CRITICAL INTENSITIES OF BOOLEAN MODELS WITH DIFFERENT UNDERLYING CONVEX SHAPES

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Abstract

We consider the Poisson Boolean model of percolation where the percolating shapes are convex regions. By an enhancement argument we strengthen a result of Jonasson (2001) to show that the critical intensity of percolation in two dimensions is minimized among the class of convex shapes of unit area when the percolating shapes are triangles, and, for any other shape, the critical intensity is strictly larger than this minimum value. We also obtain a partial generalization to higher dimensions. In particular, for three dimensions, the critical intensity of percolation is minimized among the class of regular polytopes of unit volume when the percolating shapes are tetrahedrons. Moreover, for any other regular polytope, the critical intensity is strictly larger than this minimum value.

Keywords: Poisson process; Boolean model; percolation; critical intensity

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1. Introduction

The study of the percolation and geometric properties of the Boolean continuum percolation model has primarily being restricted to the model where there are balls centred at points of an underlying spatial Poisson process, see e.g. Hall (1988) and Meester and Roy (1996). Recently, Jonasson (2001) introduced the study of the critical densities of the Boolean continuum percolation model as functions of the shape of the underlying convex region at every point of the Poisson point process comprising the model. In particular, he showed that among all two-dimensional convex shapes of unit area, the triangle minimizes the critical densities. In this paper we continue this topic and first obtain an easier understanding of Jonasson's results. Based on this we show how Jonasson's results could be partly extended to higher dimensions, thereby obtaining that the tetrahedron of unit volume is the shape which minimizes the critical density among all *regular* polytopes. Moreover, we show that the critical density of the model where the underlying shape is a triangle of unit area in two dimensions (or a tetrahedron of unit volume in three dimensions) is *strictly smaller* than that of a model where the underlying shape is a convex region, barring triangles, of unit area (or a regular polytope with five or more faces of unit volume in three dimensions). This result cannot be derived by the method used by Jonasson.

Finally, the method of enhancement we have used here leads to an easy extension to obtain that, for $d \geq 2$, the critical density of percolation for a model in $\mathbb{R}^d$ with cubes of d-dimensional Lebesgue measure 1 is strictly larger than the critical density of percolation for a model in $\mathbb{R}^{d+1}$

with cubes of $(d+1)$-dimensional Lebesgue measure 1. Sarkar (1997) obtained a similar result, but only for $d = 2$, because he needed the fact that in two dimensions there is no coexistence of an occupied region and a vacant unbounded region.

2. The model and statement of results

Let $\mathbb{X} := (X_1, X_2, \dots)$ be a Poisson point process of density λ on $\mathbb{R}^d$ and S be a fixed d-dimensional bounded convex region. The region $C := \bigcup_{i=1}^{\infty}(X_i + S)$ is the *covered region* in $\mathbb{R}^d$ and the connected component C_0 of C containing the origin is the *occupied cluster* of the origin. The critical density of percolation $\lambda_c(S)$ is given by

$$\lambda_c(S) := \inf\{\lambda : \mathbf{P}_\lambda\{C_0 \text{ is unbounded}\} > 0\}.$$

For $d = 2$ and $n \geq 3$, let P_n be the regular polygon of n sides with unit area, while for $d = 3$ and $n \geq 4$, let D_n be the regular polytope of n faces with unit volume.

Proposition 2.1. (a) *For $d = 2$, we have $\lambda_c(P_3) < \lambda_c(P_n)$ for all $n \geq 4$; and* (b) *for $d = 3$, we have $\lambda_c(D_4) < \lambda_c(D_n)$ for all $n \geq 5$.*

In the proof of the above proposition we will see that while a straightforward comparison method immediately yields $\lambda_c(P_3) < \lambda_c(P_n)$ for all $n \geq 5$ (and this method also yields part (b) of the proposition), for the relation between $\lambda_c(P_3)$ and $\lambda_c(P_4)$ we need to do more work. For this we need the following result.

