A new formulation of stress–strength reliability in a regression setup☆

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Abstract

Bayesian inference on stress–strength reliability is considered when the observations are binary in nature and the covariates affecting the stress and the strength of a component are observable. The posterior is evaluated using Gibbs sampling. The method is illustrated with a data set.

Keywords: Stress–strength reliability; Prior distribution; Posterior distribution; Gibbs sampling

1. Introduction

It is well known that the strength of a manufactured unit is a random variable. This fact is the basis of all reliability modeling. In stress–strength modeling the second source of variability that needs to be considered is the stress conditions of the operating environment. If X represents the environmental stress on a unit and Y, the strength of a unit, the stress–strength reliability (R) is then defined by

$$R = P(Y > X). \tag{1.1}$$

R represents the probability that the strength exceeds the stress, in other words, the probability that a manufactured unit works satisfactorily. The statistical formulation (1.1) appears to be given first by Birnbaum (1956). The problem he considered was to find both the point estimate and an interval estimate of R on the basis of n independent observations $X_1,\ldots,X_n$ on X and m independent observations $Y_1,\ldots,Y_m$ on Y. Birnbaum used Mann–Whitney statistic to estimate R and found the confidence interval of R in nonparametric set-up following the Hodges–Lehmann approach. This

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paper opened up the flood gates and was followed by a deluge of papers (Birnbaum and McCarthy, 1958; Owen et al., 1964; Govindarajulu, 1967, 1968; Church and Harris, 1970; Enis and Geisser, 1971; Bhattacharyya and Johnson, 1974; Guttman et al., 1988; Weerahandi and Johnson, 1992; etc.), on the same theme. For an excellent review we refer to Johnson (1988).

Most of the work subsequent to Birnbaum's assume that both X and Y have some parametric distributions of known form. In most cases either the distributions are assumed to be normal or exponential or Weibull with the same shape parameter. The problem of estimating R then naturally reduces to the estimation of a function of the unknown parameters of the distributions of X and Y. In order to get an estimate of R then the data on X and Y are used. In our view, the application of stress–strength reliability analysis in real-life situation has been somewhat limited by this classical formulation of the problem. Two reasons can be ascribed. (i) This formulation implicitly assumes that the stress is identified with a single randomly varying factor or a known function of multiple randomly varying factors (e.g., mechanical, thermal and other relevant conditions of the system and the way they interact with each other) of the operating environment and it is observable and (ii) the strength of a manufactured unit is an in-built entity which can be observed only by carrying out tests in the laboratory. In many practical situations, however, the assumption (i) does not hold. Also it is often unrealistic to accept the fact that the strength is only an inbuilt entity. It does change with time and is also dependent on the operating environment.

As pointed out by Mazumdar (1970) the stress strength reliability of a unit during a given period $(0, T]$ is taken to be the probability that its strength exceeds the maximum stress during the entire interval. With this formulation in mind is it always reasonable to assume that the strength of the unit remains same over the whole interval $(0, T]$ and is independent of its past stress condition and or conditions of the operating environment? For example, consider a biological system and suppose the unit under consideration is some organ, say, the kidney and one is interested to find its stress–strength reliability during a given period of time. It is not realistic to assume then that the strength of the kidney is independent of its past stress history and its present operating environment (viz., the state of the other related biological, clinical and the physiological factors). Also the definitions of X and Y are not clear enough in this case. Thus, the question of getting independent set of observations on X and Y does not arise and hence it is not possible to find the stress–strength reliability in a situation like this using classical formulation. However, in situations like this we can think of X and Y as latent variables which are not observable but working behind the scene in tandem and giving rise to the response variable indicating whether the organ is malfunctioning or not. Even if X and Y are not observable we could observe the covariates that are supposed to be related to the latent stress variable X and the latent strength variable Y. There may be some common covariates affecting both. In this paper, we propose a Bayesian analysis using Markov Chain Monte Carlo (MCMC) method to infer about R, the stress strength reliability, when the data are available in the above form.

