Achieving the first best in sequencing problems

Manipushpak Mitra

Department of Economics, University of Bonn, Adenauerallee 24–26, 53113 Bonn, Germany
(e-mail: m.mitra@wiwi.uni-bonn.de)

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Abstract. In a sequencing problem with linear time cost, Suijs (1996) proved that it is possible to achieve first best. By first best we mean that one can find mechanisms that satisfy efficiency of decision, dominant strategy incentive compatibility and budget balancedness. In this paper we show that among a more general and natural class of sequencing problems, sequencing problems with linear cost is the only class for which first best can be achieved.

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1 Introduction

In a sequencing problem there is a large multi-unit firm with each unit in need of the facility provided by a particular repair and maintenance unit. The repair and maintenance unit can service only one unit at any given time. Therefore, units which remain unattended, incur a cost for the time they are down. In this framework, the firm's role is that of a planner wanting to service the units by forming a queue that minimises the total cost of waiting. Each unit's cost parameter is private information. The objective of the firm is to determine the order in which the units are to be

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serviced. The presence of private information implies that the firm has an incentive problem. Sequencing, as an incentive problem, was studied by Dolan (1978). He provided a mechanism which is incentive compatible but not budget balancing.

There is a vast literature on incentive theory under incomplete information suggesting that under quasi-linear preferences the achievement of truth-telling and efficiency is possible. The pioneering works of Groves (1973); and Clarke (1971) have established the existence of a class of mechanisms, the so called Groves mechanisms, where all individuals have a dominant strategy to reveal their information. Moreover, the truth-telling outcome leads to efficiency. Green and Laffont (1977) has proved the uniqueness of Groves mechanism in the public goods problems. Holmström (1979) has proved that if the domain of preferences in the quasi-linear framework is "smoothly connected", that is, if the domain is rich enough, then Groves mechanisms are the only mechanisms that are dominant strategy incentive compatible and efficient in terms of decisions. However, Groves mechanisms are in general not balanced, that is, there are preference realizations where aggregate transfers are non-zero. The budget imbalance of Groves mechanisms, in the context of public goods problem, is shown in Hurwicz (1975); Green and Laffont (1979); Walker (1980). Hurwicz and Walker (1990) proved the impossibility result in the context of pure exchange economies, that is, economies in which there is no production and in which there are no public goods or other externalities. The damaging nature of budget imbalance of Groves mechanism, in the public goods context, was analysed by Groves and Ledyard (1977). They had shown, using a very simple model, that an alternative procedure based on majority rule voting may lead to an allocation of resources which is Pareto-superior to the one produced by Groves mechanism. However, Suijs (1996), by assuming costs to be linear over time has proved that a sequencing problem is first best implementable. A sequencing problem is first best implementable if it is possible to design a mechanism that satisfies truth-telling in dominant strategies, efficiency of decision and budget-balancedness. Further, he conjectured that linearity of the costs is crucial for this result.

In this paper, sequencing problems with more general and natural class of cost functions is analysed. The main result of this paper is that, a sequencing problem is first best implementable *only if* the cost function is linear over time. Thus, while Suijs (1996) proved that for first best implementability of a sequencing problem it is sufficient to have a linear cost, we prove its necessity. If the cost function is linear then the relative queue position of any two units is independent of the preferences revealed by the other units. Suijs (1996) conjectured that independence of this sort, which we define as *independence property* in this paper, is crucial for first best implementability. However, we show that this type of independence is not the only requirement that drives first best implementability in a sequencing problem. There exists other cost functions, like exponential costs, that satisfy *independence property*. For first best implementability it is also necessary that the cost function satisfies a nice *combinatorial structure*. Therefore, the necessity of the independence property and the combinatorial structure of the cost function together imply that a sequencing problem is first best implementable only if the cost function is linear.

This paper is arranged in the following way. In section two, the model is developed. Section three is the main section of this paper where, among other things, the necessary condition for first best implementability is derived. Section four concludes the paper.

2 The model

Let $\mathbf{N} \equiv \{1, 2, \ldots, n\}$ be the set of units of a multi-unit firm in need of the facility provided by a particular repair and maintenance unit. Each unit $j \in \mathbf{N}$ has a cost parameter $\theta_j \in \Theta$, that belongs to an interval in the non-negative orthant $\mathbf{R}_+$ of the real line $\mathbf{R}$. Each unit $j \in \mathbf{N}$ also has a servicing time s_j that belongs to the positive orthant $\mathbf{R}_{++}$ of the real line. Let $C(S_j; \theta_j) = \theta_j F(S_j) + \beta_j$ measure the cost of waiting $S_j \in \mathbf{R}_{++}$ periods in the queue for unit $j \in \mathbf{N}$ with cost parameter (or unit type) $\theta_j \in \mathbf{R}_+$. Here the mapping $F : \mathbf{R}_{++} \rightarrow \mathbf{R}_+$ is the time dependent cost function and β_j is a fixed cost to unit j. We assume that the form of time dependent cost function $F(S_j)$ is identical for all units and that F is *continuous* and *strictly increasing* in S_j. Let $\mathbf{F}$ be the class of continuous and strictly increasing time cost functions. Observe that S_j depends not only on the servicing time s_j of unit j but also on the servicing time of the units serviced before it. The firm's aim is to find an efficient queue, that is, a queue that minimises the aggregate cost. By means of a permutation σ on $\mathbf{N}$, one can describe the positions of each unit in the queue. Specifically, $\sigma_j = k$ indicates that unit j has the kth position in the queue. Let $\mathbf{\Sigma}$ be the set of all possible permutations of $\mathbf{N}$. We define $P(\sigma, j) = [p \in \mathbf{N} - \{j\} \mid \sigma_j > \sigma_p]$ to be the predecessor set of j in the queue $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{\Sigma}$. Given a servicing time vector $s = (s_1, \ldots, s_n)$ and a queue σ, the cost of waiting in the queue for unit $j \in \mathbf{N}$ is $C(S_j(\sigma); \theta_j) = \theta_j F(S_j(\sigma)) + \beta_j$, where $S_j(\sigma) = \sum_{l \in P(\sigma, j)} s_l + s_j$. The utility of unit j, with cost parameter θ_j, is given by $U_j(\sigma, t_j; \theta_j) = v_j - C(S_j(\sigma); \theta_j) + t_j$ where v_j is the benefit, derived by unit j, from the service and t_j is the transfer that it receives.

Let $\theta = (\theta_1, \ldots, \theta_{j-1}, \theta_j, \theta_{j+1}, \ldots, \theta_n)$ be a state of the world or a profile and let (θ'_j, θ_{-j}) be another profile of the form $(\theta_1, \ldots, \theta_{j-1}, \theta'_j, \theta_{j+1}, \ldots, \theta_n)$, where both θ and (θ'_j, θ_{-j}) belong to Θ^n. Consider the problem of the firm whose objective is to minimise the aggregate cost of waiting in the queue. A queue $\sigma^* \in \mathbf{\Sigma}$, given s, is *efficient* or minimises aggregate waiting cost if $\sigma^* \in argmin_{\sigma \in \mathbf{\Sigma}} \sum_{j \in \mathbf{N}} C(S_j(\sigma); \theta_j)$. Throughout this analysis, the servicing time vector $s = (s_1, \ldots, s_n)$ is assumed to be common knowledge. If the firm also knows $\theta = (\theta_1, \ldots, \theta_n)$, then it can calculate the efficient queue and service the units accordingly. However, if θ_j is private information to unit j, the firm's problem is to design a *mechanism* that will elicit this information truthfully. Formally, a mechanism $\mathbf{M}$ is a pair $\langle \sigma, \mathbf{t} \rangle$, where $\sigma : \Theta^n \rightarrow \Sigma$ and $\mathbf{t} \equiv (t_1, \ldots, t_n) : \Theta^n \rightarrow \mathbf{R}^n$. A *sequencing problem* under incomplete information is written as $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++}) \rangle$, where $\mathbf{N}$ is the number of units of a firm in need of the facility, $F \in \mathbf{F}$ represents the cost of each unit of the firm, which takes identical functional form for all units $j \in \mathbf{N}$, Θ is the type space of each unit representing the cost parameter and $\mathbf{R}_{++}$ is the space of servicing time for each unit. Under $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$, given all others' announcement

θ_{-j}, the utility of unit j of type θ_j, when its announcement is $\theta_j{}'$, is given by $U_j(\sigma(\theta_j{}', \theta_{-j}), t_j(\theta_j{}', \theta_{-j}), \theta_j) = v_j - C(S_j(\sigma(\theta_j{}', \theta_{-j})); \theta_j) + t_j(\theta_j{}', \theta_{-j})$.

