslant x_i+\varepsilon_i$. Since tied uncensored observations precede censored ties by convention, $S_{2n}(\phi)-E_{\tilde{F}}\phi$ remains unchanged. F and G now do not have common jumps. The integrability conditions remain valid and thus the representation is obtained on the extended time scale. The right side of (2.14) is the same on both time scales. This finishes the proof. ∎

Proof of Lemma 3. Note that from (3.3),

$$\bar{T}_2 = \frac{1}{n^2} \sum_{1 \leqslant i \neq j \leqslant n}\sum \phi(Z_i, Z_j)\, \delta_i, \delta_j \gamma_0(Z_i)\, \gamma_i(Z_j)\, C_{in}$$

$$= \int_{\mathbb{R}^5} g_\phi(u, v)\, I(u \neq v)\, a_4(s, u, t_1, t_2)$$

$$\times [H_n(dt_1)\, H_n(dt_2)\, H_{n0}(ds)\, H_{n1}(du)\, H_{n1}(dv)]$$

$$-3 \int_{\mathbb{R}^4} g_\phi(u, v)\, I(u \neq v)\, a_3(s, u, t)[H_n(dt)\, H_{n0}(ds)\, H_{n1}(du)\, H_{n1}(dv)]$$

$$+3 \int_{\mathbb{R}^3} g_\phi(u, v)\, I(u \neq v)\, a_2(s, u)[H_{n0}(ds)\, H_{n1}(du)\, H_{n1}(dv)]$$

$$-\int_{\mathbb{R}^2} g_\phi(u, v)\, a_1(u)\, I(u \neq v)[H_{n1}(du)\, H_{n1}(dv)]$$

$$+\int_{\mathbb{R}^3} g_\phi(u, v)\, I(u \neq v)\, I(s < u)\, R_{n0}(ds)\, H_{n1}(u)\, H_{n1}(dv)$$

$$= \bar{T}_{n1} - 3\bar{T}_{n2} + 3\bar{T}_{n3} - \bar{T}_{n4} + \bar{T}_{n5}, \qquad \text{say.}$$

Since $\sup_{\mathbb{R}} |H_n(s) - H(s)|^3 = O(n^{-3/2}(\log n)^{3/2})$ *a.s.*, we have $\bar{T}_{n5} = O(n^{-3/2}(\log n)^{3/2})$ a.s. We omit the details.

To tackle $\bar{T}_{n1}$ through $\bar{T}_{n4}$, first ignore the diagonal domains $\{t_1 = t_2\}$, $\{v = t_1 = t_2\}$ etc. We shall show that the diagonals lead to the term $n^{-1}\alpha_1(\phi)$ in the end. Without these, $\bar{T}_{nk}$, $1 \leqslant k \leqslant 4$, are U-statistics (with asymmetric kernels) again, as in (2.2). Further, because of degeneracy, each $\bar{T}_{nk}$, $1 \leqslant k \leqslant 4$, has mean zero, and all the first-order projections add up to zero. (By degeneracy,

$$\int_{\mathbb{R}} \{g_\phi(u, v) \ldots\} \, H_1(dv) \equiv 0 \qquad \forall (u, s, t_1, t_2),$$

and projections on the v-coordinate

$$\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^p} g_\phi(u, v) \ldots H(dt_1) \ldots H_1(du) \right\} H_{n1}(dv)$$

cancel out one another across $\bar{T}_{nk}$, $1 \leqslant k \leqslant 4$, because of the coefficients $(1, -3, 3, 1)$. Here $1 \leqslant p \leqslant 4$.) Therefore, we may go right up to the *second order* projections. By the well-known orthogonality of the projections, the remainder will thus be $O_P(n^{-3/2})$. Note, however, that $\bar{T}_{n4}$, having degree 2, will leave *no* remainder at all.

The proof will be complete if we show that the second-order projections of $\bar{T}_{nk}$, $1 \leqslant k \leqslant 4$, add up to the two terms on the right side of (3.4) and the diagonals contribute $n^{-1}\alpha_1(\phi)$.