The paper is organized as follows. In Section 2, using the latent variable model we formulate the stress–strength reliability problem based on the kind of data

mentioned above. This formulation is important on a number of counts. First, it offers a new formulation to the classical stress–strength reliability problem. Secondly, it offers the modeling of stress–strength reliability in terms of a certain number of covariates thus making the prediction of stress–strength reliability possible in a given situation observing the values of the covariates only. Use of covariates in finding the stress–strength reliability in the classical formulation under the normality assumption of the distributions of X and Y was first considered by Bhattacharyya and Johnson (1981). Last but not the least important is the fact that this formulation certainly widens the domain of applications of the stress–strength reliability to a wider variety of problems. In Section 3, we adopt a Bayesian approach to the inference problems in stress–strength reliability for convenience. Following Albert and Chib (1993), Tanner (1996) and Gelman and Rubin (1992) we then discuss how Gibbs Sampling can be adopted to find the posterior distribution of the stress–strength reliability R. In Section 4, we present an example to illustrate the methodology described in Section 3. In Section 5, we conclude the discussion by identifying a few extensions and problems that need to be taken care of in future.

2. Formulation of the problem

In an experiment suppose n units are observed independently for a specified period of time possibly under different operating environments. The data are represented by (U_i, x_i), $i = 1, \ldots, n$ where $U_i = 1, 0$ according as the unit operates or fails and x_i represents the covariate vector that is supposed to have influence both on the stress and strength of the unit. Denoting by Y_i and X_i the strength and stress of the ith unit we see that the stress–strength reliability

$$R = P(U_i = 1) = P(Y_i - X_i > 0),$$

where we assume without loss of generality that $(Y_i, X_i) \sim BN(x_1'\beta_1, x_2'\beta_2, 1, 1, \rho)$. Then R reduces to $R = \Phi(x'\beta)$ where $x'\beta = (x_1'\beta_1 - x_2'\beta_2)/\sqrt{2(1-\rho^2)}$ with x representing the vector consisting of only the distinct components of x_1 and x_2 and β the vector of regression coefficients with proper definitional adjustments and $\Phi(y) = (1/\sqrt{2\pi}) \int_{-\infty}^{y} e^{-t^2/2} dt$. For notational convenience we write $x_i'\beta = (x_{1i}'\beta_1 - x_{2i}'\beta_2)/\sqrt{2(1-\rho^2)}$.

Let $p(\beta)$ be a prior on β, summarizing the prior information about β. The posterior density of β is then given by

$$p(\beta/data) = \left\{ p(\beta) \prod_{i=1}^{n} \Phi(x_i'\beta)^{u_i} (1 - \Phi(x_i'\beta))^{1-u_i} \right\} \Bigg/ \left\{ \int p(\beta) \prod_{i=1}^{n} \Phi(x_i'\beta)^{u_i} (1 - \Phi(x_i'\beta))^{1-u_i} d\beta \right\}.$$

This is clearly analytically intractable. The usual asymptotic approximation to this by $N_k(\hat{\beta}, I(\hat{\beta})^{-1})$ where $\hat{\beta}$ is the posterior mode and $I(\hat{\beta})$ is the Fisher information matrix evaluated at $\hat{\beta}$. If $p(\beta)$ is uniform then $\hat{\beta}$ is the MLE. Griffiths et al. (1987) observed that MLE has significant bias for small sample sizes.

Here we adopt a simulation based approach for computing the exact posterior of β using data augmentation and Gibbs sampling known as the systematic scan Gibbs sampler (see Tanner, 1996). Once we find the posterior distribution of β we can find the posterior density of R, the stress–strength reliability, and hence the point estimate and a credible interval of R.

3. Systematic scan Gibbs sampler

We implement systematic scan Gibbs sampler (Tanner, 1996) to find the posterior density of β given the data. We introduce n latent variables Z_i, $i = 1, 2, \ldots, n$ such that $U_i = 1$ if $Z_i > 0$, and $=0$, if $Z_i \leqslant 0$ where $Z_i \sim N(x_i'\beta, 1)$, $i = 1, 2, \ldots, n$ and are independent. If Z_i's were observed the posterior distribution of β takes simple form if the prior $p(\beta)$ is either multivariate normal or diffuse. Also given U_i the conditional distribution of Z_i is truncated normal. In the applications considered in this paper the computations are done with the above two choices of the prior.