Definition 2.1. *A sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++}) \rangle$, is said to be implementable, if there exists an efficient rule $\sigma^* : \Theta^n \to \mathbf{\Sigma}$ and a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ such that, for all $j \in \mathbf{N}$, for all $(\theta_j, \theta_j{}') \in \Theta^2$ and for all $\theta_{-j} \in \Theta^{n-1}$, $U_j(\sigma^*(\theta), t_j(\theta); \theta_j) \geq U_j(\sigma^*(\theta_j{}', \theta_{-j}), t_j(\theta_j{}', \theta_{-j}); \theta_j)$.*

This definition says that for any given θ_{-j}, unit j cannot benefit by reporting anything other than it's true type. In other words, truth-telling is a dominant strategy for all units. Moreover, implementability also means that in each state the queue selected is an efficient one.

Definition 2.2. *A sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++}) \rangle$, is first best implementable, if there exists a mechanism $\mathbf{M} = \langle \sigma^*, \mathbf{t} \rangle$ which implements it and such that, for all $\theta \in \Theta^n$, $\sum_{j \in \mathbf{N}} t_j(\theta) = 0$.*

Thus, a sequencing problem is first-best implementable if, it can be implemented in a manner such that aggregate transfers are zero in every state of the world. In such problems, incomplete information does not impose any welfare loss.

We define the minimum cost function $\mathbf{C} : \Theta^n \times \Theta^n \to \mathbf{R}$. For a state θ', with announcement θ, $\mathbf{C}(\sigma^*(\theta); \theta') = \sum_{j \in \mathbf{N}} C(S_j(\sigma^*(\theta)); \theta'_j)$, where $\sigma^*(\theta) \in argmin_{\sigma \in \Sigma} \sum_{j \in \mathbf{N}} C(S_j(\sigma); \theta_j)$. For simplicity of notation, we write the minimum cost function $\mathbf{C}(\sigma^*(\theta); \theta)$ as $\mathbf{C}(\theta)$. In other words, $\mathbf{C}(\theta)$ represents the minimum cost when announced state θ is also the true state.

Definition 2.3. *A mechanism $\mathbf{M} = \langle \sigma, \mathbf{t} \rangle$ is a Groves mechanism if, for all $j \in \mathbf{N}$ and for all $\theta \in \Theta^n$,*

$$t_j(\theta) = -\mathbf{C}(\theta) + C(S_j(\sigma^*(\theta)); \theta_j) + \gamma_j(\theta_{-j}) \tag{2.1}$$

In a Groves mechanism the transfer of any unit $j \in \mathbf{N}$, in any state θ, is the negative of minimum cost $\mathbf{C}(\theta)$ less the cost of unit j upto a constant $\gamma_j(\theta_{-j})$. The utility of unit j with a Groves transfer is its benefit v_j less the minimum cost in state θ plus the constant. It is well known that such a transfer results in dominant strategy incentive compatibility because the firms's objective of minimising the aggregate cost is now an objective of unit j as well and this is true for all $j \in \mathbf{N}$.

Remark 2.1. A sequencing problem Ω is implementable *if and only if* the mechanism is a Groves mechanism. This result is not new in the literature. Under relatively weak assumptions, on the domain of preferences, Groves mechanisms have been shown, by Holmström (1979) and more recently by Suijs (1996), to be the only ones to satisfy implementability condition. The domain of any sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++}) \rangle$, with $F \in \mathbf{F}$, satisfies Holmström's definition of "convex" domains.[1] Moreover, the domain of preferences in a sequencing problem also satisfies Suijs' definition of "graph connectedness" (see Suijs (1996)). Thus it follows, from theorem 2 of Holmström (1979) and theorem 3.2 of Suijs (1996), that sequencing problems are implemented uniquely by Groves mechanism.

[1] If the domain of preferences is "convex" then it is "smoothly connected" (see Holmström 1979).

The main difficulty with Groves mechanisms is that they are not balanced for a broad class of public decision problems (see Green and Laffont (1979), Walker (1980)). The question of whether or not Groves mechanism can first best implement sequencing problems is addressed in the following section.

3 Main result

Consider a sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++})\rangle$. Let the servicing time vector be $s = (s_1, \ldots, s_n)$ and let the state be θ. Consider a particular queue $\sigma = (\sigma_1, \ldots, \sigma_n)$, with $\sigma_l = \sigma_j + 1$, in state θ. Let $P(\sigma; j, l) = [p \in \mathbf{N} - \{j, l\} \mid \sigma_p < min\{\sigma_j, \sigma_l\}]$ and $\hat{s} = \sum_{p \in P(\sigma; j,l)} s_p \geq 0$.[2] Consider a different queue σ', obtained by interchanging only the queue positions of j and l. In other words, $\sigma' = (\sigma'_1, \ldots, \sigma'_n)$ is such that, $\sigma'_m = \sigma_m$ for all $m \in \mathbf{N}/\{j, l\}$, $\sigma'_j = \sigma_l$ and $\sigma'_l = \sigma_j$. The difference in total cost is given by $\mathbf{C}(\sigma; \theta) - \mathbf{C}(\sigma'; \theta) = \theta_l\{F(\hat{s} + s_j + s_l) - F(\hat{s} + s_l)\} - \theta_j\{F(\hat{s} + s_j + s_l) - F(\hat{s} + s_j)\}$. This interchange will lead to an increase (a decrease) in total cost if, $\mathbf{C}(\sigma; \theta) < (>) \mathbf{C}(\sigma'; \theta)$, that is, if $\frac{\theta_j}{\theta_l} > (<) \frac{F(\hat{s}+s_j+s_l) - F(\hat{s}+s_l)}{F(\hat{s}+s_j+s_l) - F(\hat{s}+s_j)}$. Let $f(\hat{s}; s_j, s_l) = \frac{F(\hat{s}+s_j+s_l) - F(\hat{s}+s_l)}{F(\hat{s}+s_j+s_l) - F(\hat{s}+s_j)}$. Therefore, the ratio function $f(\hat{s}; s_j, s_l)$, plays an important role in determining the efficient queue. In particular, the numerator of the ratio function $f(\hat{s}; s_j, s_l)$ measures the increase in the time cost of unit l if unit j is served ahead of unit l given that unit l is already incurring a cost of $\hat{s} + s_l$. Similarly, the denominator of the ratio function $f(\hat{s}; s_j, s_l)$ measures the increase in the time cost of unit j if unit l is served ahead of unit j given that unit j is already incurring a cost of $\hat{s} + s_j$. *Independence property*, defined below, is a restriction on the ratio function $f(\hat{s}; s_j, s_l)$.

Definition 3.4. *A sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++})\rangle$, with cost function $F \in \mathbf{F}$, satisfies the independence property, if there exists a map $g : \mathbf{R}^2_{++} \to \mathbf{R}_{++}$ such that, for all $\hat{s} \geq 0$, for all $s_j > s_l > 0$ and for all $j, l \in \mathbf{N}$ such that $j \neq l$, $f(\hat{s}; s_j, s_l) = g(s_j, s_l)$.*

If a sequencing problem Ω, with cost function F, satisfies the independence property then the relative queue positions of any two units j and l, with given θ_j, s_j and θ_l, s_l, is independent of the preferences of all other units. In particular, if θ_j, s_j and θ_l, s_l are such that $\frac{\theta_j}{\theta_l} > g(s_j, s_l)$, then for all servicing time vectors and states with given (s_j, s_l) and (θ_j, θ_l), efficient queue σ^* will imply that $S_j(\sigma^*) < S_l(\sigma^*)$, that is, $\sigma^*_j < \sigma^*_l$. In the next paragraph, we provide some examples of sequencing problems that satisfy the independence property and one example of a sequencing problem that fails to satisfy the independence property.

