COROLLARY 3.3

Conjecture 3 is true for all positive integer n of the form $n = 2^a 3^b 5^c 7^d$ *for all* $(a, b, c, d) \in \mathbf{N}^4 \backslash \{(0, 0, 0, 0)\}$.

Remark 3.4. It must be noted that there are sequences of length $4n - 4$ in $\mathbf{Z}_n \oplus \mathbf{Z}_n$ which are made up of four distinct elements repeated $n - 1$ times each which may contain a zero-sum subsequence of length n. In other words, the candidates appearing in the conclusion of Conjecture 3 are somewhat restricted. For example, if $(0, 0)$, (a, b), $(-a, -b)$ are three of the four elements, there is always a zero-sum sequence of length n. Similarly, if $n = 5$, the elements $(0, 2)$, $(2, 0)$, $(1, 1)$ occurring four times each gives a zero-sum subsequence of length 5.

4. Zero subsequences of length n in $\mathbf{Z}_n \oplus \mathbf{Z}_n$

In this section, we shall prove results about sequences in $\mathbf{Z}_n \oplus \mathbf{Z}_n$ which must contain a zero subsequence of length n. In particular, we obtain some results pertaining to Conjecture 2 of Kemnitz for the group $\mathbf{Z}_p \oplus \mathbf{Z}_p$.

It is trivial to see that if the conjecture holds good for two integers m and n, it is also true for mn. So, if one proves it for all primes, then it holds good for all natural numbers. For our convenience, instead of writing $f(\mathbf{Z}_p \oplus \mathbf{Z}_p)$, we write simply $f(p)$.

Harborth [12] considered a function $g(n)$ which is related to $f(n)$. To define $g(n)$, let us define an element $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ to be square-free, if a_i's are pairwise distinct in $\mathbf{Z}_n \oplus \mathbf{Z}_n$. Then $g(n)$ is defined to be the least positive integer such that given any square-free $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ contains a zero subsequence of length n. Harborth proved that $g(3) = 5$ and used this to prove $f(3) = 9$. Then Kemnitz [13] utilized the special values of $g(p) = 2p - 1$ for $p = 5, 7$ to prove $f(p) = 4p - 3$ for $p = 5, 7$. A bound known for all primes p is, due to Kemnitz [13]:

$$2p - 1 \leq g(p) \leq 4p - 3.$$

We shall prove on the one hand that the lower bound $2p - 1$ is tight for many classes of sequences and, on the other hand, we improve the upper bound for many classes of sequences. In 1996, Gao [7] proved that if $f(n) = 4n - 3$ and $n \geq ((3m - 4)(m - 1)m^2 + 3)/4m$ with $m \geq 2$, then $f(nm) = 4nm - 3$. These results were improved upon by the second author of this paper in [17] where it has, in fact, been proved that if $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ with $|S| = 4n - 3$ and $T = a^s$ as its subsequence with $s \geq \lfloor n/2 \rfloor$, then S satisfies Conjecture 2 and that if $f(n) = 4n - 3$ and $n > (2m^3 - 3m^2 + 3)/4m$, with $m \geq 2$, then $f(nm) = 4nm - 3$. In 1995, Alon and Dubiner [1] gave the upper bound $f(n) \leq 6n - 5$ for all $n \in \mathbf{N}$. Later this was improved upon for all primes to $f(p) \leq 5p - 1$ by Gao [8]. In 2000, Rónyai [14] proved that $f(p) \leq 4p - 2$ for all primes p. From this bound, he concluded that $f(n) \leq (41/10)n$. Recently, Gao [11] has proved that $f(p^k) \leq 4p^k - 2$ for all primes p and $k \geq 1$. Many of these proofs use graph theory and are quite different from our methods.