Taking projections, we have, with $g_\phi(u, v)$ as in (2.1),

$$\begin{aligned}
\bar{T}_{n1} = \bar{F}_{n1} + \int_{\mathbb{R}^5} & g_\phi(u, v) \, a_4(s, u, t_1, t_2) \\
& \times [(H_n - H)(dt_1) \, H(dt_2) \, H_0(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H(dt_1)(H_n - H)(dt_2) \, H_0(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H(dt_1) \, H(dt_2)(H_{n0} - H_0)(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H(dt_1) \, H(dt_2) \, H_0(ds)(H_{n1} - H_1)(du) \, H_{n1}(dv)] + O_P(n^{-3/2})
\end{aligned}$$

$$\begin{aligned}
-3\bar{T}_{n2} = \bar{F}_{n2} - 3 \int_{\mathbb{R}^4} & g_\phi(u, v) \, a_3(s, u, t)[(H_n - H)(dt) \, H_0(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H(dt)(H_{n0} - H_0)(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H(dt) \, H_0(ds)(H_{n1} - H_1)(du) \, H_{n1}(dv)] + O_P(n^{-3/2})
\end{aligned}$$

$$\begin{aligned}
3\bar{T}_{n3} = \bar{F}_{n3} + 3 \int_{\mathbb{R}^3} & g_\phi(u, v) \, a_2(s, u)[(H_{n0} - H_0)(ds) \, H_1(du) \, H_{n1}(dv) \\
& + H_0(ds)(H_{n1} - H_1)(du) \, H_{n1}(dv)] + O_P(n^{-3/2})
\end{aligned}$$

$$-\bar{T}_{n4} = \bar{F}_{n4} - \int_{\mathbb{R}^2} g_\phi(u, v) \, a_1(u)(H_{n1} - H_1)(du) \, H_{n1}(dv).$$

Here $\bar{F}_{nk}$, $1 \leqslant k \leqslant 4$, are the first-order projections and as noted earlier, satisfy $\sum_{k=1}^{4} \bar{F}_{nk} = 0$. The second-order projections not involving the v-coordinate are zero, by degeneracy. Equation (3.4), except for $-n^{-1}\alpha_1(\phi)$, follows by adding the above expressions, since by tedious algebra, most of the second-order terms cancel out.

To tackle the diagonals, note that $\bar{T}_{n3}$ and $\bar{T}_{n4}$ are already diagonal-free (in $\bar{T}_{n3}$, the diagonal $\{s = v\}$ is zero because $\delta_j(1-\delta_j) = 0\ \forall 1 \leqslant j \leqslant n$). Write $\bar{T}_{n1}$ and $-3\bar{T}_{n2}$ as:

$$\bar{T}_{n1} = n^{-5} \sum_{i \neq j} \sum_{k} \sum_{l} \sum_{m} \{\delta_i\, \delta_j (1-\delta_k)\, g_\phi(Z_i, Z_j)$$

$$\times I(Z_k < Z_i)\, I(Z_k < Z_l)\, I(Z_k < Z_m)(1 - H(Z_k))^{-3}\}$$

$$-3\bar{T}_{n2} = -3n^{-4} \sum_{i \neq j} \sum_{k} \sum_{l} \{\delta_i\, \delta_j (1-\delta_k)\, g_\phi(Z_i, Z_j)$$

$$\times I(Z_k < Z_i)\, I(Z_k < Z_l)(1 - H(Z_k))^{-2}\}.$$

It is easy to see that, except for $\{j = l\}$ (in $\bar{T}_{n1}$ and $\bar{T}_{n2}$) and $\{j = m\}$ (in $\bar{T}_{n1}$), all other diagonals are either zero (for example, $j = k$) or $o_P(n^{-1})$ (for example, $i = l$) or $O_P(n^{-2})$ (for example, $j = k = m$), by degeneracy and/or SLLN for U-statistics, since the diagonals lead to U-statistics of smaller order. The "errant" diagonals $\{j = l\}$ and $\{j = m\}$ lead to the following three U-statistics:

$$n^{-5} \sum_{i \neq j \neq k \neq l} \{\dots\}, \qquad n^{-5} \sum_{i \neq j \neq k \neq m} \{\dots\}, \qquad -3n^{-4} \sum_{i \neq j \neq k} \{\dots\}.$$

Since all the three U-kernels have the same expectation, namely, $\alpha_1(\phi)$, the result follows from the SLLN for U-statistics. This completes the proof of Lemma 3. ∎

