Optimality of orthogonally blocked diallels with specific combining abilities ☆

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Abstract

Optimality of orthogonally blocked complete diallel crosses for estimating general combining abilities is investigated when the model also includes specific combining abilities. It is proved that these designs remain optimal even in the presence of specific combining abilities. Three new series of orthogonally blocked designs are also reported.

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1. Introduction

Optimal block designs for diallel crosses for the case where treatment differences are due only to general combining ability (gca) effects have been considered by Gupta and Kageyama (1994), Dey and Midha (1996), Mukerjee (1997), Das et al. (1998), among others. Recently, Chai and Mukerjee (1999) considered the case when specific combining ability (sca) effects are also present in the model. They defined balanced and orthogonal designs, that is designs that are (i) balanced for gca comparisons, (ii) balanced for sca comparisons, and (iii) orthogonal in the sense that the least-squares estimate of any gca contrast is uncorrelated with the least-squares estimate of any sca contrast. Throughout this paper, by balance we mean variance balance. They gave a necessary and

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sufficient condition for a design to be balanced and orthogonal. For binary designs, that is for designs in which a cross occurs at most once in any block, they further proved that a design is balanced and orthogonal if and only if, it is a triangular design.

Henceforth, by optimal we mean universally optimal (Kiefer, 1975). Furthermore, we now restrict attention to complete diallel cross designs. As shown by Chai and Mukerjee (1999), the triangular designs that were proved to be optimal for gca comparisons by Dey and Midha (1996) and Das et al. (1998) remain optimal for gca comparisons even if sca effects are present in the model. Gupta and Kageyama (1994), and Das et al. (1998), among others, gave several series of optimal designs for gca comparisons that are not triangular. However, optimality of these designs for gca comparisons when sca effects are also present in the model is not yet known. Gupta et al. (1995) investigated orthogonal blocking of diallel crosses; orthogonal in the sense that no loss of information is incurred as a result of blocking the crosses. Since complete diallel crosses are optimal for gca comparisons before blocking, orthogonally blocked complete diallel cross designs are also optimal for gca comparisons. The purpose of this paper is to prove the optimality of orthogonally blocked complete diallel cross designs for gca comparisons when sca effects are also present in the model. In particular, this establishes the optimality of orthogonally blocked designs of Gupta and Kageyama (1994), Das et al. (1998) and Choi and Gupta (2000) for gca comparisons when sca effects are also present in the model. Note that although Das et al. (1998) reported several series of optimal designs, only their Family 5 designs are orthogonally blocked. Three new series of orthogonally blocked designs are also reported.

2. Orthogonally blocked designs

We follow the notations and definitions of Chai and Mukerjee (1999). They proved that if a triangular design is optimal for the estimation of gca comparisons then it remains optimal even if sca effects are included in the model. Here we extend their result to orthogonally blocked designs for estimating gca comparisons. Note that these designs are optimal for estimating gca comparisons, but they are not necessarily triangular. Specifically, we establish that orthogonally blocked designs remain optimal for estimating gca comparisons even if sca effects are included in the model.

Gupta et al. (1995) defined a diallel cross design to be orthogonally blocked if each line occurs in every block r/b times, where r is the constant replication number of the lines and b is the number of blocks in the design. Let D be a binary orthogonally blocked complete diallel cross design with parameters $v = p(p-1)/2, b, r_c, k$, in b blocks of k crosses each with each of the v crosses replicated r_c times. Since D is orthogonally blocked, it can be seen that each line occurs $2k/p$ times in every block, where $2k/p$ is necessarily an integer for D to be orthogonal.

The model for the data is assumed to be

$$Y_{ij\ell} = \mu + \alpha_i + \alpha_j + \beta_{ij} + \gamma_\ell + \varepsilon_{ij\ell},$$

where $Y_{ij\ell}$ is the response from the cross (i,j) in the ℓth block, $i,j(i<j)=1,2,\ldots,p$; $\ell=1,2,\ldots,b$; μ is the overall mean of all the crosses, α_i, β_{ij} are, respectively, the gca and sca effects, with $\alpha_1 + \cdots + \alpha_p = 0$, and for $i = 1,2,\ldots,p$, $\beta_{1i} + \cdots + \beta_{(i-1)i} + \beta_{i(i+1)} + \cdots + \beta_{ip} = 0$, with $\beta_{(i-1)i}$ appearing only when $i > 1$. Also, γ_ℓ is the ℓth block effect, and $\varepsilon_{ij\ell}$ are independent random errors with zero expectation and constant variance σ^2. Our main interest is in the comparisons of gca parameters α_i's allowing for the possibility that at least some of the comparisons of sca parameters β_{ij}'s may not be zero. Thus, although the above model includes β_{ij} parameters, not all comparisons of sca parameters may be estimable. This is especially the case when $r_c = 1$.

