Outer inverses: Jacobi type identities and nullities of submatrices

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Abstract

According to the Jacobi identity, if A is an invertible matrix then any minor of A^{-1} equals, up to a sign, the determinant of A^{-1} times the complementary minor in the transpose of A. The identity is extended to any outer inverse, thereby generalizing several results in the literature for special generalized inverses. A permanental analog of the Jacobi identity is proved. Bounds are obtained for the difference between the nullity of a submatrix of A and that of the complementary submatrix in any generalized inverse or an outer inverse of A. The result extends earlier work of Fiedler, Markham and Gustafson for the inverse and of Robinson for the Moore–Penrose inverse.

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1. Introduction

We deal with complex matrices. The conjugate transpose of A is denoted by A^*. For an $m \times n$ matrix A consider the usual Penrose equations

(1) $AGA = A$,
(2) $GAG = G$,
(3) $(AG)^* = AG$,
(4) $(GA)^* = GA$.

Recall that the $n \times m$ matrix G is called a generalized inverse or a g-inverse of A if it satisfies (1) and an outer inverse of A if it satisfies (2). If G satisfies (1) and (2), then it is called a reflexive g-inverse of A. The Moore–Penrose inverse of A is the matrix G satisfying (1)–(4). Any matrix A admits a unique Moore–Penrose inverse, denoted A^+. If A is $n \times n$, then G is called the group inverse of A if it satisfies (1), (2) and $AG = GA$. The matrix A has group inverse, which is unique, if and only if $\text{rank}(A) = \text{rank}(A^2)$. For a square matrix A, if k is the least nonnegative integer such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$, then k is the *index* of A. There is a unique G, called the Drazin inverse of A, which satisfies $GAG = G,\ AG = GA$ and $A^{k+1}G = A^k$. Thus if A has index 1 then its Drazin inverse is the group inverse. We refer to [5,8] for basic results on g-inverses.

The Jacobi identity extends the well-known adjoint formula for the inverse of a nonsingular matrix. According to the Jacobi identity, if A is a nonsingular matrix then any minor of A^{-1} equals, up to a sign, the determinant of A^{-1} times the complementary minor in the transpose of A.

Determinantal formulae for the Moore–Penrose inverse have been obtained by various authors [2,3,6,7,10,11]. The formulae have been extended to the group inverse, a reflexive g-inverse and for minors of such g-inverses [1,15–17]. These formulae can indeed be viewed as generalizations of the Jacobi identity.

Stanimirović and Djordjević [19] gave a determinantal identity for the Drazin inverse and obtained some partial results for an arbitrary outer inverse. Their work was the motivation for the next section of the present paper, in which we obtain a Jacobi identity for any outer inverse, capturing several results in the literature as special cases. The proof technique is new, yet simple, based only on the Laplace expansion and the Cauchy–Binet formula for the determinant. Both these tools are available for the permanent as well and this observation permits us to obtain a Jacobi type identity for permanents presented in Section 3.

Section 4 deals with a different problem, motivated though as an application of the Jacobi identity. It presents bounds on the difference between the nullity of any submatrix of A and that of the complementary submatrix of a g-inverse or an outer inverse of A. The work extends results in [9,12,18].

2. Jacobi identity for an outer inverse

If $1 \leqslant k \leqslant n$, then $Q_{k,n}$ will denote the set of strictly increasing sequences of k integers chosen from $1, 2, \ldots, n$. Clearly the cardinality of $Q_{k,n}$ is $\binom{n}{k}$.

Let A be an $m \times n$ matrix and let $1 \leqslant k \leqslant m$, $1 \leqslant r \leqslant n$. Let $\alpha \in Q_{k,m}$ and $\beta \in Q_{r,n}$. The submatrix of A formed by the rows in α and columns in β will be denoted by A^{α}_{β}.

Let $\alpha \subset \beta$ where $\alpha \in Q_{p,n}, \beta \in Q_{q,n}, p \leqslant q$. Suppose α has elements $\alpha_1 < \cdots < \alpha_p$, β has elements $\beta_1 < \cdots < \beta_q$ and that $\alpha_i = \beta_{u_i}, i = 1, \ldots, p$. Then we denote by $s(\alpha|\beta)$ the sum $u_1 + \cdots + u_p$. The relevance of this definition will be clear from the following observation: Let A be an $m \times n$ matrix and let $\alpha \in Q_{q,m}$,

$\beta \in Q_{q,n}, \gamma \in Q_{p,m}, \delta \in Q_{p,n}$, where $\gamma \subset \alpha, \delta \subset \beta$. Then the coefficient of $|A^{\gamma}_{\delta}||A^{\alpha\backslash\gamma}_{\beta\backslash\delta}|$ in the expansion of $|A^{\alpha}_{\beta}|$ is $(-1)^{s(\gamma|\alpha)+s(\delta|\beta)}$.