Theorem 2.1. *Let S and B be two d-dimensional bounded convex regions with $S \subseteq B$ and $\partial S \neq \partial B$, where ∂A denotes the boundary of the region A. We have*

$$\lambda_c(B) < \lambda_c(S).$$

The comparison method used in the proof of Proposition 2.1 allows us to compare the critical densities for general convex shapes. In particular, for two dimensions, let H_α denote the regular hexagon of area α and, for three dimensions, let I_α denote the regular icosahedron (i.e. 20 faces) of volume α. We show that $\lambda_c(P_3) = \lambda_c(H_{3/2})$ and $\lambda_c(D_4) = \lambda_c(I_{5/2})$. We may view this result *vis-à-vis* the covered volume fraction question of Meester *et al.* (1994). There it was shown that the critical covered area/volume fraction is not a constant when the underlying shape is not fixed, but random. Here our results show that the critical covered area/volume fraction among different shapes is not a constant, rather it is minimized by these triangles/tetrahedrons. Indeed, the critical covered area/volume fraction for a Boolean model with fixed shapes S is given by $1 - \exp(-\lambda_c(S)\ell(S))$, where $\ell(S)$ is Lebesgue measure of S. In our case, we see that among all regular two-dimensional shapes S of unit area, the critical covered area fraction is not a constant; instead, it is minimized when S is a triangle. A similar conclusion may be drawn in three dimensions with the critical covered volume fraction being minimized by tetrahedrons among all regular three-dimensional convex shapes.

First we put $C(S) = \frac{1}{2}(S \ominus (-S))$. Note that $C(P_3) = H_{3/2}$, $C(D_4) = I_{5/2}$ and $\lambda_c(S) = \lambda_c(C(S))$. Then we have the following proposition.

Proposition 2.2. (a) *If for a two-dimensional convex bounded region S of unit area there exists an affine transformation T such that $C(TS) \subset H_{3/2}$, then $\lambda_c(P_3) < \lambda_c(S)$.*

(b) *If for a three-dimensional convex bounded region S of unit volume there exists an affine transformation T such that $C(TS) \subset I_{5/2}$, then $\lambda_c(D_4) < \lambda_c(S)$.*

The essence of the proof of Theorem 1.2 of Jonassen (2001) is that the condition in (a) above holds for all convex shapes S barring triangles. Thus we have the following theorem.

Theorem 2.2. *For any two-dimensional convex shape S of unit area, $\lambda_c(P_3) \le \lambda_c(S)$, with equality holding if and only if S is a triangle. Moreover, if the condition in Proposition 2.2(b) holds, then for any three-dimensional convex shape S of unit volume, $\lambda_c(D_4) \le \lambda_c(S)$, with equality holding if and only if S is a tetrahedron.*

In four dimensions, using the Schläfi symbol (see Coxeter (1989)) to denote four-dimensional regular polytopes, the comparison method shows that the regular polytope denoted by the Schläfi symbol $\{3, 3, 3\}$ minimizes the critical density among regular polytopes with unit four-dimensional Lebesgue measure. This could be extended to higher dimensions too.

3. The comparison argument

Let us first consider $\lambda_c(P_3)$. Let P_3 be as in Figure 1 with one vertex of it being at the origin of $\mathbb{R}^2$, one on the x-axis, and the third in the positive quadrant. Consider the Boolean model generated by the shapes $\{X_i + P_3\}_{\{i\ge 1\}}$. Given $X_i + P_3$, a shape in this model, shapes $X_j + P_3$ which have nonempty intersection with $X_i + P_3$ must satisfy the following:

$$(X_j + P_3) \cap (X_i + P_3) \neq \emptyset \iff X_j \in X_i + H_6, \tag{3.1}$$

where H_6 is the regular hexagon centred at the origin of area 6.