Proof of Lemma 4. From (3.3), and writing the variables corresponding to C_{jn} as $\bar{s}, \bar{t}, \dots$ etc., we have, with $g_\phi(u, v)$ as in (2.1),

$$\bar{T}_6 = \tfrac{1}{2} \bigg[\int_{\mathbb{R}^8} g_\phi(u, v)\, a_4(s, u, t_1, t_2)\, a_4(\bar{s}, v, \bar{t}_1, \bar{t}_2)\, H_n(dt_1)\, H_n(dt_2)$$

$$\times H_{n0}(ds)\, H_n(d\bar{t}_1)\, H_n(d\bar{t}_2)\, H_{n0}(d\bar{s})\, H_{n1}(du)\, H_{n1}(dv)$$

$$-6 \int_{\mathbb{R}^7} g_\phi(u, v)\, a_4(s, u, t_1, t_2)\, a_3(\bar{s}, v, \bar{t}_1)\, H_n(dt_1)\, H_n(dt_2)$$

$$\times H_{n0}(ds)\, H_n(d\bar{t}_1)\, H_{n0}(d\bar{s})\, H_{n1}(du)\, H_{n1}(dv)$$

$$+6\int_{\mathbb{R}^6} g_\phi(u,v)\,a_4(s,u,t_1,t_2)\,a_2(\bar{s},v)\,H_n(dt_1)\,H_n(dt_2)$$

$$\times H_{n0}(ds)\,H_{n0}(d\bar{s})\,H_{n1}(du)\,H_{n1}(dv)$$

$$-2\int_{\mathbb{R}^5} g_\phi(u,v)\,a_4(s,u,t_1,t_2)\,a_1(v)\,H_n(dt_1)\,H_n(dt_2)$$

$$\times H_{n0}(ds)\,H_{n1}(du)\,H_{n1}(dv)$$

$$+9\int_{\mathbb{R}^6} g_\phi(u,v)\,a_3(s,u,t_1)\,a_3(\bar{s},v,\bar{t}_1)\,H_n(dt_1)\,H_{n0}(ds)$$

$$\times H_n(d\bar{t}_1)\,H_{n0}(d\bar{s})\,H_{n1}(du)\,H_{n1}(dv)$$

$$-18\int_{\mathbb{R}^5} g_\phi(u,v)\,a_3(s,u,t_1)\,a_2(\bar{s},v)\,H_n(dt_1)\,H_{n0}(ds)$$

$$\times H_{n0}(d\bar{s})\,H_{n1}(du)\,H_{n1}(dv)$$

$$+6\int_{\mathbb{R}^4} g(u,v)\,a_3(s,u,t_1)\,a_1(v)\,H_n(dt_1)\,H_{n0}(ds)\,H_{n1}(du)\,H_{n1}(dv)$$

$$+9\int_{\mathbb{R}^4} g_\phi(u,v)\,a_2(s,u)\,a_2(\bar{s},v)\,H_{n0}(ds)\,H_{n0}(d\bar{s})\,H_{n1}(du)\,H_{n1}(dv)$$

$$-6\int_{\mathbb{R}^3} g_\phi(u,v)\,a_2(s,u)\,a_1(v)\,H_{n0}(ds)\,H_{n1}(du)\,H_{n1}(dv)$$

$$+\int_{\mathbb{R}^2} g_\phi(u,v)\,a_1(u)\,a_1(v)\,H_{n1}(du)\,H_{n1}(dv)\Bigg]+\tilde{R}_n$$

$$=\sum_{k=1}^{10}\tilde{T}_{nk}+\tilde{R}_n, \qquad \text{say},$$

Here $\tilde{R}_n$ is the sum of all the $R_{n0}(ds)$-integrals arising out of $\frac{1}{2n^2}\sum_{1\leqslant i<j\leqslant n}\{\dots\}\,C_{in}C_{jn}$. Hence by the same arguments as in the case $\tilde{T}_{n5}$ in the proof of Lemma 3, we clearly have

$$\tilde{R}_n = O(n^{-3/2}(\log n)^{3/2})\ \text{a.s.}, \qquad \text{as} \quad n\to\infty.$$

As for the $\tilde{T}_{nk}$, $1\leqslant k\leqslant 10$, we again separate the *diagonal* and the *nondiagonal* part. The nondiagonal parts then form U-statistics, but now there is no degeneracy, due to the presence of the factors $a_r(\cdot)$, $1\leqslant r\leqslant 4$, in the

U-kernels. Now decompose $\tilde{T}_{nk}^{(ND)}$, the nondiagonal part of $\tilde{T}_{nk}$, $1 \leqslant k \leqslant 10$, in terms of its first- and second-order projections as

$$\tilde{T}_{nk}^{(ND)} = \mu_k + \tilde{F}_{nk}^{(1)} + \tilde{F}_{nk}^{(2)} + O_p(n^{-3/2}), \qquad 1 \leqslant k \leqslant 9,$$
$$\tilde{T}_{n,10}^{(ND)} = \mu_{10} + \tilde{F}_{n,10}^{(1)} + \tilde{F}_{n,10}^{(2)},$$

where μ_k is the expectation of the kernel of $\tilde{T}_{nk}$, $1 \leqslant k \leqslant 10$. It may now be verified by brute force, thanks to massive cancellations, that

$$\sum_{k=1}^{10} \mu_k = 0 = \sum_{k=1}^{10} \tilde{F}_{nk}^{(1)},$$
$$\sum_{k=1}^{10} \tilde{F}_{nk}^{(2)} = \text{sum of first three } U\text{-statistics in (3.5)}.$$

