Waves on the surface of a falling power-law fluid film

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Abstract

Waves that occur at the surface of a falling film of thin power-law fluid on a vertical plane are investigated. Using the method of integral relations an evolution equation is derived for two types of waves equation which are possible under long wave approximation. This equation reveals the presence of both kinematic and dynamic wave processes which may either act together or singularly dominate the wave field depending on the order of different parameters. It is shown that, at a small flow rate, kinematic waves dominate the flow field and the energy is acquired from the mean flow during the interaction of the waves, while, for high flow rate, inertial waves dominate and the energy comes from the kinematic waves. It is also found that this exchange of energy between kinematic and inertial waves strongly depends on the power-law index n. Linear stability analysis predicts the contribution of different terms in the wave mechanism. Further, it is found that the surface tension plays a double role: for a kinematic wave process, it exerts dissipative effects so that a finite amplitude case may be established, but for a dynamic wave process it yields dispersion. Further, it is shown that the non-Newtonian character n plays a vital role in controlling the role of the term that contains surface tension in the above processes.

Keywords: Power-law fluid film; Waves on falling film; Stability of power-law fluid film

1. Introduction

Wave motion in a thin film can be observed when the rain water overflows the eaves trough, flows down a window pane, or when one hoses down during cleaning the windscreen of a car. The dynamics of thin film waves has received much attention from various industries due to its dramatic effect on transport rate of mass [1], heat [2,3] and momentum [4] in designing distillation and adsorption columns, evaporators, condensers, nuclear reactor emergency cooling system, etc. Knowledge of film waves is necessary in connection with the modern precision coating of photographic emulsions, magnetic material, protective paints, flow of molten metal/lava, etc. Study on the wave evolution on a falling film started with the pioneering experiment by Kapitza [5] and Kapitza and Kapitza [6]. Up to date works on this fascinating problem can be seen by the review works of Fulford [7], Lin and Wang [8], Chang [9]. It is interesting to note that most of the studies on the development of waves in the thin film on the surface of an inclined/vertical plane assumed the fluid to be Newtonian. These results of Newtonian fluid cannot completely describe the rheological behavior of the non-Newtonian fluid. Further, it is known that most of the fluids used in industry are basically non-Newtonian. Many mathematical models have been proposed to describe the characteristics of simple non-Newtonian fluids by several authors viz. Rajagopal [10], Málek et al. [11] and others. Málek et al. [12,13] studied the different models along with the existence of regularity of the solutions and the stability of the rest state for non-Newtonian fluids. Although the above references do not deal with the film flow down an inclined/vertical plane, they do discuss many important general issues concerning the non-Newtonian fluids. Some studies on linear stability of non-Newtonian liquid film flow were made by Gupta [14] considering a second-order fluid; by Liu and Mei [15] a Bingham fluid, Lai [16] for an Oldroyd-B fluid and Hwang et al. [17] and Berezin et al. [18] for power-law model. Using Benney's [19] approach, Dandapat and Gupta [20] studied the stability of a falling film of an incompressible second-order fluid with respect to two-dimensional disturbances of small but finite amplitude. They found that in the presence of surface tension the stability of flow of the falling film is supercritically stable and an initially growing monochromatic wave reaches an equilibrium state of finite amplitude. Further, they found that the equilibrium amplitude first increases with the elastic parameter M(say) of the fluid, reaches a maximum and then decreases with increase in M. In a recent study Dandapat and Gupta [21] have shown the existence and the role of the solitary wave in the finite amplitude instability of a layer of a second-order fluid flowing down an inclined plane. Ng and Mei [22] studied the roll waves on a layer of mud modelled as a power-law fluid flowing down an inclined plane. They found through linearized instability that the growth rate of unstable disturbances increases monotonically with the wave number; this prevented them from predicting any preferred wavelength for the roll wave. Further, they observed that the existence of long roll waves depends on the power-law index even if the corresponding uniform flow is stable. It is to be pointed out here that Ng and Mei [22] have neglected the surface tension term in their analysis. It is well known that the wavelength, amplitudes and their relation with the flow rate are of primary importance for the design of process devices. For a better understanding of a physical phenomenon it is therefore desirable to investigate the types of waves that occur under various conditions.

2. Mathematical formulation of the problem

Consider a two-dimensional laminar flow of a thin layer of a power-law fluid on a vertical plane. A co-ordinate system is defined with the x-axis along the direction of gravity and the z-axis normal to the plane (Fig. 1).

The governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \tag{1}$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \rho g + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \tag{2}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}, \tag{3}$$

where u, v, ρ and p are the longitudinal, transverse velocity components, density and the pressure, respectively. Here τ_{ij} is the stress tensor defined by

$$\tau_{ij} = 2\mu_n[2D_{kl}D_{kl}]^{(n-1)/2}D_{ij}, \tag{4}$$

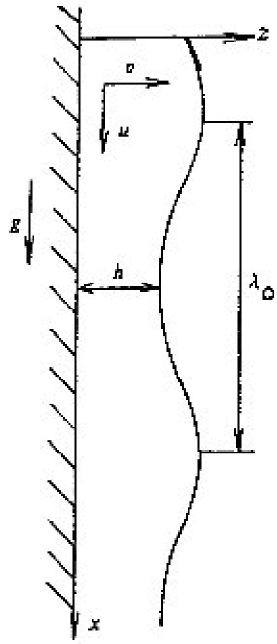

Fig. 1. Sketch of the problem.

where

$$D_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]. \tag{5}$$

denotes the strain-rate tensor, μ_n is the viscosity coefficient of dimension $[ML^{-1}T^{(n-2)}]$ and n is the power-law index which is positive. $n=1$ represents a Newtonian fluid with constant dynamic coefficient of viscosity μ, while $n<1$ and >1 correspond to the case of pseudoplastic (shear-thinning) and dilatant (shear-thickening) fluids, respectively.