Consider a sequencing problem $\Omega^l = \langle \mathbf{N}, F^l, (\Theta; \mathbf{R}_{++})\rangle$, for which the cost function is linear. Therefore, for the sequencing problem Ω^l, $F^l(x) = a_1 x + a_0$, for all $x \in \mathbf{R}_{++}$ and $a_1 > 0$. Observe that for Ω^l, $f^l(\hat{s}, s_j, s_l) = \frac{s_j}{s_l} = g^l(s_j, s_l)$, for all $\hat{s} \geq 0$ and for all $s_j > s_l > 0$. Therefore, Ω^l satisfies independence property. Suijs (1996) conjectured that the independence property of a linear cost sequencing problem Ω^l plays a crucial role in its first best implementability. Consider the

[2] Observe that $\hat{s} = 0 \Leftrightarrow P(\sigma; j, l) = \phi$.

sequencing problem $\Omega^e = \langle \mathbf{N}, F^e, (\Theta; \mathbf{R}_{++})\rangle$, where the time cost function is exponential, that is, $F^e(x) = a_1 e^x + a_0$, for all $x \in \mathbf{R}_{++}$ and $a_1 > 0$. In this problem, $f^e(\hat{s}; s_j, s_l) = \frac{e^{s_j+s_l} - e^{s_l}}{e^{s_j+s_l} - e^{s_j}} = g^e(s_j, s_l)$, for all $\hat{s} \geq 0$ and for all $s_j > s_l > 0$. Therefore, Ω^e also satisfies the independence property. In general, all sequencing problems Ω^*, with cost function of the form $F^*(x) = a_1 c^x + a_0$, where $a_1 > 0$ a nd $c > 1$, satisfy the independence property since $f^*(\hat{s}; s_j, s_l) = \frac{c^{s_j+s_l} - c^{s_l}}{c^{s_j+s_l} - c^{s_j}} = g^*(s_j, s_l)$, for all $\hat{s} \geq 0$ and for all $s_j > s_l > 0$. Thus, the set of sequencing problems, satisfying independence property, includes non-linear time cost functions. Consider a sequencing problem $\Omega^{\bar{q}} = \langle \mathbf{N}, F^{\bar{q}}, (\Theta; \mathbf{R}_{++})\rangle$, where $F^{\bar{q}}(x) = x^2$, for all $x \in \mathbf{R}_{++}$. In this problem, given $s_j > s_l > 0$, the ratio $f(\hat{s}; s_j, s_l) = \frac{s_j}{s_l}\{1 - \frac{s_j - s_l}{\hat{s} + 2s_j + s_l}\}$ is not independent of $\hat{s}$. Therefore, $\Omega^{\bar{q}}$ fails to satisfy the independence property.

Theorem 3.1. *A sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++})\rangle$, with $F \in \mathbf{F}$, is first best implementable if and only if the cost is linear and there are at least three units.*

The if part of the theorem is due to Suijs (1996) and hence we omit its proof. We now state three lemmas and prove two of them. These lemmas are necessary for proving the *only if* part of Theorem 3.1. For these lemmas, some more notations and definitions are introduced. Consider two profiles $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$. We define, for $S \subseteq \mathbf{N}$, a type $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. Therefore, for each $S \subseteq \mathbf{N}$, a state $\theta(S)$ is of the form $(\theta_1(S), \ldots, \theta_n(S))$.

Lemma 3.1. *(Walker 1980). A sequencing problem Ω, with $F \in \mathbf{F}$, is first best implementable only if, for all pairs of profiles $\{\theta, \theta'\} \in \Theta^n \times \Theta^n$, $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|}$ $\mathbf{C}(\theta(S)) = 0$.*

Given the form of the Groves transfer (2.1), balancedness requires that the minimum aggregate cost is $(n-1)$ type separable, that is, $(n-1)\mathbf{C}(\theta) = \sum_{j \in \mathbf{N}} \gamma_j(\theta_{-j})$. Thus, it follows that for all pairs $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$, $\sum_{S \subseteq \mathbf{N}}$ $(-1)^{|S|}\mathbf{C}(\theta(S)) = \frac{1}{(n-1)} \sum_{j \in \mathbf{N}} \sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \gamma_j(\theta_{-j}(S)) = 0$. We now verify that Lemma 3.1 holds for a sequencing problem Ω with three units. Observe that balanced Groves transfer implies that

$$\mathbf{C}(\hat{\theta}) = \frac{1}{2}\{\gamma_1(\hat{\theta}_2, \hat{\theta}_3) + \gamma_2(\hat{\theta}_1, \hat{\theta}_3) + \gamma_3(\hat{\theta}_1, \hat{\theta}_2)\} \tag{3.2}$$

for all $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \in \Theta^3$. Now consider any two states θ and θ' and the sum

$$\begin{aligned}\sum_{S \subseteq \mathbf{N}} (-1)^{|S|}\mathbf{C}(\theta(S)) = \mathbf{C}(\theta_1, \theta_2, \theta_3) - \mathbf{C}(\theta'_1, \theta_2, \theta_3) - \mathbf{C}(\theta_1, \theta'_2, \theta_3) \\ -\mathbf{C}(\theta_1, \theta_2, \theta'_3) + \mathbf{C}(\theta'_1, \theta'_2, \theta_3) + \mathbf{C}(\theta'_1, \theta_2, \theta'_3) \\ +\mathbf{C}(\theta_1, \theta'_2, \theta'_3) - \mathbf{C}(\theta'_1, \theta'_2, \theta'_3).\end{aligned}$$

By substituting the right hand side of (3.2), after making all the relevant adjustments, in the sum $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|}\mathbf{C}(\theta(S))$, we get the result. Now one can easily check this result for sequencing problems with other unit sizes. Lemma 3.1 will help prove the next two lemmas.

Lemma 3.2. *A sequencing problem Ω, with $F \in \mathbf{F}$, is first best implementable only if, for all $\bar{s} \in \mathbf{R}_{++}$,*

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}F(k\bar{s}) = 0 \tag{3.3}$$

Before giving a general proof of Lemma 3.2, we first provide a general criterion for calculating an efficient queue when servicing costs are identical for all units. Using this criterion, we provide the proof of Lemma 3.2 for a sequencing problem $\Omega = \langle \mathbf{N} = \{1,2,3\}, F, (\Theta; \mathbf{R}_{++})\rangle$ with three units. This will help in understanding the general proof. Consider a servicing time vector $\bar{s} = (s_1 = \bar{s}, s_2 = \bar{s}, s_3 = \bar{s})$. Given the servicing time vector $\bar{s}$, calculating the efficient queue in each state $\theta = (\theta_1, \theta_2, \theta_3) \in \Theta^3$ is quite easy if the types of different units are non-identical. For example, if a state $\theta = (\theta_1, \theta_2, \theta_3) \in \Theta^3$ is such that $\theta_3 > \theta_2 > \theta_1$, then $\frac{\theta_3}{\theta_2} > \frac{F(\hat{s}+\bar{s}+\bar{s})-F(\hat{s}+\bar{s})}{F(\hat{s}+\bar{s}+\bar{s})-F(\hat{s}+\bar{s})} = f(\hat{s};\bar{s},\bar{s}) = 1$ and $\frac{\theta_2}{\theta_1} > f(\hat{s};\bar{s},\bar{s}) = 1$, for both $\hat{s} = \bar{s}$ and $\hat{s} = 0$. Therefore, the efficient queue in state $\theta = (\theta_1, \theta_2, \theta_3)$, with given servicing cost vector $\bar{s} = (s_1 = \bar{s}, s_2 = \bar{s}, s_3 = \bar{s})$, is $\sigma^*(\theta) = (\sigma_1^*(\theta) = 3, \sigma_2^*(\theta) = 2, \sigma_3^*(\theta) = 1)$. In general, if units have identical servicing costs and non-identical types, then efficient queue, in any state θ, can be obtained from the ascending order of unit types in that state. That is, in state θ, if the types of units j and l are such that $\theta_j > \theta_l$, then $\sigma_j^*(\theta) < \sigma_l^*(\theta)$. This is the *general criterion*, with identical servicing cost and non-identical types, for finding an efficient queue in any sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++})\rangle$ with $F \in \mathbf{F}$.[3]

To prove the lemma for three units, we consider any two profiles $\theta = (\theta_1, \theta_2, \theta_3) \in \Theta^3$ and $\theta' = (\theta'_1, \theta'_2, \theta'_3) \in \Theta^3$, satisfying the following condition: $\theta'_1 > \theta'_2 > \theta'_3 > \theta_1 > \theta_2 > \theta_3$. For all $S \subseteq \mathbf{N}$, we consider profiles $\theta(S) = (\theta_1(S), \theta_2(S), \theta_3(S))$ where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. We then calculate

$$\begin{aligned}\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta(S)) = {} & \mathbf{C}(\theta_1,\theta_2,\theta_3) - \mathbf{C}(\theta'_1,\theta_2,\theta_3) - \mathbf{C}(\theta_1,\theta'_2,\theta_3)\\ & -\mathbf{C}(\theta_1,\theta_2,\theta'_3) + \mathbf{C}(\theta'_1,\theta'_2,\theta_3) + \mathbf{C}(\theta'_1,\theta_2,\theta'_3)\\ & +\mathbf{C}(\theta_1,\theta'_2,\theta'_3) - \mathbf{C}(\theta'_1,\theta'_2,\theta'_3).\end{aligned}$$