We start with the observation:

Lemma 4.1. *If* $S \in \mathcal{F}(\mathbf{Z}_p \oplus \mathbf{Z}_p)$ *with* $|S| = 4p - 3$ *such that there is no zero subsequence* T *of* S *with* $|T| = 2p$, *then* S *must contain a zero subsequence of length* p, *i.e.,* S *satisfies Conjecture 2.*

Proof. The proof follows by putting $d = 2$ in Theorem 2.1(b) and applying Proposition 2.4.

PROPOSITION 4.2

(a) *Let k be an integer such that $0 \leq k \leq \lfloor n/2 \rfloor$. Let $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ with $|S| = 4n - 3$. Suppose $T = a^{n-1-k}$ is a subsequence of S for some $a \in \mathbf{Z}_n \oplus \mathbf{Z}_n$. Then there exists a zero subsequence R of S with $|R| = n$.*

(b) *Let ℓ and k be two integers such that $0 \leq \ell < k \leq \lfloor n/2 \rfloor$. Let $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ with $|S| = 4n - 3 - \ell$. Suppose $T = (0, 0)^{n-k}$ is a subsequence of S. Then S contains a zero subsequence R with $n - \ell \leq |R| \leq n$.*

Proof of (a). Without loss of generality we can assume that $T = (0, 0)^{n-1-k}$. Let $S^* = ST^{-1}$ be the subsequence of S. Clearly $|S^*| = 4n - 3 - n + 1 + k = 3n - 2 + k$. By Proposition 2.3, there exists a zero subsequence U of S^* with $k + 1 \leq |U| \leq n$. Thus there exists a zero subsequence R of TU with $|R| = n$.

Proof of (b). Let $S^* = ST^{-1}$ be the subsequence of S with $|S^*| = 4n - 3 - \ell - n + k = 3n - 2 + (k - \ell - 1)$. Therefore by Proposition 2.3, there exists a zero subsequence T_1 of S^* with $k - \ell \leq |T_1| \leq n$. Therefore there exists a zero subsequence R of TT_1 with $n - \ell \leq |R| \leq n$.

Remark 4.3. One can prove that if $f(n) = 4n - 3$ and $n \geq (3m^3 - m^2 + 6)/8m$ for some positive integer m, then $f(nm) = 4nm - 3$. The proof of this is quite similar to the corresponding result proved in [17], except that one uses $f(n) \leq (41/10)n$ instead of $f(n) \leq 5n - 4$.

Here is a result about the group $\mathbf{Z}_m \oplus \mathbf{Z}_n$.

PROPOSITION 4.4

Let $S \in \mathcal{F}(\mathbf{Z}_m \oplus \mathbf{Z}_n)$ with $|S| = 2n + (21/10)m$ where $m|n$. Then S contains a zero subsequence of length n.

Proof. Since $2n + (21/10)m = (2n/m - 2)m + (41/10)m$ and we know $f(m) \leq (41/10)m$, we can extract $2n/m - 1$ disjoint subsequences $S_1, S_2, \ldots, S_{2n/m-1}$ of S with length m whose sum is zero in $\mathbf{Z}_m \oplus \mathbf{Z}_m$. Since we have the following exact sequence

$$0 \longrightarrow \mathbf{Z}_{n/m} \longrightarrow \mathbf{Z}_m \oplus \mathbf{Z}_n \longrightarrow \mathbf{Z}_m \oplus \mathbf{Z}_m \longrightarrow 0$$

and by the E–G–Z theorem (Corollary 2.2(a) here), we know there is a subsequence of the sequence $\{s_i\}_{i=1}^{2n/m-1}$ of length n/m where $s_i \in \mathbf{Z}_{n/m}$ such that $s_i := 1/m \sum_{j=1, a_{ij} \in S_i}^{m} a_{ij}$ under the exact sequence. Let $s_1, s_2, \ldots, s_{n/m}$ be the zero subsequence of $\{s_i\}_{i=1}^{2n/m-1}$ of length n/m. This means

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} = 0$$

in $\mathbf{Z}_m \oplus \mathbf{Z}_n$ where $a_{ij} \in S_i$ for $j = 1, 2, \ldots, m$ and for $i = 1, 2, \ldots, n/m$.

