# Deformation Theory of Dialgebras

ANITA MAJUMDAR* and GOUTAM MUKHERJEE
*Stat-Math Division, Indian Statistical Institute, 203 B.T. Road, Calcutta 700 108, India.*
*e-mail: {anita_r, goutam}@isical.ac.in*

(Received: January 2002)

**Abstract.** We develop deformation theory of (associative) dialgebras and show that perturbation of algebraic structures of a dialgebra is controlled by the dialgebra cohomology.

**Mathematics Subject Classifications (2000):** 13D03, 13D10, 14D15.

**Key words:** Dialgebras, dialgebra cohomology, planar binary trees, deformations of dialgebras, pre-Lie system.

## 1. Introduction

The study of deformations of algebraic structures was initiated by M. Gerstenhaber [4–8]. He introduced deformation theory for associative algebras. His theory was extended to Lie algebras by A. Nijenhuis and R. Richardson [12–14]. The deformation theory of bialgebras, which relates to quantum groups, was studied by M. Gerstenhaber and S. D. Schack [9].

The aim of this paper is to develop an algebraic deformation theory for a class of binary quadratic algebras whose structure is determined by two associative operations intertwined by some relations, called associative dialgebras (or simply dialgebras in this paper), discovered by J.-L. Loday [10].

The notion of Leibniz algebras and dialgebras was discovered by Loday while studying periodicity phenomena in algebraic $K$-theory [11]. Leibniz algebras are a non-commutative variation of Lie algebras and dialgebras are a variation of associative algebras. Recall that any associative algebra gives rise to a Lie algebra by $[x, y] = xy - yx$. The notion of dialgebras was invented in order to build analogue of the couple

$$\text{Lie algebras} \leftrightarrow \text{associative algebras},$$

when Lie algebras are replaced by Leibniz algebras.

In the majority of the available cases of deformation theory, experience provides one with a natural candidate for the cohomology controlling the deformations.

*The first author is supported by NBHM research fellowship.

For example, one knows that Hochschild cohomology captures the deformation of associative algebras and the Chevalley–Eilenberg cohomology controls the deformation of Lie algebras.

A (co)homology theory associated to dialgebras was developed by Loday, called the dialgebra cohomology, where planar binary trees play a crucial role in the construction. Dialgebra cohomology with coefficients was studied by A. Frabetti [2]. In the present paper, we develop a deformation theory following [5] and show that in this case, the dialgebra cohomology is a natural candidate for the cohomology controlling the deformation. It may be mentioned that D. Balavoine, in [1], studied formal deformations of algebras over a quadratic operad in general, and showed that the cohomology theory which is involved is the one given by the Koszul dual operad. The triple cohomology and the enriched cohomology can also be used to develop deformation theory (see [3]). An interesting feature of our approach is to show that the cochain modules associated to dialgebra admit the structure of a pre-Lie system which we use to develop the deformation theory.

The paper is organized as follows. In Section 2, we summarise the basic facts about dialgebras and their cohomology. In Section 3, we define formal deformation of dialgebras, obstruction cochains, prove few basic properties and state one of the two main theorems about obstruction cochains. In Section 4, we study the notion of equivalent and trivial deformations in this context and prove that the free dialgebra $\mathrm{Dias}(V)$ is rigid. In Section 5, we introduce the notion of infinitesimal of an automorphism, define obstruction to integrability of 1-cocycles and state the other theorem about obstruction cochains. Recall that in the deformation theory of associative algebras, the existence of $\circ_i$-products on the Hochschild complex play a crucial role in proving that obstruction cochains are cocycles. Interestingly enough, using planar binary trees, we show in Section 6 that, $\circ_i$-products exist on the cochain modules $CY^*(D, D)$ of a dialgebra $D$. It turns out that equipped with these $\circ_i$-products, $CY^*(D, D)$ admits the structure of a pre-Lie system. We then use $\circ_i$-products to define a pre-Lie product $\circ$ on $CY^*(D, D)$ which makes $CY^*(D, D)$ a pre-Lie ring. There is also defined an associative product $*$ on $CY^*(D, D)$ and we establish a relation connecting the pre-Lie product $\circ$, the associative product $*$ and the coboundary operators of $CY^*(D, D)$. Finally, in Section 7, we interpret the obstruction cochains in terms of $\circ$ and $*$ and prove the two main theorems stated in Sections 3 and 5.

## 2. Dialgebras and Dialgebra Cohomology

In this section, we recall the definition of dialgebras and dialgebra cohomology [2,10].

DEFINITION 2.1. Let $k$ be a field. A dialgebra $D$ over $k$ is a vector space over $k$ along with two $k$-linear maps $\dashv : D \otimes D \longrightarrow D$ called left and $\vdash : D \otimes D \longrightarrow D$

called right satisfying the following axioms :

$$x \dashv (y \dashv z) \overset{1}{=} (x \dashv y) \dashv z \overset{2}{=} x \dashv (y \vdash z),$$

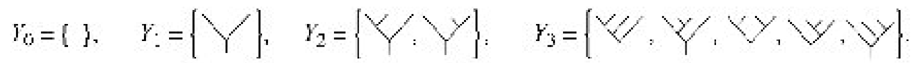
$$(x \vdash y) \dashv z \overset{3}{=} x \vdash (y \dashv z),$$

$$(x \dashv y) \vdash z \overset{4}{=} x \vdash (y \vdash z) \overset{5}{=} (x \vdash y) \vdash z, \tag{1}$$

for all $x, y, z \in D$.

A morphism of dialgebras from $D$ to $D'$ is a $k$-linear map $f: D \longrightarrow D'$ such that $f(x \dashv y) = f(x) \dashv f(y)$ and $f(x \vdash y) = f(x) \vdash f(y)$ for all $x, y \in D$. Let Dias denote the category of dialgebras.

A planar binary tree with $n$ vertices (in short, $n$-tree) is a planar tree with $(n+1)$ leaves, one root and each vertex trivalent. Let $Y_n$ denote the set of all $n$-trees. Let $Y_0$ be the singleton set consisting of a root only. Some low-dimensional trees are given by the following diagrams:




For any $y \in Y_n$, the $(n+1)$ leaves are labelled by $\{0, 1, \ldots, n\}$ from left to right and the vertices are labelled $\{1, 2, \ldots, n\}$ so that the $i$th vertex is between the leaves $(i-1)$ and $i$. Recall from [10] that the only element | of $Y_0$ is denoted by [0]. The only element of $Y_1$ is denoted by [1]. The grafting of a $p$-tree $y_1$ and a $q$-tree $y_2$ is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$ which is obtained by joining the roots of $y_1$ and $y_2$ and creating a new root from that vertex. This is denoted by $[y_1\ p+q+1\ y_2]$ with the convention that all zeros are deleted except for the element in $Y_0$. With this notation, the trees pictured above from left to right are [0], [1], [12], [21], [123], [213], [131], [312], [321]. Throughout this paper we shall use these notations to represent elements of $Y_n$, $0 \leqslant n \leqslant 3$. For any $i$, $0 \leqslant i \leqslant n$, there is a map, called the face map, $d_i: Y_n \longrightarrow Y_{n-1}$, $y \mapsto d_i y$ where $d_i y$ is obtained from $y$ by deleting the $i$th leaf. The face maps satisfy all the classical presimplicial relations $d_i d_j = d_{j-1} d_i$, $0 \leqslant i < j \leqslant n$.

Let $D$ be a dialgebra over a field $k$. For any $n \geqslant 0$, let $k[Y_n]$ denote the $k$-vector space spanned by $Y_n$. The dialgebra cohomology $HY^n(D, D)$ is defined by the cochain complex $\{CY^*(D, D), \delta\}$, where

$$CY^n(D, D) := \mathrm{Hom}_k(k[Y_n] \otimes D^{\otimes n}, D)$$

and $\delta: CY^n(D, D) \longrightarrow CY^{n+1}(D, D)$ is defined as the $k$-linear map $\delta = \sum_{i=0}^{n+1} (-1)^i \delta^i$, with

$$(\delta^i f)(y; a_1, a_2, \ldots, a_{n+1}) := \begin{cases} a_1 o_0^y f(d_0 y; a_2, \ldots, a_{n+1}), & \text{if } i = 0, \\ f(d_i y; a_1, \ldots, a_i o_i^y a_{i+1}, \ldots, a_{n+1}), & \\ \quad \text{if } 1 \leqslant i \leqslant n, & \\ f(d_{n+1} y; a_1, \ldots, a_n) o_{n+1}^y a_{n+1}, & \\ \quad \text{if } i = n+1, & \end{cases}$$

for any $y \in Y_{n+1}$; $a_1, \ldots, a_{n+1} \in D$ and $f: k[Y_n] \otimes D^{\otimes n} \longrightarrow D$. Here, for any $i$, $0 \leqslant i \leqslant n+1$, the maps $o_i: Y_{n+1} \longrightarrow \{\dashv, \vdash\}$, are defined by

$$o_0(y) = o_0^y := \begin{cases} \dashv & \text{if } y \text{ is of the form } | \vee y_1, \text{ for some } n\text{-tree } y_1, \\ \vdash & \text{otherwise,} \end{cases}$$

$$o_i(y) = o_i^y := \begin{cases} \dashv & \text{if the } i\text{th leaf of } y \text{ is oriented like `}\backslash\text{'} \\ \vdash & \text{if the } i\text{th leaf of } y \text{ is oriented like `/'} \end{cases} \text{ for } 1 \leqslant i \leqslant n$$

$$o_{n+1}(y) = o_{n+1}^y := \begin{cases} \vdash & \text{if } y \text{ is of the form } y_1 \vee |, \text{ for some } n\text{-tree } y_1, \\ \dashv & \text{otherwise.} \end{cases}$$

The submodule of $n$-cocycles is denoted by $ZY^n(D, D)$.

## 3. Deformations of Dialgebras

We begin this section with the definition of formal deformations of dialgebras.

DEFINITION 3.1. Let $D$ be a dialgebra over a field $k$ with left product $\dashv$ and right product $\vdash$. Let $V$ be the underlying vector space of $D$, $k[[t]]$ denote the power series ring in one variable and $K = k((t))$ denote the quotient power series field. Let $V_K$ denote the $K$-vector space $V \otimes_k k((t))$. Note that any $k$-bilinear map $V \times V \longrightarrow V$ extends to a $K$-bilinear map $V_K \times V_K \longrightarrow V_K$ in a natural way and any $K$-bilinear map $V_K \times V_K \longrightarrow V_K$ which is such an extension will be called 'defined over $k$'. Suppose there be given two bilinear maps $f_t^\ell$, $f_t^r: V_K \times V_K \longrightarrow V_K$, which are expressible in the form

$$f_t^\ell(a,b) = F_0^\ell(a,b) + F_1^\ell(a,b)t + F_2^\ell(a,b)t^2 + \cdots, \tag{2}$$

$$f_t^r(a,b) = F_0^r(a,b) + F_1^r(a,b)t + F_2^r(a,b)t^2 + \cdots, \tag{3}$$

for all $a, b \in V_K$, where $F_i^\ell$ and $F_i^r$ are bilinear maps $V_K \times V_K \longrightarrow V_K$ defined over $k$, and $F_0^\ell$ and $F_0^r$ are induced by $\dashv$ and $\vdash$, respectively. Moreover, assume that $V_K$ equipped with the products $f_t^\ell$ and $f_t^r$ is a dialgebra which we denote by $D_t$. Then $D_t$ is called a one-parameter family of formal deformations of $D$.