Let A be an $m \times n$ matrix and let $1 \leqslant r \leqslant \min(m, n)$. The rth compound matrix of A, denoted by $C_r(A)$, is an $\binom{m}{n} \times \binom{r}{r}$ matrix whose elements are $|A^{\alpha}_{\beta}|$, $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ arranged lexicographically in α and β. It follows from the Cauchy–Binet formula that $C_r(AB) = C_r(A)C_r(B)$ if A and B are matrices such that AB is defined.

The main result of this section which gives a Jacobi identity for any outer inverse is stated next.

Theorem 1. *Let A be an $m \times n$ matrix of* rank r *and let G be an $n \times m$ matrix of* rank k $(\leqslant r)$ *such that $GAG = G$. Let $1 \leqslant p < q \leqslant k$. Then for any $\alpha \in Q_{p,n}$, $\beta \in Q_{p,m}$,*

$$|G^{\alpha}_{\beta}| = \frac{1}{\begin{pmatrix} k-p \\ k-q \end{pmatrix}} \sum_{\gamma \in Q_{q,n}, \alpha \subset \gamma} \sum_{\delta \in Q_{q,m}, \beta \subset \delta} (-1)^{s(\alpha|\gamma)+s(\beta|\delta)} |G^{\gamma}_{\delta}||A^{\delta\backslash\beta}_{\gamma\backslash\alpha}|. \quad (1)$$

Proof. Let $G = UV$ be a rank factorization so that U is $n \times k$, V is $k \times m$ and rank U = rank $V = k$.

By the Cauchy–Binet formula, for any $\gamma \in Q_{q,n}, \delta \in Q_{q,m}$,

$$|G^{\gamma}_{\delta}| = \sum_{\rho \in Q_{q,k}} |U^{\gamma}_{\rho}||V^{\rho}_{\delta}|. \quad (2)$$

If $\alpha \in Q_{p,k}$, $\alpha \subset \gamma$, then by the Laplace expansion,

$$|U^{\gamma}_{\rho}| = \sum_{\tau \in Q_{p,k}, \tau \subset \rho} (-1)^{s(\alpha|\gamma)+s(\tau|\rho)} |U^{\alpha}_{\tau}||U^{\gamma\backslash\alpha}_{\rho\backslash\tau}|. \quad (3)$$

Similarly, if $\beta \in Q_{p,k}$, $\beta \subset \delta$, then by the Laplace expansion,

$$|V^{\rho}_{\delta}| = \sum_{\psi \in Q_{p,k}, \psi \subset \rho} (-1)^{s(\psi|\rho)+s(\beta|\delta)} |V^{\psi}_{\beta}||V^{\rho\backslash\psi}_{\delta\backslash\beta}|. \quad (4)$$

It follows from (2)–(4) that

$$\sum_{\gamma \in Q_{q,n}, \alpha \subset \gamma} \sum_{\delta \in Q_{q,m}, \beta \subset \delta} (-1)^{s(\alpha|\gamma)+s(\beta|\delta)} |G^{\gamma}_{\delta}||A^{\delta\backslash\beta}_{\gamma\backslash\alpha}|$$

equals

$$\sum_{\gamma \in Q_{q,n}, \alpha \subset \gamma} \sum_{\delta \in Q_{q,m}, \beta \subset \delta} (-1)^{s(\alpha|\gamma)+s(\beta|\delta)}$$
$$\times \sum_{\rho \in Q_{q,k}} \sum_{\tau \in Q_{p,k}, \tau \subset \rho} \sum_{\psi \in Q_{p,k}, \psi \subset \rho} (-1)^{s(\alpha|\gamma)+s(\tau|\rho)} (-1)^{s(\psi|\rho)+s(\beta|\delta)}$$
$$\times |U^{\alpha}_{\tau}||U^{\gamma\backslash\alpha}_{\rho\backslash\tau}||V^{\psi}_{\beta}||V^{\rho\backslash\psi}_{\delta\backslash\beta}||A^{\delta\backslash\beta}_{\gamma\backslash\alpha}|. \quad (5)$$

Note that if $\gamma \in Q_{q-p,n}$ and if $\alpha \cap \gamma$ is nonempty, then

$$\sum_{\tau \in Q_{p,k}, \tau \subset \rho} (-1)^{s(\tau|\rho)} |U_\tau^\alpha| |U_{\rho\setminus\tau}^\gamma| = 0. \tag{6}$$

Similarly, if $\delta \in Q_{q-p,m}$ and if $\beta \cap \delta$ is nonempty, then

$$\sum_{\psi \in Q_{p,k}, \psi \subset \rho} (-1)^{s(\psi|\rho)} |V_\beta^\psi| |V_\delta^{\rho\setminus\psi}| = 0. \tag{7}$$