Remark 4.5. If $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ is square free with $|S| = 2n - 1$, then all the first (or second) co-ordinates of the a_j's cannot be distinct in $\mathbf{Z}_n$. Also, none of the first (second) co-ordinates can be repeated more than n times, since the corresponding second (first) co-ordinates run through 0 to $n - 1$. If n is odd and, one of the first (second) co-ordinate repeats exactly n times, then the corresponding second (first) co-ordinate runs through 0 to $n - 1$ and we pick up those a_j in S to produce a zero subsequence of length n. Hence we can always assume that if n is odd, then, in any such sequence, a single residue class modulo n is repeated at most $n - 1$ times among the first (second) co-ordinates.

Now, we can prove two qualitative results both of which exemplify the tightness of the lower bound $g(p) \geq 2p - 1$.

PROPOSITION 4.6

(a) *Let n be a prime and let $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| = 2n - 1$. Suppose the first co-ordinates of the a_j's run through all the different n residue classes modulo n such that $n - 1$ different residue classes modulo n are repeated exactly twice. Then there exists a zero subsequence T of S with $|T| = n$.*

(b) *Let n be a prime and let $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| = 2n - 1$. Suppose the first co-ordinates of the a_j run through three distinct residue classes modulo n such that two of the residue classes repeat $n - 1$ times. Then there exists a zero subsequence T of S with $|T| = n$.*

The following lemma will be used in the proof of (a) as well as later in the proof of Proposition 4.9.

Lemma 4.7. *Let n be a prime and let $S = \prod_i a_j \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| = 2n - 1$. Let $a_i = (x_i, y_i)$ and $a_{i+n-1} = (x_i, z_i)$ for $i = 1, 2, \ldots n - 1$ where $y_i \not\equiv z_i \pmod{n}$ for all i and $a_{2n-1} = (b, c)$. If $x_1 + x_2 + \cdots + x_{n-1} + b \equiv 0 \pmod{n}$, then, there exists a zero subsequence T of S with $|T| = n$.*

Proof. Let $K \equiv y_1 + y_2 + \cdots + y_{n-1} + c \pmod{n}$ and $e_\ell = z_\ell - y_\ell \pmod{n}$ for all $\ell = 1, 2, \ldots, n - 1$. Clearly, $e_\ell \not\equiv 0 \pmod{n}$ because $y_i \not\equiv z_i \pmod{n}$ for all i. If we form all the partial sums of e_ℓ's we get all the distinct residue classes modulo n (This can be done by simple induction, see for instance [6]). Therefore, there exists a positive integer m such that $K + e_{i_1} + e_{i_2} + \cdots + e_{i_m} \equiv 0 \pmod{n}$ which implies

$$\begin{aligned} &y_1 + \cdots + y_{i_1 - 1} + z_{i_1} + y_{i_1 + 1} + \cdots + y_{i_m - 1} + z_{i_m} \\ &\quad + y_{i_m + 1} + \cdots + y_{n-1} + c \equiv 0 \pmod{n}. \end{aligned}$$

Then, the following subsequence of S

$$(x_1, y_1), \ldots, (x_{i_1 - 1}, y_{i_1 - 1}), (x_{i_1}, z_{i_1}), (x_{i_1 + 1}, y_{i_1 + 1}), \ldots, (x_{n-1}, y_{n-1}), (b, c)$$

produces the required zero subsequence of length n

Proof of Proposition 4.6(a). Let $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be the given square-free element satisfying the hypothesis. Let us list the elements of S as follows:

$$a_i = (x_i, y_i) \quad \text{for all } i = 1, 2, \ldots, n - 1$$

and

$$a_{i+n-1} = (x_i, z_i) \quad \text{for all } i = 1, 2, \ldots, n-1$$

where $z_i \not\equiv y_i \pmod{n}$ for all $i = 1, 2, \ldots, n-1$ and $x_i \not\equiv x_j \pmod{n}$ for every $i \neq j$. Also, let $a_{2n-1} = (b, c)$ such that $b \not\equiv x_i \pmod{n}$ for every $i = 1, 2, \ldots, n-1$. Clearly, we have a zero-sum of length n as follows:

$$x_1 + x_2 + \cdots + x_{n-1} + b \equiv 0 \pmod{n}.$$

Now, the result follows from lemma 4.7.