Note that there is a canonical inclusion $V \hookrightarrow V_K$, $a \mapsto a \otimes 1$. Thus in order to check that the identities (2) and (3) hold for all $a, b \in V_K$, it is enough to check that they hold for all $a, b \in V$, as all the maps involved are defined over $k$.

DEFINITION 3.2. The 'infinitesimal' or 'differential' of this family of formal deformations is the function $F_1: k[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ defined by

$$F_1(y; a_1, a_2) = \begin{cases} F_1^\ell(a_1, a_2), & \text{if } y = [21], \\ F_1^r(a_1, a_2), & \text{if } y = [12], \end{cases}$$

where $F_1^\ell$ and $F_1^r$ are considered as $k$-bilinear functions from $V \times V$ to $V$. More generally, if $F_i^\ell = 0 = F_i^r$, $1 \leqslant i \leqslant n-1$, with either $F_n^\ell$ or $F_n^r$ nonzero, then the function $F_n\colon k[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ defined by $F_n^\ell$ and $F_n^r$ as above is called the $n$-infinitesimal of this family of deformations.

Thus infinitesimal of a family is simply 1-infinitesimal.

Note that $D_t$ is a dialgebra if and only if $f_t^\ell$ and $f_t^r$ satisfy the dialgebra axioms (1), that is,

$$
\begin{aligned}
&f_t^\ell(a, f_t^\ell(b, c)) = f_t^\ell(f_t^\ell(a, b), c),\\
&f_t^\ell(f_t^\ell(a, b), c) = f_t^\ell(a, f_t^r(b, c)),\\
&f_t^\ell(f_t^r(a, b), c) = f_t^r(a, f_t^\ell(b, c)),\\
&f_t^r(f_t^\ell(a, b), c) = f_t^r(a, f_t^r(b, c)),\\
&f_t^r(a, f_t^r(b, c)) = f_t^r(f_t^r(a, b), c),
\end{aligned}
\tag{4}
$$

hold for all $a, b, c \in V$. Now expanding both sides of each of the equations in (4) and collecting coefficients of $t^\nu$ we see that (4) is equivalent to the system of equations

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^\ell(b, c)) = 0, \tag{$5_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^r(b, c)) = 0, \tag{$6_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}} F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)) = 0, \tag{$7_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}} F_\lambda^r(F_\mu^\ell(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) = 0, \tag{$8_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu\geqslant 0}} F_\lambda^r(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^r(b, c)) = 0, \tag{$9_\nu$}$$

for all $a, b, c \in V$ and for $\nu = 0, 1, 2 \ldots$. The above equations reduce to axioms (1) of the dialgebras for $\nu = 0$. The following lemma relates deformations of a dialgebra $D$ to the dialgebra cohomology $HY^*(D, D)$.

LEMMA 3.3. *The infinitesimal $F_1$ of a formal deformation $D_t$ of a dialgebra $D$ is a 2-cocycle.*

*Proof.* First note that the equations ($5_\nu$) to ($9_\nu$), for $\nu = 1$ are, respectively,

$$F_1^\ell(a \dashv b, c) - F_1^\ell(a, b \dashv c) + F_1^\ell(a, b) \dashv c - a \dashv F_1^\ell(b, c) = 0,$$

$$F_1^\ell(a \dashv b, c) - F_1^\ell(a, b \vdash c) + F_1^\ell(a, b) \dashv c - a \dashv F_1^r(b, c) = 0,$$

$$F_1^\ell(a \vdash b, c) - F_1^r(a, b \dashv c) + F_1^r(a, b) \dashv c - a \vdash F_1^\ell(b, c) = 0,$$

$$F_1^r(a \dashv b, c) - F_1^r(a, b \vdash c) + F_1^\ell(a, b) \vdash c - a \vdash F_1^r(b, c) = 0,$$

$$F_1^r(a \vdash b, c) - F_1^r(a, b \vdash c) + F_1^r(a, b) \vdash c - a \vdash F_1^r(b, c) = 0.$$

Now in order to prove that $\delta F_1 = 0$, by definition we have to show that $(\delta F_1)(y; a, b, c) = 0$ for all $a, b, c \in D$ and for $y = [123], [213], [131], [312], [321]$. Let us prove it for $y = [321]$. By the definition of the coboundary map we have,

$$\begin{aligned}
&(\delta F_1)([321]; a, b, c) \\
&\quad = a \dashv F_1([21]; b, c) - F_1([21]; a \dashv b, c) + F_1([21]; a, b \dashv c) - \\
&\qquad - F_1([21]; a, b) \dashv c \\
&\quad = a \dashv F_1^\ell(b, c) - F_1^\ell(a \dashv b, c) + F_1^\ell(a, b \dashv c) - F_1^\ell(a, b) \dashv c \\
&\quad = 0
\end{aligned}$$

(by $(5_1)$) for all $a, b, c \in D$. The other cases follow similarly by using Equations $(6_1)$ to $(9_1)$. Thus $F_1$ is a 2-cocycle. □

Similarly, one can prove in general, that the $n$-infinitesimal is a cocycle.

DEFINITION 3.4. Any 2-cocycle $F$ need not be the infinitesimal of a deformation. If it be so, then we call $F$ integrable.

Therefore, $F$ is integrable if $F = F_1$ can be extended to a sequence $F_2, F_3, \ldots, F_\nu, \ldots$, where $F_\nu: k[Y_2] \otimes D^{\otimes 2} \longrightarrow D$ is defined by

$$F_\nu(y; a, b) = \begin{cases} F_\nu^\ell(a, b), & \text{if } y = [21], \\ F_\nu^r(a, b), & \text{if } y = [12], \end{cases} \tag{10}$$

for some $D_t$, along with $f_t^\ell$ and $f_t^r$, denoting a one parameter family of deformations of $D$, as defined in Equations (2) and (3) and satisfying

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^\ell(b, c)) = \delta F_\nu([321]; a, b, c), \tag{$11_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^\ell(a, b), c) - F_\lambda^\ell(a, F_\mu^r(b, c)) = \delta F_\nu([312]; a, b, c), \tag{$12_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda^\ell(F_\mu^r(a, b), c) - F_\lambda^r(a, F_\mu^\ell(b, c)) = \delta F_\nu([131]; a, b, c), \tag{$13_\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}} F^r_\lambda(F^\ell_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)) = \delta F_\nu([213]; a, b, c), \qquad (14_\nu)$$

$$\sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}} F^r_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)) = \delta F_\nu([123]; a, b, c), \qquad (15_\nu)$$

for all $a, b, c \in D$. This is because by the definition of coboundary we have

$$\begin{aligned}&(\delta F_\nu)(y; a, b, c)\\ &\quad = a o^y_0 F_\nu(d_0 y; b, c) - F_\nu(d_1 y; a o^y_1 b, c) + F_\nu(d_2 y; a, b o^y_2 c) -\\ &\qquad - F_\nu(d_3 y; a, b) o^y_3 c,\end{aligned}$$

for all $y \in Y_3$ and $a, b, c \in D$. Note that in particular, for $y = [321]$, the above equation yields

$$\begin{aligned}(\delta F_\nu)(y; a, b, c) &= a \dashv F^\ell_\nu(b,c) - F^\ell_\nu(a \dashv b, c) + F^\ell_\nu(a, b \dashv c) -\\ &\quad - F^\ell_\nu(a,b) \dashv c\\ &= - \sum_{\substack{\lambda+\mu=\nu\\ \lambda=0, or\, \mu=0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a, F^\ell_\mu(b,c)).\end{aligned}$$

Similarly for the other trees.

Now suppose we are given 2-cochains $F_\nu$, $1 \leqslant \nu \leqslant n-1$. Define $F^\ell_\nu, F^r_\nu$: $D^{\otimes 2} \longrightarrow D$ by

$$F^\ell_\nu(a,b) = F_\nu([21]; a, b) \quad \text{and} \quad F^r_\nu(a,b) = F_\nu([12]; a, b),$$

for all $a, b \in D$. Moreover suppose that $F^\ell_\nu$, $F^r_\nu$ and $F_\nu$ satisfy Equations $(11_\nu)$–$(15_\nu)$, $1 \leqslant \nu \leqslant n-1$, then define a 3-cochain $G$: $k[Y_3] \otimes D^{\otimes 3} \longrightarrow D$ as follows:

$$\begin{aligned}G([321]; a, b, c) &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a, F^\ell_\mu(b,c)),\\
G([312]; a, b, c) &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F^\ell_\lambda(F^\ell_\mu(a,b),c) - F^\ell_\lambda(a, F^r_\mu(b,c)),\\
G([131]; a, b, c) &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F^\ell_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a, F^\ell_\mu(b,c)),\\
G([213]; a, b, c) &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F^r_\lambda(F^\ell_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)),\\
G([123]; a, b, c) &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F^r_\lambda(F^r_\mu(a,b),c) - F^r_\lambda(a, F^r_\mu(b,c)),\end{aligned}$$

for all $a, b, c \in D$.

By introducing a pre-Lie product $\circ$ on $CY^*(D, D)$ in Section 6, we shall show in Section 7, that the above data is equivalent to giving 2-cochains $F_\nu$, $1 \leqslant \nu \leqslant n-1$, satisfying $\delta F_\nu = \sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu$, for all $1 \leqslant \nu \leqslant n-1$. It turns out that the 3-cochain $G$ as defined above can be expressed as $G = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda \circ F_\mu$. Moreover, we shall prove the following theorem:

THEOREM 3.5. *Let $D$ be a dialgebra and $F_1, F_2, \ldots, F_{n1}$ be elements of $CY^2(D, D)$ with $F_1$ a 2-cocycle, such that*

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} F_\lambda \circ F_\mu = \delta F_\nu, \tag{16}$$

*for all $\nu = 1, 2, \ldots, n-1$. If $G \in CY^3(D, D)$ is given by*

$$G = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} F_\lambda \circ F_\mu,$$

*then $\delta G = 0$, that is, $G$ is a 3-cocycle. The cohomology class of $G$ must vanish in order to extend the given sequence to a sequence $F_1, F_2, \ldots, F_n$ satisfying Equation (16) for all $\nu = 1, 2, \ldots, n$.*

DEFINITION 3.6. The cohomology class of $G$ is called the $(n-1)$th obstruction to extend the sequence $F_1, F_2, \ldots, F_{n-1}$ satisfying equations $(11_\nu)$ to $(15_\nu)$, $1 \leqslant \nu \leqslant n-1$ to a sequence $F_1, F_2, \ldots, F_n$ satisfying Equations $(11_\nu)$ to $(15_\nu)$, $1 \leqslant \nu \leqslant n$, with $F_i^\ell$'s and $F_i^r$'s obtained from $F_i$ as described above.

COROLLARY 3.7. *If $HY^3(D, D) = 0$ for a dialgebra $D$, then all the obstructions vanish and hence any 2-cocyle is integrable.*

EXAMPLE 3.8. Let $k[x, y, x^{-1}, y^{-1}]$ denote the vector space of all Laurent polynomials in two variables $x$ and $y$ over a field $k$ with basis $x^p y^q$ with $p, q \in \mathbb{Z}$. Define two operations $\dashv$ and $\vdash$ on the basis elements by

$$x^m y^n \dashv x^r y^s = x^m y^{n+r+s} \quad \text{and} \quad x^m y^n \vdash x^r y^s = x^{m+n+r} y^s.$$

Extending these two operations on $k[x, y, x^{-1}, y^{-1}]$ by bilinearity, we get linear maps

$$\dashv, \vdash : k[x, y, x^{-1}, y^{-1}] \otimes k[x, y, x^{-1}, y^{-1}] \longrightarrow k[x, y, x^{-1}, y^{-1}].$$

It is straightforward to verify that $k[x, y, x^{-1}, y^{-1}]$ equipped with the operations $\dashv$ and $\vdash$ is a dialgebra.