The basic idea behind the implementation of systematic scan Gibbs sampler in this case is to augment the observed data $U = (U_1, U_2, \ldots, U_n)$ by $Z = (Z_1, Z_2, \ldots, Z_n)$, the latent data. This is done by drawing Z_i from the conditional distribution $p(Z_i/U, \beta)$ for $i = 1, \ldots, n$ at the first stage starting with some initial value of β. Note in this case $p(Z_i/U, \beta)$ is $n(x_i'\beta, 1)$ truncated at the left by 0 if $U_i = 1$ and truncated at right by 0 if $U_i = 0$. At the second stage, β is drawn from $p(\beta/U, Z)$ where Z is kept fixed at the values obtained at the first stage. This completes one cycle or replication of the algorithm. After a sufficiently large number of replications the generated values of β and Z are approximately a random sample from the distribution $p(\beta, Z/U)$ under certain regularity conditions.

Now given both U and Z the conditional distribution of β is of relatively simple form and is given by

$$p(\beta/U, Z) = p(\beta/Z) \propto p(\beta) \prod_{i=1}^{n} \pi(Z_i; x_i'\beta, 1), \tag{3.1}$$

where $\pi(.; \mu, \sigma^2)$ is the probability density of a normal distribution with mean μ and variance σ^2. The posterior density given in (3.1) is the same for the regression parameter β under the usual normal linear model setup with dispersion matrix equal to the identity matrix. If the prior $p(\beta)$ is diffuse, then $p(\beta/U, Z)$ is $N_k(\hat{\beta}_z, (X'X)^{-1})$, where $\hat{\beta}_z = (X'X)^{-1}(X'Z)$. If prior $p(\beta)$ is the proper conjugate, say, $N(\beta^*, \Sigma^*)$, then $p(\beta/U, Z)$ is $N_k(\tilde{\beta}, \tilde{\Sigma})$, where $\tilde{\beta} = (\Sigma^{*-1} + X'X)^{-1}(\Sigma^{*-1}\beta^* + X'Z)$ and $\tilde{\Sigma} = (\Sigma^{*-1} + X'X)^{-1}$.

To obtain an approximation to the posterior density $p(\beta/U)$, one computes the average of $p(\beta/U, Z)$ over the retained values of the imputations i.e., $\hat{p}(\beta/U) \approx$

$\frac{1}{100}\sum_{i=101}^{200} p(\beta/U,Z_i)$. Of course our ultimate interest lies in finding both the point estimate and the posterior probability interval of $R=\Phi(x^{*T}\beta)$, the stress–strength reliability, for any given value of the covariate x say, x^*. Simply applying the transformation of variable technique on the multivariate normal density $p(\beta/U,Z)$ the estimate of R comes out as $\hat{p}(R/U)=(1/100)\sum_{i=101}^{200}\phi(\Phi^{-1}(R);\mu,\sigma^2)/\phi(\Phi^{-1}(R))$, where $\phi(.;\mu,\sigma^2)$ is the probability density function of normal distribution with mean μ and variance σ^2, $\mu=x^{*\prime}\hat{\beta}$ and $\sigma^2=x^{*\prime}(X'X)^{-1}x^*$ and ϕ the normal density corresponding to mean 0 and variance 1.

4. A numerical example

As an illustrative example we consider the following data set (see Weichung and Weisberg, 1986) with certain modifications to suit our purpose. Endogenous creatinine (CR) clearance is an important measure of renal function. So also is serum creatanine concentration (SC). Low value of CR and high value of SC is associated with renal malfunction. For the purpose of illustrating the methodology we classify somewhat arbitrarily all the patients with SC greater than 1.5 and CR less than 70 as having suffering from a renal malfunction and thus for these patients we take the value of the response variable as one and for the others equal to zero. CR can be thought of as a covariate associated with the strength of the renal system and SC a covariate associated with the stress on the renal system although both are associated. The data are shown in the Table 1. We implement Gibbs sampler to find the posterior based on this data using both the diffused prior and the normal priors. For normal priors with moderately large variance the results look similar to what is obtained by diffused prior and so we have reported the results for diffused prior only. For implementing Gibbs sampler we have used BUGS (see Spigelhalter et al., 1996).