Using (6) and (7) we may express (5) as

$$\sum_{\gamma \in Q_{q-p,n}} \sum_{\delta \in Q_{q-p,m}} \sum_{\rho \in Q_{q,k}} \sum_{\tau \in Q_{p,k}, \tau \subset \rho} \sum_{\psi \in Q_{p,k}, \psi \subset \rho} (-1)^{s(\tau|\rho)+s(\psi|\rho)} \times |U_\tau^\alpha| |U_{\rho\setminus\tau}^\gamma| |V_\beta^\psi| |V_\delta^{\rho\setminus\psi}| |A_\gamma^\delta|. \tag{8}$$

Since $GAG = G$, then $UVAUV = UV$ and hence $VAU = I$. Therefore $C_{q-p}(VAU) = I$. Hence

$$\sum_{\gamma \in Q_{q-p,n}} \sum_{\delta \in Q_{q-p,m}} |V_\delta^{\rho\setminus\psi}| |A_\gamma^\delta| |U_{\rho\setminus\tau}^\gamma| = \begin{cases} 1 & \text{if } \psi = \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Using (9), the expression (8) equals

$$\sum_{\rho \in Q_{q,k}} \sum_{\tau \in Q_{p,k}, \tau \subset \rho} |U_\tau^\alpha| |V_\beta^\tau| = \binom{k-p}{k-q} \sum_{\tau \in Q_{p,k}} |U_\tau^\alpha| |V_\beta^\tau|. \tag{10}$$

Finally, by the Cauchy–Binet formula, the expression in (10) equals $\binom{k-p}{k-q} |G_\beta^\alpha|$ and (1) is proved. □

Example. Consider the 3×3 matrix A of rank 3 and G, the inverse of A:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 7 & 1 & 3 \end{pmatrix}, \quad G = A^{-1} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 3 & -1 \\ -5 & -8 & 3 \end{pmatrix}.$$

A list of the 2×2 submatrices of G and the corresponding determinants is given in Table 1.

Take $p = 1, q = 2$. Since $k = 3$, then $\binom{k-p}{k-q} = 2$. Suppose $\alpha = \beta = 1$. Then the right-hand side of (1) can be computed as

$$\begin{aligned} &\tfrac{1}{2}\left(|G_{1,2}^{1,2}||A_2^2| + |G_{1,3}^{1,2}||A_3^2| + |G_{1,2}^{1,3}||A_2^3| + |G_{1,3}^{1,3}||A_3^3|\right) \\ &= \tfrac{1}{2}((3)(1) + (-1)(1) + (-1)(1) + (1)(3)) = 2 = |G_1^1| = g_{11}, \end{aligned}$$

thereby confirming (1).

Table 1

γ	δ	G_δ^γ	$\lvert G_\delta^\gamma\rvert$
12	12	$\begin{pmatrix}2 & 3\\ 1 & 3\end{pmatrix}$	3
12	13	$\begin{pmatrix}2 & -1\\ 1 & -1\end{pmatrix}$	−1
12	23	$\begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix}$	0
13	12	$\begin{pmatrix}2 & 3\\ -5 & -8\end{pmatrix}$	−1
13	13	$\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}$	1
13	23	$\begin{pmatrix}3 & -1\\ -8 & 3\end{pmatrix}$	1
23	12	$\begin{pmatrix}1 & 3\\ -5 & -8\end{pmatrix}$	7
23	13	$\begin{pmatrix}1 & -1\\ -5 & 3\end{pmatrix}$	−2
23	23	$\begin{pmatrix}3 & -1\\ -8 & 3\end{pmatrix}$	1

Now if $\alpha = 1, \beta = 2$, then the right-hand side of (1) equals

$$\begin{aligned}&\tfrac{1}{2}\big(- |G_{1,2}^{1,2}||A_2^1| + |G_{2,3}^{1,2}||A_2^3| - |G_{1,2}^{1,3}||A_3^1| + |G_{2,3}^{1,3}||A_3^3|\big)\\ &\quad = \tfrac{1}{2}(-(3)(-1) + (0)(1) - (-1)(0) + (1)(3)) = 3 = |G_2^1| = g_{12},\end{aligned}$$

again confirming (1).

We now describe some results in the literature which can be seen as special cases of Theorem 1. It may be remarked that all these results have $q = k$.

(i) If $G = A^+ = (a_{ij}^+)$, $p = 1$ and $q = k = r$, then (1) reduces to

$$a_{ij}^+ = \sum_{\gamma\in Q_{r,n}, i\in\gamma}\ \sum_{\delta\in Q_{r,m}, j\in\delta} (-1)^{s(i|\gamma)+s(j|\delta)} |G_\delta^\gamma| |A_{\gamma\setminus i}^{\delta\setminus j}|. \tag{11}$$

It is well-known [1] that the $r \times r$ minors of A^+ are proportional to the corresponding minors of A^*. More specifically, if $\gamma \in Q_{r,n}$, $\delta \in Q_{r,m}$, then

$$|G_\delta^\gamma| = \frac{|A_\gamma^\delta|}{\sum_{\rho\in Q_{r,m},\,\tau\in Q_{r,n}} |A_\tau^\rho||A_\tau^\rho|^*}, \tag{12}$$

where the denominator in (12) is the square of the *volume* of A [4]. Thus (12) reduces to the following determinantal formula for A^+.