Define linear maps

$$F_\nu^\ell, F_\nu^r : k[x, y, x^{-1}, y^{-1}] \otimes k[x, y, x^{-1}, y^{-1}] \longrightarrow k[x, y, x^{-1}, y^{-1}]$$

by

$$\begin{aligned} F_\nu^\ell(x^m y^n, x^r y^s) &= x^m y^{n-\nu} \dashv x^{r-\nu} y^s \\ &= x^m y^{n+r+s-2\nu}, \\ F_\nu^r(x^m y^n, x^r y^s) &= x^m y^{n-\nu} \vdash x^{r-\nu} y^s \\ &= x^{m+n+r-2\nu} y^s. \end{aligned}$$

Then one checks that the linear maps $F_\nu^\ell$, $F_\nu^r$ satisfy Equations ($5_\nu$) to ($9_\nu$). For this it is enough to verify that for $a = x^m y^n$, $b = x^r y^s$ and $c = x^u y^v$ each term of the left-hand side of Equations ($5_\nu$) to ($9_\nu$) except ($7_\nu$) is zero. For ($7_\nu$), the term corresponding to $(\lambda, \mu)$, $\lambda + \mu = \nu$, $\lambda \neq \mu$ cancels with the term corresponding to $(\mu, \lambda)$, $\lambda + \mu = \nu$, $\lambda \neq \mu$ and any term with $\lambda = \mu$ is again zero. Hence

$$D_t = k[x, y, x^{-1}, y^{-1}] \otimes_k k((t))$$

with $f_t^\ell$ and $f_t^r$ is a deformation of the dialgebra $D = k[x, y, x^{-1}, y^{-1}]$, where

$$f_t^\ell = \sum_{\nu \geqslant 0} F_\nu^\ell t^\nu \quad \text{and} \quad f_t^r = \sum_{\nu \geqslant 0} F_\nu^r t^\nu.$$

## 4. Equivalent and Trivial Deformations

In this section we study the isomorphisms between deformations of a dialgebra $D$ which keeps $D$ fixed, define trivial deformations, rigidity and prove that the free dialgebra $\mathrm{Dias}(V)$ is rigid.

DEFINITION 4.1. Let $D_t(f)$ and $D_t(g)$ be two deformations of a dialgebra $D$ given by

$$f_t^\ell = \sum_{\nu \geqslant 0} F_\nu^\ell t^\nu, \quad f_t^r = \sum_{\nu \geqslant 0} F_\nu^r t^\nu \quad \text{and} \quad g_t^\ell = \sum_{\nu \geqslant 0} G_\nu^\ell t^\nu, \quad g_t^r = \sum_{\nu \geqslant 0} G_\nu^r t^\nu$$

respectively. By a formal isomorphism $D_t(f) \longrightarrow D_t(g)$ we mean a $K$-linear automorphism $\Psi_t\colon V_K \longrightarrow V_K$ of the form $\Psi_t(a) = a + \psi_1(a)t + \psi_2(a)t^2 + \cdots$, where each $\psi_i\colon V_K \longrightarrow V_K$ is a linear map 'defined over $k$' such that

$$f_t^*(a, b) = \Psi_t^{-1}(g_t^*(\Psi_t(a), \Psi_t(b))) \tag{17}$$

for all $a, b \in V_K$ (or equivalently for all $a, b \in V$, since all the maps involved are defined over $k$) and $* = \ell, r$. If such a $\Psi_t$ exists, then the deformations $D_t(f)$ and $D_t(g)$ are said to be equivalent. A formal automorphism of a deformation $D_t(f)$ is simply a formal isomorphism $D_t(f) \longrightarrow D_t(f)$.

DEFINITION 4.2. A deformation $D_t(f)$ is said to be trivial if it is equivalent to the identity deformation, where the identity deformation is the dialgebra $D_K = D \otimes_k K$ with the underlying vector space $V_K$ and with multiplications $g_t^\ell(a, b) = a \dashv b$ and $g_t^r(a, b) = a \vdash b$ for all $a, b \in V_K$, induced by the products of $D$.

With the above notations we have the following proposition:

PROPOSITION 4.3. *If $D_t(f)$ and $D_t(g)$ are equivalent deformations of $D$ given by the isomorphism $\Psi_t$: $D_t(f) \longrightarrow D_t(g)$, then the infinitesimals of $D_t(f)$ and $D_t(g)$ determine the same cohomology class.*

*Proof.* From (17), we have

$$\Psi_t(f_t^\ell(a,b)) = g_t^\ell(\Psi_t(a), \Psi_t(b)), \tag{18}$$
$$\Psi_t(f_t^r(a,b)) = g_t^r(\Psi_t(a), \Psi_t(b)), \tag{19}$$

for all $a, b \in V$. Expanding both sides of Equations (18) and (19) and collecting the coefficients of $t^n$, we get

$$\sum_{i+j+k=n} G_i^*(\psi_j(a), \psi_k(b)) = \sum_{i+j=n} \psi_i(F_j^*(a,b)),$$

where $* = \ell, r$. Taking $n = 1$, we get

$$\begin{aligned} F_1^\ell(a,b) &= G_1^\ell(a,b) + a \dashv \psi_1(b) + \psi_1(a) \dashv b - \psi_1(a \dashv b), \\ F_1^r(a,b) &= G_1^r(a,b) + a \vdash \psi_1(b) + \psi_1(a) \vdash b - \psi_1(a \vdash b), \end{aligned}$$

for all $a, b \in V$. Now since $\mathrm{Hom}_k(k[Y_1] \otimes_k D, D) \cong \mathrm{Hom}_k(D, D)$, $\psi_1$ can be identified with a unique 1-cochain again denoted by $\psi_1$ where $\psi_1([1]; a) = \psi_1(a)$ for all $a \in D$. Observe that

$$\begin{aligned} \delta\psi_1([21]; a, b) &= a \dashv \psi_1(b) - \psi_1(a \dashv b) + \psi_1(a) \dashv b \\ &= F_1^\ell(a,b) - G_1^\ell(a,b), \\ \delta\psi_1([12]; a, b) &= a \vdash \psi_1(b) - \psi_1(a \vdash b) + \psi_1(a) \vdash b \\ &= F_1^r(a,b) - G_1^r(a,b). \end{aligned}$$

Hence, $\delta\psi_1 = F_1 - G_1$. This completes the proof. □

*Remark* 4.4. It follows that the integrability of an element of $ZY^2(D, D)$ depends only on its cohomology class. For if $D_t$ is a deformation given by $f_t^\ell$ and $f_t^r$ with infinitesimal $F_1$ and if $G_1 = F_1 + \delta\varphi$, then $G_1$ is the infinitesimal of the deformation $D_t'$ given by $g_t^\ell$ and $g_t^r$ where

$$g_t^\ell(a,b) = \Psi_t^{-1} f_t^\ell(\Psi_t(a), \Psi_t(b)) \quad \text{and} \quad g_t^r(a,b) = \Psi_t^{-1} f_t^r(\Psi_t(a), \Psi_t(b)),$$

where $\Psi_t$: $D_t' \longrightarrow D_t$ is the isomorphism given by $\Psi_t(a) = a + \psi(a)t$.

THEOREM 4.5. *A nontrivial deformation of a dialgebra is equivalent to a deformation whose infinitesimal is not a coboundary.*

*Proof.* Let $D_t$ be a deformation of $D$ with multiplications $f_t^\ell = \sum_{\nu \geqslant 0} F_\nu^\ell t^\nu$ and $f_t^r = \sum_{\nu \geqslant 0} F_\nu^r t^\nu$. Let $F_n$ (the unique 2-cochain defined by $F_n^\ell$ and $F_n^r$ as in (10)) be the $n$-infinitesimal of the deformation, for $n \geqslant 1$. Then as in the case of $n = 1$, Equations $(5_\nu)$ to $(9_\nu)$ imply that $\delta F_n = 0$. Now suppose that $F_n$ is a coboundary, say $F_n = -\delta\psi_n$ for some $\psi_n \in CY^1(D, D)$ (which is isomorphic to $\mathrm{Hom}_k(D, D)$). Let $\Psi_t$ be the formal automorphism of $V_K$ defined by $\Psi_t(a) = a + \psi_n(a)t^n$. Then setting

$$g_t^\ell(a, b) = \Psi_t^{-1} f_t^\ell(\Psi_t(a), \Psi_t(b)) = \sum_{\nu \geqslant 0} G_\nu^\ell(a, b)t^\nu,$$

$$g_t^r(a, b) = \Psi_t^{-1} f_t^r(\Psi_t(a), \Psi_t(b)) = \sum_{\nu \geqslant 0} G_\nu^r(a, b)t^\nu,$$

we get a deformation $D_t'$ isomorphic to $D_t$. Explicitly, $g_t^\ell$ and $g_t^r$ are given by

$$\begin{aligned} g_t^\ell(a, b) &= a \dashv b - \{\psi_n(a \dashv b) - \psi_n(a) \dashv b - a \dashv \psi_n(b) - F_n^\ell(a, b)\}t^n + \\ &\quad + F_{n+1}^\ell t^{n+1} + \cdots, \\ g_t^r(a, b) &= a \vdash b - \{\psi_n(a \vdash b) - \psi_n(a) \vdash b - a \vdash \psi_n(b) - F_n^r(a, b)\}t^n + \\ &\quad + F_{n+1}^r t^{n+1} + \cdots. \end{aligned}$$

Suppose $F_n^\ell \neq 0$. Then as $F_n = -\delta\psi_n$, we see that

$$\begin{aligned} F_n^\ell(a, b) &= F_n([21]; a, b) = -\delta\psi_n([21]; a, b) \\ &= -\{a \dashv \psi_n([1]; b) - \psi_n([1]; a \dashv b) + \psi_n([1]; a) \dashv b\} \\ &= -\{a \dashv \psi_n(b) - \psi_n(a \dashv b) + \psi_n(a) \dashv b\}. \end{aligned}$$

Thus the coefficient of $t^n$ in $g_t^\ell(a, b)$ is zero. In case $F_n^\ell = 0$, then $\delta\psi_n([21]; a, b) = 0$ and hence coefficient of $t^n$ is again zero. By a similar argument the coefficient of $t^n$ in the expression of $g_t^r$ is also zero. Thus $G_i^* = 0$ for $1 \leqslant i \leqslant n$ where $* = \ell, r$. Hence, we can repeat our argument to kill off an infinitesimal that is a coboundary and the process must stop if the deformation $D_t$ is nontrivial. □

DEFINITION 4.6. A dialgebra $D$ is said to be rigid if every deformation is equivalent to the trivial deformation.

COROLLARY 4.7. *If* $HY^2(D, D) = 0$, *then* $D$ *is rigid*. □

It is well known that the tensor algebra $T(V)$ which is the free object in the category of associative algebras is rigid in the sense of deformation theory of associative algebras. Here we show that the free object in the category Dias, that is, the free dialgebra $\mathrm{Dias}(V)$ over a vector space $V$ [10], is rigid in the sense of deformation theory of dialgebras.

