On commutativity of rings

D.S.Nagaraj & B.Sury

Introduction

Here we discuss how certain identities on a ring force it to be commutative under some mild hypotheses. Let us assume that A is a nonzero associative ring with unity 1 and satisfies an identity of the form

$$x^{a_1}y^{b_1}\cdots x^{a_r}y^{b^r} = x^{c_1}y^{d_1}\cdots x^{c_s}y^{d_s} \quad \forall \quad x, y \in A.$$

Here a_i, b_i, c_i, d_i are fixed positive integers. Note that identities like $(xy)^n = x^n y^n$ or $(xy)^n = (yx)^n$ give rise to special cases of the above identity. We prove some general commutativity results assuming the ring is N-torsion free for a suitable integer N. Here, A is said to be N-torsion free for an integer N if $Na = 0$ for some $a \in A$ implies $a = 0$. We also give some examples to show that some assumption on torsion is necessary. Some commutativity results appear in $[A]$, $[ABY]$, $[Aw]$ and $[JOY]$.

Theorem.
Assume that A is a nonzero associative ring with unity 1 and satisfies an identity of the form

$$x^{a_1}y^{b_1}\cdots x^{a_r}y^{b_r} = x^{c_1}y^{d_1}\cdots x^{c_s}y^{d_s} \quad \forall \quad x, y \in A.$$

Further, assume that $(\sum_{i=1}^r a_i)(\sum_{i=1}^r b_i) = (\sum_{j=1}^s c_i)(\sum_{j=1}^s d_i)$ and that the integer $u = \sum_{i=1}^r a_i(b_i + b_{i+1} + \cdots + b_r) - \sum_{j=1}^s c_j(d_j + d_{j+1} + \cdots + d_s) \neq 0$. Then, there is an integer N depending only on a_i, b_i, c_i, d_i such that if A is N-torsion free, then it must necessarily be commutative.

Remarks
(i) If $M = \text{Max}\ (\ \sum_{i=1}^r a_i, \sum_{j=1}^s c_j, \sum_{i=1}^r b_i, \sum_{j=1}^s d_j)$, then one may take N to be the least common multiple of $M!$ and u where u is as in the theorem.
(ii) Note that $r = n, s = 1, a_i = b_i = 1, c_1 = d_1 = n$ gives the identity $(xy)^n = x^n y^n$ and the corresponding $u = -n(n-1)/2$.
(iii) The papers $[A], [ABY]$ prove theorems of the following type:
Let R be a ring satisfying the following hypotheses: (1) for each $x \in R$ there exists an integer $k = k(x) \geq 1$ and a polynomial with integer coefficients

$f(\lambda)$ such that $x^k = x^{k+1}f(x)$; (2) for every $x, y \in R$, $(xy)^n - y^n x^n$ and $(xy)^{n+1} - y^{n+1}x^{n+1}$ are central elements, where n is fixed integer; (3) R is $n(n+1)$-torsion free; (4) the nilpotent elements of R commute. Then R is commutative.
An n-torsion-free ring R with identity such that, for all x, y in R, $x^n y^n = y^n x^n$ and $(xy)^{n+1} - x^{n+1}y^{n+1}$ is central, must be commutative. Further, a periodic n-torsion free ring (not necessarily with identity) for which $(xy)^n - (yx)^n$ is always in the centre is commutative provided that the nilpotents of R form a commutative set.
(iv) The papers $[JOY]$ and $[Aw]$ prove some commutativity theorems of the following type without assuming associativity :
If R is a ring (associative or not) with identity such that $(xy)^2 = x^2y^2$, then R is commutative.
Let R be a non-associative ring with unity $1 \neq 0$, such that $(xy)^n = (yx)^n$ for some fixed positive integer $n \geq 1$ and for all x, y in R; further, let the additive group of R be p-torsion free for every prime integer $p \leq n$; then R is commutative.

Proof of theorem.
Applying the identity to $1 + tx$ and y where t is a positive integer, we have

$$(1+tx)^{a_1}y^{b_1}\cdots(1+tx)^{a_r}y^{b_r} = (1+tx)^{c_1}y^{d_1}\cdots(1+tx)^{c_s}y^{d_s} \quad \forall \quad x, y \in A.$$