$$a_{ij}^{+} = \frac{1}{\sum_{\rho\in Q_{r,m},\tau\in Q_{r,n}} |A_{\tau}^{\rho}||A_{\tau}^{\rho}|^{*}} \times \sum_{\gamma\in Q_{r,n}, i\in\gamma} \sum_{\delta\in Q_{r,m}, j\in\delta} (-1)^{s(i|\gamma)+s(j|\delta)} |A_{\gamma}^{\delta}||A_{\gamma\setminus i}^{\delta\setminus j}|. \tag{13}$$

Formula (13) has been independently obtained by a number of authors and has inspired much research (see, for example, [2,6,7,10,11]).

(ii) The special case of Theorem 1 when $p = 1$, G is any reflexive g-inverse of A and $q = k = r$ has been proved in [15]. In the same context the case $p > 1$ has been dealt with in [16,17]. These two papers provide formulas for the minors of the Moore–Penrose inverse, the group inverse and of any reflexive g-inverse.

(iii) In Theorem 1 if $m = n$, if G is the Drazin inverse of A and if $p = 1, q = k < r$, then we obtain a recent result in [19]. This paper also gives a statement to the effect that when $p = 1$ and $q = k < r$, Theorem 1 is valid for a "subclass" of outer inverses (see [19, Theorem 3.3]). However, as shown in Theorem 1 the result is true for any outer inverse, even in the general case when $1 \leqslant p < q \leqslant k$.

Let R be an integral domain, i.e., a commutative ring with multiplicative identity and with no zero divisors. An element $a \in R$ is called a unit if it has a multiplicative inverse. The rank of a matrix over R is defined as the maximal order of a nonvanishing minor. Let F be the quotient field of R. If A is a matrix over R, then its rank over R obviously coincides with its rank as a matrix over F.

Let G and H be matrices over R with rank $H = k$. We say that G and H have proportional minors if there exists a unit θ of R such that $|G_{\beta}^{\alpha}| = \theta|H_{\beta}^{\alpha}|$ for all $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$. Note that if G and H have proportional minors and if rank $H = k$, then $C_k(G) = \theta C_k(H)$. Since $C_k(G)$ has rank 1, it follows that $C_k(G)$ has rank 1 and therefore G has rank k.

As an application of Theorem 1 the following result can be proved, extending the work in [1]. The proof is omitted since it closely follows that in [1].

Theorem 2. *Let R be an integral domain and let F be the quotient field of R. Let A and H be matrices over R of order $m \times n$ and $n \times m$, respectively. Let* rank $A = r$, rank $H = k$ $(\leqslant r)$ *and let $H = EF$ be a rank factorization over F. Then the following conditions are equivalent*:

(i) *A admits an outer inverse G over R such that the column (row) space of G equals the column (row) space of H (regarding G and H as matrices over F).*
(ii) rank $FAE = k$.
(iii) rank $AH =$ rank $HA = k$.
(iv) trace $C_k(AH)$ *is a unit.*
(v) *A admits an outer inverse G over R such that G and H have proportional minors.*

3. A permanental Jacobi identity

The permanent of an $n \times n$ matrix A is defined as [13]

$$\operatorname{per}(A) = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

where the summation is over all permutations σ of $1, \ldots, n$.

For positive integers k and n, $F_{k,n}$ will denote the set of nondecreasing sequences of k integers chosen from $1, \ldots, n$. Note that the cardinality of $F_{k,n}$ is $\binom{n+k-1}{k}$. If $\alpha \in F_{k,n}$ and $1 \leqslant t \leqslant n$, then $m_t(\alpha)$ will denote the multiplicity of t in α. Clearly $m_t(\alpha) = 0$ if t does not feature in α. Observe that $\sum_{t=1}^{n} m_t(\alpha) = k$. We set

$$\mu(\alpha) = \prod_{t=1}^{n} m_t(\alpha)!.$$

If $\alpha \in F_{p,n}, \beta \in F_{q,n}, p < q$, then we write $\alpha \prec \beta$ if $m_t(\alpha) \leqslant m_t(\beta)$ for $t = 1, \ldots, n$. If $\alpha \prec \beta$, then we define $\beta - \alpha$ to be the sequence in $F_{q-p,n}$ obtained by taking t with multiplicity $m_t(\beta) - m_t(\alpha)$ for $t = 1, \ldots, n$. Furthermore, we set

$$\left(\binom{\beta}{\alpha}\right) = \frac{\mu(\beta)}{\mu(\alpha)\mu(\beta - \alpha)}.$$

If A is an $m \times n$ matrix and if k is a positive integer then the kth induced matrix of A, designated by $P_k(A)$, is the

$$\binom{m+k-1}{k} \times \binom{n+k-1}{k}$$

matrix whose entries are

$$\frac{\operatorname{per}\left(A_{\beta}^{\alpha}\right)}{\sqrt{\mu(\alpha)\mu(\beta)}}$$

for $\alpha \in F_{k,m}$ and $\beta \in F_{k,n}$ arranged lexicographically.