PROPOSITION 4.8. *The free dialgebra* $\mathrm{Dias}(V)$, *over the vector space* $V$, *is rigid*.

*Proof.* Let us denote $\mathrm{Dias}(V)$ by $D$. By Corollary 4.7, it is enough to show that $HY^2(D, D) = 0$. Let $f \in ZY^2(D, D)$. Let $\bar{D}$ denote the underlying vector space of $D$. Consider the short exact sequence of dialgebras

$$0 \longrightarrow \bar{D} \xrightarrow{i} \bar{D} \oplus \bar{D} \xrightarrow{\pi} D \longrightarrow 0,$$

where $\bar{D}$ on the left is considered as a dialgebra with Abelian products, that is, $a \dashv b = a \vdash b = 0$ for all $a, b \in \bar{D}$, and the dialgebra structure on $\bar{D} \oplus \bar{D}$ is defined by

$$(a_1, b_1) \dashv (a_2, b_2) = (a_1 \dashv b_2 + b_1 \dashv a_2 - f([21]; b_1, b_2), b_1 \dashv b_2),$$
$$(a_1, b_1) \vdash (a_2, b_2) = (a_1 \vdash b_2 + b_1 \vdash a_2 - f([12]; b_1, b_2), b_1 \vdash b_2),$$

$i$ being the inclusion into the first factor and $\pi$ the projection onto the second factor. This sequence splits as a sequence of vector spaces. So there exists a $k$-linear map $\sigma\colon D \longrightarrow \bar{D} \oplus \bar{D}$ such that $\pi \circ \sigma = \mathrm{id}_D$. Hence, $\sigma$ must be of the form $(g, \mathrm{id})$, where $g\colon \bar{D} \longrightarrow \bar{D}$ is $k$-linear. Let $\sigma' = \sigma/V\colon V \longrightarrow \bar{D} \oplus \bar{D}$. Universal property of $D = \mathrm{Dias}(V)$ gives a dialgebra map $\widetilde{\sigma}\colon D \longrightarrow \bar{D} \oplus \bar{D}$ with $\widetilde{\sigma} \circ j = \sigma'$, $j$ being the inclusion $V \hookrightarrow D$. Since $\pi$ is a dialgebra map we have $\pi \circ \widetilde{\sigma} = \mathrm{id}$. Hence $\widetilde{\sigma}$ is of the form $(\varphi, \mathrm{id})$ for some $k$-linear map $\varphi\colon \bar{D} \longrightarrow \bar{D}$. Now as $\widetilde{\sigma}$ is a dialgebra map, we deduce that $f(y; a, b) = \delta\varphi(y; a, b)$ for $y = [21], [12]$ and where $\varphi$ has been interpreted as a 1-cochain. Thus $f = \delta\varphi$. This completes the proof. □

## 5. Automorphisms of the Dialgebra $D_K$

Let $D_K = D \otimes_k K$, $K = k((t))$ be the dialgebra denoting the identity deformation as introduced in Definition 4.2. In this section we study the automorphisms of $D_K$ and define the obstructions to integrability of derivations of $D$. According to Definition 4.1, a formal automorphism of the identity deformation $D_K$ is given by a $k$-linear map $\Psi_t\colon D_K \longrightarrow D_K$ of the form

$$\Psi_t(a) = \psi_0(a) + \psi_1(a)t + \psi_2(a)t^2 + \cdots, \tag{20}$$

for all $a \in D$, where each $\psi_i\colon D_K \longrightarrow D_K$ is a linear map defined over $k$ and $\psi_0$ is the identity map. Moreover the following hold

$$\Psi_t(a \dashv b) = \Psi_t(a) \dashv \Psi_t(b), \quad \Psi_t(a \vdash b) = \Psi_t(a) \vdash \Psi_t(b)$$

for all $a, b \in D$.

DEFINITION 5.1. The first nonzero coefficient $\psi_n$ in (20) is called the infinitesimal of $\Psi_t$.

LEMMA 5.2. *The infinitesimal of an automorphism of $D_K$ is a cocycle.*

*Proof.* Substituting $\Psi_t$ as is given in Equation (20) and equating coefficients of $t^\nu$ we get

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu \geqslant 0}} \psi_\lambda(a) \dashv \psi_\mu(b) = \psi_\nu(a \dashv b), \tag{21$\nu$}$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu \geqslant 0}} \psi_\lambda(a) \vdash \psi_\mu(b) = \psi_\nu(a \vdash b), \tag{22$\nu$}$$

for all $a, b \in D$, and for all $\nu = 0, 1, 2, \ldots$. Equations ($21_\nu$) and ($22_\nu$) can be rewritten as

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda(a) \dashv \psi_\mu(b) = -\psi_\nu(a) \dashv b + \psi_\nu(a \dashv b) - a \dashv \psi_\nu(b), \qquad (23\nu)$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda(a) \vdash \psi_\mu(b) = -\psi_\nu(a) \vdash b + \psi_\nu(a \vdash b) - a \vdash \psi_\nu(b), \qquad (24\nu)$$

for all $a, b \in D$, and for all $\nu = 0, 1, 2, \ldots$. We identify the linear map $\psi_i$ with the corresponding 1-cochain as mentioned in Section 4. Then Equations ($23_\nu$) and ($24_\nu$) reduce to

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b) = -\delta\psi_\nu([21]; a, b), \qquad (25\nu)$$

$$\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b) = -\delta\psi_\nu([12]; a, b), \qquad (26\nu)$$

for all $a, b \in D$ and for all $\nu = 0, 1, \ldots$. Thus Equations ($25_\nu$) and ($26_\nu$) give a necessary and sufficient condition for a linear automorphism $\Psi_t$ as in (20) to be a dialgebra automorphism of $D_K$. It follows from ($25_\nu$) and ($26_\nu$) that $\delta\psi_n = 0$. Hence, the infinitesimal of an automorphism is a derivation of $D$. □

We may ask when a derivation of $D$ may be extended to an automorphism of $D_K$. Suppose that a derivation $\psi_1$ has been extended to a truncated automorphism $\Psi_t = \sum_0^{n-1} \psi_i t^i$ so that $\psi_i$'s satisfy ($25_\nu$) and ($26_\nu$) for all $\nu = 0, 1, \ldots, n-1$. Define a 2-cochain $F$ by

$$F(y; a, b) = \begin{cases} \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b), & \text{if } y = [21], \\ \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b), & \text{if } y = [12], \end{cases}$$

for all $a, b \in D$. We shall show in Section 7 that given a truncated automorphism $\Psi_t = \sum_0^{n-1} \psi_i t^i$ with 1-cochains $\psi_i$ satisfying ($25_\nu$) and ($26_\nu$) for all $\nu = 0, 1, \ldots, n-1$ and extending the derivation $\psi_1$, is equivalent to giving 1-cochains $\psi_2, \psi_3 \ldots, \psi_{n-1}$ satisfying

$$\delta\psi_\nu = -\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu,$$

for all $\nu = 0, \ldots, n-1$, where $*$ is the graded associative product on $CY^*(D, D)$ induced by the pre-Lie system as described in Section 6. It turns out that the 2-cochain $F$ as defined above can be expressed as

$$F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu.$$

Moreover, we shall prove the following theorem in Section 7.

THEOREM 5.3. *Let $D$ be a dialgebra and $\psi_1, \psi_2, \ldots, \psi_{n-1}$ be 1-cochains, with $\psi_1$ a 1-cocycle, such that*

$$-\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu = \delta\psi_\nu,$$

*for all $\nu = 0, 1, \ldots, n-1$. If $F \in CY^2(D, D)$ be given by*

$$F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu,$$

*then $\delta F = 0$, or $F$ is a 2-cocycle. The cohomology class of $F$ must vanish if the truncated automorphism is to be extended.*

DEFINITION 5.4. The cohomology class of $F$ is called the $(n-1)$th obstruction to extend the sequence $\psi_1, \psi_2, \ldots, \psi_{n-1}$ satisfying $(25_\nu)$ and $(26_\nu)$, $1 \leqslant \nu \leqslant n-1$, to a sequence $\psi_1, \psi_2, \ldots, \psi_n$ satisfying $(25_\nu)$ and $(26_\nu)$ for $1 \leqslant \nu \leqslant n$.

Corollary 5.5 follows, from the above theorem.

COROLLARY 5.5. *If $HY^2(D, D) = 0$, then every derivation of $D$ may be extended to an automorphism of $D_K$.*

We end this section with the following theorem.

THEOREM 5.6. *If every derivation of $D$ extends to an automorphism of $D_K$, then every trivial deformation of $D$ has a trivial infinitesimal.*

*Proof.* Suppose that

$$\begin{aligned} f_t^\ell(a, b) &= a \dashv b + F_1^\ell(a, b)t + F_2^\ell(a, b)t^2 + \cdots, \\ f_t^r(a, b) &= a \vdash b + F_1^r(a, b)t + F_2^r(a, b)t^2 + \cdots, \end{aligned}$$

define a trivial deformation of a dialgebra $D$. Let $F_n$ defined by

$$F_n([21]; a, b) = F_n^\ell(a, b), \quad F_n([12]; a, b) = F_n^r(a, b),$$

be the $n$-infinitesimal of this deformation. We have seen in the proof of Theorem 4.5 that $F_n$ is a cocycle. Suppose that $F_n$ is not a coboundary. Let

$$\Psi_t(a) = a + \psi_1(a)t + \psi_2(a)t^2 + \cdots$$

be the isomorphism from $D_t$ to $D_K$, where $D_t$ denotes the deformation defined by $f_t^\ell$ and $f_t^r$. Thus, we have

$$\Psi_t(f_t^\ell(a, b)) = \Psi_t(a) \dashv \Psi_t(b) \text{ and } \Psi_t(f_t^r(a, b)) = \Psi_t(a) \vdash \Psi_t(b),$$

for all $a, b \in D$. Substituting the expression for $\Psi_t$ and equating the coefficients of $t^n$, we get

$$\delta\psi_n([21]; a, b) - F_n^\ell(a, b) = -\sum_{\substack{i+j=n \\ i,j\neq n}} \psi([1]; a) \dashv \psi_j([1]; b),$$

$$\delta\psi_n([21]; a, b) - F_n^r(a, b) = -\sum_{\substack{i+j=n \\ i,j\neq n}} \psi([1]; a) \vdash \psi_j([1]; b).$$

Now if we define a 2-cochain $F$ by

$$F(y; a, b) = \begin{cases} -\sum_{\substack{i+j=n \\ i,j\neq n}} \psi([1]; a) \dashv \psi_j([1]; b), & \text{if } y = [21], \\ -\sum_{\substack{i+j=n \\ i,j\neq n}} \psi([1]; a) \vdash \psi_j([1]; b), & \text{if } y = [12], \end{cases}$$

then $F$ defines the obstruction to extending $\sum_i \psi_i t^i$, $i < n$ as an automorphism of $D_K$ and, since $F_n$ is not a coboundary, this obstruction fails to vanish. □

## 6. Structure of a Pre-Lie System on $CY^*(D, D)$

In this section, we show that the cochain modules $CY^*(D, D)$ of a dialgebra $D$ admit a structure of a pre-Lie system by introducing certain operations on the set of planar binary trees. As a consequence, there is a 'pre-Lie product' defined on $CY^*(D, D)$ making it a pre-Lie ring. It follows that there exists an associative product $*$ on $CY^*(D, D)$ induced by the pre-Lie system. Finally we establish an important relationship connecting the associative product, the pre-Lie product and the coboundary maps.