A crucial fact in this verification is that for $2 \leqslant i \leqslant k$, $k \geqslant 3$,

$$\int_{\mathbb{R}} a_k(z_1, z_2, ..., z_{i-1}, t, z_{i+1}, ..., z_k)\, H(dt)$$
$$= a_{k-1}(z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_k).$$

As for the diagonal parts, the diagonals involving three or more indices (coordinates) are obviously of the order $O(n^{-2})$ almost surely, by the SLLN for U-statistics as before. For any two coordinates (z_{i_1}, z_{i_2}), denote by $\{z_{i_1} = z_{i_2}\}$ the *sum* of *all* the U-statistics obtained from $\tilde{T}_{nk}^{(ND)}$, $1 \leqslant k \leqslant 8$, by restricting to the diagonal $\{z_{i_1} = z_{i_2}\}$. Note that $\tilde{T}_{n,9}$ and $\tilde{T}_{n,10}$ are diagonal-free. Now verify again by brute force that

(i) $\{s = \bar{s}\} = n^{-1}(\frac{1}{2}\,\alpha_1(\phi) + o(1))$ a.s.

(ii) $\{t_1 = \bar{t}_1\} + \{t_2 = \bar{t}_2\} + \{t_1 = \bar{t}_2\} + \{t_2 = \bar{t}_1\} = n^{-1}(\frac{1}{2}\,\alpha_2(\phi) + o(1))$ a.s.

(iii) $\{t_1 = \bar{s}\} + \{t_2 = \bar{s}\} + \{\bar{t}_1 = s\} + \{\bar{t}_2 = s\} = n^{-1}(-\alpha_{12}(\phi) + o(1))$ a.s.

(iv) $\sum_{\{(z_{i_1}, z_{i_2}) \text{ not of types (ii)–(iii)}\}} \{z_{i_1} = z_{i_2}\} = n^{-1}(0 + o(1))$ a.s.

This completes the proof of Lemma 4. ∎

Proof of Theorem 2. Note that by (3.2) and Lemmas 3–5, (3.9) holds for a $\phi(\cdot,\cdot)$ satisfying (2.7) and (2.8). For a given $\phi(\cdot,\cdot)$ satisfying (N_1) through (N_{21}) and $\varepsilon > 0$, we can get a *degenerate* $\bar{\phi}(\cdot,\cdot)$ satisfying (2.7) and (2.8) and

$$\|\phi - \bar{\phi}\|_1 < \varepsilon,\ \|\phi - \bar{\phi}\|_{12} < \varepsilon,$$
$$\|\phi - \bar{\phi}\|_2 < \varepsilon,\ \|\phi - \bar{\phi}\|_{21} < \varepsilon. \qquad \text{(A3)}$$

For instance, one could take a degenerate $\bar{\phi}$ of the form

$$\bar{\phi}_T(z_1, z_2) = \Bigg[\phi(z_1, z_2) - \frac{\begin{pmatrix} E\{\phi(Z_1, Z_2)\, \delta_2 \gamma_0(Z_2)\, I(Z_2 \leqslant T)\} \\ \times E\{|\phi(Z_1, z_2)|\, \delta_1 \gamma_0(Z_1)\, I(Z_1 \leqslant T)\} \end{pmatrix}}{\mu_T(|\phi|)}$$
$$- \frac{\begin{pmatrix} E\{\phi(Z_1, z_2)\, \delta_1 \gamma_0(Z_1)\, I(Z_1 \leqslant T)\} \\ \times E\{|\phi(z_1, Z_2)|\, \delta_2 \gamma_0(Z_2)\, I(Z_2 \leqslant T)\} \end{pmatrix}}{\mu_T(|\phi|)}$$
$$+ \frac{\begin{pmatrix} E\{\phi(Z_1, Z_2)\, \delta_1\, \delta_2 \gamma_0(Z_1) \\ \times \gamma_0(Z_2)\, I(Z_1 \leqslant T)\, I(Z_2 \leqslant T)\} \end{pmatrix}}{\mu_T(|\phi|)} |\phi(z_1, z_2)| \Bigg]$$
$$\times I(z_1 \leqslant T)\, I(z_2 \leqslant T),$$