We use the general criterion with identical servicing cost and non-identical types, specified in the previous paragraph, to find the efficient queue in the eight different states. The minimum costs in these states are: $C(\theta_1, \theta_2, \theta_3) = \theta_1 F(\bar{s}) + \theta_2 F(2\bar{s}) + \theta_3 F(3\bar{s})$, $C(\theta'_1, \theta_2, \theta_3) = \theta'_1 F(\bar{s}) + \theta_2 F(2\bar{s}) + \theta_3 F(3\bar{s})$, $C(\theta_1, \theta'_2, \theta_3) = \theta_1 F(2\bar{s}) + \theta'_2 F(\bar{s}) + \theta_3 F(3\bar{s})$, $C(\theta_1, \theta_2, \theta'_3) = \theta_1 F(2\bar{s}) + \theta_2 F(3\bar{s}) + \theta'_3 F(\bar{s})$, $C(\theta'_1, \theta'_2, \theta_3) = \theta'_1 F(\bar{s}) + \theta'_2 F(2\bar{s}) + \theta_3 F(3\bar{s})$, $C(\theta'_1, \theta_2, \theta'_3) = \theta'_1 F(\bar{s}) + \theta_2 F(3\bar{s}) +$

[3] It is quite easy to observe that if units have identical types then one can easily impose a tie breaking rule to calculate the efficient queue. It is important to note that finding the efficient queue is quite difficult if the servicing cost is different for different units and if the cost function does not satisfy the independence property. For such costs one cannot find an algorithm for calculating the efficient queue.

$\theta'_3 F(2\bar{s})$, $C(\theta_1, \theta'_2, \theta'_3) = \theta_1 F(3\bar{s}) + \theta'_2 F(\bar{s}) + \theta'_3 F(2\bar{s})$ and finally, $C(\theta'_1, \theta'_2, \theta'_3) = \theta'_1 F(\bar{s}) + \theta'_2 F(2\bar{s}) + \theta'_3 F(3\bar{s})$. After appropriate substitution and simplification in $\sum_{S\subseteq N}(-1)^{|S|}\mathbf{C}(\theta(S))$, we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta(S)) = \{\theta_1 - \theta'_3\}\sum_{k=1}^{3}(-1)^{k-1}\binom{3-1}{k-1}F(k\bar{s}) \tag{3.4}$$

From the construction of the profiles θ and θ', we know that $\theta_1 \neq \theta'_3$. From Lemma 3.1 we know that for first best implementability it is necessary that $\sum_{S\subseteq \mathbf{N}} (-1)^{|S|}\mathbf{C}(\theta(S)) = 0$. Therefore, from Eq. 3.4, it follows that $\sum_{k=1}^{3} (-1)^{k-1}\binom{3-1}{k-1}F(k\bar{s}) = 0$. Thus, we have proved Lemma 3.2 for a sequencing problem with three units.

Proof of Lemma 3.2. To prove the lemma, we first construct two profiles and then apply Lemma 3.1. Consider a servicing time vector where the servicing time of each unit is the same, that is, consider $\bar{s} = (s_1 = \bar{s}, \ldots, s_n = \bar{s})$. Let $\theta = (\theta_1, \ldots, \theta_n)$ and $\theta' = (\theta'_1, \ldots, \theta'_n)$ be any two profiles satisfying the following condition: $\theta'_1 > \theta'_2 > \ldots > \theta'_n > \theta_1 > \theta_2 > \ldots > \theta_n$. For all $S \subseteq \mathbf{N}$, we consider profiles $\theta(S) = (\theta_1(S), \ldots, \theta_j(S), \ldots, \theta_n(S))$, where $\theta_j(S) = \theta_j$ if $j \notin S$ and $\theta_j(S) = \theta'_j$ if $j \in S$. Since the servicing costs are taken to be identical for all $j \in \mathbf{N}$ and unit types are non-identical, we use the general criterion to derive the efficient queue in each state. Observe that, for all $S \subseteq \mathbf{N} - \{1\}$, with profiles $(\theta_1, \theta_{-1}(S))$, $\sigma^*_1(\theta_1, \theta_{-1}(S)) = |S|+1$. Thus, $\sum_{S\subseteq \mathbf{N}/\{1\}}(-1)^{|S|}C(S_1(\sigma^*(\theta_1, \theta_{-1}(S))); \theta_1) = \sum_{|S|=0}^{n-1}(-1)^{|S|}\binom{n-1}{|S|}F((|S|+1)\bar{s})\theta_1$. Moreover, for all $S \subseteq \mathbf{N}$ such that $n \in S$, that is, for profiles $(\theta'_n, \theta_{-n}(S))$, $\sigma^*_n(\theta'_n, \theta_{-n}(S)) = |S|$ and hence, it follows that, $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}C(S_n(\sigma^*(\theta'_n, \theta_{-n}(S))); \theta'_n) = \sum_{|S|=1}^{n}(-1)^{|S|}\binom{n-1}{|S|} F(|S|\bar{s})\theta'_n$. Finally, for all other types $x_j \in \{\theta_2, \ldots, \theta_n, \theta'_1, \ldots, \theta'_{n-1}\} \equiv \tilde{\mathbf{T}}$, if the sets $\{m_1, \ldots, m_p\}$, all subsets of $\mathbf{N} - \{j\}$, are such that $\sigma^*_j(x_j, \theta_{-j}(m_q)) = k$, for all $q \in \{1, \ldots, p\}$, then $\sum_{q=1}^{p}(-1)^{m_q}F(k\bar{s}) = 0$ because $\sum_{q=1}^{p}(-1)^{m_q} = 0$. Thus, for all $x_j \in \tilde{\mathbf{T}}$, $\sum_{S\subseteq \mathbf{N}/\{j\}}(-1)^{|S|}C(S_j(\sigma^*(x_j, \theta_{-j}(S))); x_j) = 0$. Therefore, the sum $\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta(S))$ is independent of all $x_j \in \tilde{\mathbf{T}}$. Combining all these observations we get

$$\sum_{S\subseteq \mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta(S)) = \{\theta_1 - \theta'_n\}\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}F(k\bar{s}). \tag{3.5}$$

Applying Lemma 3.1 and using $\theta_1 \neq \theta'_n$ in condition (3.5) we get condition (3.3). □

Condition (3.3) in Lemma 3.2 is a combinatorial condition on the time cost function F. The meaning of this condition will become explicit from the following discussion. Define $\Delta(h)F(x)$ as $\Delta(h)F(x) = F(x+h) - F(x)$. Thus, $\Delta(h)F(x)$ measures the increase in time cost as one moves from time x to time $x+h$. Using this definition observe that $\Delta(x)F(x) = F(2x) - F(x)$. Similarly, $\Delta^2(x)F(x)$ is given by $\Delta^2(x)F(x) = \Delta(x)[\Delta(x)F(x)] = \Delta(x)[F(2x) - F(x)] = F(3x) - 2F(2x) + F(x)$. We can similarly derive $\Delta^3(x)F(x)$, $\Delta^4(x)F(x)$ and so on. It is now quite easy to verify that condition (3.3) can be rewritten as $\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}F(kx) = \Delta^{n-1}(x)F(x) = 0$. Thus, F satisfies Lemma 3.2 if the $(n-1)$th order difference is zero.

Remark 3.2. The most obvious implication of Lemma 3.2 is that for a sequencing problem with two units, condition (3.3) holds only if, for all $x > 0, F(x) - F(2x) = 0$, that is, $F(x) = c$. However, $F(x) = c$ for all $x > 0$, is a violation of our assumption that F is strictly increasing in $\mathbf{R}_{++}$. Therefore, there does not exist a sequencing problem $\Omega = \langle \mathbf{N} = \{1, 2\}, F, (\Theta; \mathbf{R}_{++})\rangle$, with strictly increasing F, that satisfies condition (3.3) in Lemma 3.2. Thus, *a sequencing problem with two units is not first best implementable*.