DEFINITION 6.1. Given a pair of integers, $p, q \geqslant 1$ with $p + q = n + 1$, we define two maps $R_1^i(n; p, q)\colon Y_n \longrightarrow Y_p$ and $R_2^i(n; p, q)\colon Y_n \longrightarrow Y_q$ for each $i$, $0 \leqslant i \leqslant p - 1$ as follows. For $y \in Y_n$

$$R_1^i(n; p, q)(y) = \begin{cases} d_{i+1}d_{i+2}\cdots d_{i+q-1}(y), & \text{if } p, q \geqslant 2 \text{ and } 0 \leqslant i \leqslant p-1, \\ y, & \text{if } p = n, q = 1 \\ & \quad \text{and } 0 \leqslant i < p-1, \\ d_1 d_2 \cdots d_{n-1}(y), & \text{if } p = 1, q = n \text{ and } i = 0 \end{cases}$$

and

$$R_2^i(n;\, p, q)(y) = \begin{cases} d_0 d_1 \cdots d_{i-1} d_{i+q+1} \cdots d_{p+q-1}(y), \\ \quad \text{if } p, q \geqslant 2 \text{ and } 0 < i < p-1, \\ d_{q+1} \cdots d_{p+q-1}(y), \\ \quad \text{if } p, q \geqslant 2 \text{and } i = 0, \\ d_0 d_1 \cdots d_{p-2}(y), \\ \quad \text{if } p, q \geqslant 2 \text{ and } i = p-1, \\ d_0 d_1 \cdots d_{i-1} d_{i+2} \cdots d_n(y), \\ \quad \text{if } p = n, q = 1 \text{ and } 0 \leqslant i \leqslant p-1, \\ y, \quad \text{if } p = 1, q = n \text{ and } i = 0. \end{cases}$$

To simplify notation, we shall denote the maps $R_1^i(m; r, s)$ and $R_2^i(m; r, s)$, $0 \leqslant i \leqslant r-1$, corresponding to any triple of integers $m$, $r$ and $s$ with $m+1 = r+s$, as defined above, simply by $R_1^i$ and $R_2^i$.

Recall from [4] that

DEFINITION 6.2. A right pre-Lie system $\{V_m, \circ_i\}$ is a sequence $\ldots, V_{-1}, V_0, V_1, \ldots$ of $k$-modules, equipped with a linear map $\circ_i = \circ_i(m, n)\colon V_m \otimes V_n \longrightarrow V_{m+n}$ for every triple of integers $m, n, i \geqslant 0$ with $i \leqslant m$ satisfying the following properties

$$(f^m \circ_i g^n) \circ_j h^p = \begin{cases} (f^m \circ_j h^p) \circ_{i+p} g^n & \text{if } 0 \leqslant j \leqslant i-1 \\ f^m \circ_i (g^n \circ_{j-i} h^p) & \text{if } i \leqslant j \leqslant n+1,\ i \neq 0 \\ & \quad \text{and } 0 \leqslant j < n+1, \text{if } i = 0 \end{cases}$$

where $f \in V_m$ is written as $f^m$ to indicate its degree and $f \circ_i g = \circ_i(f \otimes g)$.

Let $D$ be a dialgebra over a field $k$.

DEFINITION 6.3. For all $i$, $0 \leqslant i \leqslant p-1$ the maps

$$\circ_i\colon CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D)$$

are defined in the following way. Given $f \in CY^p(D, D)$ and $g \in CY^q(D, D)$,

$$\begin{aligned} &(f \circ_i g)(y; a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q-1}) \\ &\quad = f(R_1^i(y); a_1, \ldots, a_i, g(R_2^i(y); a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1}) \end{aligned}$$

where $y \in Y_{p+q-1}$, $R_1^i\colon Y_{p+q-1} \longrightarrow Y_p$ and $R_2^i : Y_{p+q-1} \longrightarrow Y_q$ are maps as in 6.1.

PROPOSITION 6.4. *The maps*

$$\circ_i\colon CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D), \quad 0 \leqslant i \leqslant p-1,$$

*as defined above form a pre-Lie system on* $CY^*(D, D)$.

To prove this proposition we need the following lemmas the proofs of which involve the simplicial identity $d_i d_j = d_{j-1} d_i$, $i < j$.

LEMMA 6.5. *Let* $n + 2 = p + q + r$. *For* $0 \leqslant j \leqslant p + q - 2$, $0 \leqslant i \leqslant p - 1$ *and* $j \leqslant i - 1$, *the following maps*:

$$
\begin{aligned}
R_1^j &= R_1^j(n;\, p+q-1, r)\colon Y_n \longrightarrow Y_{p+q-1},\\
R_2^j &= R_2^j(n;\, p+q-1, r)\colon Y_n \longrightarrow Y_r,\\
R_1^i &= R_1^i(p+q-1;\, p, q)\colon Y_{p+q-1} \longrightarrow Y_p,\\
R_2^i &= R_2^i(p+q-1;\, p, q)\colon Y_{p+q-1} \longrightarrow Y_q,\\
R_1^{i+r-1} &= R_1^{i+r-1}(n;\, p+r-1, q)\colon Y_n \longrightarrow Y_{p+r-1},\\
R_2^{i+r-1} &= R_2^{i+r-1}(n;\, p+r-1, q)\colon Y_n \longrightarrow Y_q,\\
R_1^j &= R_1^j(p+r-1;\, p, r)\colon Y_{p+r-1} \longrightarrow Y_p,\\
R_2^j &= R_2^j(p+r-1;\, p, r)\colon Y_{p+r-1} \longrightarrow Y_r
\end{aligned}
$$

*satisfy*

$$
\text{(i) } R_1^i R_1^j = R_1^j R_1^{i+r-1}, \qquad \text{(ii) } R_2^i R_1^j = R_2^{i+r-1}, \qquad \text{(iii) } R_2^j = R_2^j R_1^{i+r-1},
$$

*where the terms on the either side of the equalities* (i), (ii) *and* (iii) *are suitable composition of maps, for example,* $R_1^i R_1^j$ *at the left-hand side of the equality* (i) *denotes the composition of the maps* $R_1^j(n;\, p+q-1, r)$ *and* $R_1^i(p+q-1;\, p, q)$.

*Proof.* We prove the lemma for the case $p, q, r \geqslant 2$, $0 < j < p + q - 2$, $0 < i < p - 1$ and $j \leqslant i - 1$. The proofs for the other cases are similar. We have

$$
R_1^i R_1^j = d_{i+1} d_{i+2} \cdots d_{i+q-1} d_{j+1} d_{j+2} \cdots d_{j+r-1}, \tag{27}
$$

$$
R_1^j R_1^{i+r-1} = d_{j+1} d_{j+2} \cdots d_{j+r-1} d_{i+r} d_{i+r+1} \cdots d_{i+r+q-2}. \tag{28}
$$

Since $j \leqslant i-1$, $j+r-1 < i+r$. Hence, the simplicial identities imply that the adjacent terms $d_{j+r-1} d_{i+r}$ in the right-hand side of Equation (28) can be replaced by $d_{i+r-1} d_{j+r-1}$. We apply this argument again to the term $d_{j+r-1} d_{i+r+1}$. Continuing the process, (28) reduces to

$$
R_1^j R_1^{i+r-1} = d_{j+1} d_{j+2} \cdots d_{j+r-2} d_{i+r-1} d_{i+r} \cdots d_{i+r+q-3} d_{j+r-1}.
$$

Next, we repeat the argument starting with $d_{j+r-2} d_{i+r-1}$. Proceeding this way the string $d_{j+1} \cdots d_{j+r-1}$ in (28) can be pushed off to the right to get (27). This proves (i).

To prove (ii), note that

$$
R_2^{i+r-1} = d_0 d_1 \cdots d_{i+r-2} d_{i+r+q} \cdots d_{p+q+r-2} \quad \text{and} \quad j + r - 1 \leqslant i + r - 2,
$$

as $j \leqslant i - 1$. If $j + r - 1 < i + r - 2$, then in the above expression of $R_2^{i+r-1}$ we can replace $d_{j+r-1} d_{j+r}$ by $d_{j+r-1} d_{j+r-1}$ and then replace $d_{j+r-1} d_{j+r+1}$ by $d_{j+r} d_{j+r-1}$.

Repeating this process, we can make $d_{j+r-1}$ and $d_{i+r+q}$ adjacent and, hence, can replace $d_{j+r-1}d_{i+r+q}$ by $d_{i+r+q-1}d_{j+r-1}$. Then starting with $d_{j+r-1}d_{i+r+q+1}$ and successively applying the simplicial identities we get

$$R_2^{i+r-1} = d_0d_1\cdots d_{j+r-2}d_{j+r-1}d_{j+r}\cdots d_{i+r-3}d_{i+r+q-1}\cdots d_{p+q+r-3}d_{j+r-1}.$$

Next, we apply the above argument again starting with the terms $d_{j+r-2}d_{j+r-1}$ to get

$$R_2^{i+r-1} = d_0d_1\cdots d_{j+r-3}d_{j+r-2}\cdots d_{i+r-4}d_{i+r+q-2}\cdots d_{p+q+r-4}d_{j+r-2}d_{j+r-1}.$$

We repeat the process $(r-1)$ times to obtain

$$R_2^{i+r-1} = d_0d_1\cdots d_{i-1}d_{i+q+1}\cdots d_{p+q-1}d_{j+1}d_{j+2}\cdots d_{j+r-1} = R_2^iR_1^j.$$

To prove (iii) we note that

$$R_2^j = d_0d_1\cdots d_{j-1}d_{j+r+1}\cdots d_{p+q+r-2}$$

and

$$R_2^jR_1^{i+r-1} = d_0d_1\cdots d_{j-1}d_{j+r+1}\cdots d_{p+r-1}d_{i+r}d_{i+r+1}\cdots d_{i+r+q-2}.$$

As $p > i$, $d_{p+r-1}d_{i+r}$ can be replaced by $d_{i+r}d_{p+r}$. Next we consider $d_{p+r}d_{i+r+1}$ and replace it by $d_{i+r+1}d_{p+r+1}$. Repeating this $(q-1)$ times we get

$$R_2^jR_1^{i+r-1} = d_0d_1\cdots d_{j-1}d_{j+r+1}\cdots d_{p+r-2}d_{i+r}d_{i+r+1}\cdots d_{i+r+q-2}d_{p+q+r-2}.$$

Next, apply the above argument to the adjacent terms $d_{p+r-2}d_{i+r}$ to get

$$\begin{aligned} R_2^jR_1^{i+r-1} &= d_0d_1\cdots d_{j-1}d_{j+r+1}\cdots d_{p+r-3}\times \\ &\quad\times d_{i+r}d_{i+r+1}\cdots d_{i+r+q-2}d_{p+q+r-3}d_{p+q+r-2}. \end{aligned}$$

Continuing this process $(p-i-1)$ times we obtain

$$\begin{aligned} R_2^jR_1^{i+r-1} &= d_0d_1\cdots d_{j-1}d_{j+r+1}\cdots d_{i+r-1}d_{i+r}d_{i+r+1}\cdots \\ &\quad\cdots d_{i+r+q-2}d_{i+r+q-1}\cdots d_{p+q+r-3}d_{p+q+r-2} \\ &= R_2^j. \end{aligned}$$