Let A be an $m \times n$ matrix and let $\alpha \in F_{k,m}, \beta \in F_{k,n}$. We continue to use the notation A_{β}^{α} for the matrix formed by choosing the rows of A corresponding to indices in α and the columns of A corresponding to indices in β. Thus A_{β}^{α} may have a row or column appearing several times. For example if α is the sequence $1, 2, 2$ and β is the sequence $3, 3, 4$, then

$$A_{\beta}^{\alpha} = \begin{bmatrix} a_{13} & a_{13} & a_{14} \\ a_{23} & a_{23} & a_{24} \\ a_{23} & a_{23} & a_{24} \end{bmatrix}.$$

We will use the Cauchy–Binet formula and the Laplace expansion for permanents. The important multiplicative property of the induced matrix, $P_k(AB) = P_k(A)P_k(B)$ whenever AB is defined, will also be needed, see [13].

The following result will be used in the sequel.

Lemma 3. *Let p, q and k be positive integers with $p < q$. Then for any $\phi \in F_{p,k}$,*

$$\sum_{\rho \in F_{q,k}, \phi \prec \rho} \left(\binom{\rho}{\phi} \right) = \binom{q+k-1}{q-p}. \tag{14}$$

Proof. Note that the coefficient of x^{q-p} in

$$\prod_{i=1}^{k} (1-x)^{-m_i(\phi)-1}$$

equals the left-hand side of (14). Since

$$\prod_{i=1}^{k} (1-x)^{-m_i(\phi)-1} = (1-x)^{-p-k}$$
$$= \sum_{r=0}^{\infty} \binom{p+k-1+r}{r} x^r,$$

the same coefficient equals

$$\binom{p+k-1+q-p}{q-p} = \binom{q+k-1}{q-p},$$

which is the right-hand side of (14). □

The next result provides a permanental analog of the Jacobi identity.

Theorem 4. *Let A be an $m \times n$ matrix of rank r and let G be an $n \times m$ matrix of* rank k $(\leqslant r)$ *such that $GAG = G$. Let $1 \leqslant p < q$. Then for any $\alpha \in F_{p,n}$, $\beta \in F_{p,m}$,*

$$\operatorname{per}(G^{\alpha}_{\beta}) = \frac{1}{\binom{q+k-1}{q-p}} \sum_{\gamma \in F_{q,n}, \alpha \prec \gamma} \sum_{\delta \in F_{q,m}, \beta \prec \delta} \frac{\operatorname{per}(G^{\gamma}_{\delta}) \operatorname{per}(A^{\delta-\beta}_{\gamma-\alpha})}{\mu(\gamma-\alpha)\mu(\delta-\beta)}. \tag{15}$$

Proof. Let $G = UV$ be a rank factorization so that U is $n \times k$, V is $k \times m$ and rank $U =$ rank $V = k$.

By the Cauchy–Binet formula, for any $\gamma \in F_{q,n}$, $\delta \in F_{q,m}$,

$$\operatorname{per}(G^{\gamma}_{\delta}) = \sum_{\rho \in F_{q,k}} \frac{\operatorname{per}(U^{\gamma}_{\rho}) \operatorname{per}(V^{\rho}_{\delta})}{\mu(\rho)}. \tag{16}$$

If $\alpha \in F_{p,k}, \alpha \prec \gamma$, then by the Laplace expansion,

$$\operatorname{per}(U^{\gamma}_{\rho}) = \sum_{\tau \in F_{p,k}, \tau \prec \rho} \left(\binom{\rho}{\tau} \right) \operatorname{per}(U^{\alpha}_{\tau}) \operatorname{per}(U^{\gamma-\alpha}_{\rho-\tau}). \tag{17}$$

Similarly, if $\beta \in F_{p,k}$, $\beta \prec \delta$, then by the Laplace expansion,

$$\operatorname{per}\left(V_\delta^\rho\right) = \sum_{\psi \in F_{p,k}, \psi \prec \rho} \left(\binom{\rho}{\psi}\right) \operatorname{per}\left(V_\beta^\psi\right) \operatorname{per}\left(V_{\delta-\beta}^{\rho-\psi}\right). \tag{18}$$