Remark 3.3. Another implication of Lemma 3.2 is that for a sequencing problem $\Omega = \langle \mathbf{N} = \{1, 2, 3\}, F, (\Theta; \mathbf{R}_{++})\rangle$, $F(\in \mathbf{F})$ must be linear. For a sequencing problem with three units, Lemma 3.2 implies that, for all $x \in \mathbf{R}_{++}$, $F(x) + F(3x) = 2F(2x)$. From this condition it is obvious that, if $F \in \mathbf{F}$, then $F = F^l$, where $F^l(x) = a_1x + a_0$ for all $x \in \mathbf{R}_{++}$. Therefore, *Lemma 3.2 proves that if there are three units then a sequencing problem is first best implementable only if the cost function is linear.*

Lemma 3.2 is not enough to prove the only if part of Theorem 3.1 for a sequencing problem with more than three units. For example, consider $\Omega^q = \langle \mathbf{N} = \{1, 2, 3, 4\}, F^q, (\Theta; \mathbf{R}_{++})\rangle$, such that the time cost function $F^q \in \mathbf{F}$ is quadratic. Thus, the function F^q is such that, for all $x \in \mathbf{R}_{++}$, $F^q(x) = a_0 + a_1x + a_2x^2$ and $F^q(x+h) > F^q(x)$, for all $h > 0$. It is obvious that for a sequencing problem with four units, $F^q \in \mathbf{F}$ satisfies condition (3.3) of Lemma 3.2.[4] To prove the only if part of Theorem 3.1 for sequencing problems with more than four units, we need another result which is summarised in the next lemma.

Lemma 3.3. *A sequencing problem $\Omega = \langle \mathbf{N}, F, (\Theta; \mathbf{R}_{++})\rangle$ with at least four units is first best implementable only if it satisfies the independence property.*

Before proving Lemma 3.3 we introduce some more relevant definitions and observations. Consider the servicing time vector $s = (s_1, \ldots, s_n)$. We define the sum of servicing times of all units $p \in \mathbf{N} - \{j, l\}$ as $M(jl) = \sum_{p \neq \{j,l\}} s_p$. For units j and l, with servicing time s_j and s_l respectively, let $\hat{m}_{jl} \in [0, M(jl)]$ be a number such that $f(\hat{m}_{jl}; s_j, s_l) \geq f(y; s_j, s_l)$, for all $y \in [0, M(jl)]$. Note that $\hat{m}_{jl}$ always exists since $f(y; s_j, s_l)$ is continuous and we are considering all y in the closed and bounded interval $[0, M(jl)]$. We define $H(s) \geq f(\hat{m}_{jl}; s_j, s_l)$ for all $j \in \mathbf{N}$ and for all $l \in \mathbf{N} - \{j\}$. Therefore, $H(s)$ is at least as large as the highest value that the ratio function f, corresponding to F, can take given the servicing time vector s. Similarly, let $\bar{m}_{jl} \in [0, M(jl)]$ be a number such that $f(\bar{m}_{jl}; s_j, s_l) \leq f(y; s_j, s_l)$

[4] In general for a sequencing problem $\Omega^m = \langle \mathbf{N}, F^m, (\Theta; \mathbf{R}_{++})\rangle$ having polynomial time cost functions of order m (that is $F^m(x) = \sum_{i=0}^{m} a_i x^i$) where $m \leq n - 2$ satisfies condition (3.3). Thus if we have a sequencing problem with three units, then polynomial of order $m = 1$, that is, $F = F^l$ satisfies condition (3.3). If we have a sequencing problem with four units, then polynomials of order $m = \{1, 2\}$, that is, $F = F^l$ and $F = F^q$ satisfies condition (3.3) and so on.

for all $y \in [0, M(jl)]$ and we define $L(s) \leq f(\bar{m}_{jl}; s_j, s_l)$ for all $j \in \mathbf{N}$ and for all $l \in \mathbf{N} - \{j\}$. $L(s)$ is at most as small as the lowest value that the ratio function f can take given the servicing time vector s. Using the numbers $H(s)$ and $L(s)$, we make the following observations about the efficient ordering.

Observation [1]. Given the servicing time vector $s = (s_1, \ldots, s_n)$, if for any two units j and l, $\frac{\theta_j}{\theta_l} > H(s)$, then from the construction of $H(s)$ we know that $H(s) \geq f(\hat{s}; s_j, s_l)$, for all $\hat{s} \in [0, M(jl)]$. Therefore, $\frac{\theta_j}{\theta_l} > f(\hat{s}; s_j, s_l)$, for all $\hat{s} \in [0, M(jl)]$ since $\frac{\theta_j}{\theta_l} > H(s)$. Thus, given s, *if* $\frac{\theta_j}{\theta_l} > H(s)$, *then* $\sigma_j^*(\theta_j, \theta_l, \tilde{\theta}_{-j-l}) < \sigma_l^*(\theta_j, \theta_l, \tilde{\theta}_{-j-l})$, *for all* $\tilde{\theta}_{-j-l} \in \Theta^{n-2}$.

Observation [2]. Given the servicing time vector $s = (s_1, \ldots, s_n)$, if for any two units j and l, $\frac{\theta_j}{\theta_l} < L(s)$, then from the construction of $L(s)$ we know that $L(s) \leq f(\hat{s}; s_j, s_l)$, for all $\hat{s} \in [0, M(jl)]$. Therefore, $\frac{\theta_j}{\theta_l} < f(\hat{s}; s_j, s_l)$, for all $\hat{s} \in [0, M(jl)]$ since $\frac{\theta_j}{\theta_l} < L(s)$. Thus, given s, *if* $\frac{\theta_j}{\theta_l} < L(s)$ *then* $\sigma_j^*(\theta_j, \theta_l, \tilde{\theta}_{-j-l}) > \sigma_l^*(\theta_j, \theta_l, \tilde{\theta}_{-j-l})$, *for all* $\tilde{\theta}_{-j-l} \in \Theta^{n-2}$.

These two observations will be used in proving Lemma 3.3.

Proof of Lemma 3.3. Consider a sequencing problem Ω, with at least four units, that fails to satisfy the independence property. To prove the lemma, we show that for such a sequencing problem Ω, there exists a servicing time vector s and there exist profiles, for which the condition in Lemma 3.1 is violated. Since Ω, does not satisfy the independence property, there exist $s_1 > s_2 > 0$ and an appropriate selection of interval $[0, A]$, such that, the ratio function $f(y; s_1, s_2)$ is monotonic and non-constant in y, whenever $y \in [0, A]$. Given that F is continuous and strictly increasing and the denominator of $f(y; s_1, s_2)$ is non-zero, for all y, it follows that $f(y; s_1, s_2)$ is continuous, monotonic and non-constant in $y \in [0, A]$. Therefore, on $y \in [0, A]$, $f(y; s_1, s_2)$ satisfies at least one of the following four conditions:

1. There exists $B \in (0, A)$, such that $f(y; s_1, s_2)$ is constant in y, if $y \in [0, B)$ and strictly increasing in y, whenever $y \in [B, A]$.
2. $f(y; s_1, s_2)$ is strictly increasing in $y \in [0, A]$.
3. There exists $\bar{B} \in (0, A)$, such that $f(y; s_1, s_2)$ is constant in y, if $y \in [0, \bar{B})$ and strictly decreasing in y, whenever $y \in [\bar{B}, A]$.
4. $f(y; s_1, s_2)$ is strictly decreasing in $y \in [0, A]$.

We consider each of these conditions in different steps.

Step 1. In this step we consider condition 1. Since the sequencing problem under consideration is the one that fails to satisfy the independence property, we construct profiles in such a way that the relative queue position of two particular units, with given types, change as the type of other units change. This type of construction leads to a violation of Lemma 3.1.