where

$$\mu_T(|\phi|) = E\{|\phi(Z_1, Z_2)|\, \delta_1\, \delta_2 \gamma_0(Z_1)\, \gamma_0(Z_2)\, I(Z_1 \leqslant T)\, I(Z_2 \leqslant T)\}$$

and $0 < T < \infty$ is chosen large enough (or close enough to τ_H if $\tau_H < \infty$) so that (A3) holds. Note that *degeneracy* of ϕ is crucially used in constructing $\bar{\phi}_T$ in the fact that

$$\lim_{T \to \tau_H} E\{\phi(z_1, Z_2)\, \delta_2 \gamma_0(Z_2)\, I(Z_2 \leqslant T)\} = 0 \qquad \forall z_1 \in \mathbb{R}.$$

After choosing $\bar{\phi}$ satisfying (A3), let

$$\phi_\varepsilon := \phi - \bar{\phi}.$$

Then ϕ_ε is also degenerate, in particular

$$\int \phi_\varepsilon(x_1, x_2)\, \tilde{F}(dx_1)\, \tilde{F}(dx_2) = 0,$$

and (3.9) holds for $\bar{\phi}$.

It follows that, in order to establish (3.9) for ϕ, it suffices to show

$$n S_{2n}(\phi_\varepsilon) = O_P(\varepsilon), \qquad \text{as} \quad n \to \infty. \tag{A4}$$

(Note that the kernels $\delta_1\, \delta_2\, g_\phi$, $\gamma_{ij}(\phi)$, $2 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 3$, are all square-integrable by our assumptions (N_1)–(N_{21}), hence by degeneracy,

$$n\bar{U}_{2n}(\phi_\varepsilon) = O_P(\varepsilon), \qquad n\gamma_{21}(\phi_\varepsilon)(H_{n0} - H_{01}, H_{n1}) = O_P(\varepsilon),$$

and so on, as $n \to \infty$).

Now from Lemma 1,

$$
\begin{aligned}
nS_{2n}(\phi_\varepsilon) &= \frac{1}{n} \sum_{1 \leqslant i < j \leqslant n} \phi_\varepsilon(Z_i, Z_j)\, \delta_i\, \delta_j \gamma_0(Z_i)\, \gamma_0(Z_j) \{\exp(A_{in} + A_{jn}) - 1\} + n.\bar{U}_{2n}(\phi_\varepsilon) \\
&= n\tilde{U}_{2n}^{(1)}(\phi_\varepsilon) + n\tilde{U}_{2n}^{(2)}(\phi_\varepsilon) + n\bar{U}_{2n}(\phi_\varepsilon), \qquad \text{say,} \qquad (A5)
\end{aligned}
$$

where using the identity $e^x . e^y - 1 \equiv (e^x - 1)(e^y - 1) + \{(e^x - 1) + (e^y - 1)\}$.

$$
\tilde{U}_{2n}^{(1)}(\phi_\varepsilon) = \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} \phi_\varepsilon(Z_i, Z_j)\, \delta_i\, \delta_j \gamma_0(Z_i)\, \gamma_0(Z_j)(e^{A_{in}} - 1).(e^{A_{jn}} - 1),
$$

$$
\tilde{U}_{2n}^{(2)}(\phi_\varepsilon) = \frac{1}{n^2} \sum_{i \neq j} \phi_\varepsilon(Z_i, Z_j)\, \delta_i\, \delta_j \gamma_0(Z_i)\, \gamma_0(Z_j)(e^{A_{in}} - 1).
$$

Consider $\tilde{U}_{2n}^{(2)}(\phi_\varepsilon)$ first. We have

$$
\begin{aligned}
n\, |\tilde{U}_{2n}^{(2)}(\phi_\varepsilon)| &\leqslant \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i)\, |e^{A_{in}} - 1| \left| \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&\leqslant \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i)(|B_{in}| + |C_{in}|) \left| \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| .O_P(1) \\
&= (\beta_n + \theta_n).O_P(1), \qquad \text{say,} \qquad (A6)
\end{aligned}
$$

using the facts that $|e^{A_{in}} - 1| \leqslant |A_{in}|\, e^{|A_{in}|}$, $A_{in} = B_{in} + C_{in}$, and $e^{|A_{in}|} = O_P(1)$ (see the proof of Theorem 1.1, Stute (1995), p. 436).