It follows from (16)–(18) that

$$\sum_{\gamma \in F_{q,n}, \alpha \prec \gamma} \sum_{\delta \in F_{q,m}, \beta \prec \delta} \frac{\operatorname{per}\left(G_\delta^\gamma\right) \operatorname{per}\left(A_{\gamma-\alpha}^{\delta-\beta}\right)}{\mu(\gamma-\alpha)\mu(\delta-\beta)}$$

equals

$$\begin{aligned}
&\sum_{\gamma \in F_{q,n}, \alpha \prec \gamma} \sum_{\delta \in F_{q,m}, \beta \prec \delta} \frac{1}{\mu(\gamma-\alpha)\mu(\delta-\beta)} \sum_{\rho \in F_{q,k}} \frac{1}{\mu(\rho)} \\
&\quad \times \sum_{\tau \in F_{p,k}, \tau \prec \rho} \sum_{\psi \in F_{p,k}, \psi \prec \rho} \left(\binom{\rho}{\tau}\right)\left(\binom{\rho}{\psi}\right) \\
&\quad \times \operatorname{per}\left(U_\tau^\alpha\right) \operatorname{per}\left(U_{\rho-\tau}^{\gamma-\alpha}\right) \operatorname{per}\left(V_\beta^\psi\right) \operatorname{per}\left(V_{\delta-\beta}^{\rho-\psi}\right) \operatorname{per}\left(A_{\gamma-\alpha}^{\delta-\beta}\right)
\end{aligned} \tag{19}$$

The expression in (19) equals

$$\begin{aligned}
&\sum_{\gamma \in F_{q-p,n}} \sum_{\delta \in F_{q-p,m}} \frac{1}{\mu(\gamma)\mu(\delta)} \sum_{\rho \in F_{q,k}} \frac{1}{\mu(\rho)} \\
&\quad \times \sum_{\tau \in F_{p,k}, \tau \prec \rho} \sum_{\psi \in F_{p,k}, \psi \prec \rho} \left(\binom{\rho}{\tau}\right)\left(\binom{\rho}{\psi}\right) \\
&\quad \times \operatorname{per}\left(U_\tau^\alpha\right) \operatorname{per}\left(V_\beta^\psi\right) \operatorname{per}\left(V_\delta^{\rho-\psi}\right) \operatorname{per}\left(A_\gamma^\delta\right) \operatorname{per}\left(U_{\rho-\tau}^\gamma\right)
\end{aligned} \tag{20}$$

Since $GAG = G$, then $UVAUV = UV$ and hence $VAU = I$. Therefore $P_{q-p}(VAU) = I$. Hence

$$\begin{aligned}
&\sum_{\gamma \in F_{q-p,n}} \sum_{\delta \in F_{q-p,m}} \frac{\operatorname{per}\left(V_\delta^{\rho-\psi}\right) \operatorname{per}\left(A_\gamma^\delta\right) \operatorname{per}\left(U_{\rho-\tau}^\gamma\right)}{\mu(\delta)\mu(\gamma)} \\
&\quad = \begin{cases} \mu(\rho-\tau) & \text{if } \psi = \tau, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{21}$$

Using (21), we can express (20) as

$$\sum_{\rho \in F_{q,k}} \frac{1}{\mu(\rho)} \sum_{\tau \in F_{p,k}, \tau \prec \rho} \left(\binom{\rho}{\tau}\right)^2 \mu(\rho-\tau) \operatorname{per}\left(U_\tau^\alpha\right) \operatorname{per}\left(V_\beta^\tau\right),$$

which equals

$$\sum_{\tau \in F_{p,k}} \frac{\operatorname{per}\left(U_\tau^\alpha\right) \operatorname{per}\left(V_\beta^\tau\right)}{\mu(\tau)} \sum_{\rho \in F_{q,k}, \tau \prec \rho} \left(\binom{\rho}{\tau}\right). \tag{22}$$

Table 2

γ	δ	G^{γ}_{δ}	$\text{per}\left(G^{\gamma}_{\delta}\right)$
11	11	$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$	8
11	12	$\begin{pmatrix} 2 & -7 \\ 2 & -7 \end{pmatrix}$	−28
11	22	$\begin{pmatrix} -7 & -7 \\ -7 & -7 \end{pmatrix}$	98
12	11	$\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$	−4
12	12	$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}$	15
12	22	$\begin{pmatrix} -7 & -7 \\ 4 & 4 \end{pmatrix}$	−56
22	11	$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$	2
22	12	$\begin{pmatrix} -1 & 4 \\ -1 & 4 \end{pmatrix}$	−8
22	22	$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$	32

In view of Lemma 3 and the Cauchy–Binet formula, (22) simplifies to

$$\binom{q+k-1}{q-p}\text{per}\left(G^{\alpha}_{\beta}\right)$$

and thus (15) is proved. □

Example. Consider the 2×2 matrix A of rank 2 and G, the inverse of A:

$$A = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}, \quad G = A^{-1} = \begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix}.$$

A list of the 2×2 submatrices of G, with repetitions of rows and columns allowed, and the corresponding permanents is given in Table 2.