The boundary conditions are

$$u=0, \quad v=0 \text{ at } z=0, \tag{6}$$

$$\tau_{zx}\left[1-\left(\frac{\partial h}{\partial x}\right)^2\right]-(\tau_{xx}-\tau_{zz})\frac{\partial h}{\partial x}=0 \quad \text{at } z=h, \tag{7}$$

and

$$-p+\left[\tau_{xx}\left(\frac{\partial h}{\partial x}\right)^2-2\tau_{zx}\frac{\partial h}{\partial x}+\tau_{zz}\right]\left[1+\left(\frac{\partial h}{\partial x}\right)^2\right]^{-1}+p_0=\sigma\frac{\partial^2 h}{\partial x^2}\left[1+\left(\frac{\partial h}{\partial x}\right)^2\right]^{-3/2} \quad \text{at } z=h, \tag{8}$$

where σ is the surface tension, p_0 is the atmospheric pressure and h is the deflection from the mean depth h_0.

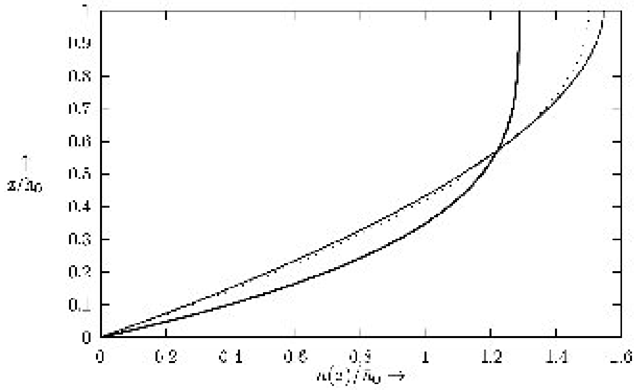

Fig. 2. Velocity profiles for different values of n. Thin, dotted and thick lines correspond to $n = 1.2, 1, 0.4$, respectively.

The kinematic condition at the free surface is

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \quad \text{at } z = h. \tag{9}$$

The basic velocity $[u(z), 0]$ in the steady flow down the plane is

$$u = \frac{(1+2n)}{(1+n)}\bar{u}_0\left[1 - \left(1 - \frac{z}{h_0}\right)^{(1+n)/n}\right]. \tag{10}$$

To obtain Eq. (10) we have used Eqs. (4) and (5) in the momentum equation and the no-slip condition $u(0) = 0$ and the condition of zero shear stress at the free surface $z = h_0$ which is the undisturbed layer thickness. Here, $\bar{u}_0$ is the depth-averaged characteristic velocity defined by

$$\bar{u}_0 = \frac{1}{h_0}\int_0^{h_0} u(z)\,\mathrm{d}z = \frac{n}{(1+2n)}\left[\frac{\rho g}{\mu_n}\right]^{1/n} h_0^{(1+n)/n}. \tag{11}$$

Variation of the steady uniform flow with z/h_0 defined in (10) is plotted in Fig. 2. It is clear from the figure that the power-law index n has a strong effect on the shape of the velocity profile. The steady discharge rate per unit width is

$$\bar{Q} = \bar{u}_0 h_0 = \left(\frac{n}{1+2n}\right)\left(\frac{\rho g}{\mu_n}\right)^{1/n} h_0^{(1+n)/n}. \tag{12}$$

We assume the characteristic longitudinal length scale to be l_0 whose order may be considered the same as the wavelength λ_0 and the mean film thickness h_0 as the length scale in the transverse direction. We define the dimensionless quantities as

$$x = l_0 x^*, \quad (h, z) = h_0(h^*, z^*), \quad t = \left(\frac{l_0}{\bar{u}_0}\right)t^*, \quad u = \bar{u}_0 u^*, \quad v = \left(\frac{h_0}{l_0}\right)\bar{u}_0 v^*, \quad p = \rho\bar{u}_0^2 p^*,$$

$$(\tau_{xx}, \tau_{zz}) = \mu_n(\bar{u}_0/h_0)^{n-1}(\bar{u}_0/l_0)(\tau_{xx}^*, \tau_{zz}^*) \quad \text{and} \quad (\tau_{xz}, \tau_{zx}) = \mu_n(\bar{u}_0/h_0)^n(\tau_{xz}^*, \tau_{zx}^*). \tag{13}$$

Using (13) in (1)–(3) and in (6)–(9), after dropping the asterisk we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0, \tag{14}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\varepsilon \mathrm{Fr}}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial \tau_{xx}}{\partial x}+\frac{1}{\varepsilon \mathrm{Re}}\frac{\partial \tau_{xz}}{\partial z}, \tag{15}$$

$$\varepsilon^2\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial z}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial \tau_{zx}}{\partial x}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial \tau_{zz}}{\partial z}, \tag{16}$$

$$u=0, \quad v=0 \text{ at } z=0, \tag{17}$$

$$\tau_{zx}\left[1-\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right]-\varepsilon^2(\tau_{xx}-\tau_{zz})\frac{\partial h}{\partial x}=0 \quad \text{at } z=h, \tag{18}$$

$$-p+\left[\frac{\varepsilon^3}{\mathrm{Re}}\tau_{xx}\left(\frac{\partial h}{\partial x}\right)^2-\frac{2\varepsilon}{\mathrm{Re}}\tau_{zx}\frac{\partial h}{\partial x}+\frac{\varepsilon}{\mathrm{Re}}\tau_{zz}\right]\left[1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right]^{-1}+p_0$$

$$=\varepsilon^2\frac{\mathrm{W}}{\mathrm{Fr}}\frac{\partial^2 h}{\partial x^2}\left[1+\left(\frac{\partial h}{\partial x}\right)^2\right]^{-3/2} \quad \text{at } z=h, \tag{19}$$

and

$$v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} \quad \text{at } z=h, \tag{20}$$

where Re is the Reynolds number

$$\mathrm{Re}=\frac{\rho \bar{u}_0^{(2-n)}h_0^n}{\mu_n},$$

W is the Weber number

$$\mathrm{W}=\frac{\sigma}{\rho g h_0^2},$$

Fr is the Froud number

$$\mathrm{Fr}=\frac{\bar{u}_0^2}{g h_0}=\left(\frac{n}{1+2n}\right)^n \mathrm{Re}$$

and ε is the aspect ratio

$$\varepsilon=\frac{h_0}{l_0}\ll 1$$

for long wave length approximation. Using the dimensionless form of (4) and (5) in (14)–(20) under usual boundary-layer approximations for long-wave expansions, we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0, \tag{21}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\varepsilon \mathrm{Fr}}+\frac{1}{\varepsilon \mathrm{Re}}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^n, \tag{22}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{23}$$