To implement and monitor the convergence of Gibbs sampler we follow the basic approach of Gelman and Rubin (1992). Starting from some initial value of β, say, β_0, we generate 4 chains of values of Z and β, each chain being generated starting from an over-dispersed distribution and with a sample size 20,000. We delete 10,000 replications as "burning" samples to minimize the effect of initial values and retain the values of the next 10,000 replications to approximate the posterior distribution. To monitor the convergence we focus our attention to R, the primary parameter of our interest.

Following Gelman and Rubin (1992), we compute the between and the within chain mean squares of the retained values of R, say, B and W, respectively. Then we find

$$s^2=(10,000-1)W/10,000+B/10,000, \quad v=s^2+((4)(10,000))^{-1}B.$$

and finally the *potential scale reduction factor* $r=v/W$. If the potential scale reduction factor is nearly 1, then this suggests that the desired convergence is achieved in the Gibbs sampler. For convergence diagnostics CODA (see Best et al., 1995) has been used. We have taken two sets of values of covariates viz (i) $SC=2.7$, $CR=40$ and (ii) $SC=1.53$, $CR=70$ and found the posterior distributions of R, say distributions of

Table 1
Data showing renal malfunction, CR, SC for 29 patients

Patient serial number	Response	SC	CR
1	0	0.71253	132.0
2	0	1.48161	53.0
3	1	2.20545	50.0
4	0	1.42505	82.0
5	0	0.67860	110.0
6	0	0.75777	100.0
7	0	1.11969	68.0
8	0	0.91611	92.0
9	1	1.54947	60.0
10	0	0.93873	94.0
11	0	0.99528	105.0
12	0	1.07445	98.0
13	0	0.70122	112.0
14	0	0.71253	125.0
15	0	0.99528	108.0
16	1	2.52212	30.0
17	0	1.13100	111.0
18	0	1.11969	130.0
19	0	1.37982	94.0
20	0	1.11969	130.0
21	0	0.97266	59.0
22	1	1.60602	38.0
23	1	1.58339	65.0
24	0	1.40244	85.0
25	0	0.67860	140.0
26	0	1.19886	80.0
27	1	2.10001	43.2
28	0	1.35719	75.0
29	0	1.05183	41.0

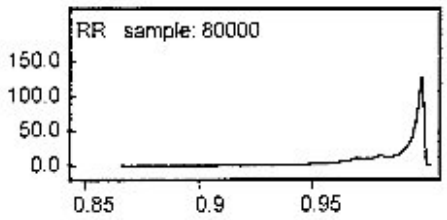

Fig. 1. Kernel density for R(i).

R(i) and R(ii). These are shown in Figs. 1 and 2, respectively. For (i) the posterior mean comes out to be 0.9983 and 95% credible interval as 0.9948–1.000. For (ii) the posterior mean is 0.49 and the 95% credible interval is 0.426–0.552.

In Figs. 3 and 4, the convergence of Gelman–Rubin shrink factor to one are shown for R(i) and R(ii), respectively. From Figs. 3 and 4 it is clearly evident that the

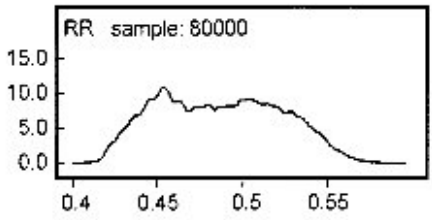

Fig. 2. Kernel density for R(ii)