Take $p = 1, q = 2$. Since $k = 2$, then $\binom{q+k-1}{q-p} = 3$. Suppose $\alpha = \beta = 1$. Then the right-hand side of (15) can be computed as

$$\begin{aligned} &\frac{1}{3}\left(\text{per}\, G^{1,1}_{1,1}\text{per}\, A^1_1 + \text{per}\, G^{1,1}_{1,2}\text{per}\, A^2_1 + \text{per}\, G^{1,2}_{1,1}\text{per}\, A^1_2 + \text{per}\, G^{1,2}_{1,2}\text{per}\, A^2_2\right) \\ &\quad = \frac{1}{3}\left((8)(4) + (-28)(1) + (-4)(7) + (15)(2)\right) = 2 = \text{per}\, G^1_1, \end{aligned}$$

thereby confirming (15).

Now if $\alpha = 1, \beta = 2$, then the right-hand side of (15) equals

$$\frac{1}{3}\left(\operatorname{per} G^{1,1}_{1,2} \operatorname{per} A^1_1 + \operatorname{per} G^{1,1}_{2,2} \operatorname{per} A^2_1 + \operatorname{per} G^{1,2}_{1,2} \operatorname{per} A^1_2 + \operatorname{per} G^{1,2}_{2,2} \operatorname{per} A^2_2\right)$$
$$= \frac{1}{3}((-28)(4) + (98)(1) + (15)(7) + (-56)(2)) = -\frac{21}{3} = -7 = \operatorname{per} G^1_2,$$

again confirming (15).

4. Nullities of submatrices

The following simple consequence of Theorem 1 will be used in the sequel.

Lemma 5. *Let A be an $m \times n$ matrix of rank r and let G be an $n \times m$ matrix of* rank k $(\leqslant r)$ *such that $GAG = G$. Let $\alpha \in Q_{t,n}$, $\beta \in Q_{t,m}$ be such that G^{α}_{β} is nonsingular. Then*

$$\operatorname{rank}\left(A^{\{1,\dots,m\}\setminus\beta}_{\{1,\dots,n\}\setminus\alpha}\right) \geqslant k - t.$$

Proof. Since G^{α}_{β} is nonsingular, it follows from (1) of Theorem 1, with $q = k$, that there exist $\gamma \in Q_{k,n}, \delta \in Q_{k,m}$ such that $\alpha \subset \gamma, \beta \subset \delta$ and that $A^{\delta\setminus\beta}_{\gamma\setminus\alpha}$ is nonsingular. Thus $A^{\{1,\dots,m\}\setminus\beta}_{\{1,\dots,n\}\setminus\alpha}$ has a nonsingular submatrix of order $k - t$ and the result is proved. □

In the remainder of this section we will assume that A and G are matrices of order $m \times n$ and $n \times m$, respectively, partitioned as follows:

$$A = \begin{matrix} & q_1 & q_2 \\ p_1 & A_{11} & A_{12} \\ p_2 & A_{21} & A_{22} \end{matrix} \quad \text{and} \quad G = \begin{matrix} & p_1 & p_2 \\ q_1 & G_{11} & G_{12} \\ q_2 & G_{21} & G_{22} \end{matrix} \tag{23}$$

where $p_1 + p_2 = m$ and $q_1 + q_2 = n$.

By $\eta(A)$ we denote the row nullity of A, which by definition is the number of rows minus the rank of A.

If $m = n$, A is nonsingular, $G = A^{-1}$ and if A and G are partitioned as in (23) then it was proved by Fiedler and Markham [9] and independently by Gustafson [12] that

$$\eta(A_{11}) = \eta(G_{22}). \tag{24}$$

It is not difficult to derive this result using the Jacobi identity. The motivation for the work presented in this section was to use the general Jacobi identity, presented in Theorem 1, to get stronger results than (24).

Robinson [18] generalized (24) to the Moore–Penrose inverse. Specifically, he showed that if G is the Moore–Penrose inverse of A and if A and G are as in (23), then

$$-(m-r) \leqslant \eta(G_{22}) - \eta(A_{11}) \leqslant n-r. \tag{25}$$

In this section we show that (25) holds for *any* g-inverse G of A.

The following result is well-known. We include a proof for completeness.

Lemma 6. *Let A be an $m \times n$ matrix partitioned as in* (23). *Then*

$$p_1 + q_1 - \operatorname{rank}(A_{11}) \leqslant m + n - \operatorname{rank}(A).$$

Proof. We have

$$\begin{aligned} \operatorname{rank}(A) &\leqslant \operatorname{rank}[A_{11}\ A_{12}] + \operatorname{rank}[A_{21}\ A_{22}] \\ &\leqslant \operatorname{rank}(A_{11}) + \operatorname{rank}(A_{12}) + p_2 \\ &\leqslant \operatorname{rank}(A_{11}) + q_2 + p_2 \end{aligned}$$

and the result follows. □

Lemma 7. *Let A and G be matrices of order $m \times n$ and $n \times m$, respectively, partitioned as in* (23). *Suppose $GAG = G$, $\operatorname{rank}(A) = r \geqslant k = \operatorname{rank}(G)$. Then*

(i) $\operatorname{rank}(A_{11}) \geqslant \operatorname{rank}(G_{22}) + k - p_2 - q_2$
(ii) $\eta(A_{11}) - \eta(G_{22}) \leqslant m - k$.