Let s_1 and s_2, in condition 1, represent the servicing time of units 1 and 2 respectively. Let the servicing time of unit 3 be $s_3 = A - B$ and the servicing time of all other units $j \in \mathbf{N} - \{1, 2, 3\}$ be $s_j = \frac{B}{n-3} \equiv \bar{s}$. Thus, the servicing time vector is given by $\bar{s} = (s_1, s_2, s_3, \bar{s}, \ldots, \bar{s})$. Note that the servicing time vector is constructed in such a way that $(n-3)\bar{s}+s_3=A$. From condition 1 and from the construction of $\bar{s}$ it follows that $f(k\bar{s};s_1,s_2)\leq f(k\bar{s}+s_3;s_1,s_2)$ and $f(k\bar{s};s_1,s_2)<f(A;s_1,s_2)$ for all $k\in\{0,\ldots,n-3\}$ and $f(k\bar{s}+s_3;s_1,s_2)<f(A;s_1,s_2)$ for all $k\in\{0,\ldots,n-4\}$. Let $D=(max[f((n-3)\bar{s};s_1,s_2),f((n-4)\bar{s}+s_3;s_1,s_2)],f(A;s_1,s_2))$. Note that the interval always exists since $max\{(n-3)\bar{s},(n-4)\bar{s}+s_3\}<A$ implies that $max[f((n-3)\bar{s};s_1,s_2),f((n-4)\bar{s}+s_3;s_1,s_2)]<f(A;s_1,s_2))$. Consider two numbers $x_a(1)$ and $x_b(1)$, such that, $x_a(1)\in D$, $x_b(1)\in D$ and $x_a(1)<x_b(1)$. Observe that since $s_1>s_2$, $x_r(1)>1$, for all $r\in\{a,b\}$. Using the numbers $x_a(1)$ and $x_b(1)$, we define $x(2)=\frac{1}{2}x_a(1)L(\bar{s})$ and $x(3)=x_b(1)+x_b(1)H(\bar{s})$. Consider two profiles $\theta^r=(\theta_1^r=x_r(1),\theta_2^r=1,\theta_3^r=x(3),\ldots,\theta_n^r=x(3))$ and $\theta'=(\theta'_1=1,\theta'_2=x(2),\ldots,\theta'_n=x(2))$ where $r\in\{a,b\}$. Define, for $S\subseteq\mathbf{N}$, a type $\theta_j^r(S)=\theta_j^r$ if $j\notin S$ and $\theta_j^r(S)=\theta'_j$ if $j\in S$. Therefore, for $S\subseteq\mathbf{N}$, a state $\theta^r(S)$ is of the form $(\theta_1^r(S),\ldots,\theta_n^r(S))$. Now we consider the terms containing $\theta_1^r(=x_r(1))$ in the sum $\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^r(S))$. From the construction of $x(2)$ it follows that $\frac{x(2)}{x_r(1)}<L(\bar{s})$. Therefore, from Observation [2] it is obvious that, for all $S\subseteq\mathbf{N}-\{1\}$, such that $j\in\mathbf{N}-\{1\}$ and $j\in S$, $\sigma_1^*(\theta_1^r,\theta_{-1}^r(S))<\sigma_j^*(\theta_1^r,\theta_{-1}^r(S))$. Similarly, from the construction of $x(3)$ we get $\frac{x(3)}{x_r(1)}>H(\bar{s})$. Therefore, from Observation [1] it is obvious that, for all $S\subseteq\mathbf{N}-\{1\}$, such that $j\in\mathbf{N}-\{1,2\}$ and $j\notin S$, $\sigma_1^*(\theta_1^r,\theta_{-1}^r(S))>\sigma_j^*(\theta_1^r,\theta_{-1}^r(S))$. Now consider all possible $S\subseteq\mathbf{N}-\{1\}$, such that $2\notin S$. We start with $S=\phi$, that is, $\theta^r(S)=\theta^r$. Note that since $x(3)=x_r(1)+x_r(1)H(\bar{s})$ and $x_r(1)>1$, $x(3)>H(\bar{s})$. Therefore, for all $j\in\mathbf{N}-\{1,2\}$, $\frac{\theta_j^r}{\theta_2^r}=x(3)>f(\hat{s};s_j,s_2)$, for all $\hat{s}\in[0,M(j2)]$. Thus $\sigma_2^*(\theta^r)>\sigma_j^*(\theta^r)$, for all $j\in\mathbf{N}-\{1,2\}$. Again, from the construction of $x_r(1)$, it follows that $\frac{\theta_1^r}{\theta_2^r}=x_r(1)<f(A;s_1,s_2)$. Therefore, given $\sigma_2^*(\theta^r)>\sigma_j^*(\theta^r)$ and $\sigma_1^*(\theta^r)>\sigma_j^*(\theta^r)$, for all $j\in\mathbf{N}-\{1,2\}$ and $x_r(1)<f(A;s_1,s_2)$, we obtain that $\sigma_1^*(\theta^r)=n>\sigma_2^*(\theta^r)=n-1$. From the construction of $x_r(1)$, we also know that, $\frac{\theta_1^r}{\theta_2^r}=x_r(1)>max[f((n-3)\bar{s};s_1,s_2),f((n-4)\bar{s}+s_3;s_1,s_2)]$. Therefore, if $S\subseteq\mathbf{N}-\{1\}$, $S\neq\phi$ and $2\notin S$, then $\sigma_1^*(\theta^r(S))<\sigma_2^*(\theta^r(S))$. Combining all these observations and simplifying the sum $\sum_{S\subseteq\mathbf{N}-\{1\}}(-1)^{|S|}C(S_1(\sigma^*(\theta_1^r,\theta_{-1}^r(S)));\theta_1^r)$, we get for all $r\in\{a,b\}$,

$$\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^r(S))=x_r(1)[F(s_1+s_2+A)-F(s_1+A)]+Z \quad (3.6)$$

where Z is the sum of terms that are independent of $x_r(1)$, in the sum $\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^r(S))$. From Lemma 3.1 we know that for first best implementability of Ω, it is necessary that $\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^r(S))=0$, for all $r\in\{a,b\}$. Therefore, it is obvious that for first best implementability it is necessary that

$$\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^a(S))-\sum_{S\subseteq\mathbf{N}}(-1)^{|S|}\mathbf{C}(\theta^b(S))=0 \quad (3.7)$$

Simplifying condition (3.7) using (3.6), we get

$$\{x_a(1)-x_b(1)\}[F(s_1+s_2+A)-F(s_1+A)]=0 \quad (3.8)$$

Condition (3.8) cannot hold since from the construction of $x_r(1)$ and from our assumption that F is strictly increasing in $\mathbf{R}_{++}$, it follows that the left hand side of condition (3.8) is strictly negative. Therefore, if condition 1 holds then, given $\bar{s}$, there exist profiles for which Lemma 3.1 is not satisfied. Observe that when unit 1's type is $\theta_1^r = x_r(1)$ and that of unit 2 is $\theta_2^r = 1$, the construction of profiles $\theta_{-1-2}(S)$, for all $S \subseteq \mathbf{N} - \{1,2\}$ are such that:

1. $\sigma_1^*(\theta_1^r, \theta_2^r, \theta_{-1-2}^r(S)) > \sigma_2^*(\theta_1^r, \theta_2^r, \theta_{-1-2}^r(S))$, if $[S = \phi]$ (that is, if the state is $\theta^r(S) = \theta^r$) and
2. $\sigma_1^*(\theta_1^r, \theta_2^r, \theta_{-1-2}^r(S)) < \sigma_2^*(\theta_1^r, \theta_2^r, \theta_{-1-2}^r(S))$, for all $[S \subseteq \mathbf{N} - \{1,2\}$ and $S \neq \phi]$.

Therefore, the change in relative queue positions of units 1 and 2, given their types θ_1^r and θ_2^r respectively, with the change in the types of other units is crucial for the result.

Step 2. In this step we consider condition 2. Let s_1 and s_2 in condition 2 represent the servicing time of units 1 and 2 respectively. Let the servicing time of all other units $j \in \mathbf{N} - \{1,2\}$ be $s_j = \frac{A}{n-2} \equiv \bar{s}$. Therefore, the servicing time vector is given by $\bar{s} = (s_1, s_2, \bar{s}, \bar{s}, \ldots, \bar{s})$. By replacing s_3 in Step [1] with $\bar{s}$ and by following the same steps we get the result.