For the Boolean model obtained from $\{X_i + H_{3/2}\}_{i\ge 1}$, $H_{3/2}$ being the regular hexagon 'centred' at the origin and of area $\frac{3}{2}$, we observe, as in (3.1), that

$$(X_j + H_{3/2}) \cap (X_i + H_{3/2}) \neq \emptyset \iff X_j \in X_i + H_6. \tag{3.2}$$

Thus, for the processes $\{X_i + P_3\}_{\{i\ge 1\}}$ and $\{X_i + H_{3/2}\}_{\{i\ge 1\}}$, we see from (3.1) and (3.2) that

$$\begin{aligned}&\bigcup_{i=1}^{\infty}(X_i + P_3) \text{ admits an unbounded connected component}\\ &\qquad\Longrightarrow \bigcup_{i=1}^{\infty}(X_i + H_{3/2}) \text{ admits an unbounded connected component.}\end{aligned} \tag{3.3}$$

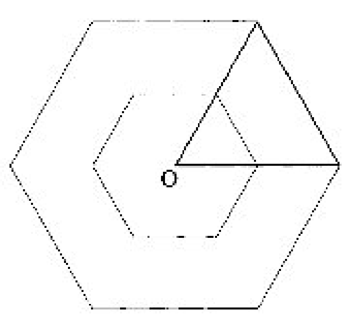

Figure 1: The finite cluster for large λ. The bold line outlines the triangle P_3. The hexagon which contains P_3 is H_6. The inner hexagon is $H_{3/2}$.

Remark 3.1. The above statement is valid for all spatial point processes $\mathbb{X}$ and not only for the Poisson point process.

From (3.3), we have that, for the Poisson Boolean model,

$$\lambda_c(P_3) = \lambda_c(H_{3/2}).$$

Now, simple calculations and geometry show that, for every $n \geq 5$, $\text{int}(H_{3/2}) \supseteq \text{cl}(P_n)$, where int and cl denote the topological interior and closure respectively. Moreover, we obtain that, for every $n \geq 5$, there exist constants $c_n > 1$ such that $\text{int}(H_{3/2}) \supseteq \text{cl}(c_n P_n)$. This observation immediately yields that

$$\lambda_c(P_3) = \lambda_c(H_{3/2}) \leq \lambda_c(c_n P_n) \quad \text{for all } n \geq 5. \tag{3.4}$$

However, the scaling properties of the Boolean model (see Meester and Roy (1996, Chapter 2)) imply that, for $\kappa > 0$, $\lambda_c(\kappa S) = \kappa^{-2}\lambda_c(S)$ for the two-dimensional Boolean model; thus (3.4) gives

$$\lambda_c(P_3) < \lambda_c(P_n) \quad \text{for all } n \geq 5.$$

For the three-dimensional case, this comparison method yields

$$\lambda_c(D_4) = \lambda_c(I_{5/2}),$$

where $I_{5/2}$ is the regular icosahedron of volume $\frac{5}{2}$. Moreover, as in the two-dimensional case, we observe that, for every $n \geq 5$, there exist constants $c_n \geq 1$ such that $\text{int}(I_{5/2}) \supseteq \text{cl}(c_n D_n)$. This, along with the three-dimensional scaling relation, gives

$$\lambda_c(D_4) = \lambda_c(I_{5/2}) < \lambda_c(D_n) \quad \text{for all } n \geq 5.$$

Proposition 2.1(b) is thus proved in its entirety, while for Proposition 2.1(a) we still need to consider the case $n = 4$.

Remark 3.2. It may be easily observed that the disc of unit area, or the sphere of unit volume, both satisfy the 'strong' set inclusion with respect to the hexagon $H_{3/2}$ or the icosahedron $I_{5/2}$. So the strict inequality between the corresponding critical intensities is obtained.

To obtain the relation between $\lambda_c(P_3)$ and $\lambda_c(P_4)$, we first note an observation made by Jonassen (2001) that the percolation properties of the Boolean model remain invariant under area-preserving (volume-preserving in three dimensions) affine transformations. Thus $\lambda_c(P_4)$ equals $\lambda_c(R)$ for any rectangle of unit area. In particular, taking R to be the rectangle of size $3^{-1/4} \times 3^{1/4}$, we see that an affine transformation $\tilde{R}$ of R is contained in $H_{3/2}$. Unfortunately, the strong set inclusion is not true, and, on the contrary, we have $\tilde{R} \subseteq H_{3/2}$, but $\partial\tilde{R} \cap \partial H_{3/2} \neq \varnothing$. Thus, although we have $\lambda_c(P_3) \leq \lambda_c(\tilde{R}) = \lambda_c(P_4)$, the absence of the scaled inclusion relation prohibits us from drawing any conclusion regarding the strict inequality.