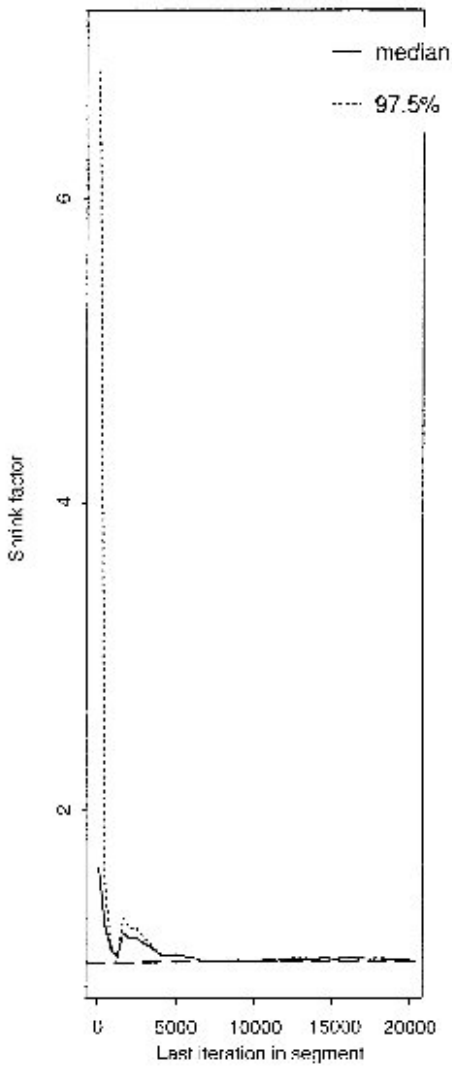

Fig. 3. Gelman–Rubin shrink factors for R(i).

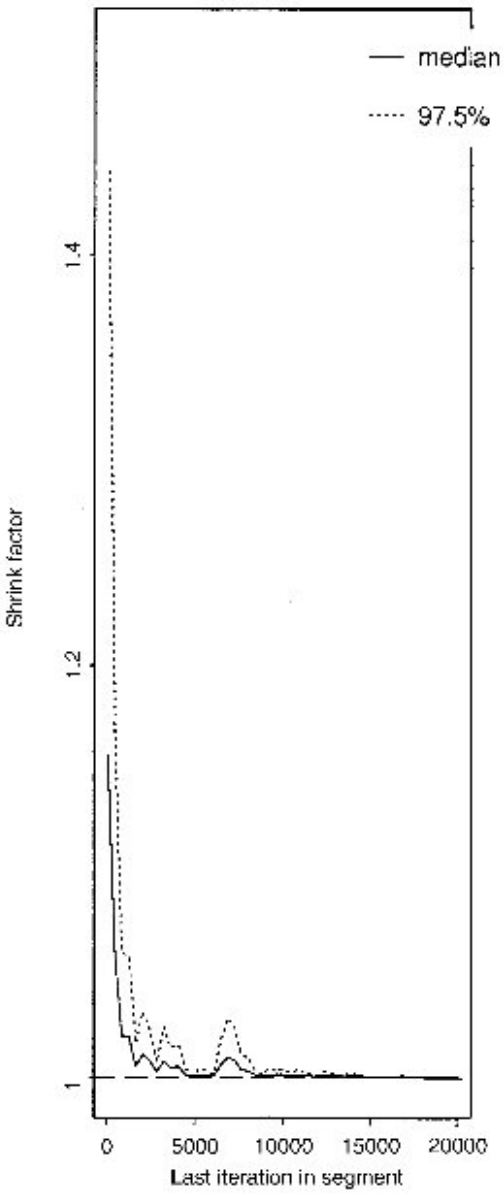

Fig. 4. Gelman–Rubin shrink factors for R(ii).

convergence is more or less achieved after 100 iterations. Also Fig. 5 clearly shows the autocorrelations between replications of the values of R in each of the four chains are very small. The results obtained are heuristically consistent. In case (i) the values of SC and CR are clearly much above and much below the threshold levels and so