Further,

$$
\begin{aligned}
\beta_n :&= \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i)\, |B_{in}| \left| \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&\leqslant \frac{1}{n^2} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i) \left(\int I(s < Z_i) \frac{H_{n0}(ds)}{(1 - H_n(s))^2} \right) \left| \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&= \frac{1}{n^2} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i) \left(\int I(s < Z_i) \frac{H_{n0}(ds)}{(1 - H(s))^2} \right) \\
&\quad \times \left| \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| O_P(1) \\
&= \frac{1}{n^{5/2}} \sum_{i \neq k} \delta_i (1 - \delta_k)\, \gamma_0(Z_i) \frac{I(Z_k < Z_i)}{(1 - H(Z_k))^2} \\
&\quad \times \left| \frac{1}{\sqrt{n}} \sum_{j \neq i, k} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| .O_P(1), \qquad (A7)
\end{aligned}
$$

using $|B_{in}| \leqslant (2n)^{-1} \int I(s < Z_i)(1-H_n(s))^{-2} H_{n0}(ds)$, by Eq. (2.4) of Stute (1995) and the fact that

$$\sup_{s < Z_{n:n}} (1-H(s)/(1-H_n(s)) = O_P(1), \tag{A8}$$

as on p. 436 of Stute (1995).

Now by a symmetry argument and the degeneracy of ϕ_ε,

$$\begin{aligned}
E&\left\{\frac{1}{n^{5/2}} \sum_{i \neq k} \delta_i(1-\delta_k)\, \gamma_0(Z_i)\, \frac{I(Z_k < Z_i)}{(1-H(Z_k))^2} \left| \frac{1}{\sqrt{n}} \sum_{j \neq i,k} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \right\} \\
&= \frac{1}{\sqrt{n}} E\left\{ \delta_1(1-\delta_2)\, \gamma_0(Z_1)\, \frac{I(Z_2 < Z_1)}{(1-H(Z_2))^2} \left| \frac{1}{\sqrt{n}} \sum_{j \neq 1,2} \phi_\varepsilon(Z_1, Z_j)\, \delta_j \gamma_0(Z_j) \right| \right\} \\
&\leqslant \frac{1}{\sqrt{n}} E\left\{ \delta_1(1-\delta_2)\, \gamma_0(Z_1)\, \frac{I(Z_2 < Z_1)}{(1-H(Z_2))^2} \right. \\
&\qquad \left. \times \left[\frac{n-2}{n} E(\phi_\varepsilon^2(Z_1, Z_3)\, \delta_3 \gamma_0^2(Z_3) \mid Z_1) \right]^{1/2} \right\} \\
&\leqslant \frac{1}{\sqrt{n}} E\, \delta_1 \{ E(\delta_3 g_{\phi_\varepsilon}^2(Z_1, Z_3) \mid Z_1) \}^{1/2}\, C(Z_1) \\
&= \frac{1}{\sqrt{n}} \|\phi_\varepsilon(\cdot, \cdot)\|_{12} \leqslant \frac{\varepsilon}{\sqrt{n}},
\end{aligned} \tag{A9}$$

using Hölders inequality and the fact that

$$E\left((1-\delta_2)\, \frac{I(Z_2 < Z_1)}{(1-H(Z_2))^2} \,\middle|\, Z_1 \right) = C(Z_1).$$

By (A7) and (A9),

$$\beta_n = O_P(\varepsilon/\sqrt{n}) \qquad \text{as} \quad n \to \infty. \tag{A10}$$

Next, to deal with θ_n, write

$$\begin{aligned}
C_{in} &= \int I(s < Z_i)\big((1-H_n(s))^{-1} H_{n0}(ds) - (1-H(s))^{-1} H_0(ds)\big) \\
&= \int I(s < Z_i)(1-H(s))^{-1} (H_{n0}(ds) - H_0(ds)) \\
&\quad + \int I(s < Z_i)\, \frac{(H_n(s) - H(s))}{(1-H_n(s))(1-H(s))} H_{n0}(ds).
\end{aligned} \tag{*}$$

Then

$$\theta_n := \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i)\, |C_{in}| \left| \sum_{j \neq i} \phi_s(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right|$$

$$\leqslant \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i) \left(\left| \int \frac{I(s < Z_i)}{(1 - H(s))} (H_{n0}(ds) - H_0(ds)) \right| \right.$$

$$\left. + \int \frac{I(s < Z_i)\, |H_n(s) - H(s)|}{(1 - H(s))^2} H_{n0}(ds) . O_p(1) \right)$$

$$\times \left| \sum_{j \neq i} \phi_s(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right|$$

$$= \theta_n^{(1)} + \theta_n^{(2)} . O_p(1), \qquad \text{say, using (*).} \tag{A11}$$