4. An enhancement argument for Theorem 2.1

In this section we will prove Theorem 2.1. The proof will be based on an adaptation of the enhancement argument of Aizenman and Grimmett (1991) to the Boolean model. Sarkar (1997) has also used an enhancement argument, but our method is significantly different from his, in that we consider a more general enhancement technique and we do not 'discretize' the problem.

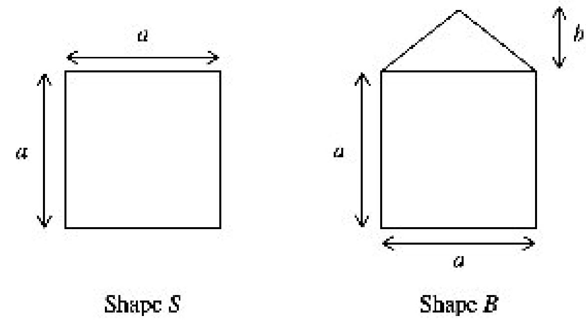

FIGURE 2: The shapes S and B.

For convenience in writing the technical details, we restrict ourselves to two dimensions, as well as taking specific shapes S and B as in Figure 2.

Shape S is a square with sides of length a, centred at the origin and B is the square S capped by an isosceles triangle of height b. This choice of shapes was made with the inequality $\lambda_c(H_{3/2}) < \lambda_c(\tilde{R})$ in mind; however, as can be seen from the proof, the argument extends to arbitrary convex shapes.

On $\mathbb{R}^2$ we construct two independent processes $(\mathbb{X}_1, \mu, S)$ and $(\mathbb{X}_2, \nu, B)$, i.e. Poisson Boolean models with underlying shapes S and B respectively. Let $\mathbb{X}$ denote the superposition of these two processes.

Let A_m be the event that $C_0 \cap ([-m, m]^2)^c \neq \varnothing$, where C_0 is the connected component containing the origin of the covered region of the plane in the superposed process $\mathbb{X}$. Let $\mathbf{P}_{\mu,\nu}$ denote the probability measure governing the superposed process $\mathbb{X}$. By Russo's formula (see Tanemura (2000)) we have

$$\frac{\partial}{\partial \mu}\mathbf{P}_{\mu,\nu}(A_m) = \mathbf{E}_{\mu,\nu}(\text{area of the } S\text{-pivotal region for } A_m), \tag{4.1}$$

$$\frac{\partial}{\partial \nu}\mathbf{P}_{\mu,\nu}(A_m) = \mathbf{E}_{\mu,\nu}(\text{area of the } B\text{-pivotal region for } A_m). \tag{4.2}$$

Here a point $\boldsymbol{x} \in \mathbb{R}^2$ is said to be S-pivotal for (ω, A_m) for a configuration ω if $\omega \notin A_m$, but $\omega' \in A_m$, where ω' is the configuration which agrees with ω for all $\boldsymbol{y} \in \mathbb{R}^2$ with $\boldsymbol{y} \neq \boldsymbol{x}$ and, at $\boldsymbol{x}$, ω' assigns a point there with S as the underlying shape, i.e.

$$C(\omega) = \bigcup_{X \in \mathbb{X}_1(\omega)} (X + S) \cup \bigcup_{X \in \mathbb{X}_2(\omega)} (X + B)$$

does *not* contain any connected component which contain the origin and has nonempty intersection with $([-m, m]^2)^c$, while

$$C(\omega') = \bigcup_{X \in \mathbb{X}_1(\omega')} (X + S) \cup \bigcup_{X \in \mathbb{X}_2(\omega')} (X + B) = \bigcup_{X \in \mathbb{X}_1(\omega)} (X + S) \cup \bigcup_{X \in \mathbb{X}_2(\omega)} (X + B) \cup (\boldsymbol{x} + S)$$

does contain such a component. We define B-pivotal points similarly.