The boundary conditions are

$$u = v = 0 \quad \text{at } z = 0, \tag{24}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at } z = h, \tag{25}$$

$$p = p_0 - \varepsilon^2 \frac{\mathrm{W}}{\mathrm{Fr}} \frac{\partial^2 h}{\partial x^2} \quad \text{at } z = h, \tag{26}$$

and the kinematic condition is

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = v \quad \text{at } z = h. \tag{27}$$

The z-momentum equation (23) and the normal stress boundary condition (26) are used to eliminate $\partial p / \partial x$ in Eq. (22) and the resulting system reduces to

$$u_x + v_z = 0, \tag{28}$$

$$u_t + uu_x + vu_z = \frac{1}{\varepsilon \mathrm{Re}} \left[\left(\frac{1+2n}{n} \right)^n (1 + \varepsilon^3 \mathrm{W} h_{xxx}) + \{(u_z)^n\}_z \right], \tag{29}$$

$$u = 0 = v \quad \text{at } z = 0, \tag{30}$$

$$u_z = 0 \quad \text{at } z = h(x,t), \tag{31}$$

$$h_t + uh_x = v \quad \text{at } z = h(x,t), \tag{32}$$

here the subscripts denote the derivative of the respective variables t, x and z.

Integrating (28) and (29) with respect to z from 0 to h, we get

$$h_t + q_x = 0, \tag{33}$$

$$q_t + \beta \left(\frac{q^2}{h} \right)_x = \frac{1}{\varepsilon \mathrm{Re}} \left(\frac{1+2n}{n} \right)^n \left[(1 + \varepsilon^3 \mathrm{W} h_{xxx}) h - \left(\frac{q}{h^2} \right)^n \right], \tag{34}$$

where the flow rate per unit film width is

$$q = \int_0^h u \, \mathrm{d}z, \tag{35}$$

and the shape factor β is defined as

$$\beta = \frac{1}{h\overline{u}_0^2} \int_0^h u^2 \, \mathrm{d}z = \frac{2(1+2n)}{(2+3n)}. \tag{36}$$

For shear-thinning fluids, $0 < n \leqslant 1$, the range of β is $1 \leqslant \beta \leqslant \frac{6}{5}$. $\bar{u}$ is defined as the depth-averaged velocity

$$\overline{u} = h^{-1} \int_0^h u \, \mathrm{d}z = \frac{q}{h}. \tag{37}$$

It is to be noted here that the momentum integral method has been used by earlier researchers in connection with the boundary layer theory of Schlichting [23] and on stability theory starting from Kapitza [5] in connection with the wave film. Although used later by Maurin and Sorokin [24], Alekseenko et al. [25], Jurman and McCready [26] and others, Shkadov [27] used first for vertical fluid film for Newtonian fluid.

3. Derivation of two-wave equation

To study the slightly non-linear waves, let us assume that

$$h = 1 + H(x,t), \quad q = 1 + Q(x,t), \quad H, Q \ll 1, \tag{38}$$

where H and Q are dimensionless perturbations of the film thickness and flow rate, respectively. Substituting (38) into (33) and (34) and retaining the terms up to second order fluctuations, the continuity and momentum equations reduce to

$$H_t + Q_x = 0, \tag{39}$$

$$Q_t + \beta(2Q_x - H_x) - \left(\frac{1+2n}{n}\right)^n [(1+2n)H - nQ + \varepsilon^3 \mathrm{W} H_{xxx}]/(\varepsilon\,\mathrm{Re}) = \frac{n^2}{\varepsilon\,\mathrm{Re}}\left(\frac{1+2n}{n}\right)^{1+n} H^2$$
$$-2nHQ_t - 2\beta[QQ_x - (1-2n)HQ_x - QH_x + (1-n)HH_x]$$
$$+\left(\frac{1+2n}{n}\right)^n \left[\frac{n(1-n)}{2}Q^2 + (1+2n)\varepsilon^3 \mathrm{W} HH_{xxx}\right] \Big/ (\varepsilon\,\mathrm{Re}). \tag{40}$$

Eqs. (39) and (40) can be expressed in a single equation for the film height disturbance H by differentiating Eq. (40) with respect to x and eliminating Q and its derivative according to the procedure described below:

1. To eliminate the linear derivative of Q use Eq. (39) and
2. for the non-linear terms, approximation methods of quasistationary process are to be used.