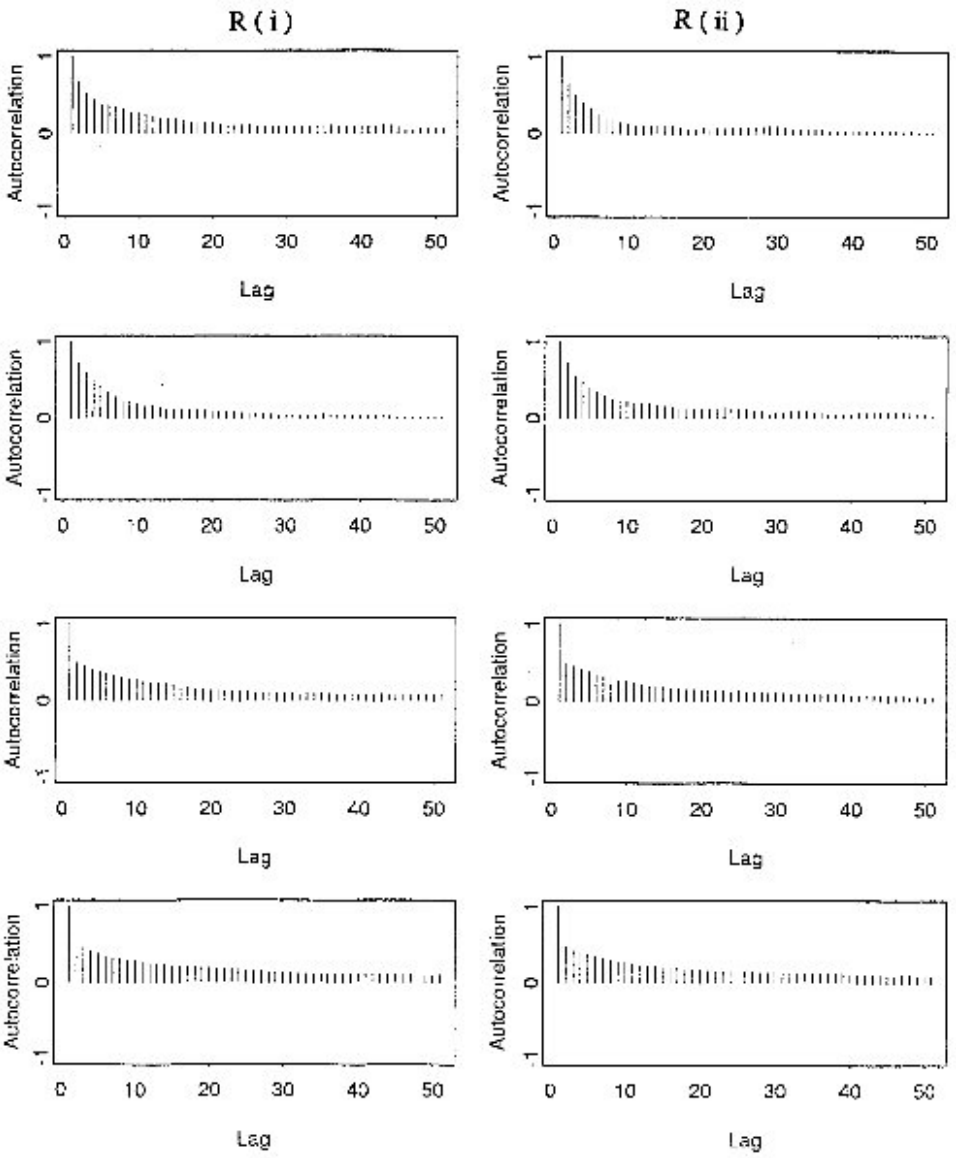

Fig. 5. Autocorrelations for four chains for both R(i) and R(ii).

the length of the credible interval is shorter compared to that in case (ii). In case (ii) the values of SC and CR are at the border lines of the threshold levels and thus one is more uncertain about the malfunction of the kidney and hence the length of the credible interval is larger.

5. Discussion

It is noted in the Introduction that the formulation of the stress–strength reliability analysis in this paper is somewhat different from the usual. To make a few additional comments, first, we would like to mention about its potential application to finding the stress strength reliability of a manufactured unit using feed back data after its marketing. When the units are marketed they are expected to operate under different environments and environment have an effect on both the stress and the strength of the unit. It is important for an industry to have feedback data on the performance of the unit marketed for further improvement of the product. The feed back data in most of the situations are not expected to provide observations on stress and strength. On the other hand one can get observations like what we have considered in this paper. The covariates may be some environmental factors affecting both stress and strength. Secondly, using this formulation there is further scope of extending the analysis to the case when a cluster effect is present without much difficulty. Cluster effect may arise in different ways. For example in the kidney malfunction data the patients with same dietary habit or belonging to the same racial group may form a cluster. In the same way the units marketed that are produced in a single factory or shipped in a single batch may form a cluster. Thirdly, one can apply the latent variable approach adopted here to the case when the stress and the strength variables are non-normal. It is our intention to take up this aspect of stress–strength reliability analysis in a future communication.

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