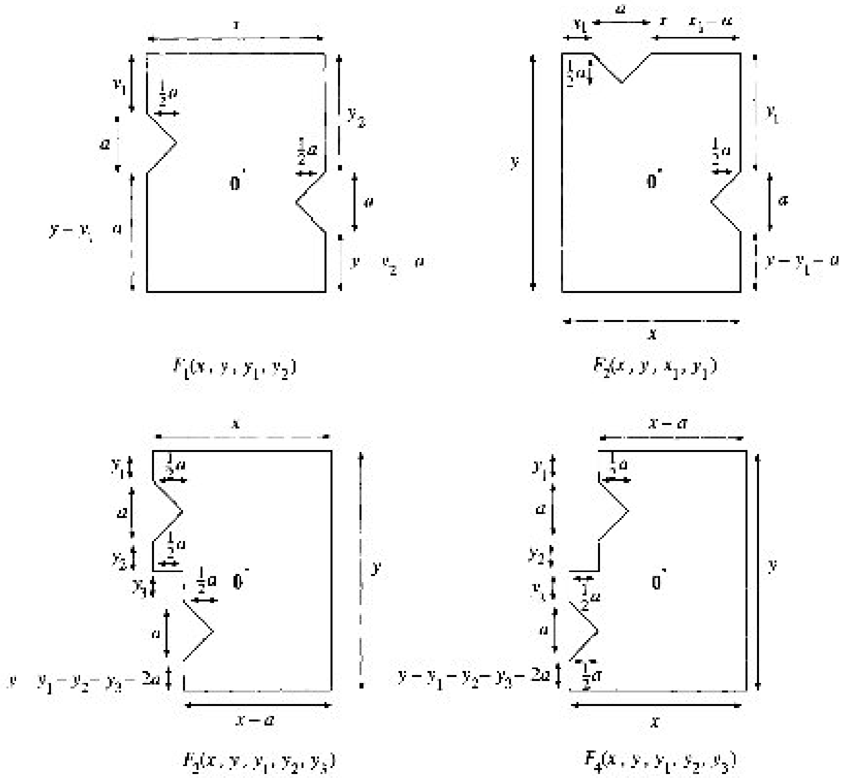

FIGURE 3: Faces.

Lemma 4.1. *For* $(\mu, \nu) \in [\alpha, \beta]^2$ *with* $0 < \alpha < \beta$, *there exists a constant* $0 < c < 1$ *independent of* μ, ν *and* m, *such that*

$$\mathrm{E}_{\mu,\nu}(\textit{area of the } S\textit{-pivotal region for } A_m) \le c\mathrm{E}_{\mu,\nu}(\textit{area of the } B\textit{-pivotal region for } A_m).$$

Proof. Clearly if $x \in \mathbb{R}^2$ is S-pivotal for (ω, A_m), then it is also B-pivotal for (ω, A_m), because $S \subseteq B$.

Consider the lattice $(a\mathbb{Z})^2$ and, without loss of generality, assume that m is divisible by a (otherwise, instead of A_m, we would look at A_{am}). Let $D_1, D_2, \ldots, D_{l(m)}$ be the cells of this lattice which lie in $[-m, m]^2$.

We will show that if a cell D_i admits an S-pivotal point, then the probability of the configurations for which there is a region of area larger than some $A > 0$ in D_i which is B-pivotal, but not S-pivotal, is larger than $p > 0$, where A and p are constants not depending on the cell D_i or on m. This will ensure that the lemma holds with $c = (1 + pAa^{-2})^{-1}$.