Alekseenko et al. [28] have used this method for a vertical film. In this method the basic assumption used in conformation with the experimental observation that the wave generally evolves in shape rather slowly with the downstream distance. Following Alekseenko et al. [28], we assume the system of co-ordinate moving with velocity c, which allows the co-ordinate transformation $(t,x) \to (t, \xi = x - ct)$. It is further assumed that the phase velocity c is approximately constant for quasistationary waves in the interval Δt. Under this transformation Eq. (39) gives

$$H_t - cH_\xi + Q_\xi = 0. \tag{41}$$

The wave profile in a moving co-ordinate system is deformed slightly in the quasistationary process; this approximates Eq. (41) as $cH_\xi = Q_\xi$ from which the following relations are obtained:

$$Q = cH, \tag{42}$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}. \tag{43}$$

After using rule (i) and substituting relations (42) and (43) where needed in Eq. (33), we get

$$(\partial_t + c_0\partial_x)H + \frac{\varepsilon\,\mathrm{Re}}{n}\left(\frac{n}{1+2n}\right)^n (\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)H + \frac{\mathrm{W}}{n}\varepsilon^3 H_{xxxx}$$
$$= -\left[(1+2n) + \frac{1-n}{2}c^2\right](H^2)_x + \frac{2(\beta - n)}{n}\left(\frac{n}{1+2n}\right)^n (\varepsilon\,\mathrm{Re})(HH_t)_t$$
$$+\frac{2\beta(1-n)}{n}\left(\frac{n}{1+2n}\right)^n (\varepsilon\,\mathrm{Re})[2(HH_x)_t + (HH_x)_x] - \frac{1+2n}{n}\varepsilon^3 \mathrm{W}(HH_{xxx})_x, \tag{44}$$

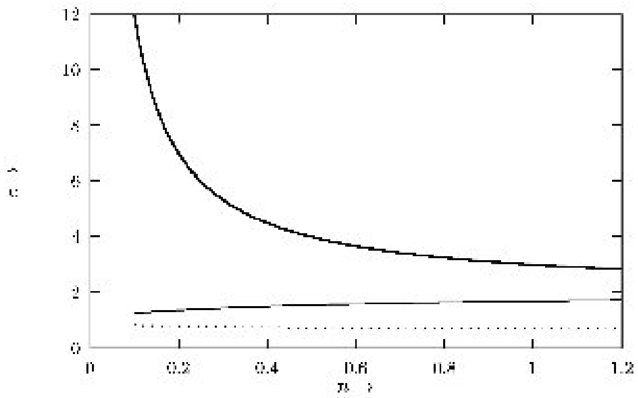

Fig. 3. Variation of wave velocities c_0, c_1 and c_2 with n. Thick, thin and dotted lines correspond to c_0, c_1 and c_2, respectively.

where

$$c_0 = \frac{2n+1}{n} \quad \text{and} \quad c_{1,2} = \beta \pm \sqrt{\beta^2 - \beta} \tag{45}$$

are the respective characteristic wave velocities. It is clear from Fig. 3, that c_0 decreases from a very large value to an asymptotic value 2 as n increases, whereas $c_1(c_2)$ increases(decreases) with n but their (c_1, c_2) variation with n is very slow compared to that of c_0.

It should be noted here that weakly nonlinear waves are small in curvature; therefore, the contribution from the higher order derivatives of the quadratic terms on the right hand side of (44) is very small and hence may be neglected. Therefore Eq. (44) consists of two-wave structure which reveals that two-wave processes occur simultaneously on thin liquid film. They are according to [29] (i) *Kinematic waves*: this is the lower order wave with characteristic velocity c_0. This wave is non-dispersive and is expected to be a low frequency disturbance. This wave is responsible for the transport of fluids. It is clear from Fig. 3 that the kinematic wave velocity c_0 strongly depends on the power-law index n. (ii) *Dynamic waves*: These are higher order waves with characteristic velocities approximated by c_1 and c_2. These waves are dispersive, their speeds in general consist of fluid inertia, gravity, surface tension and power-law index n. No net transport of the fluid is associated with the motion of these types of waves. On the other hand, these waves may be called inertial waves, since Eq. (44) has appeared due to the inertial term of the Navier–Stokes equation.

4. Case-I: small flow rate: Re $\sim$ 1, W$e \sim 1/\varepsilon^2$

Inspecting the non-linear wave equation (44), it can be seen that the Kinematic waves, associated with the first order terms derivative, dominate the wave field for Re $\sim$ 1 and are described in the first approximation by the differential equation of the first order

$$\left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x}\right) H = 0.$$

In this limit kinematic waves are expected to dominate the wave field. Following Alekseenko et al. [20], the time derivative of higher order terms is replaced by $-c_0(\partial/\partial x)$ in Eq. (44) and neglecting the non-linear quadratic terms we get

$$H_t + c_0 H_x + \frac{\varepsilon\,\mathrm{Re}}{n}\left(\frac{n}{1+2n}\right)^n (c_1 - c_0)(c_2 - c_0)H_{xx} + \frac{\mathrm{W}}{n}\varepsilon^3 H_{xxxx} = 0. \tag{46}$$

A linear stability analysis of Eq. (46) by assuming that the perturbation is of the form

$$H = \delta \exp[\mathrm{i}(k\tilde{x} - \omega\tilde{t})],$$

where $\omega(=\omega_r + \omega_i)$ is the complex wave speed and δ is the amplitude assumed to be real, with $x = \varepsilon\tilde{x}$ and $t = \varepsilon\tilde{t}$. Equating real and imaginary parts we get

$$\omega_r = c_0 k,$$

and

$$\omega_i = \frac{\mathrm{Re}}{n}\left(\frac{n}{1+2n}\right)^n (c_0^2 - 2\beta c_0 + \beta)k^2 - \frac{\mathrm{W}}{n}k^4. \tag{47}$$

This gives the phase velocity

$$c = \omega_r/k = c_0 = \frac{1+2n}{n},$$

independent of the wave number k, implying non-dispersive waves. It should be pointed out here that the phase velocity c_0 will be larger for pseudoplastic $(n<1)$ fluids than for both Newtonian $(n=1)$ and dilatant $(n>1)$ fluids. Further, Eq. (47) shows that ω_i is different from zero and has two terms of which the first term is always positive in the entire range of Re, yields the energy pumping into the perturbations and results in instability while the second term, which is due to surface tension, is always negative implying dissipation of the perturbation.