Proof. Let $\operatorname{rank}(G_{22}) = t$. Then G_{22} has a nonsingular $t \times t$ submatrix. By Lemma 5 there exists $\alpha \subset \{p_1+1, \ldots, m\}$ and $\beta \subset \{q_1+1, \ldots, n\}$, where $|\alpha| = p_2 - t$, $|\beta| = q_2 - t$ such that

$$\operatorname{rank}\left(A^{\{1,\ldots,p_1\}\cup\alpha}_{\{1,\ldots,q_1\}\cup\beta} \right) \geqslant k - t. \tag{26}$$

It follows by Lemma 6 and (26) that

$$\begin{aligned} p_1 + q_1 - \operatorname{rank}(A_{11}) &\leqslant p_1 + p_2 - t + q_1 + q_2 - t - \operatorname{rank}\left(A^{\{1,\ldots,p_1\}\cup\alpha}_{\{1,\ldots,q_1\}\cup\beta} \right) \\ &\leqslant m - t + n - t - (k - t) \\ &= m + n - t - k. \end{aligned}$$

Thus $\operatorname{rank}(A_{11}) \geqslant k + t - p_2 - q_2 = \operatorname{rank}(G_{22}) + k - p_2 - q_2$ and the proof of (i) is complete. Part (ii) easily follows from (i). □

Lemma 8. *Let A and G be matrices of order $m \times n$ and $n \times m$, respectively, partitioned as in* (23). *Suppose $AGA = A$ and $\operatorname{rank}(A) = r$. Then*

$$\eta(G_{22}) - \eta(A_{11}) \geqslant -(m-r). \tag{27}$$

Proof. According to a result on bordered matrices and g-inverse (see, for example, [14]) there exist matrices X, Y and Z of order $m \times (m-r)$, $(n-r) \times n$ and $(n-r) \times (m-r)$, respectively, such that the matrix

$$S = \begin{bmatrix} A & X \\ Y & Z \end{bmatrix}$$

is nonsingular and the submatrix formed by the first n rows and the first m columns of $T = S^{-1}$ is G. Thus we may write

$$S = \begin{matrix} \\ p_1 \\ p_2 \\ n-r \end{matrix} \begin{matrix} q_1 & q_2 & m-r \\ \left(\begin{matrix} A_{11} \\ A_{21} \\ Y_1 \end{matrix}\right. & \begin{matrix} A_{12} \\ A_{22} \\ Y_2 \end{matrix} & \left.\begin{matrix} X_1 \\ X_2 \\ Z \end{matrix}\right) \end{matrix}, \quad T = \begin{matrix} \\ q_1 \\ q_2 \\ m-r \end{matrix} \begin{matrix} p_1 & p_2 & n-r \\ \left(\begin{matrix} G_{11} \\ G_{21} \\ V_1 \end{matrix}\right. & \begin{matrix} G_{12} \\ G_{22} \\ V_2 \end{matrix} & \left.\begin{matrix} U_1 \\ U_2 \\ W \end{matrix}\right) \end{matrix}.$$

Since S is nonsingular, we have, using (24),

$$\begin{aligned} \eta(A_{11}) &= \eta \begin{bmatrix} G_{22} & U_2 \\ V_2 & W \end{bmatrix} \\ &= q_2 + m - r - \operatorname{rank} \begin{bmatrix} G_{22} & U_2 \\ V_2 & W \end{bmatrix} \\ &\leqslant q_2 + m - r - \operatorname{rank}(G_{22}) \\ &= \eta(G_{22}) + m - r. \end{aligned}$$

It follows that $\eta(G_{22}) - \eta(A_{11}) \geqslant -(m-r)$ and the proof is complete. □

We remark that 7 can be proved using the technique of bordered matrices as in the proof of Lemma 8, thereby avoiding the use of the Jacobi identity. We have retained the present proof since it illustrates an interesting application of Lemma 5 and hence of Theorem 1. Besides, Lemma 5 may be of independent interest.

We conclude with the main result of this section.

Theorem 9. *Let A and G be matrices of order $m \times n$ and $n \times m$, respectively, partitioned as in* (23). *Let* $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(G) = k$. *Then the following assertions are true.*

(i) *If $AGA = A$, then*

$$-(m-r) \leqslant \eta(G_{22}) - \eta(A_{11}) \leqslant n - r$$

(ii) *If $GAG = G$, then*

$$-(n-k) \leqslant \eta(A_{11}) - \eta(G_{22}) \leqslant m - k.$$

Proof. The lower bound in (i) follows from Lemma 8 while the upper bound in (ii) follows from Lemma 7. The remaining two bounds are obtained by interchanging the roles of A and G and using the first two bounds. □

As remarked earlier, (i) is contained in [18] for the case when G is the Moore–Penrose inverse of A. Similar results may of course be presented for the column nullity of a matrix, which is the number of columns minus the rank.

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