Let $T = [-\frac{11}{2}a, \frac{11}{2}a] \times [-\frac{1}{2}(11a + b), \frac{1}{2}(11a + b)]$. Let F_1, F_2, F_3 and F_4 be as in Figure 3. A region F will be called a *face* if F is isomorphic to a rotation $R_k F_i$ of F_i, where R_k denotes a rotation by an angle $\frac{1}{2}k\pi$, $k = 0, 1, 2, 3$, $i = 1, 2, 3, 4$, for some $0 \le x_1 < x < \infty$ and $0 \le y_1, y_2, y_3 < y < \infty$. The two triangular cuts on a face are its *noses*, while the front $\frac{1}{4}a$ part of the nose is its *tip* (see Figure 4).

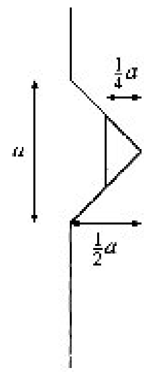

FIGURE 4: A nose with the shaded area being its tip.

Let $\mathcal{F}$ be the collection of all faces F such that $F \subseteq T$ and, for $j = 1, 2, 3, 4$, $d(T_j, \partial F) \leq a$, where T_1, T_2, T_3 and T_4 denote the top, bottom, left and right edges of the rectangle T.

Let ω be a configuration admitting an S-pivotal point in a cell D for the event A_m, for some D such that

$$c(D) + T \subseteq [-m, m]^2 \quad \text{and} \quad \mathbf{0} \notin c(D) + T, \tag{4.3}$$

where $c(D)$ is the 'centre' of the cell D and $\mathbf{0}$ denotes the origin $(0, 0)$.

Let $F \in \mathcal{F}$ be such that the configuration ω outside $c(D) + F$ admits

(i) a connected component of the covered region which contains the origin and contains a Poisson point situated in the tip of a nose of F, and,

(ii) a connected component of the covered region which intersects $\partial([-m, m]^2)$ and contains a Poisson point situated in the tip of the other nose of F.

Observe that F is determined by the configuration *outside* $c(D) + F$, so the configuration *inside* $c(D) + F$ is determined by a Poisson point process of intensity λ which is independent of the configuration outside $c(D) + F$.

Since D contains a pivotal point for (ω, A_m), there must exist two disjoint connected components of the covered region such that both of them have nonempty intersection with $c(D) + ([-\frac{3}{2}a, \frac{3}{2}a] \times [-\frac{3}{2}a, \frac{3}{2}a])$ and one of them contains the origin while the other has nonempty intersection with $\partial([-m, m]^2)$. Note that there is a lot of freedom in choosing the face F; in particular, we only require that the boundary of the smallest rectangle containing the face lies in the region $T \setminus [-\frac{9}{2}a, \frac{9}{2}a] \times [-\frac{1}{2}(9a + b), \frac{1}{2}(9a + b)]$. This ensures that we can obtain a face with properties (i) and (ii) described above.

Now divide the plane by the lattice $(\delta\mathbb{Z})^2$, where $\delta = \min\{(\frac{1}{32}a), (\frac{1}{32}b)\}$. For D as given in (4.3), we first choose two δ-cells Δ_{top} and Δ_{bot} such that

(i) $c(D) + (0, a + \frac{1}{4}b - (\frac{1}{32}\min\{a, b\})) \in \Delta_{\text{top}}$,

(ii) $c(D) + (0, -a - \frac{1}{4}b + (\frac{1}{32}\min\{a, b\})) \in \Delta_{\text{bot}}$.

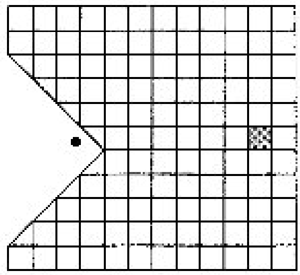

FIGURE 5: The cell Δ_{11}. The point near the nose denotes the site of a Poisson point which is 'connected' to the origin.

Note that these cells Δ_{top} and Δ_{bot} have been chosen such that if a shape S were to be centred anywhere in Δ_{top} and another shape S were to be centred anywhere in Δ_{bot}, then

(a) no single shape shape S centred inside D will have nonempty intersection with both the shapes; however,

(b) there is a region in D of area at least $\frac{1}{1024}ab$ such that a shape B placed anywhere in this region will have nonempty intersection with both the shapes.