Proof of (b). Let $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| = 2n-1$ satisfying the hypothesis. We shall list the elements of S in the following manner. Let

$$a_i = (x, y_i) \quad \text{for } i = 1, 2, \ldots, n-1 \quad \text{where } y_i \not\equiv y_j \pmod{n}$$

and

$$a_{i+n-1} = (y, z_i) \quad \text{for } i = 1, 2, \cdots, n-1 \quad \text{where } z_i \not\equiv z_j \pmod{n}$$

and $x \not\equiv y \pmod{n}$. Also, we let $a_{2n-1} = (b, c)$ where $b \not\equiv x \pmod{n}$ and $b \not\equiv y \pmod{n}$. Consider $R = x^{n-1}y^{n-1}b \in \mathcal{F}(\mathbf{Z}_n)$ with $|R| = 2n-1$. Therefore, by the Erdős–Ginzburg–Ziv theorem, there exists a zero subsequence T_1 of R with $|T_1| = n$. Clearly, b appears in T_1. Thus, we have, $T_1 = x^m y^\ell b \in \mathcal{F}(\mathbf{Z}_n)$ such that $\ell + m + 1 = n$ where $\ell, m \geq 1$.

Suppose $\{y_i\}_{i=1}^{n-1}$ and $\{z_i\}_{i=1}^{n-1}$ miss r and s residue classes modulo n respectively. If $r \equiv s \equiv c \pmod{n}$, then we can choose, by relabeling indices, $y_1, y_2, \ldots, y_\ell, z_1, z_2, \ldots, z_m$ such that $y_i \not\equiv z_j \pmod{n}$ for all $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, m$. We are in the following situation:

$$(x, y_1), \ldots, (x, y_\ell), (y, z_1), \ldots, (y, z_m), (b, c)$$

such that its sum is zero modulo n, since $y_1, \ldots, y_\ell, z_1, \ldots z_m, c$ runs through all distinct residue modulo n.

If $r \not\equiv s \pmod{n}$, then we can choose $y_1, \ldots, y_\ell, z_1, \ldots z_m, c$ runs through all distinct residue modulo n. Therefore again we can produce a zero-sum subsequence of S of length n.

If $r \equiv s \not\equiv c \pmod{n}$, then we do the following. Let $r \equiv s \equiv a \pmod{n}$. Let us take

$$\mathbf{Z}_n = \{0, 1, 2, \ldots, a-1, a, a+1, \ldots, \ell, \ell+1, \ldots, c-1, c, \ldots, n-1\}.$$

Then we choose the sequences

$$\{y_i\}_{i=1}^{\ell} : 0, 2, 3, \ldots, a-1, a+1, a+2, \ldots, \ell+1$$

and

$$\{z_j\}_{j=1}^{m} : a+1, \ell+2, \ell+3, \ldots, c-1, c+1, \ldots, n-2, n-1.$$

Then we see that

$$y_1 + y_2 + \cdots + y_\ell + z_1 + z_2 + \cdots + z_m + c \equiv 0 \pmod{n}.$$

Thus, we have the following zero subsequence T of S of length n

$$(x, y_1), (x, y_2), \ldots, (x, y_\ell), (y, z_1), \ldots, (y, z_m), (b, c)$$

in $\mathbf{Z}_n \oplus \mathbf{Z}_n$.

Our last two results go to indicate that the upper bound $g(p) \leq 4p - 3$ can be strengthened in some cases. In the proof, we shall need to use the so-called:

Cauchy–Davenport Inequality. Let A and B be two nonempty subsets of $\mathbf{Z}_p$. If we denote the cardinality of A by $|A|$ and of B by $|B|$, then

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

where $A + B$ stands for the sum-set of these two subsets.