Flow instability is determined by the condition $\omega_i > 0$ and $\omega_i = 0$ gives the neutral state. For neutral perturbation we have two relations

$$k = 0,$$

and

$$k_{\mathrm{N}} = \sqrt{\frac{1}{n}\left(\frac{1+2n}{n}\right)^{1-n}\frac{\mathrm{Re}}{\mathrm{W}}}. \tag{48}$$

This gives two branches of the neutral curve and the flow instability takes place in between them. The wave number of the wave with maximum growth can be obtained from the relation $\mathrm{d}\omega_i/\mathrm{d}k = 0$ and it gives

$$k_m = \sqrt{\frac{1}{2n}\left(\frac{1+2n}{n}\right)^{1-n}\frac{\mathrm{Re}}{\mathrm{W}}} = \frac{k_{\mathrm{N}}}{\sqrt{2}}, \tag{49}$$

where k_{N} is given by relation (48b). It should be pointed out here that in Fig. 4, the flow instability takes place between the regions bounded by the curves defined in Eqs. (48a) and (48b). Further, it is clear that this unstable region increases with the decrease of n. It is clear from Fig. 4 that the flow becomes unstable at $\mathrm{Re} = 0$ for all n.

5. Case-II: high flow rate:

We shall study the type of waves that dominate in the high flow rate implied in the range of large Reynolds number, $\mathrm{Re} \sim 1/\varepsilon^2 \gg 1$. In this range dynamic or inertial waves have a controlling power over the kinematic waves. Different limiting cases are considered depending on the relative order of the parameter W.

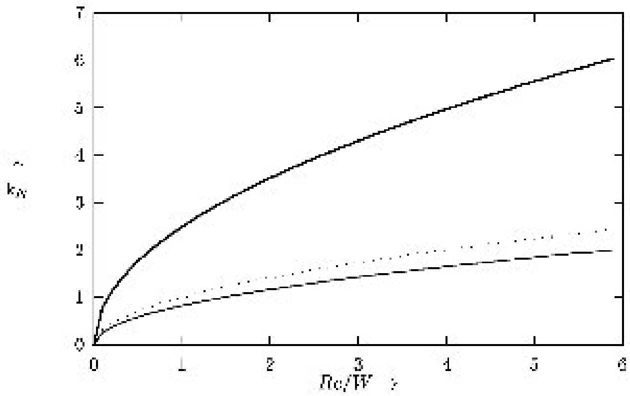

Fig. 4. Variation of neutral curves (Eq. (48)) with Re/W for different values of the power-law index n. The horizontal axis Re/W coincides with the curve $k=0$. Thin, dotted and thick lines denote $n=1.2$, 1 and 0.4, respectively.

5.1. Case-(i) : $Re \sim 1/\varepsilon^2$, $W \sim 1/\varepsilon^3$

Under this limit, Eq. (44) will be guided by the dynamic or inertial wave field. Keeping the leading order terms we have

$$(\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)H = 0. \tag{50}$$

The equivalent forms of the above equation are

$$\left(\frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x}\right)H = 0, \tag{51}$$

$$\left(\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)H = 0, \tag{52}$$

describing the propagation of the travelling waves in the mean flow direction with velocities c_1 and c_2, given in (45) above. It is clear from Eqs. (51), (52) and (45) that the wave described by (51) moves faster than the mean flow whereas the wave represented by (52) moves slower than it. Factorization of the classical wave equation shows two waves moving in opposite directions with the same velocity. The same result may be obtained if system (51)–(52) is transformed through the system of coordinates moving with velocity $(c_1+c_2)/2$ and it yields

$$\frac{\partial H}{\partial t} + \frac{(c_1-c_2)}{2}\frac{\partial H}{\partial \xi} = 0,$$

$$\frac{\partial H}{\partial t} - \frac{(c_1-c_2)}{2}\frac{\partial H}{\partial \xi} = 0,$$

where $\xi = x - (c_1+c_2)t/2$.

Following the procedure described above, the time derivatives in Eq. (44) are replaced by the relation $\partial/\partial t = -c_1\partial/\partial x$, except for, naturally, the operator $\partial/\partial t + c_1(\partial/\partial x)$. The time scale c_1 is chosen because it corresponds to the wave in the direction of shear and should be the primary disturbance. The linear equation after integrating once with respect to x gives

$$H_t + c_1H_x - n\left(\frac{1+2n}{n}\right)^n \frac{(c_0-c_1)}{(c_1-c_2)}\frac{1}{\varepsilon \mathrm{Re}}H - \left(\frac{1+2n}{n}\right)^n \frac{1}{(c_1-c_2)}\frac{\varepsilon^2 \mathrm{W}}{\mathrm{Re}}H_{xxx} = 0. \tag{53}$$

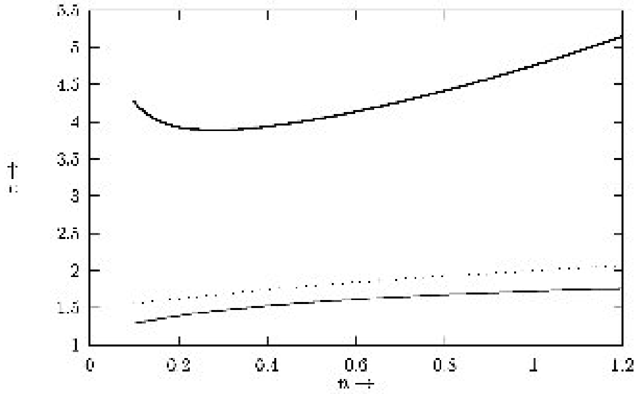

Fig. 5. Variation of phase velocity c with power-law index n for different values of Wk^2/Re (Eq. (55)). Thin, dotted and thick lines correspond to $Wk^2/\mathrm{Re} = 0.01$, 0.1 and 1, respectively.

To obtain (53), we have assumed that the amplitude of thickness perturbation $\delta \sim \varepsilon$ and we have retained the terms up to $O(\varepsilon)$.