This completes the proof of the lemma. □

LEMMA 6.6. *Let $n+2 = p+q+r$. For $0 \leqslant j \leqslant p+q-2$, $0 \leqslant i \leqslant p-1$ and $i \leqslant j \leqslant q$ if $i > 0$ and $0 \leqslant j < q$ if $i = 0$ the maps*

$$\begin{aligned}
&R_1^j = R_1^j(n;\, p+q-1, r)\colon Y_n \longrightarrow Y_{p+q-1},\\
&R_2^j = R_2^j(n;\, p+q-1, r)\colon Y_n \longrightarrow Y_r,\\
&R_1^i = R_1^i(n;\, p, q+r-1)\colon Y_n \longrightarrow Y_p,\\
&R_2^i = R_2^i(n;\, p, q+r-1)\colon Y_n \longrightarrow Y_{q+r-1},\\
&R_1^i = R_1^i(p+q-1;\, p, q)\colon Y_{p+q-1} \longrightarrow Y_p,\\
&R_2^i = R_2^i(p+q-1;\, p, q)\colon Y_{p+q-1} \longrightarrow Y_q,\\
&R_1^{j-i} = R_1^{j-i}(q+r-1;\, q, r)\colon Y_{q+r-1} \longrightarrow Y_q,\\
&R_2^{j-i} = R_2^{j-i}(q+r-1;\, q, r)\colon Y_{q+r-1} \longrightarrow Y_r,
\end{aligned}$$

*satisfy*

$$\text{(i) } R_1^i R_1^j = R_1^i, \qquad \text{(ii) } R_2^i R_1^j = R_1^{j-i} R_2^i, \qquad \text{(iii) } R_2^j = R_2^{j-i} R_2^i.$$

The proof of this lemma involves ideas similar to the proof of the previous lemma, and hence is omitted. □

*Proof of Proposition 6.4.* Let

$$f \in CY^p(D, D), \qquad g \in CY^q(D, D) \quad \text{and} \quad h \in CY^r(D, D)$$

and assume that $0 \leqslant j \leqslant i - 1$. Then, for $y \in Y_{p+q+r-2}$,

$$\begin{aligned}
&(f \circ_i g) \circ_j h(y; a_1, \ldots, a_{p+q+r-2})\\
&\quad = (f \circ_i g)(R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1}, \ldots,\\
&\qquad \ldots a_{p+q+r-2})\\
&\quad = f(R_1^i R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1}, \ldots,\\
&\qquad \ldots a_{i+r-1}, g(R_2^i R_1^j(y); a_{i+r}, \ldots, a_{i+r+q-1}), a_{i+r+q}, \ldots, a_{p+q+r-2})
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(f \circ_j h) \circ_{i+r-1} g(y; a_1, \ldots, a_{p+q+r-2})\\
&\quad = (f \circ_j h)(R_1^{i+r-1}(y); a_1, \ldots, a_{i+r-1}, g(R_2^{i+r-1}(y); a_{i+r}, \ldots, a_{i+r+q-1}),\\
&\qquad a_{i+r+q}, \ldots, a_{p+q+r-2})\\
&\quad = f(R_1^j R_1^{i+r-1}(y); a_1, \ldots, a_j, h(R_2^j R_1^{i+r-1}(y); a_{j+1}, \ldots, a_{j+r}), a_{j+r+1},\\
&\qquad \ldots a_{i+r-1}, g(R_2^{i+r-1}(y); a_{i+r}, \ldots, a_{i+r+q-1}), a_{i+r+q}, \ldots, a_{p+q+r-2}).
\end{aligned}$$

It now follows from Lemma 6.5 that $(f \circ_i g) \circ_j h = (f \circ_j h) \circ_{i+r-1} g$ for $0 \leqslant j \leqslant i - 1$.

Suppose now that $i \leqslant j \leqslant q$ if $i > 0$ and $0 \leqslant j < q$ if $i = 0$. Then

$$\begin{aligned}
&(f \circ_i g) \circ_j h(y; a_1, \ldots, a_{p+q+r-2})\\
&\quad = (f \circ_i g)(R_1^j(y); a_1, \ldots, a_j, h(R_2^j(y); a_{j+1}, \ldots, a_{j+r}),\\
&\qquad a_{j+r+1}, \ldots, a_{p+q+r-2})\\
&\quad = f(R_1^i R_1^j(y); a_1, \ldots, a_i, g(R_2^i R_1^j(y); a_{i+1}, \ldots, a_j, h(R_2^j(y); a_{j+1},\\
&\qquad \ldots a_{j+r}), a_{j+r+1} \ldots, a_{q+r+i-1}), a_{i+r+q} \ldots, a_{p+q+r-2}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&f \circ_i (g \circ_{j-i} h)(y; a_1, \ldots, a_{p+q+r-2})\\
&\quad = f(R_1^i(y); a_1, \ldots, a_i, (g \circ_{j-i} h)(R_2^i(y); a_{i+1}, \ldots, a_{q+r+i-1}),\\
&\qquad a_{q+r+i}, \ldots, a_{p+q+r-2})\\
&\quad = f(R_1^i(y); a_1, \ldots, a_i, g(R_1^{j-i} R_2^i(y); a_{i+1}, \ldots, a_j, h(R_2^{j-i} R_2^i(y); a_{j+1},\\
&\qquad \ldots a_{j+r}), a_{j+r+1}, \ldots, a_{q+r+i-1}), a_{q+r+i} \ldots, a_{p+q+r-2}).
\end{aligned}$$

Lemma 6.6 now implies that $(f \circ_i g) \circ_j h = f \circ_i (g \circ_{j-i} h)$ for $i \leqslant j \leqslant q$ if $i > 0$ and $0 \leqslant j < q$ if $i = 0$. Thus considering elements of $CY^p(D, D)$ to be of degree $(p-1)$ we see that the maps

$$\circ_i : CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D)$$

for $i \leqslant p-1$ as defined above make $CY^*(D, D)$ into a pre-Lie system. □

DEFINITION 6.7. The 'pre-Lie product'

$$\circ : CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q-1}(D, D)$$

on $CY^*(D, D)$ is defined by

$$f \circ g = \sum_{i=0}^{p-1} (-1)^{i(q-1)} f \circ_i g$$

for $f \in CY^p(D, D)$ and $g \in CY^q(D, D)$.

Then $CY^*(D, D)$ equipped with the pre-Lie product becomes a pre-Lie ring (cf. [4]).

Next we define a product $*$ on the graded modules $CY^*(D, D)$ as follows.

DEFINITION 6.8. For $f \in CY^p(D, D)$ and $g \in CY^q(D, D)$,

$$* : CY^p(D, D) \times CY^q(D, D) \longrightarrow CY^{p+q}(D, D)$$

is given by $f * g = (\pi \circ_0 f) \circ_p g$, where $\pi \in CY^2(D, D)$ is the 2-cochain defined by

$$\pi([21]; a, b) = a \dashv b, \qquad \pi([12]; a, b) = a \vdash b$$

for all $a, b \in D$.

Explicitly, for $y \in Y_{p+q}, a_1, a_2, \ldots, a_{p+q} \in D$,

$$\begin{aligned}
&(f * g)(y; a_1, a_2, \ldots, a_{p+q}) \\
&\quad = (\pi \circ_0 f) \circ_p g(y; a_1, a_2, \ldots, a_{p+q}) \\
&\quad = \pi(R_1^0 R_1^p(y); f(R_2^0 R_1^p(y); a_1, \ldots, a_p), g(R_2^p(y); a_{p+1}, \ldots, a_{p+q})) \\
&\quad = f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \bowtie g(R_2^p(y); a_{p+1}, \ldots, a_{p+q})
\end{aligned}$$

where $\bowtie$ is either $\dashv$ or $\vdash$ according as $R_1^0 R_1^p(y)$ is [21] or [12], respectively, and $R_1^i$, $R_2^i$ are the maps as defined in 6.1.

It is easy to see that $\pi$ is a cocycle by the dialgebra axiom. It is infact a coboundary, $\pi = \delta\varphi$, where $\varphi([1]; a) = a$ for all $a \in D$.

LEMMA 6.9. *The graded product $*$ on $CY^*(D, D)$ is associative.*

*Proof.* Suppose

$$f \in CY^p(D, D), \qquad g \in CY^q(D, D) \quad \text{and} \quad h \in CY^r(D, D).$$

Then, for $y \in Y_{p+q+r}$ and $a_1, a_2, \ldots, a_{p+q+r} \in D$, it is easy to see that

$$\begin{aligned}
&(f * g) * h(y; a_1, \ldots, a_{p+q+r}) \\
&\quad = \pi(R_1^0 R_1^{p+q}(y); \pi(R_1^0 R_1^p R_2^0 R_1^{p+q}(y); \\
&f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p), g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})), \\
&\quad h(R_2^{p+q}(y); a_{p+q+1}, \ldots a_{p+q+r})),
\end{aligned}$$

where

$$\begin{aligned}
&R_1^{p+q}\colon Y_{p+q+r} \longrightarrow Y_{p+q+1}, \qquad R_2^{p+q}\colon Y_{p+q+r} \longrightarrow Y_r, \\
&R_1^0\colon Y_{p+q+1} \longrightarrow Y_2, \qquad R_2^0\colon Y_{p+q+1} \longrightarrow Y_{p+q}, \\
&R_1^p\colon Y_{p+q} \longrightarrow Y_{p+1}, \qquad R_2^p\colon Y_{p+q} \longrightarrow Y_q, \\
&R_1^0\colon Y_{p+1} \longrightarrow Y_2, \qquad R_2^0\colon Y_{p+1} \longrightarrow Y_p,
\end{aligned}$$

are the maps involved in the above equation. On the other hand,

$$\begin{aligned}
&f * (g * h)(y; a_1, \ldots, a_{p+q+r}) \\
&\quad = \pi(R_1^0 R_1^p(y); f(R_2^0 R_1^p(y); a_1, \ldots, a_p), \\
&\qquad \pi(R_1^0 R_1^q R_2^p(y); g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}), \\
&\qquad h(R_2^q R_2^p(y); a_{p+q+1}, \ldots, a_{p+q+r}))),
\end{aligned}$$

where

$$\begin{aligned}
&R_1^p\colon Y_{p+q+r} \longrightarrow Y_{p+1}, \qquad R_2^p\colon Y_{p+q+r} \longrightarrow Y_{q+r}, \\
&R_1^0\colon Y_{p+1} \longrightarrow Y_2, \qquad R_2^0\colon Y_{p+1} \longrightarrow Y_p, \\
&R_1^q\colon Y_{q+r} \longrightarrow Y_{q+1}, \qquad R_2^q\colon Y_{q+r} \longrightarrow Y_r, \\
&R_1^0\colon Y_{q+1} \longrightarrow Y_2, \qquad R_2^0\colon Y_{q+1} \longrightarrow Y_q.
\end{aligned}$$

Note that, according to the convention, following Definition 6.1, we are using the same symbol to denote different maps. For example, in the expression of $(f*g)*h$, $R_1^0$ denotes the map $Y_{p+q+1} \longrightarrow Y_2$ as well as the map $Y_{p+1} \longrightarrow Y_2$. Now to prove that the right-hand sides of equalities given above are the same, we proceed as follows.