An induction argument easily gives: *If $A_1, A_2, \ldots, A_h$ are nonempty subsets of $\mathbf{Z}_p$, then*

$$|A_1 + A_2 + \cdots + A_h| \geq \min(p, \sum_{i=1}^{h} |A_i| - h + 1).$$

Remark 4.8. Let $S \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| > 3n - 3$. We know that if n is odd and S does not contain a zero subsequence of length n, then no single residue class can occur as the first co-ordinate more than $n - 1$ times. Therefore, the first co-ordinates of the elements of S run through at least four distinct residue classes modulo n in such a case.

PROPOSITION 4.9

Let s be an integer such that $4 \leq s \leq p$. Let $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_p \oplus \mathbf{Z}_p)$ be a square-free element with $|S| = 4p - 2 - s$. Assume that the first co-ordinates of the a_j's run through exactly s different residue classes modulo p and that each different residue class modulo p repeats an odd number of times. Then there is a zero subsequence T of S with $|T| = p$.

Proof. Let $S = \prod_j a_j \in \mathcal{F}(\mathbf{Z}_p \oplus \mathbf{Z}_p)$ be the given element satisfying the hypothesis. By hypothesis, the first co-ordinates of the elements a_j run through s different residue classes modulo p and each of these residue classes repeats an odd number of times. Some of the residues may appear only once. The number of such residues is at most s. Now, let us list the elements of S as follows if necessary by relabeling the indices

$$a_i = (b_i, c_i) \quad \text{for } i = 1, 2, \ldots, s$$

where $b_i \not\equiv b_j \pmod{p}$ for $i \neq j$. Also among the b_i's we put those residues which appear only once in S. Therefore the remaining residues will be appearing as pairs. So, let

$$a_{i+s} = (x_i, y_i) \quad \text{for } i = 1, 2, \ldots, 2p - 1 - s$$

and

$$a_{i+2p-1} = (x_i, z_i) \quad \text{for } i = 1, 2, \ldots, 2p - 1 - s$$

where $y_i \not\equiv z_i \pmod{p}$ for all $i = 1, 2, \ldots, 2p - 1 - s$. This kind of listing is possible because of the assumption on the first co-ordinates of the elements $a_i \in \mathbf{Z}_p \oplus \mathbf{Z}_p$.

Now we partition the x_i; $i = 1, 2, \ldots, 2p-1-s$ into nonempty classes $A_1, A_2, \ldots, A_{p-1}$ such that each A_i consists of different residues modulo p. This is possible because no single residue class can be repeated more than $p-1$ times. Set

$$A_p = \{b_1, b_2, \ldots, b_s\}.$$

Clearly $A_i \subset \mathbf{Z}_p$ for $i = 1, 2, \ldots, p$. Consider the sum $A_1 + A_2 + \cdots + A_p$. Cauchy–Davenport inequality implies now that

$$|A_1 + \cdots + A_p| \geq \min\left(p, \sum_{i=1}^{p} |A_i| - p + 1\right) = \min(p, (2p-1-s+s-p+1)) = p.$$

This means, $0 \in \mathbf{Z}_p$ can be written as sum of p elements, i.e., $x_1 + x_2 + \cdots + x_{p-1} + b_r = 0$ where $x_i \in A_i$ for $i = 1, 2, \ldots, p-1$ and $b_r \in A_p$ (Here we have relabeled the indices of x_i.)

Now we have the following situation.

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{p-1}, y_{p-1}), (b_r, c_r)$$

and

$$(x_1, z_1), (x_2, z_2), \ldots, (x_{p-1}, z_{p-1})$$

where $x_1 + x_2 + \cdots + x_{p-1} + b_r \equiv 0 \pmod p$ and $y_i \not\equiv z_i$ for all $i = 1, 2, \ldots, p-1$. An application of Lemma 4.7 now yields the result.