This can be rewritten as $\sum_{i=0}^{M} \alpha_i t^i = 0$ where $\alpha_i \in A$ are independent of t and $M = \text{Max }(\sum a_i, \sum c_i)$. Let us write these down for $t = 1, 2, \ldots, M+1$. We have a matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^M \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & M+1 & (M+1)^2 & \cdots & (M+1)^M \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_M \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix on the left hand side is a Vandermonde matrix whose determinant is $M!(M-1)!\cdots 1!$. First, note that A is $M!$-torsion free since by assumption A is N-torsion free for a multiple N of $M!$. So, if $M!(M-1)!\cdots 1!a = 0$ for some $a \neq 0$, then $(M-1)!(M-2)!\cdots 1!a = 0$ since A is $M!$-torsion free. Multiplying by M, we again have $(M-2)!(M-3)!\cdots 1!a = 0$. Proceeding in this manner, we obtain $a = 0$. Now $M!(M-1)!\cdots 1!\alpha_i = 0$ for all

$i = 0, \ldots, M$. Therefore, $\alpha_i = 0$ for each $i = 0, \ldots, M$. In particular, $\alpha_1 = 0$ gives us

$$a_1 xy^{b_1+\cdots+b_r} + a_2 y^{b_1} xy^{b_2+\cdots+b_r} + \cdots + a_r y^{b_1+\cdots+b_{r-1}} xy^{b_r}$$

$$= c_1 xy^{d_1+\cdots+d_s} + c_2 y^{d_1} xy^{d_2+\cdots+d_s} + \cdots + c_s y^{d_1+\cdots+d_{s-1}} xy^{d_s}.$$

This is an identity for each $x, y \in A$. Now, we apply it to x and $1 + ty$ for natural numbers t. Writing it for $t = 1, 2, \ldots, \text{Max}\,(\sum b_i, \sum d_i) + 1$ and using once again the Vandermonde argument with y in this identity, we get

$$\sum_{i=1}^{r} a_i(b_i + b_{i+1} + \cdots + b_r)xy + \sum_{i=2}^{r} a_i(b_1 + \cdots + b_{i-1})yx$$

$$= \sum_{j=1}^{s} c_j(d_j + d_{j+1} + \cdots + d_s)xy + \sum_{j=2}^{s} c_j(d_1 + \cdots + d_{j-1})yx.$$

Thus, we have

$$(\sum_{i=1}^{r} a_i(b_i + b_{i+1} + \cdots + b_r) - \sum_{j=1}^{s} c_j(d_j + d_{j+1} + \cdots + d_s))xy$$

$$= (\sum_{j=2}^{s} c_j(d_1 + \cdots + d_{j-1}) - \sum_{i=2}^{r} a_i(b_1 + \cdots + b_{i-1}))yx$$

for all $x, y \in A$. Now, note that the assumption that $(\sum_{i=1}^{r} a_i)(\sum_{i=1}^{r} b_i) = (\sum_{j=1}^{s} c_i)(\sum_{j=1}^{s} d_i)$ means that the coefficients of xy and yx above are equal and equal the integer denoted by u in the theorem. As A is u-torsion free, we get $xy = yx$. This completes the proof.

Corollary.
Let A be a non-zero associative ring which contains 1 and let n be a natural number ≥ 2 such that A is $n!$-torsion free. If A has the property that

$$(xy)^n = x^n y^n \quad \forall \quad x, y \in A,$$

then, A is necessarily commutative.

Remarks. There is another way of proving commutativity in the case of some special identities like the ones in corollary. This depends on a non-commutative polynomial identity which may be of independent interest. We merely state this and do not discuss it in detail. For convenience, let us denote by S, the polynomial in n noncommuting variables given by

$$S(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Note that $S(x_1, x_2) = x_1 x_2 + x_2 x_1 = (x_1 + x_2)^2 - x_1^2 - x_2^2$.
Our contention is that $S(x_1, \ldots, x_n)$ can be written as a sum or difference of n-th powers of certain polynomials. To state it, we introduce one last notation.
For $1 \leq r \leq n$, there are $\binom{n}{r}$ ways to choose r of the x_i's. Call $S_{r,1}, \ldots, S_{r,\binom{n}{r}}$, the corresponding sums of the $x's$. In particular, $S_{1,i} = x_i$ and $S_{n,1} = x_1 + \cdots + x_n$. Then, one can prove :

$$S(x_1, \cdots, x_n) = S_{n,1}^n - (S_{n-1,1}^n + \cdots + S_{n-1,n}^n)$$

$$+(S_{n-2,1}^n + \cdots + S_{n-2,\binom{n}{2}}^n) + \cdots + (-1)^{n-1}(S_{1,1}^n + \cdots + S_{1,n}^n).$$

The identity can be deduced from the inclusion-exclusion principle. Note that the special case when the variables commute leads us to the familiar elementary identity

$$n! = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r)^n.$$

We now give some examples to show that there are noncommutative rings in which identities such as we have been discussing hold good. These possess torsion.