Now

$$\theta_n^{(1)} := \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i) \left| \frac{1}{n} \sum_{k=1}^{n} ((1 - \delta_k)\, a_2(Z_k, Z_i) - a_1(Z_i)) \right|$$

$$\times \left| \sum_{j \neq i} \phi_s(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right|$$

$$\leqslant \frac{1}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i) \left| \frac{1}{n} \sum_{k \neq j,\, k \neq i,\, j \neq i} ((1 - \delta_k)\, a_2(Z_k, Z_i) \right.$$

$$\left. - a_1(Z_i))\, \phi_s(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right|$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \delta_i \gamma_0(Z_i)\, a_1(Z_i) \left| \frac{1}{n} \sum_{j \neq i} \phi_s(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right|, \tag{A12}$$

since $a_2(Z_i, Z_i) = 0$, $\delta_j(1 - \delta_j) = 0$. Further, since

$$E((1 - \delta_k)\, a_2(Z_k, Z_i) \mid Z_i) = a_1(Z_i), \qquad k \neq i$$

$$E((1 - \delta_k)\, a_2^2(Z_k, Z_i) \mid Z_i) = C(Z_i), \qquad k \neq i$$

$$a_1(Z_1) = \int I(s < Z_i)(1 - H(s))\, \frac{H_0(ds)}{(1 - H(s))^2} \leqslant C(Z_i),$$

$$\sqrt{C(Z_1)} \leqslant C(Z_1),$$

we see from (A12) that, by arguments similar to those in (A9),

$$\theta_n^{(1)} = O_P(\varepsilon) + O_P(\varepsilon/\sqrt{n}) = O_P(\varepsilon). \tag{A13}$$

Next

$$\begin{aligned}
\theta_n^{(2)} &:= \sum_{i \neq k} \delta_i (1-\delta_k)\, \gamma_0(Z_i)\, \frac{I(Z_k < Z_i)}{(1-H(Z_k))^2} \left| \frac{1}{n} \sum_{l=1}^{n} (I(Z_l \leqslant Z_k) - H(Z_k)) \right| \\
&\qquad \times \left| \sum_{j \neq i, j \neq k} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&\leqslant \frac{1}{n^2} \sum_{i \neq k} \delta_i (1-\delta_k)\, \gamma_0(Z_i)\, \frac{I(Z_k < Z_i)}{(1-H(Z_k))^2} \\
&\qquad \times \left| \frac{1}{n} \sum_{i \neq j,\, (l,j) \in \{i,k\}^c} (I(Z_l \leqslant Z_k) - H(Z_k))\, \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&\qquad + \frac{3}{n^2} \sum_{i \neq k} \delta_i (1-\delta_k)\, \gamma_0(Z_i)\, \frac{I(Z_k < Z_i)}{(1-H(Z_k))^2} \left| \frac{1}{n} \sum_{j \neq i} \phi_\varepsilon(Z_i, Z_j)\, \delta_j \gamma_0(Z_j) \right| \\
&= O_P(\varepsilon) + O_P(\varepsilon/\sqrt{n}) = O_P(\varepsilon).
\end{aligned} \tag{A14}$$

From (A11), (A13), and (A14), $\theta_n = O_P(\varepsilon)$, so that, by (A6), (A10) and the above

$$n\tilde{U}_{2n}^{(2)}(\phi_\varepsilon) = O_P(\varepsilon). \tag{A15}$$

As for $\tilde{U}_{2n}^{(1)}(\phi_\varepsilon)$, we have, by arguments leading to (A6),

$$\begin{aligned}
&n\, |\tilde{U}_{2n}^{(1)}(\phi_\varepsilon)| \\
&\quad \leqslant \frac{1}{n} \sum_{i<j} |\phi_\varepsilon(Z_i, Z_j)|\, \delta_i\, \delta_j \gamma_0(Z_i)\, \gamma_0(Z_j) (|B_{in}| + |C_{in}|)(|B_{jn}| + |C_{jn}|)\, O_P(1) \\
&\quad = \left[\frac{1}{n} \sum_{i<j} \{\ldots\}\, |B_{in}|\, |B_{jn}| + \frac{1}{n} \sum_{i \neq j} \{\ldots\}\, |B_{in}| . |C_{jn}| \right. \\
&\qquad \left. + \frac{1}{n} \sum_{i<j} \{\ldots\}\, |C_{in}| . |C_{jn}| . \right] . O_P(1).
\end{aligned}$$