To obtain the dispersive relation of Eq. (53), we assume that the perturbation is of the form

$$H = \delta \exp[\mathrm{i}(k\tilde{x} - \omega\tilde{t})],$$

where δ, ω and k have their usual meaning as defined earlier and we have used $x = \varepsilon\tilde{x}$ and $t = \varepsilon\tilde{t}$. Substituting into Eq. (53), we get

$$\omega_r = c_1 k + \left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}} \frac{1}{(c_1 - c_2)} k^3$$

and

$$\omega_i = n \left(\frac{1+2n}{n}\right)^n \frac{(c_0 - c_1)}{(c_1 - c_2)} \frac{1}{\mathrm{Re}}. \tag{54}$$

The phase speed becomes

$$c = c_1 + \left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}} \frac{1}{(c_1 - c_2)} k^2. \tag{55}$$

It is clear from (55) that the surface tension yields dispersion in this case. Again, it is that term which becomes prominent due to kinematic waves which is responsible for the low frequency energy pumping resulting in instability of the film flow at high Reynolds number since $\omega_i > 0$. A general comment on the wave process described by Eq. (44) can be made as follows: (i) The lower order (kinematic) waves obtain energy from the mean flow through the wave mechanism of higher order and regulate the process with small Reynolds number. (ii) The higher order (dynamic) waves dominate the mechanism with high Reynolds number and obtain energy due to kinematic wave process. (iii) The surface tension plays a double role: in the first case, it exerts dissipative effects or in other words it helps in stabilizing the flow, so that a finite amplitude case may be established, but for the later case it yields dispersion. To study the non-Newtonian effects if one observes Fig. 5, then it will be evident that phase speed c increases with n as long as $Wk^2/\mathrm{Re} \leqslant 0.1$. At this range, c increases more or less linearly with n, but for $Wk^2/\mathrm{Re} > 0.1$, c first decreases and then increases with n. It is obvious that c increases with Wk^2/Re.

5.2. Case-(ii): $Re \sim 1/\varepsilon^2$, $W \sim 1/\varepsilon^4$

At this order the approximate Eq. (44) will reduce to the form

$$(\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)H + \left(\frac{1+2n}{n}\right)^n \frac{\varepsilon^2 \mathrm{W}}{\mathrm{Re}} H_{xxxx} = 0. \tag{56}$$

In order to get dispersive relation for (56), we assume

$$H = \delta \exp[\mathrm{i}(k\tilde{x} - \omega\tilde{t})],$$

where $\omega(=\omega_r + \mathrm{i}\omega_i)$ is the complex wave speed, the amplitude δ is real; with $x = \varepsilon\tilde{x}$ and $t = \varepsilon\tilde{t}$ the final relation becomes

$$\omega^2 - 2\beta\omega k + \beta k^2 - \left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}} k^4 = 0. \tag{57}$$

Equating real and imaginary parts, we have

$$\omega_i(\omega_r - \beta k) = 0, \tag{58}$$

$$\omega_r^2 - \omega_i^2 - 2\beta\omega_r k + \beta k^2 - \left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}} k^4 = 0. \tag{59}$$

It can be shown that $\omega_r = \beta k$ will lead to a contradiction that ω_i is real, hence

$$\omega_i = 0,$$

and

$$\omega_r = \beta k \pm k\sqrt{\beta^2 - \beta + \left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}} k^2}.$$

The phase speed becomes

$$c = \frac{2(1+2n)}{(2+3n)} \pm \frac{\sqrt{2n(1+2n)}}{(2+3n)} \left[1 + \frac{1}{2}\left(\frac{2+3n}{n}\right)^2 \left(\frac{1+2n}{n}\right)^{n-1} \frac{\mathrm{W}}{\mathrm{Re}} k^2\right]^{1/2}. \tag{60}$$

For $k \sim 10\varepsilon$, it can be shown that

$$\frac{1}{2}\left(\frac{2+3n}{n}\right)^2 \left(\frac{1+2n}{n}\right)^{n-1} \frac{\mathrm{W}}{\mathrm{Re}} k^2 \sim 10^2 \frac{1}{2}\left(\frac{2+3n}{n}\right)^2 \left(\frac{1+2n}{n}\right)^{n-1} \gg 1$$

for all practical values of n. Hence, by neglecting the unity in the radical expression of (60), one can get

$$c = \beta \pm k\sqrt{\left(\frac{1+2n}{n}\right)^n \frac{\mathrm{W}}{\mathrm{Re}}}. \tag{61}$$

For a Newtonian fluid ($n = 1$), Eq. (61) reduces to

$$c = 1.2 \pm k\sqrt{3\mathrm{W}/\mathrm{Re}},$$

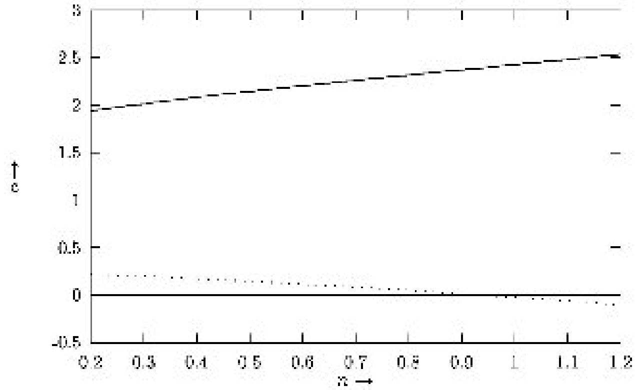

Fig. 6. Variation of the phase speed c with power-law index n (Eq. (61)). Taking $\mathrm{W}k^2/\mathrm{Re}=0.5$.

which coincides with Eq. (36) of Alekseenko et al. [20]. It is evident from Fig. 6 that the phase speed c changes with n.