Example. Consider any commutative ring A with identity and let M be the free module of rank 2 with an A-basis e_1, e_2. Form the tensor A-algebra

$$T_A(M) := \bigoplus_{n \geq 0} T^n(M)$$

where $T^n(M)$ is the n-fold tensor product $M \otimes \cdots \otimes M$ of the A-module M. Look at the two-sided ideal I_3 of $T(M)$ generated by $T^3(M)$; then $R_A := T(M)/I_3$ is a noncommutative, associative A-algebra. Note that any $x \in R_A$ is the image of an element $x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 \otimes e_1 + x_4 e_2 \otimes e_2 + x_{12} e_1 \otimes e_2 + x_{21} e_2 \otimes e_1 \in T_A(M)$. For any prime number $p \geq 3$, we look at the further quotient ring S_A of R_A by the two-sided ideal generated by all elements $(xy)^p - x^p y^p$ for $x, y \in R_A$. It is evident that elements of S satisfy the identity $(xy)^p = x^p y^p$. We claim that the ring $S_{\mathbf{Z}}$ is noncommutative and has $p(p-1)/2$-torsion.

Let us consider the images f_1, f_2 in $S_{\mathbf{Z}}$ of e_1, e_2 in $T_A(M)$. The identity

$$(1+f_1)^p(1+f_2)^p = ((1+f_1)(1+f_2))^p$$

gives

$$(1 + pf_1 + \binom{p}{2} f_1^2)(1 + pf_2 + \binom{p}{2} f_2^2) = (1 + f_1 + f_2 + f_1 f_2)^p$$

$$= 1 + pf_1 + pf_2 + pf_1 f_2 + \binom{p}{2}(f_1^2 + f_2^2 + f_1 f_2 + f_2 f_1)$$

since $p \geq 3$ and all products of f_i's of length ≥ 3 are zero in $S_{\mathbf{Z}}$. This reduces to

$$\binom{p}{2}(f_1 f_2 - f_2 f_1) = 0.$$

We have not used until now that p is a prime. To show that $S_{\mathbf{Z}}$ indeed has $\binom{p}{2}$-torsion, it suffices to show that $f_1 f_2 \neq f_2 f_1$ in $S_{\mathbf{Z}}$. To do this, we take p to be prime. Note that $f_1 f_2 = f_2 f_1$ if, and only if, $S_{\mathbf{Z}}$ is commutative. Therefore, let us show that $S_{\mathbf{Z}}$ is noncommutative. Let us look at the construction of R_A and S_A when $A = \mathbf{Z}/p$. In this case, if $x \in R_A$ is the image of $x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 \otimes e_1 + x_4 e_2 \otimes e_2 + x_{12} e_1 \otimes e_2 + x_{21} e_2 \otimes e_1 \in T_A(M)$, then $x^p = x_0$ since $p \geq 3$ and p as well as $\binom{p}{2}$ are zero in $\mathbf{Z}/p$. Therefore, the identity $(xy)^p = x^p y^p$ is automatically satisfied in R_A when $A = \mathbf{Z}/p$. Note that S_A is noncommutative when $A = \mathbf{Z}/p$ as $S_A = R_A$ here. Finally, since $S_{\mathbf{Z}}$ has this noncommutative ring as a quotient by the ideal generated by p, the ring $S_{\mathbf{Z}}$ itself is noncommutaive.

References.

[A] H.Abu-Khuzam, *Commutativity results for periodic rings*, Acta Math.Hungar. 58 (1991), no.3-4, 273-277.

[ABY] H.Aby-Khuzam, H.Bell & A.Yaquib, *Commutativity of rings satisfying certain polynomial identities*, Bull. Austral.Math.Soc.44 (1991), no.1, 63-69.

[*Aw*] Ram Awtar, *On the commutativity of nonassociative rings*, Publ. Math. Debrecen 22 (1975), no. 3-4, 177–188.

[*JOY*] E.C.Johnsen, D.L.Outcalt & A.Yaqub, *An elementary commutativity theorem for rings*, Amer. Math. Monthly 75 1968 288–289.

Acknowledgements

We thank the referee for some questions and remarks which helped clarify the results here.

D.S.Nagaraj
Institute of Mathematical Sciences
C.I.T.Campus, Taramani
Chennai - 600113
India.
dsn@imsc.res.in

B.Sury
Statistics & Mathematics Unit
Indian Statistical Institute
8th Mile, Mysore Road
Bangalore - 560 059
India.
sury@isibang.ac.in