We now get more such δ-cells to 'connect' Δ_{top} to one of the noses and Δ_{bot} to the other nose in the face. For this, we first obtain two δ-cells Δ_{11} and Δ_{21} inside the face such that two S shapes placed one at each of these cells will be disjoint and if a nose is on the left or right (respectively, top or bottom) side of the face, then the horizontal (respectively, vertical) distance of Δ_{11} from the end of the nose is between $\frac{3}{4}a$ and $\frac{31}{32}a$ (see Figure 5). Similarly we take Δ_{21} at a horizontal or vertical distance between $\frac{3}{4}a$ and $\frac{31}{32}a$ from the end of the other nose.

Having chosen Δ_{11} and Δ_{21}, we now choose more δ-cells $\Delta_{12}, \dots, \Delta_{1k}$ and $\Delta_{22}, \dots, \Delta_{2l}$ for some k and l such that

(a) two S shapes, one placed in $\Delta_{i,j}$ and the other in $\Delta_{i,j+1}$, have nonempty intersection,

(b) $\Delta_{i,j} \not\subseteq c(D) + [-\frac{3}{2}a, \frac{3}{2}a]^2$ and $d(\partial F, \Delta_{i,j}) > 2a$,

(c) for any $1 \leq j_1 \leq k$ and $1 \leq j_2 \leq l$, two S shapes, one placed anywhere in Δ_{1,j_1} and the other placed anywhere in Δ_{2,j_2}, are disjoint, and

(d) two sets of two S shapes, one placed in $\Delta_{1,k}$ (respectively, $\Delta_{2,l}$) and the other placed in Δ_{top} (respectively, Δ_{bot}) have nonempty intersection.

Since the total number of δ-cells in a face is at most $1024 \times 11a(11a + b)(\min\{a, b\})^{-2}$, k and l above are bounded.

Now we ensure that each of the cells $\Delta_{i,j}$ and both Δ_{top} and Δ_{bot}, have

(i) at least one Poisson point each, with a shape S centred at each of them, and,

(ii) no Poisson points with a shape B centred there.

This occurs with a probability $p(\mu, \nu) > 0$, which depends on a and b, but more importantly is continuous in μ and ν.

As seen earlier, the construction of the δ-cells ensures a region in D of area at least $\frac{1}{1024}ab$ which is B-pivotal, but not S-pivotal for A_m. Thus, given D satisfying (4.3) and containing an S-pivotal point for (ω, A_m), we obtain that the region in D which is B-pivotal, but not S-pivotal, has an expected area at least $\frac{1}{1024}p(\mu, \nu)ab$.

For D not satisfying (4.3), if D is near the boundary of $[-m, m]^2$, then we consider a face with only one nose, through which a connected component of the covered region containing the origin intersects the face. We place δ-cells inside this face, so that

(i) a set of δ-cells goes from near D (i.e. Δ_{top} or Δ_{bot}) to the nose, and

(ii) another set of δ-cells goes from near D (i.e. Δ_{top} or Δ_{bot}) to the boundary of $[-m, m]^2$ (unless D is at a distance less than $\frac{1}{2}(a+b)$ (respectively, $\frac{1}{2}a$) from the top (respectively, bottom) edge of $[-m, m]^2$, in which case we obtain a pivotal region in D where if a shape B is centred there, it will itself intersect $\partial([-m, m]^2)$ and the constructed path from the origin, but a shape S centred there will not intersect $\partial([-m, m]^2)$).

However, for D close to the origin, we will get two sets of δ-cells—one going from near D to the origin, and the other going from near D to the nose through which passes a connected component of the covered region which has nonempty intersection with $\partial[-m, m]^2$.

All these cases ensure that the given D contains an S-pivotal point for (ω, A_m), the region in D which is B-pivotal, but not S-pivotal, has an expected area of at least $\frac{1}{2048}p'(\mu, \nu)ab$, for $p(\mu, \nu) > 0$, and is continuous in μ and ν.