6. Case-III: moderate flow rate: Re $\sim 1/\epsilon$, W $\sim 1/\epsilon^3$

Under the above approximation, the linearized form of Eq. (44) reduces to

$$(\partial_t + c_0\partial_x)H + \frac{\varepsilon\,\mathrm{Re}}{n}\left(\frac{n}{1+2n}\right)^n(\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)H + \frac{\varepsilon^3\mathrm{W}}{n}H_{xxxx} = 0. \tag{62}$$

To study the stability of this film flow on the basis of two-wave equation (62), introduce the time varying perturbations of the film height

$$H = \delta\exp[ik(\tilde{x} - c\tilde{t}) + \lambda\tilde{t}]. \tag{63}$$

Here $\tilde{x} = x/\varepsilon$, $\tilde{t} = t/\varepsilon$, k is the real wave number, c is the real part of the phase velocity and λ is the temporal growth (the imaginary part of the frequency). Using (63) in (62), we have, after equating real and imaginary parts of the dispersion relation, that

$$\lambda\,\mathrm{Re} = -\frac{n}{2}\left(\frac{1+2n}{n}\right)^n\frac{c-c_0}{c-\beta}, \tag{64}$$

$$\lambda + \left(\frac{n}{1+2n}\right)^n[\lambda^2 - (c^2 - 2\beta c + \beta)k^2]\frac{\mathrm{Re}}{n} + \frac{\mathrm{W}}{n}k^4 = 0. \tag{65}$$

Elimination of λ from (65) by using (64) gives a quadratic relation with respect to $(k\,\mathrm{Re})^2$ as

$$(k\,\mathrm{Re})^4 - \left(\frac{n}{1+2n}\right)n\frac{\mathrm{Re}^3}{\mathrm{W}}[(c-c_1)(c-c_2)](k\,\mathrm{Re})^2$$
$$-\frac{\mathrm{Re}^3}{\mathrm{W}}\left(\frac{n}{2}\right)^2\left(\frac{1+2n}{n}\right)^n\left(\frac{c-c_0}{c-\beta}\right)\left(\frac{c+c_0-2\beta}{c-\beta}\right) = 0. \tag{66}$$

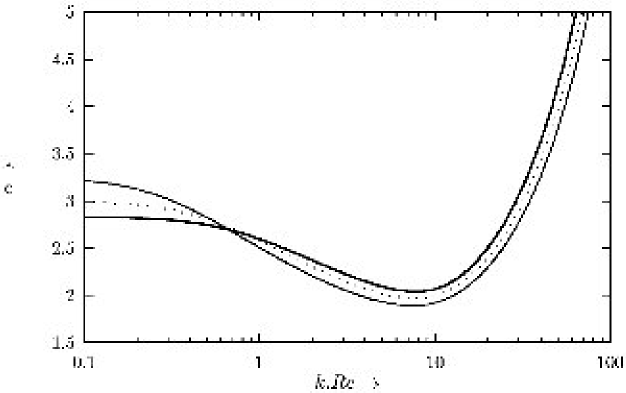

Fig. 7. Variation of dispersion curve for different values of n keeping $\mathrm{W/Re}^3 = 0.001$. Thick, dotted and thin lines represent $n = 1.2$, 1 and 0.8, respectively.

Solution of Eq. (66) reduces to

$$(\mathrm{Re}\,k)^2 = \frac{1}{2}\left(\frac{n}{1+2n}\right)^n \frac{\mathrm{Re}^3}{\mathrm{W}}(c-c_1)(c-c_2)$$

$$\left[1 \pm \sqrt{1 + \frac{\mathrm{W}}{\mathrm{Re}^3} n^2 \left(\frac{1+2n}{n}\right)^{3n} \frac{(c-c_0)(c+c_0-2\beta)}{(c-c_1)^2(c-c_2)^2(c-\beta)^2}}\,\right]. \tag{67}$$

From relations (64) and (67) we can find that on the neutral curve ($\lambda = 0$), the phase velocity is

$$c = c_0,$$

which gives

$$k = 0,$$

$$k = \left[\frac{1}{n}\left(\frac{1+2n}{n}\right)^{1-n} (\mathrm{Re/W})\right]^{1/2}. \tag{68}$$

It is clear from Eq. (64) that the perturbations will decay as long as $c > c_0$. But for the growth of perturbations the phase velocity c must lie in $\beta < c < c_0$. It is clear from the Fig. 7 that as k.Re increases, the phase speed decreases continuously and reaches a minimum and then increases with k.Re. This trend of variation between c and k.Re remains for different values of $\mathrm{W/Re}^3$ in Fig. 8. A careful scrutiny of Figs. 3 and 7 will show that the unstable range ($c < c_0$) along k.Re direction increases as the power-law index n decreases. This result is evident in Figs. 9 and 10, in which λ the growth rate parameter increases to reach a maximum and then decreases as k.Re increases. Further, it can be seen from Fig. 9 that the unstable area increases as n decreases.This shows that the temporal growth rate λ also strongly depends on the non-Newtonian character n of the fluid.

Hence, the fastest growing wave line will intersect the dispersion curve at the points of minimum phase velocities. To accurately determine the characteristics of the fastest growing waves, let us consider Eqs. (64)–(65). We rewrite Eq. (64) as

$$c = \beta + \frac{c_0 - \beta}{\phi}, \tag{69}$$

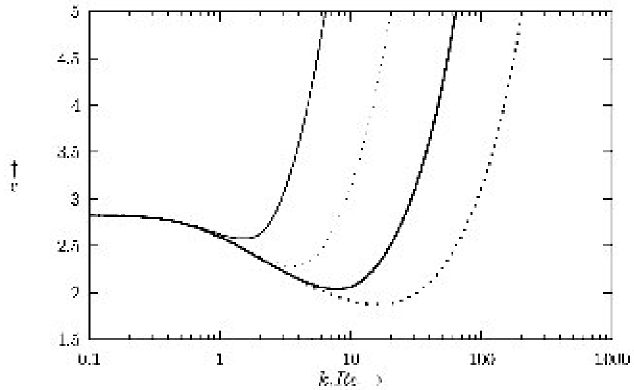

Fig. 8. Variation of dispersion curve with the parameter W/Re^3, for $n=1.2$. Thin solid, faint dotted, thick solid and bold dotted lines represent $W/Re^3=0.1$, 0.01, 0.001 and 0.0001, respectively.