Using the preceding estimates and methods, it may be established, after considerable algebra, that the above three terms are of the order $O_P(\varepsilon/n)$, $O_P(\varepsilon/\sqrt{n})$, and $O_P(\varepsilon)$, respectively. Hence

$$n\tilde{U}_{2n}^{(1)}(\phi_\varepsilon) = O_P(\varepsilon). \tag{A16}$$

The theorem follows from (A15), (A16), and (A5), since it is already noted that $n\bar{U}_{2n}(\phi_\varepsilon) = O_P(\varepsilon)$. ▌

Proof of Theorem 3. (a) Follows from Theorem 2 by the result of Dynkin and Mandelbaum (1983), using the isometry in the definition of W_2 and approximation by product functions.

(b) Follows directly from (a) by specializing to $\phi(Z_1, Z_2) = \tilde{\phi}(Z_1)\,\tilde{\phi}(Z_2)$, since for this ϕ

$$\bar{g}_\phi = \tilde{\phi}(Z_1)\,\delta_1\gamma_0(Z_1)\,\tilde{\phi}(Z_2)\,\delta_2\gamma_0(Z_2)$$

$$\bar{\gamma}_{21}(\phi)(Z_1, Z_2) = ((1-\delta_1)\,\gamma_1(\tilde{\phi})(Z_1) - E(1-\delta)\,\gamma_1(\tilde{\phi}))\,\phi(Z_2)\,\delta_2\gamma_2(Z_2)$$

and so on. Note that the result also follows directly by writing $\sum_{i \neq j} a_i b_j = (\sum_i a_i)(\sum_j b_j) - \sum_i a_i b_i$ and using the i.i.d. CLT and SLLN term by term. ▌

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REFERENCES

A. Bose, Bahadur reprensentaion of M_m estimates, *Ann. Statist.* **26** (1998), 771–777.

A. Bose and A. Sen, The strong law of large numbers for Kaplan–Meier U-statistics, *J. Theoret. Probab.* **12** (1999), 181–200.

A. Bose and A. Sen, "The Central Limit Theorem for Kaplan–Meier U-Statistics," Technical Report 28/96, Stat-Math Unit, Indian Statistical Institute, Calcutta, December 1996.

A. Bose and A. Sen, "Asymptotic Distribution of the Kaplan–Meier U-Statistics," Technical Report 16/97, Stat-Math Unit, Indian Statistical Institute, Calcutta, August 1997.

E. B. Dynkin and A. Mandelbaum, Symmetric statistics, Poisson point processes and multiple Wiener integrals, *Ann. Statist.* **11** (1983), 739–745.

I. Gijbels and N. Veraverbeke, Almost sure asymptotic representation for a class of functionals of the Kaplan–Meier estimator, *Ann. Statist.* **19** (1991), 1457–1470.

H. L. Koul and V. Susarla, A limit theorem for testing with randomly censored data, *in* "Survival Analysis" (J. Crowley and R. A. Johnson, Eds.), IMS Lecture Notes-Monograph Series, Vol. 2, pp. 189–205, I.M.S., Hayward, 1982.

J. A. Koziol and S. B. Green, A Cramer–von Mises statistic for randomly censored data, *Biometrika* **63** (1976), 465–474.

W. A. Light and E. W. Cheney, "Approximation Theory in Tensor Product Spaces," Lecture Notes in Mathematics, Vol. 1169, Springer-Verlag, Berlin, 1985.

K. R. Parthasarthy, "An Introduction to Quantum Stochastic Calculus," Birkhäuser, Basel, 1992.

B. L. S. Prakasa Rao and A. Sen, Limit distributions of conditional *U*-statistics, *J. Theoret. Probab.* **8** (1995), 261–301.

W. Serfling, "Approximation Theorems of Mathematical Statistics," Wiley, New York, 1980.

W. Stute, The bias of Kaplan–Meier integrals, *Scand J. Statist.* **21**, 475–484.

W. Stute, The central limit theorem under random censorship, *Ann. Statist.* **23** (1995), 422–439.

W. Stute and J. L. Wang, The strong law under random censorship, *Ann. Statist.* **21** (1993), 1591–1607.

V. Susarla and J. Van Ryzin, Large sample theory for an estimator of the mean survival time from censored samples, *Ann. Statist.* **8** (1980), 1002–1016.