Step 3. In this step we consider condition 3. Let s_1 and s_2, in condition 3, represent the servicing time of units 1 and 2 respectively. Let the servicing time of unit 3 be $s_3 = A - B$ and the servicing time of all other units $j \in \mathbf{N} - \{1,2,3\}$ be $s_j = \frac{B}{n-3} \equiv \bar{s}$. Thus, the servicing time vector is given by $\bar{s} = (s_1, s_2, s_3, \bar{s}, \ldots, \bar{s})$. From condition (3) and from the construction of $\bar{s}$, it follows that, $f(k\bar{s}; s_1, s_2) \geq f(k\bar{s} + s_3; s_1, s_2)$ and $f(k\bar{s}; s_1, s_2) > f(A; s_1, s_2)$, for all $k \in \{0, \ldots, n-3\}$ and $f(k\bar{s} + s_3; s_1, s_2) > f(A; s_1, s_2)$, for all $k \in \{0, \ldots, n-4\}$. Let $D = (f(A; s_1, s_2), min[f((n-3)\bar{s}; s_1, s_2), f((n-4)\bar{s} + s_3; s_1, s_2)])$. Note that the interval always exists since $min\{(n-3)\bar{s}, (n-4)\bar{s} + s_3\} < A$ implies that $min[f((n-3)\bar{s}; s_1, s_2), f((n-4)\bar{s} + s_3; s_1, s_2)] > f(A; s_1, s_2))$. Consider two numbers $x_a(1)$ and $x_b(1)$, such that, $x_a(1) \in D$, $x_b(1) \in D$ and $x_a(1) < x_b(1)$. Note that since $s_1 > s_2$, $x_r(1) > 1$, for all $r \in \{a, b\}$. Using the numbers $x_a(1)$ and $x_b(1)$, we define, $x(2) = \frac{1}{2}x_a(1)L(\bar{s})$ and $x(3) = x_b(1) + x_b(1)H(\bar{s})$. Consider two profiles $\theta^r = (\theta_1^r = x_r(1), \theta_2^r = x(3), \theta_3^r = x(3), \ldots, \theta_n^r = x(3))$ and $\theta' = (\theta_1' = 1, \theta_2' = 1, \theta_3' = x(2), \ldots, \theta_n' = x(2))$ where $r \in \{a, b\}$. Define, for $S \subseteq \mathbf{N}$, a type $\theta_j^r(S) = \theta_j^r$ if $j \notin S$ and $\theta_j^r(S) = \theta_j'$ if $j \in S$. Therefore, for each $S \subseteq \mathbf{N}$, a state $\theta^r(S)$ is of the form $(\theta_1^r(S), \ldots, \theta_n^r(S))$. Now we consider the terms containing $\theta_1^r (= x_r(1))$, in the sum $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^r(S))$. Consider all possible $S \subseteq \mathbf{N} - \{1\}$. From the construction of $x(2)$ and from Observation [2], it follows that, for all $S \subseteq \mathbf{N} - \{1\}$, such that, $j \in \mathbf{N} - \{1,2\}$ and $j \in S$, $\sigma_1^*(\theta_1^r, \theta_{-1}^r(S)) < \sigma_j^*(\theta_1^r, \theta_{-1}^r(S))$. Similarly, from the construction of $x(3)$ and from Observation [1], it follows that, for all $S \subseteq \mathbf{N} - \{1\}$, such that, $j \in \mathbf{N} - \{1\}$ and $j \notin S$, $\sigma_1^*(\theta_1^r, \theta_{-1}^r(S)) > \sigma_j^*(\theta_1^r, \theta_{-1}^r(S))$. Now consider $S \in \mathbf{N} - \{1\}$, such that, $2 \in S$. First we consider $\bar{S} = \{2\}$, that is, we consider the state $\theta^r(\bar{S}) = (\theta_1^r = x_r(1), \theta_2' = 1, \theta_3^r = x(3), \ldots, \theta_n^r = x(3))$. Observe that, for all $j \in \mathbf{N} - \{1,2\}$, $\frac{\theta_j^r}{\theta_2'} = \frac{x(3)}{1} > H(\bar{s})$, since $x(3) = x_b(1) +$

$x_b(1)H(\bar{s})$ and $x_b(1) > 1$. Therefore, if $\bar{S} = \{2\}$, then $\sigma_2^*(\theta^r(\bar{S})) > \sigma_j^*(\theta^r(\bar{S}))$, for all $j \in \mathbf{N} - \{1, 2\}$. We have already established that if $\bar{S} = \{2\}$, then $\sigma_1^*(\theta^r(\bar{S})) > \sigma_j^*(\theta^r(\bar{S}))$, for all $j \in \mathbf{N} - \{1, 2\}$. From the construction of $x_r(1)$, we get, $\frac{\theta_1^r}{\theta_2^r} = \frac{x_r(1)}{1} = x_r(1) > f(A; s_1, s_2)$. Therefore, if $\bar{S} = \{2\}$, then given $x_r(1) > f(A; s_1, s_2)$, $\sigma_1^*(\theta^r(\bar{S})) > \sigma_j^*(\theta^r(\bar{S}))$ and $\sigma_2^*(\theta^r(\bar{S})) > \sigma_j^*(\theta^r(\bar{S}))$, for all $j \in \mathbf{N} - \{1, 2\}$, we get $\sigma_1^*(\theta^r(\bar{S})) = n - 1 < \sigma_2^*(\theta^r(\bar{S})) = n$. From the construction of $\theta_1^r = x_r(1)$, we know that $x_r(1) < min[f((n-3)\bar{s}; s_1, s_2), f((n-4)\bar{s} + s_3; s_1, s_2)]$. Therefore, for all $S \subseteq \mathbf{N} - \{1\}$, such that, $S \neq \bar{S}$ and $2 \in S$, we get $\sigma_1^*(\theta^r(S)) > \sigma_2^*(\theta^r(S))$. Combining all these observations and simplifying the sum $\sum_{S \subseteq \mathbf{N} - \{1\}} (-1)^{|S|} C(S_1(\sigma^*(\theta_1^r, \theta_{-1}^r(S))); \theta_1^r)$ we get, for all $r \in \{a, b\}$,

$$\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^r(S)) = x_r(1)[F(s_1 + s_2 + A) - F(s_1 + A)] + Z \tag{3.9}$$

where Z is the sum of terms that are independent of $x_r(1)$ in the sum $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^r(S))$. From Lemma 3.1 we know that for first best implementability of Ω, it is necessary that $\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^r(S)) = 0$, for all $r \in \{a, b\}$. Therefore, it is obvious that for first best implementability it is necessary that

$$\sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^a(S)) - \sum_{S \subseteq \mathbf{N}} (-1)^{|S|} \mathbf{C}(\theta^b(S)) = 0 \tag{3.10}$$

Simplifying condition (3.10) using (3.9) we get

$$\{x_a(1) - x_b(1)\}[F(s_1 + s_2 + A) - F(s_1 + A)] = 0 \tag{3.11}$$

Condition (3.11) is not true since from the construction of $x_r(1)$ and from our assumption that F is strictly increasing in $\mathbf{R}_{++}$, it follows that, the left hand side of condition (3.11) is strictly negative. Therefore, if condition 3 holds then, given $\bar{s}$, there exist profiles for which Lemma 3.1 is not satisfied. Observe that when unit 1's type is $\theta_1^r = x_r(1)$ and that of unit 2 is $\theta_2' = 1$, the construction of profiles $\theta_{-1-2}(S)$, for all $[S \subseteq \mathbf{N} - \{1\}$ and $2 \in S]$ are such that:

1. $\sigma_1^*(\theta_1^r, \theta_2', \theta_{-1-2}^r(S)) < \sigma_2^*(\theta_1^r, \theta_2', \theta_{-1-2}^r(S))$, if $[S = \{2\}]$ and
2. $\sigma_1^*(\theta_1^r, \theta_2', \theta_{-1-2}^r(S)) > \sigma_2^*(\theta_1^r, \theta_2', \theta_{-1-2}^r(S))$, for all $[S \subseteq \mathbf{N} - \{1\}$ and $2 \in S]$.

Therefore, the construction that leads to the result in this step is similar to that in Step [1]. Here, the change in relative queue positions of units 1 and 2, given their types θ_1^r and θ_2' respectively, with the change in the types of other units is crucial for the result.

Step 4. In this step we consider condition 4. Let s_1 and s_2 in condition 4 represent the servicing time of units 1 and 2 respectively. Let the servicing time of all other units $j \in \mathbf{N} - \{1, 2\}$ be $s_j = \frac{A}{n-2} \equiv \bar{s}$. Therefore, the servicing cost vector is given by $\bar{s} = (s_1, s_2, \bar{s}, \bar{s}, \ldots, \bar{s})$. By replacing s_3 in Step [3] with $\bar{s}$ and by following the same steps we get the result. □

Finally, we come to the proof of the only if part of Theorem 3.1. We show that the necessity of condition (3.3) in Lemma 3.2 and the necessity of the independence property (as derived in Lemma 3.3), for first best implementability of a sequencing problem together imply that the cost function F must be linear.