Let $\tau_{ij}=\mu+\alpha_i+\alpha_j+\beta_{ij}$ and let $\tau=(\tau_{ij})$ be the column vector of treatment parameters. The $v=p(p-1)/2$ crosses being considered in the order given by $(1,2),(1,3),\ldots,(1,p),(2,3),\ldots,(2,p),\ldots,(p-1,p)$. Let $n_u=(n_{iu})$ be the $v\times 1$ incidence vector for the uth block, where n_{iu} is the number of times the ith cross occurs in the uth block, $i=1,2,\ldots,v$; $u=1,2,\ldots,b$, and let $N=[n_1\ n_2\ \cdots\ n_b]$ be the incidence matrix of D. Let I_v be the identity matrix and let 1_v be the column vector of 1's, both of order v. If we take (i,j) as being a treatment, then the usual intra-block matrix C for estimating the vector of treatment parameters τ is given by

$$C = r_c I - \frac{1}{k} NN',$$

$$= r_c I - \frac{1}{k}\sum_{u=1}^{b} M_u, \tag{2.1}$$

where

$$M_u = n_u n_u' \tag{2.2}$$

is a $v\times v$ symmetric matrix with rows and columns indexed by the pairs (i,j), $i,j(i<j)=1,2,\ldots,p$, such that: (a) (i_1,j_1)th diagonal element of M_u equals unity if the cross (i_1,j_1) occurs in the uth block, and zero otherwise, and (b) $\{(i_1,j_1),(i_2,j_2)\}$th element of M_u equals unity if the crosses (i_1,j_1) and (i_2,j_2) occur together in the uth block and zero otherwise, $u=1,2,\ldots,b$. Note that $M_u 1_v = kn_u$. Now, let $g'\tau$ be a contrast among treatment parameters where $g=(g_{ij})$ with $g'1_v=0$ is a $v\times 1$ vector with its elements indexed by the pairs $(i,j),(i<j)=1,2,\ldots,p$. Then we have the following lemma which will be helpful in proving the main result of this paper.

Lemma 1. *For $g'\tau$ to be a contrast among gca parameters, it is necessary that $g_{ij}=1/(p-2)\{e_i+e_j\}$, $i,j(i<j)=1,2,\ldots,p$, where e_i are such that $e_1+e_2+\cdots+e_p=0$.*

Proof. Let $e'\alpha$ be a contrast among gca parameters where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_p)'$, and $e=(e_1,e_2,\ldots,e_p)'$ with $e'1_p=0$. Using Eqs. (1.8) and (1.9) of Chai and Mukerjee (1999) we get

$$e'\alpha = \frac{1}{p-2} e'Q\tau.$$

The lemma now follows from the definition of Q as given by Chai and Mukerjee (1999).

Chai and Mukerjee (1999) considered designs that are balanced and orthogonal. In Theorem 1, we prove that orthogonally blocked designs remain optimal for the estimation of gca comparisons even in the presence of sca effects in the model. Obviously, such designs are balanced for gca comparisons, but the designs are not necessarily balanced for sca comparisons. This is especially the case when $r_c = 1$ where not all sca comparisons are even estimable. Furthermore, as shown in Corollary 1 for orthogonally blocked designs, the least squares estimate of any gca contrast is uncorrelated with the least-squares estimate of any estimable sca contrast. We now present the main result.

Theorem 1. *Consider a binary diallel cross design D with parameters* $v = p(p-1)/2, r_c, b, k$ *such that it is orthogonally blocked with respect to the lines, that is, each line occurs in every block* r/b *times, where* $r = r_c(p-1)$ *is the constant number of replications of the lines. Then D is optimal for the estimation of gca comparisons even in the presence of sca effects in the model.*

Proof. Let the k distinct crosses of the uth block of D be denoted by $(h_1,w_1),(h_2,w_2),\ldots,(h_k,w_k)$, and let $m_u\{(i_1,j_1),(i_2,j_2)\}$ denote the $\{(i_1,j_1),(i_2,j_2)\}$th element of M_u as defined by Eq. (2.2). Then from the definition of M_u it follows that $m_u\{(i_1,j_1),(i_2,j_2)\}$ equals unity if both the crosses (i_1,j_1) and (i_2,j_2) occur in the uth block of D, i.e. if $\{(i_1,j_1),(i_2,j_2)\} \subset \{(h_1,w_1),(h_2,w_2),\ldots,(h_k,w_k)\}$, otherwise it equals zero. This also implies that (a) the diagonal elements of M_u that correspond to the crosses in the uth block equal unity, and (b) the (i_1,j_1)th row of M_u consists of only zeroes if the cross (i_1,j_1) does not occur in the uth block.

Now let $g'\tau$ be a contrast among gca parameters and let $M_u g = f$, where the elements of $f = (f_{st})$ are also indexed by the pairs $s,t (s<t) = 1,2,\ldots,p$. It thus follows from (b) above that $f_{st} = 0$ if the cross (s,t) does not occur in the uth block, and if the cross (s,t) occurs in the uth block, i.e. if $(s,t) \in \{(h_1,w_1),(h_2,w_2),\ldots,(h_k,w_k)\}$, then

$$f_{st} = \sum_{\ell=1}^{k} (e_{i_\ell} + e_{j_\ell}) \tag{2.3}$$

using Lemma 1 and the definition of M_u. Now, since each line occurs r/b times in every block of D, $\{i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k\}$ constitute r/b replications of $\{1,2,\ldots,p\}$. Thus, using this fact and Lemma 1, Eq. (2.3) simplifies to