For general n, with an additional assumption on the first co-ordinates, we prove:

PROPOSITION 4.10

Let $0 \leq s \leq [(n-1)/2]$ be an integer. Let $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ with $|S| = 3n-2+s$ be a square-free element. Assume that the first co-ordinates of the a_j's run through $n-s$ different residue classes modulo n and each residue class occurs an odd number of times with at least $s+1$ different residue classes modulo n which are repeated at least three times. Then there exists a zero subsequence T of S with $|T| = n$.

Proof. Let $S = \prod_j a_j \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be the given square-free element satisfying the hypothesis. By our assumption, all the first co-ordinates of the a_j's appear an odd number of times as different residues modulo n. It is clear that the number of residues which appear exactly once cannot exceed n-s-3, since any residue modulo n can be repeated at most $n-1$ times. Therefore other than these residues, every other residue is repeated at least three times.

Now, let us list the elements of the given sequence S as follows, if necessary by relabeling the indices

$$a_i = (x_i, y_i) \quad \text{for } i = 1, 2, \ldots, n-1+s$$

and

$$a_{i+n-s} = (x_i, z_i) \quad \text{for } i = 1, 2, \ldots, n-1+s$$

where $y_i \not\equiv z_i \pmod n$ for all $i = 1, 2, \ldots, n-1+s$. Also,

$$a_{i+2(n-1+s)} = (b_i, c_i) \quad \text{for } i = 1, 2, \ldots, n-s$$

where $b_i \not\equiv b_j \pmod{n}$ for $i \neq j$. Any residue that is repeated only once has been put in the class of the b_i's. This kind of listing is possible because of the assumption over the first co-ordinates of the elements $a_i \in \mathbf{Z}_n \oplus \mathbf{Z}_n$.

Since $s + 1$ distinct residue classes modulo n repeat at least three times, we can take them to be $x_{n-1}, x_n, \ldots, x_{n-1+s}$. Other than these x_i's for $i = 1, 2, \ldots, n - 1 + s$, we have b_i's which run through $n - s$ different residue classes modulo n.

Let $\sum_{i=1}^{n-2} x_i + x_j = d_j$ for $j = n - 1, n, \ldots, n - 1 + s$. Since the sequence $\{-d_j\}$ of length $s + 1$ is such that $d_j \not\equiv d_k \pmod{n}$ for $j \neq k$, there exists one b_r among the b_i's such that $-d_j = b_r$ for some j, since the sequence $\{b_j\}$ cannot miss $s + 1$ different residue class modulo n. Hence we have

$$x_1 + x_2 + \cdots + x_{n-2} + x_j + b_r \equiv 0 \pmod{n}.$$

Suppose, by relabeling, we let $x_j = x_{n-1}$ for our convenience. Now we have the following situation:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (b_r, c_r)$$

and

$$(x_1, z_1), (x_2, z_2), \ldots, (x_{n-1}, z_{n-1})$$

where $x_1 + x_2 + \cdots + x_{n-1} + b_r \equiv 0 \pmod{n}$ and $y_i \not\equiv z_i \pmod{n}$ for all $i = 1, 2, \ldots, n - 1$. Once again, an application of Lemma 4.7 proves the result.

COROLLARY 4.11

Let r be an integer such that $0 \leq r \leq 3$. Let $S = \prod_i a_i \in \mathcal{F}(\mathbf{Z}_n \oplus \mathbf{Z}_n)$ be a square-free element with $|S| = 3n - 2 + r$. Suppose the first co-ordinates of a_j's run through $n - r$ different residue classes modulo n such that each residue class is repeated an odd number of times. Then there exists a zero subsequence T of S with $|T| = n$.

Proof. It is enough to prove that there exist $r + 1$ different residue classes modulo n which are repeated at least three times. Then, the corollary follows from the theorem. Since we have totally $n - r$ different residue classes modulo n, at least four different residue classes modulo n have to repeat a minimum of three times. Hence the corollary is proved.

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