*Step* (i). First note that the composition $R_2^0 R_1^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f * g) * h$ is same as $R_2^0 R_1^p$ appearing in that of $f * (g * h)$. For,

$$\begin{aligned}
R_2^0 R_1^p R_2^0 R_1^{p+q} &= d_{p+1}(d_{p+1} \cdots d_{p+q-1}) d_{p+q+1}(d_{p+q+1} \cdots d_{p+q+r-1}) \\
&= (d_{p+1} \cdots d_{p+q-1}) d_{p+q} d_{p+q+1}(d_{p+q+1} \cdots d_{p+q+r-1}) \\
&= d_{p+1} d_{p+1} \cdots d_{p+q-1} d_{p+q} d_{p+q+1} \cdots d_{p+q+r-1} \\
&= R_2^0 R_1^p
\end{aligned}$$

(first applying $d_i d_j = d_{j-1} d_i, i < j, q - 1$ times starting with the operators $d_{p+1} d_{p+1}$ at the left, then shifting the $(q+1)^{th}$ operator $d_{p+q+1}$ to the left by using $d_i d_j = d_{j-1} d_i, i < j, q$ times).

*Step* (ii). Next observe that $R_2^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f*g)*h$ is the same as $R_2^0 R_1^q R_2^p$ appearing in that of $f * (g * h)$. This can be seen easily using a similar idea as in Step (i) above.

*Step* (iii). Next note that the map $R_2^{p+q}$ appearing in the expression of $(f*g)*h$ is the same as $R_2^q R_2^p$ of $f * (g * h)$. The proof is similar to the previous cases.

*Step* (iv). Let $S: Y_{p+q+r} \longrightarrow Y_3$ be the operator

$$S = d_1 d_2 \cdots d_{p-1} d_{p+1} \cdots d_{p+q-1} d_{p+q+1} \cdots d_{p+q+r-1}.$$

It is easy to see that the maps $R_1^0 R_1^{p+q}$ and $R_1^0 R_1^p R_2^0 R_1^{p+q}$ appearing in the expression of $(f * g) * h$ can be written as $d_1 S$ and $d_3 S$, respectively. Similarly the maps $R_1^0 R_1^p$ and $R_1^0 R_1^q R_2^p$ appearing in $f * (g * h)$ are, respectively, $d_2 S$ and $d_0 S$. Since $y \in Y_{p+q+r}$, $S(y) \in Y_3$ and there could be five possible cases for $S(y)$. For each of these five cases the result will follow from the five axioms of dialgebras. We illustrate the case when $S(y) = [131]$. In this case,

$$\begin{aligned} d_1 S(y) &= [21], \qquad d_3 S(y) = [12], \\ d_2 S(y) &= [12] \quad \text{and} \quad d_0 S(y) = [21]. \end{aligned}$$

Hence by definition of $\pi$, we get

$$\begin{aligned} &(f * g) * h(y; a_1, \ldots, a_{p+q+r}) \\ &\quad = (f(R_2^0 R_1^p R_2^0 R_1^{p+q}(y); a_1, \ldots, a_p) \vdash \\ &\quad\quad \vdash g(R_2^p R_2^0 R_1^{p+q}(y); a_{p+1}, \ldots, a_{p+q})) \dashv \\ &\quad\quad \dashv h(R_2^{p+q}(y); a_{p+q+1}, \ldots a_{p+q+r})), \end{aligned}$$

and

$$\begin{aligned} &f * (g * h)(y; a_1, \ldots, a_{p+q+r}) \\ &\quad = f(R_2^0 R_1^p(y); a_1, \ldots, a_p) \vdash \\ &\quad\quad \vdash (g(R_2^0 R_1^q R_2^p(y); a_{p+1}, \ldots, a_{p+q}) \dashv \\ &\quad\quad \dashv h(R_2^q R_2^p(y); a_{p+q+1}, \ldots a_{p+q+r})), \end{aligned}$$

where $y = [131]$. It now follows from the dialgebra axiom 3 of (1) and Steps (i)–(iii), that

$$f * (g * h)(y; a_1, \ldots, a_{p+q+r}) = (f * g) * h(y; a_1, \ldots, a_{p+q+r}),$$

where $y = [131]$. This completes the proof of the lemma. □

We shall need the following lemma:

LEMMA 6.10. *If $f, g \in CY^1(D, D)$, then $\delta(f * g) = \delta f * g - f * \delta g$.*

*Proof.* By definition, we have to prove that

$$(\delta(f * g) - \delta f * g + f * \delta g)(y; a, b, c) = 0,$$

for all $y \in Y_3$, $a, b, c \in D$. This identity is equivalent to the dialgebra axioms. □

*Remark* 6.11. It may be remarked that the above lemma does not hold for $f \in CY^p(D, D)$ and $g \in CY^q(D, D)$ with arbitrary $p$ and $q$, so that the product $*$ does not behave like a cup product.

LEMMA 6.12. *For any* $f \in CY^p(D, D)$,

$$\delta f = -f \circ \pi + (-1)^{p-1}\pi \circ f = (-1)^{p-1}(\pi \circ f - (-1)^{p-1} f \circ \pi),$$

*where* $\pi$ *is the* 2*-cochain as defined in* 6.8.

*Proof.* Let $y \in Y_{p+1}$ and $a_1, a_2, \ldots, a_{p+1} \in D$. Then

$$\begin{aligned}
&\delta f(y; a_1, a_2, \ldots, a_{p+1})\\
&\quad= a_1 o_0^y f(d_0 y; a_2, \ldots, a_{p+1}) +\\
&\qquad+ \sum_{i=1}^{p} (-1)^i f(d_i y; a_1, \ldots, a_i o_i^y a_{i+1}, \ldots, a_{p+1}) +\\
&\qquad+ (-1)^{p+1} f(d_{p+1} y; a_1, \ldots, a_p) o_{p+1}^y a_{p+1}
\end{aligned}$$

and

$$\begin{aligned}
&(-f \circ \pi + (-1)^{p-1}\pi \circ f)(y; a_1, \ldots, a_{p+1})\\
&\quad= -\sum_{i=0}^{p-1} f(R_1^i(y); a_1 \ldots a_i, \pi(R_2^i(y); a_{i+1}, a_{i+2}),\\
&\qquad \ldots, a_{p+1}) + (-1)^{p-1}[\pi(R_1^0(y); f(R_2^0(y); a_1, \ldots, a_p), a_{p+1}) +\\
&\qquad + (-1)^{p-1}\pi(R_1^1(y); a_1, f(R_2^1(y); a_2, \ldots, a_p, a_{p+1}))]\\
&\quad= \sum_{j=1}^{p} f(R_1^{j-1}(y); a_1 \ldots a_{j-1}, \pi(R_2^{j-1}(y); a_j, a_{j+1}),\\
&\qquad \ldots, a_{p+1}) + (-1)^{p-1}[\pi(R_1^0(y); f(R_2^0(y); a_1, \ldots, a_p), a_{p+1}) +\\
&\qquad + (-1)^{p-1}\pi(R_1^1(y); a_1, f(R_2^1(y); a_2, \ldots, a_p, a_{p+1}))].
\end{aligned}$$

To complete the proof observe the following:

(a) $R_1^{j-1}(y) = d_j(y)$, which follows from the definition of $R_1^{j-1}$.
(b) $R_2^{j-1}(y)$ is the tree [21] or the tree [12] according as $\circ_j^y$ is $\dashv$ or $\vdash$.

We prove (b) by induction on the degree of $y$, where degree of $y$ is $n$ if $y \in Y_n$. Let deg $y = 2$. Then $j$ can take value 1 only and $R_2^{j-1} = R_2^0$ is the identity map. Moreover, if $o_1^y = \dashv$, then $y$ must be [21] and if $o_1^y = \vdash$, then $y$ must be [12]. Hence

(b) is true for deg $y = 2$. Assume that (b) holds for all $y$ with deg $y \leqslant m$ and for all $j$, $1 \leqslant j \leqslant m-1$. Let deg $y = m+1$ and $1 \leqslant j \leqslant m$. Let $y = x_1 \vee x_2$, where $p = \deg x_1 < m+1$, $q = \deg x_2 < m+1$ and $p+q = m$. Let $o_j^y = \dashv$. Two cases arise. The $j$th leaf of $y$ is either a leaf of $x_1$ or a leaf of $x_2$. Suppose that it is a leaf of $x_1$. In this case, it must be an interior leaf of $x_1$, that is, not those numbered 0 and $p$ as $1 \leqslant j$ and $o_j^y = \dashv$. Note that $R_2^{j-1}$: $Y_{m+1} \longrightarrow Y_2$ is given by $R_2^{j-1} = d_0 d_1 \cdots d_{j-2} d_{j+2} \cdots d_{m+1}$ and, in the present case, we also have the map $R_2^{j-1}$: $Y_p \longrightarrow Y_2$ given by $R_2^{j-1} = d_0 d_1 \cdots d_{j-2} d_{j+2} \cdots d_p$. Note that the effect of applying the operator $d_{p+1} \cdots d_{m+1}$ on $y$ is to delete the leaves of $x_2$ one after another, the leaves of $x_1$ remaining untouched during the process. In other words, $d_{p+1} \cdots d_{m+1}(y) = x_1$. Hence,

$$\begin{aligned} R_2^{j-1}(y) &= d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_p d_{p+1} \cdots d_{m+1}(y) \\ &= d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_p(x_1) \\ &= R_2^{j-1}(x_1). \end{aligned}$$

Moreover, $o_j^y = o_j^{x_1}$ by definition, as $j \neq p$. Hence, by induction, the result follows. Now if the $j$th leaf is a leaf of $x_2$ and an interior one the case is settled as above. Suppose now that the $j$th leaf in the 0th leaf of $x_2$, so that $j = p+1$. We know that $R_2^{j-1} = d_0 d_1 \cdots d_{j-1} d_{j+2} \cdots d_{m+1}$. Observe that if we apply the operator $d_{j+2} \cdots d_{m+1}$ on $y = x_1 \vee x_2$, it does not alter the leaves of $x_1$ and there are two leaves of $x_2$ which survive in the resulting tree and more over these are not deleted by applying the operator $d_0 d_1 \cdots d_{j-2}$ on the result. Since $j = p+1$, it is now clear that $R_2^{j-1}(y)$ must be of the form $[0] \vee [1] = [21]$. The case $o_j^y = \vdash$ is similar.

(c) $R_2^0(y) = d_{p+1}(y)$, is immediate from the definition.
(d) $R_1^0(y)$ is [21] or [12] according to as $o_{p+1}^y$ is $\dashv$ or $\vdash$.

To see (d), let $o_{p+1}^y = \dashv$. Then $y$ is not of the form $y_1 \vee [0]$. Thus, if $y = x_1 \vee x_2$, then $deg\ x_2 \geqslant 1$ and the last two leaves of $x_2$ cannot be deleted by applying $R_1^0 = d_1 d_2 \cdots d_{m-1}$ on $y$. It follows that $R_1^0$ must be of the form $[0] \vee [1]$ as $R_1^0(y) \in Y_2$. Thus $R_1^0(y) = [21]$. The case $R_1^0(y) = \vdash$ is dealt with similarly.

(e) $R_2^1(y) = d_0 y$, is again immediate from the definition.
(f) $R_1^1(y)$ is [21] or [12] according as $o_0^y = \dashv$ or $o_0^y = \vdash$. The proof of this is similar to that of (d). □

The lemma now follows from the above observations.