By the continuity of $p(\mu, \nu)$ and $p'(\mu, \nu)$, take $p := \min\{p(\mu, \nu), p'(\mu, \nu); (\mu, \nu) \in [\alpha, \beta]^2\} > 0$ and take $A = \frac{1}{2048}ab$ to obtain the lemma.

To complete the proof of Theorem 2.1, we now fix $\varepsilon > 0$, such that $2\varepsilon < \lambda_c(B)$. This is indeed possible, because $\lambda_c(B) > 0$ (see Hall (1985)). We take $\alpha = \frac{1}{2}\lambda_c(B) - \varepsilon$ and $\beta = \frac{1}{2}\lambda_c(B) - \varepsilon$ in Lemma 4.1 and obtain $0 < c < 1$ from the lemma from this choice of α and β. For $t \in [0, 1]$, consider $\mu(t) := \frac{1}{2}\lambda_c(B) - \gamma\varepsilon + \varepsilon t$ and $\nu(t) := \frac{1}{2}\lambda_c(B) - c\varepsilon t$, where $\gamma > 0$ is chosen such that $1 - \gamma - c > 0$.

Then from (4.1) and (4.2), we have, for $t \in [0, 1]$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}_{\mu(t),\nu(t)}(A_m) &= \varepsilon \mathrm{E}_{\mu(t),\nu(t)}(\text{area of } S\text{-pivotal region for } A_m)\\ &\quad - c\varepsilon \mathrm{E}_{\mu(t),\nu(t)}(\text{area of } B\text{-pivotal region for } A_m)\\ &\le 0. \end{aligned} \tag{4.4}$$

Now,

$$\begin{aligned}\mathrm{P}_{\lambda_c(B)+(1-\gamma-c)\varepsilon,0}(A_m) &\le \mathrm{P}_{\lambda_c(B)/2-(1-\gamma)\varepsilon,\lambda_c(B)/2-c\varepsilon}(A_m) = \mathrm{P}_{\mu(1),\nu(1)}(A_m)\\ &\le \mathrm{P}_{\mu(0),\nu(0)}(A_m) = \mathrm{P}_{\lambda_c(B)/2-\gamma\varepsilon,\lambda_c(B)/2}(A_m)\\ &\le \mathrm{P}_{0,\lambda_c(B)-\gamma\varepsilon}(A_m) = 0\end{aligned}$$

(the second inequality follows on integrating (4.4) from 0 to 1).

Thus $\lambda_c(S) \ge \lambda_c(B) + (1 - \gamma - c)\varepsilon > \lambda_c(B)$, thereby proving the theorem.

5. Conclusion

Theorem 2.1 along with the observation that $\lambda_c(P_3) = \lambda_c(H_{3/2})$ immediately yields Proposition 2.2. Moreover, Theorem 1.2 of Jonasson (2001) contains a proof of the fact that, for a two-dimensional convex bounded region S, other than a triangle, of unit area, there exists an affine transformation such that $TS \subsetneq H_{3/2}$. This proves Theorem 2.2.

In the proof of Theorem 2.1, by a different choice of $\mu(t)$ and $\nu(t)$, we could obtain the following more general result. Let $\lambda_c(p, S, B)$ be the critical intensity of percolation in a Poisson Boolean model where each Poisson point is the centre of a shape S with probability p or a shape B with probability $1 - p$, independently of other points as well as the process, with S and B as in Theorem 2.1.

Theorem 5.1. *For* $0 \leq p < p' \leq 1$, $\lambda_c(p, S, B) < \lambda_c(p', S, B)$.

Moreover, the enhancement method of Theorem 2.1 can be easily modified to consider the critical intensities of processes on $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d \times [-1, 1]$. This will yield our last theorem.

Theorem 5.2. *If S_d denotes a d-dimensional cube of unit d-dimensional Lebesgue measure, then, for $d \geq 2$, $\lambda_c^d(S_d) > \lambda_c^{d+1}(S_{d+1})$, where $\lambda_c^d(S_d)$ denotes the critical intensity of the Poisson Boolean model $(\mathbb{X}, \lambda, S_d)$ on $\mathbb{R}^d$.*

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