$$f_{st} = \frac{r}{b}\sum_{i=1}^{p} e_i = 0.$$

This establishes that if $g'\tau$ is a contrast among gca parameters then $M_u g = 0$, i.e. g is an eigenvector of M_u with zero eigenvalue. Hence, using Eq. (2.1) it follows that g is an eigenvector of the intra-block matrix C with eigenvalue r_c. Now, if g is an eigenvector of C with eigenvalue r_c, then g is also an eigenvector of C^- with eigenvalue $1/r_c$, where '−' denotes generalized inverse. This implies that $\mathrm{var}(g'\hat{\tau}) = (g'g/r_c)\sigma^2$. In other words this means that $g'\hat{\tau}$ is estimated without any loss of information. Hence, the theorem.

For orthogonally blocked designs, Corollary 1 below follows by noting that for two contrasts $g'\tau$ and $s'\tau$, with $g'\tau$ being a gca contrast, $\mathrm{cov}(g'\hat{\tau}, s'\hat{\tau}) = g'C^- s\sigma^2 = (1/r_c)g's\sigma^2$.

Corollary 1. *Let* $g'\tau$ *be a contrast among gca parameters and let* $s'\tau$ *be an estimable contrast among sca parameters. Then, for an orthogonally blocked diallel cross design,* $\mathrm{cov}(g'\hat{\tau}, s'\hat{\tau}) = 0$.

In particular, Theorem 1 establishes the optimality of the following three series of designs:

Series 1 (Gupta and Kageyama, 1994): For even p, there exists an optimal design with parameters $v = p(p-1)/2$, $b = p-1$, $r_c = 1$, $k = p/2$. The ℓth block of the design is given by $\{(j+\ell, p-1-j+\ell),(\ell,p),\ j = 1,2,\ldots,(p-2)/2\}$, $\ell = 1,2,\ldots,p-2$, where the addition is done $\mathrm{mod}(p-1)$, the p lines are coded as $0, 1, \ldots, p-2, p$, and line coded as p remains fixed while developing the blocks.

Series 2 (Das et al., 1998, Family 5): For $p = 2t + 1,\ t \geqslant 1$, there exists an optimal design with parameters $v = t(2t + 1),\ b = t,\ r_c = 1,\ k = 2t + 1$. The ℓth block of the design is given by $\{(\ell, 2t + 1 - \ell), (1 + \ell, 1 - \ell), (2 + \ell, 2 - \ell), \ldots, (2t + \ell, 2t - \ell)\},\ \ell = 1, 2, \ldots, t$.

Series 3 (Choi and Gupta, 2000): There exists an optimal diallel cross design with parameters $v = p(p-1)/2,\ b = p - 1,\ r_c = 2,\ k = p$. The ℓth block of the design is obtained by cyclically developing the cross $(0, \ell)\mathrm{mod}(p),\ \ell = 1, 2, \ldots, b$.

Three new series of orthogonally blocked designs involving only one replication of each of the crosses are given below. Since the designs of these series are orthogonally blocked, the designs remain optimal even in the presence of sca effects in the model. Series 4 and 5 are derived, respectively, from Families 1 and 2 of Das et al. (1998), by taking columns as blocks instead of the rows when the sets of crosses are cyclically developed.

Series 4: Let $p = 4t + 1,\ t \geqslant 1$, be a prime or a prime power and let x denote a primitive element of GF(p). Then cyclically developing the two sets of t crosses each $\{(x^i, x^{i+2t}),\ i = 0, 1, \ldots, t-1\}$ and $\{(x^{i+t}, x^{i+3t}), i = 0, 1, \ldots, t-1\}$, the crosses obtained from each set forming a block, yields an optimal diallel cross design with parameters $v = 2t(4t + 1),\ b = 2,\ r_c = 1,\ k = t(4t + 1)$.

Series 5: Let $p = 6t + 1,\ t \geqslant 1$, be a prime or a prime power and let x denote a primitive element of GF(p). Then, cyclically developing the three sets of t crosses each $\{(x^i, x^{i+3t}),\ i = 0, 1, \ldots, t-1\}$, $\{(x^{i+t}, x^{i+4t}),\ i = 0, 1, \ldots, t-1\}$ and $\{(x^{i+2t}, x^{i+5t}),\ i = 0, 1, \ldots, t-1\}$, the crosses obtained from each set forming a block, yields an optimal diallel cross design with parameters $v = 3t(6t + 1),\ b = 3,\ r_c = 1,\ k = t(6t + 1)$.

Series 6: Let $p = 2tn + 1, t$ and n both > 1, be a prime or a prime power and let x be a primitive element of GF(p). For $j = 0, 1, \ldots, t-1$ the initial crosses of the jth block are:

$$\{(x^{nj+i}, x^{nt+nj+i});\ i = 0, 1, \ldots, n-1\}.$$

Cyclically developing the initial crosses in each block yields an optimal diallel cross design with parameters $v = tn(2tn + 1),\ b = t,\ r_c = 1,\ k = n(2tn + 1)$.

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