We shall use the following result from [4]:

THEOREM 6.13 ([4]). *Let $\{V_m, o_i\}$ be a pre-Lie system and $f^m, g^n, h^p$ be elements of $V_m, V_n, V_p$ respectively. Then*

(i) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = \sum (-1)^{ni+pj} (f^m \circ_i g^n)) \circ_j h^p$, *where the sum extends over those* $i, j$ *with either* $0 \leqslant j \leqslant i-1$ or $n+i+1 \leqslant j \leqslant m+n$.

(ii) $(f^m \circ g^n) \circ h^p - f^m \circ (g^n \circ h^p) = (-1)^{np} [(f^m \circ h^p) \circ g^n - f^m \circ (h^p \circ g^n)$. □

THEOREM 6.14. *Let* $D$ *be a dialgebra over a field* $k$. *If* $f \in CY^p(D, D)$ *and* $g \in CY^q(D, D)$, *then*

$$f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g = (-1)^{q(p-1)} f * g + (-1)^{q-1} g * f.$$

*Proof.* From Lemma 6.12, we have

$$\begin{aligned} & f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g \\ &= [(-1)^{q-1} f \circ (\pi \circ g) - f \circ (g \circ \pi)] - \\ &\quad - [(-1)^{p+q} \pi \circ (f \circ g) - (f \circ g) \circ \pi] + \\ &\quad + (-1)^{q-1} [(-1)^{p-1} (\pi \circ f) \circ g - (f \circ \pi) \circ g]. \end{aligned}$$

As $(CY^*(D, D), \circ)$ is a pre-Lie ring, we have

$$\begin{aligned} & f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g \\ &= (-1)^{p+q} [(\pi \circ f) \circ g - \pi \circ (f \circ g)] \\ &= (-1)^{p+q} \left[ \sum (-1)^{(p-1)i + (q-1)j} (\pi \circ_i f) \circ_j g \right], \end{aligned}$$

where the sum is over those $i, j$ such that $0 \leqslant j \leqslant i-1$ or $j = p$, corresponding to $i = 0$. The last equality follows from Theorem 6.13 stated above. Note that the degrees of $\pi, f, g$ are respectively 1, $p-1$ and $q-1$. Hence

$$\begin{aligned} & f \circ \delta g - \delta(f \circ g) + (-1)^{q-1} \delta f \circ g \\ &= (-1)^{p+q} [(-1)^{q-1} p (\pi \circ_0 f) \circ_p g + \\ &\quad + (-1)^{p-1} (\pi \circ_1 f) \circ_0 g)] \\ &= (-1)^{q(p-1)} (\pi \circ_0 f) \circ_p g + (-1)^{q-1} (\pi \circ_0 g) \circ_q f \\ &= (-1)^{q(p-1)} f * g + (-1)^{q-1} g * f. \end{aligned}$$

This completes the proof of the theorem. □

## 7. Obstruction Cocycles

The purpose of this final section is to prove Theorems 3.5 and 5.3 using the results of Section 6.

Let $F_\lambda$ and $F_\mu$ be any two 2-cochains. First observe that by definition of the pre-Lie product

$$(F_\lambda \circ F_\mu)(y; a, b, c) = \begin{cases} F_\lambda^\ell (F_\mu^\ell (a, b), c) - F_\lambda^\ell (a, F_\mu^\ell (b, c)), & \text{if } y = [321], \\ F_\lambda^\ell (F_\mu^\ell (a, b), c) - F_\lambda^\ell (a, F_\mu^r (b, c)), & \text{if } y = [312], \\ F_\lambda^\ell (F_\mu^r (a, b), c) - F_\lambda^r (a, F_\mu^\ell (b, c)), & \text{if } y = [131], \\ F_\lambda^r (F_\mu^\ell (a, b), c) - F_\lambda^r (a, F_\mu^r (b, c)), & \text{if } y = [213], \\ F_\lambda^r (F_\mu^r (a, b), c) - F_\lambda^r (a, F_\mu^r (b, c)), & \text{if} y = [123], \end{cases}$$

for all $a, b, c \in D$. Thus equation $(11_\nu)$–$(15_\nu)$ can be rewritten as

$$\delta F_\nu = \sum_{\substack{\lambda+\mu=\nu\\ \lambda,\mu>0}} F_\lambda \circ F_\mu$$

and the obstruction cochain $G \in CY^3(D, D)$ as defined in Section 3 is given by

$$G = \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} F_\lambda \circ F_\mu.$$

*Proof of Theorem* 3.5. By Theorem 6.14,

$$\delta(F_\lambda \circ F_\mu) = F_\lambda \circ \delta F_\mu - \delta F_\lambda \circ F_\mu + (\pi \circ_0 F_\mu) \circ_2 F_\lambda - (\pi \circ_0 F_\lambda) \circ_2 F_\mu.$$

Hence,

$$\begin{aligned} \delta G &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} \delta(F_\lambda \circ F_\mu) \\ &= \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} (F_\lambda \circ \delta F_\mu - \delta F_\lambda \circ F_\mu) \\ &= \sum_{\substack{\alpha+\beta+\lambda=n\\ \alpha,\lambda,\mu>0}} [F_\alpha \circ (F_\beta \circ F_\lambda) - (F_\alpha \circ F_\beta) \circ F_\lambda]. \end{aligned}$$

By Lemma 1 of [5], we may assume that $\beta \neq \lambda$ in the term

$$F_\alpha \circ (F_\beta \circ F_\lambda) - (F_\alpha \circ F_\beta) \circ F_\lambda.$$

Now as in Proposition 3 of [5], the above sum can be written as a sum of terms of the form

$$[F_\alpha \circ (F_\beta \circ F_\lambda + F_\lambda \circ F_\beta) - ((F_\alpha \circ F_\beta) \circ F_\lambda + (F_\alpha \circ F_\lambda) \circ F_\beta)],$$

where $\alpha + \beta + \lambda = n$, $\alpha, \beta, \lambda > 0$ and each of these term vanishes by (ii) of Theorem 6.13. Hence $\delta G = 0$. Note that the cohomology class of $G$ is zero if and only if $G = \delta F_n$ for some $F_n \in CY^2(D, D)$. Hence the last statement follows. □

Next, consider the case of a derivation. Recall from Section 5 that if a derivation $\psi_1$ has been extended to a truncated automorphism $\Psi_t = \sum_{i=0}^{n-1} \psi_i t^i$ of $D_K$, then the $(n-1)$th obstruction is the 2-cochain $F$ defined by

$$F(y; a, b) = \begin{cases} \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} \varphi_\lambda([1]; a) \dashv \psi_\mu([1]; b) & \text{if } y = [21] \\ \sum_{\substack{\lambda+\mu=n\\ \lambda,\mu>0}} \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b) & \text{if } y = [12]. \end{cases}$$

*Proof of Theorem* 5.3. Observe that

$$\begin{aligned}(\psi_\lambda * \psi_\mu)([21]; a, b) &= (\pi \circ_0 \psi_\lambda \circ_1 \psi_\mu([21]; a, b) \\ &= \pi(R_1^0 R_1^1[21], \psi_\lambda([1]; a), \psi_\mu([1]; a)) \\ &= \psi_\lambda([1]; a) \dashv \psi_\mu([1]; b),\end{aligned}$$

where $\pi$ is the 2-cochain as defined in 6.8. Similarly,

$$(\psi_\lambda * \psi_\mu)([12]; a, b) = \psi_\lambda([1]; a) \vdash \psi_\mu([1]; b).$$

Thus, $F = \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu$ and Equations $(25_\nu)$ and $(26_\nu)$ of Section 5 can be written as

$$\delta\psi_\nu = -\sum_{\substack{\lambda+\mu=\nu \\ \lambda,\mu>0}} \psi_\lambda * \psi_\mu, \quad \text{for } \nu = 0, 1, 2, \ldots, n-1.$$

Hence

$$\begin{aligned}\delta F &= \sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \delta(\psi_\lambda * \psi_\mu) \\ &= -\sum_{\substack{\lambda+\mu=n \\ \lambda,\mu>0}} \left\{ \left( \sum_{\substack{\alpha+\beta=\lambda \\ \alpha,\beta>0}} \psi_\alpha * \psi_\beta \right) * \psi_\mu - \psi_\lambda * \left( \sum_{\substack{\alpha+\beta=\mu \\ \alpha,\beta>0}} \psi_\alpha * \psi_\beta \right) \right\} \\ &= -\sum_{\substack{\alpha+\beta+\mu=n \\ \alpha,\beta,\mu>0}} \{(\psi_\alpha * \psi_\beta) * \psi_\mu - \psi_\alpha * (\psi_\beta * \psi_\mu)\} \\ &= 0\end{aligned}$$

as $*$ is associative. The last statement is clear. □

## Acknowledgements

We would like to thank Professor J.-L. Loday for his comments and for pointing out some references. We would also like to thank the referee for making various suggestions to improve the presentation of the paper.

## References

1. Balavoine, D.: Deformations of algebras over a quadratic operad, In: J.-L. Loday, J. D. Stasheff and A. A. Voronov (eds), *Operads: Proce. Renaissance Conferences*, Contemp. Math. 202, Amer. Math. Soc., Providence, 1997, pp. 207–234.
2. Frabetti, A.: Dialgebra (co)homology with coefficients, In: J.-M. Morel, F. Takens and B. Teissier (eds), *Dialgebras and Related Operads*, Lecture Notes in Math. 1763, Springer, New York, 2001, pp. 67–103.

3. Fox, T. F.: An introduction to algebraic deformation theory, *J. Pure and Appl. Algebra* **84** (1993), 17–41.
4. Gerstenhaber, M.: The cohomology structure of an associative ring, *Ann. of Math.* **78** (1963), 267–288.
5. Gerstenhaber, M.: On the deformations of rings and algebras, *Ann. of Math.* **79** (1964), 59–103.
6. Gerstenhaber, M.: On the deformations of rings and algebras, *Ann. of Math.* **84** (1966), 1–19.
7. Gerstenhaber, M.: On the deformations of rings and algebras, *Ann. of Math.* **88** (1968), 1–34.
8. Gerstenhaber, M.: On the deformations of rings and algebras, *Ann. of Math.* **99** (1974), 257–276.
9. Gerstenhaber, M. and Schack, S. D.: Bialgebra cohomology, deformations and quantum groups, *Proc. Natl. Acad. Sci. USA* **87** (1990), 478–481.
10. Loday, J.-L.: *Dialgebras*, In: J.-M. Morel, F. Takens, and B. Teissier (eds), *Dialgebras and Related Operads*, Lecture Notes in Math. 1763, Springer, Berlin, 2001, pp. 7–66.
11. Loday, J.-L.: Une version non commutative des algèbres de Lie; Les algèbres de Leibniz, *Enseign. Math.* **39** (1993), 269–293.
12. Nijenhuis, A. and Richardson, R.: Cohomology and deformations in graded lie algebras, *Bull. Amer. Math. Soc.* **72** (1966), 1–29.
13. Nijenhuis, A. and Richardson, R.: Commutative algebra cohomology and deformation of lie and associative algebras, *J. Algebra* **9** (1968), 42–105.
14. Nijenhuis, A. and Richardson, R.: Deformations of Lie algebra structures, *J. Math. Mech.* **17** (1967), 89–105.