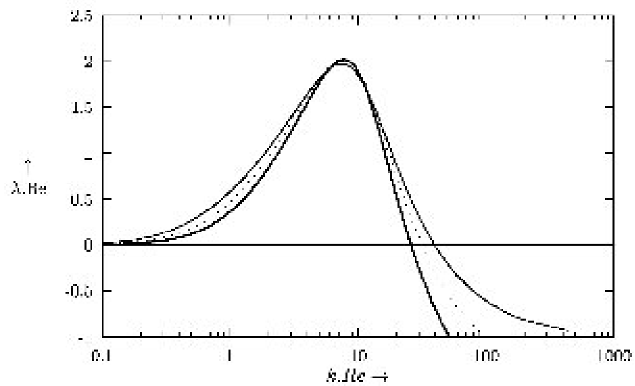

Fig. 9. Variation of temporal growth rate of a film for different values of n keeping $W/Re^3=0.001$. Thick, dotted, and thin lines represent for $n=1.2$, 1 and 0.8, respectively.

where

$$\phi = 1 + \left(\frac{2}{n}\right)\left(\frac{n}{1+2n}\right)^n (\lambda\,\mathrm{Re}).$$

Substituting (69) into (65) and differentiating the expression with respect to k and allowing for the extremum condition $\partial\phi/\partial k = 0$, we have

$$k\,\mathrm{Re} = \sqrt[4]{\left(\frac{n}{2}\right)^2\left(\frac{1+2n}{n}\right)^n\left(\frac{\mathrm{Re}^3}{\mathrm{W}}\right)(\phi^2-1)}. \tag{70}$$

Finally, after substituting (69) and (70) into (65) we obtain

$$\frac{\mathrm{Re}^3}{\mathrm{W}} = \left(\frac{n}{2}\right)^2\left(\frac{2+3n}{n}\right)^4\left(\frac{1+2n}{n}\right)^{3n-2}\frac{\phi^4(\phi^2-1)}{[\phi^2-\frac{1}{2}((1+2n)/n)((2+n)/n)^2]^2}. \tag{71}$$

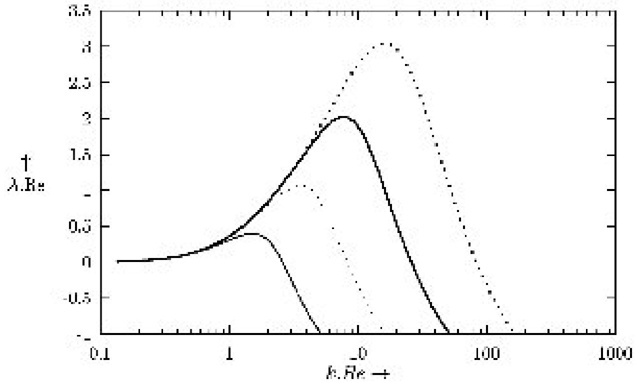

Fig. 10. Variation of temporal growth rate of a film with the parameter W/Re^3, for $n = 1.2$. Thin solid, faint dotted, thick solid and bold dotted lines represent $W/Re^3 = 0.1$, 0.01, 0.001 and 0.0001, respectively.

The increment of the maximum growth rate for waves becomes

$$\lambda = \frac{n}{2}\left(\frac{1+2n}{n}\right)^n (\phi - 1)/\mathrm{Re}. \tag{72}$$

By using Gaster's relation it is possible to determine the spatial growth rate χ from the temporal growth rate as

$$-\chi = \frac{\lambda}{c + k\partial c/\partial k} = \frac{\lambda}{c}, \tag{73}$$

taking into account $\partial c/\partial k = 0$ for the fastest growing waves.

It may be interesting to look into the grouping Re^3/W considered in studying the dispersion relation and subsequent analysis in moderate flow rate case. A close scrutiny will show that

$$\frac{\mathrm{Re}^3}{\mathrm{W}} = \left(\frac{1+2n}{n}\right)^{(2n(2-n))/(2+n)} (\mathrm{Re}/\mathrm{Fi}^{1/(6+5n)})^{(6+5n)/(2+n)},$$

where

$$\mathrm{Fi} = \sigma^{2+n}/(\rho^{2+n}v_n^4 g^{3n-2}) = \left(\frac{1+2n}{n}\right)^{2n(2-n)} \mathrm{W}^{2+n}\mathrm{Re}^{2n}$$

is the Film number of the fluid, which depends only on the fluid property and $v_n = \mu_n/\rho$ is the kinematic viscosity of the fluid. Hence the grouping is represented by the flow condition only.

7. Conclusion

In this section, we shall summarize some of the results of this study. We have analysed the waves that occur at the surface of a vertical falling thin power-law fluid film. To do this, we have derived an evolution equation representing two waves equations under long wave approximations. Based on the different ranges of the physical parameters, it is shown that different types of waves are possible on the surface of the film. Further, it is found that the result of the interaction of these different types of waves are either the exchange of energy or dispersion among them. For example, at a small flow rate, kinematic waves dominate the flow field and the energy is acquired from the mean flow during interaction of the waves, while for high flow rate, inertial waves dominate and the energy comes from the kinematic waves. It is also shown that this exchange of energy between kinematic and inertial waves strongly depends on the non-Newtonian character n for power-law fluid. Further, in both the cases, surface tension plays a double role: for a kinematic wave process, it exerts dissipative effects so that a finite amplitude case may be established, but for a dynamic wave process it yields dispersion. It should be pointed out here that the degree of non-Newtonian character n also plays a vital role in controlling the role of the term that contains surface tension in the above process. We therefore, summarize on the basis of the above analysis that the waves that occur on the surface of a vertical falling film of power-law fluid under long wave approximation are a result of nonlinear interaction between kinematic and inertial/dynamic waves and these wave characteristics strongly depend on the power-law index n.

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