Proof of Theorem 3.1. From Remark 3.2, it follows that for first best implementability of a sequencing problem, it is necessary that there are at least three units. For a sequencing problem Ω with three units the proof of the only if part of theorem follows from Remark 3.3. To prove the theorem for sequencing problems Ω with at least four units, we show that Lemmas 3.2 and 3.3 imply that $F = F^l$. By rewriting condition (3.3) in Lemma 3.2 in terms of first difference $\Delta(x)F(kx)$, we get

$$\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n-2}{k-1}\Delta(x)F(kx) = 0 \tag{3.12}$$

Consider the ratio function $f(y; 2x, x)$. From Lemma 3.3, we know that, for first best implementability of the sequencing problem Ω, $f(y; 2x, x) = g(2x, x)$, for all $y \geq 0$. After simplifying the relation $f(kx; 2x, x) = g(2x, x)$ we get, for all $k \in \{1, \dots n-1\}$ and for all $x > 0$,

$$\Delta(x)F(kx) = r^{k-1}\Delta(x)F(x) \tag{3.13}$$

where $g(2x, x) = \frac{1}{r} + 1$.[5] By substituting (3.13) in (3.12) and then simplifying it, we get, $(1-r)^{n-2}\Delta(x)F(x) = 0$, for all $x > 0$. Therefore, $r = 1$ simply because F is strictly increasing implies that $\Delta(x)F(x) > 0$. By substituting $r = 1$ in (3.13) we get

$$F((k+1)x) - F(kx) = F(2x) - F(x) \tag{3.17}$$

for all $k \in \{1, \dots, n-1\}$ and for all $x > 0$. From condition (3.17) it is obvious that $F = F^l$. □

We have thus derived that a sequencing problem with continuous and strictly increasing time cost function is first best implementable only if the cost function is linear and there are at least three units. The sufficiency part of the theorem, that is,

[5] For all $k \in \{1, \dots, n-2\}$ consider,

$$f((k-1)x; 2x, x) = \frac{\Delta(2x)F(kx)}{\Delta(x)F((k+1)x)} \tag{3.14}$$

By rewriting $\Delta(2x)F(kx)$ as $\Delta(x)F((k+1)x) + \Delta(x)F(kx)$ in (3.14) and substituting $f((k-1)x; 2x, x) = g(2x, x) = \frac{1}{r} + 1$ we get

$$\frac{\Delta(x)F(kx)}{\Delta(x)F((k+1)x)} = \frac{1}{r} \tag{3.15}$$

for all $k \in \{1, \dots, n-2\}$. Solving (3.15) recurssively we get

$$\Delta(x)F(kx) = r^{k-1}\Delta(x)F(x) \tag{3.16}$$

for all $k \in \{1, \dots n-1\}$ and for all $x > 0$.

if there are three units and if the cost function is linear then a sequencing problem is first best implementable was established by Suijs (1996).

To complete our analysis, we now provide a Groves transfer that first best implements a sequencing problem with linear cost. Consider a linear cost sequencing problem Ω^l with at least three units. In this problem, $F^l(x) = a_0 + a_1 x$ and $a_1 > 0$. Consider a servicing time vector $s = (s_1, \ldots, s_n)$ and a state $\theta \in \mathbf{\Theta}^n$. Let the transfer for unit $j \in \mathbf{N}$ be

$$t_j(\theta) = a_1 \sum_{p \in P(\sigma^*(\theta), j)} \theta_p s_j - \frac{a_1}{n-2} \sum_{l \neq j} \theta_l \left\{ \sum_{q \in Q_l(\theta, j)} s_q \right\} \tag{3.18}$$

where $Q_l(\theta, j) = \{q \in \mathbf{N} - \{j, l\} \mid q \notin P(\sigma^*(\theta), l)\}$.

Before providing an explicit form of transfer (3.18), we propose an algorithm to calculate an efficient queue. Since a sequencing problem with linear cost satisfies the independence property, calculation of the efficient queue is very transparent. It can be obtained by considering the *urgency index* $\mathrm{u_j} = \frac{\theta_j}{s_j}$, for all $j \in \mathbf{N}$. In particular, if $\mathrm{u_j} = \frac{\theta_j}{s_j} > \mathrm{u_l} = \frac{\theta_l}{s_l}$, then $\sigma_j^*(\theta_j, \theta_l, \theta_{-j-l}) < \sigma_l^*(\theta_j, \theta_l, \theta_{-j-l})$, for all $\theta_{-j-l} \in \Theta^{n-1}$. Ties can be broken by considering the natural ordering, that is, if $\mathrm{u_j} = \mathrm{u_l}$ then $\sigma_j^*(\theta_j, \theta_l, \theta_{-j-l}) < \sigma_l^*(\theta_j, \theta_l, \theta_{-j-l})$ if $j < l$ (See Curiel et al. 1989). For example, given s, if state θ is such that $\frac{\theta_1}{s_1} \geq \frac{\theta_2}{s_2} \geq \ldots \geq \frac{\theta_n}{s_n}$, then $\sigma^*(\theta) = (\sigma_1^*(\theta) = 1, \sigma_2^*(\theta) = 2, \ldots, \sigma_n^*(\theta) = n)$. To write an explicit form of the transfer for each state $\theta \in \Theta^n$, we consider the "inverse" of the order σ^*, suppose μ is a permutation such that $\frac{\theta_{\mu(1)}}{s_{\mu(1)}} \geq \frac{\theta_{\mu(2)}}{s_{\mu(2)}} \geq \ldots \geq \frac{\theta_{\mu(n)}}{s_{\mu(n)}}$. Therefore, the transfer (3.18) can be rewritten in terms of the inverse ordering μ as

$$t_{\mu(k)}(\theta) = a_1 s_{\mu(k)} \left(\sum_{h<k} \theta_{\mu(h)} \right) - \frac{a_1}{n-2} \sum_{q \neq k} \left(\theta_{\mu(q)} \left\{ \sum_{\substack{r>q \\ r \neq k}} s_{\mu(r)} \right\} \right) \tag{3.19}$$

for all $k \in \{1, 2, \ldots, n\}$. It is quite simple but tedious to show that these transfers add up to zero, that is, $\sum_{k=1}^{n} t_{\mu(k)}(\theta) = 0$. Instead of proving this result formally, we concentrate on other important aspects. In the discussion that follows, we provide a detailed analysis of balancedness and incentive compatibility of the transfer scheme (3.19) for a linear cost sequencing problem with three units.

Consider a sequencing problem $\Omega^l = \langle \mathbf{N} = \{1, 2, 3\}, F^l, (\Theta; \mathbf{R}_{++}) \rangle$. In this problem, for each state $\theta \in \Theta^3$, with $\frac{\theta_{\mu(1)}}{s_{\mu(1)}} \geq \frac{\theta_{\mu(2)}}{s_{\mu(2)}} \geq \frac{\theta_{\mu(3)}}{s_{\mu(3)}}$, we get using the Eq. (3.19), $t_{\mu(1)}(\theta) = -a_1 \theta_{\mu(2)} s_{\mu(3)}$, $t_{\mu(2)}(\theta) = a_1 \theta_{\mu(1)} (s_{\mu(2)} - s_{\mu(3)})$ and $t_{\mu(3)}(\theta) = a_1 (\theta_{\mu(1)} + \theta_{\mu(2)}) s_{\mu(3)} - a_1 \theta_{\mu(1)} s_{\mu(2)}$. We now verify that this transfer scheme is incentive compatible. Consider a servicing vector $s = (s_1 = 3, s_2 = 2, s_3 = 1)$ and let the type vector be $\theta = (\theta_1 = 1, \theta_2 = 2, \theta_3 = 3)$. Observe that, given s and θ, $\mathrm{u_1} = \frac{1}{3} < \mathrm{u_2} = \frac{2}{2} = 1 < \mathrm{u_3} = \frac{3}{1} = 3$.[6] Therefore, $\sigma^*(\theta) = (\sigma_1^*(\theta) = 3, \sigma_2^*(\theta) = 2, \sigma_3^*(\theta) = 1)$ and the transfers

[6] Recall that $\mathrm{u_j} = \frac{\theta_j}{s_j}$, for all $j \in \mathbf{N}$.

are $t_1(\theta) = t_{\mu(3)}(\theta) = a_1(\theta_3 + \theta_2)s_1 - a_1\theta_3 s_2 = 9a_1$, $t_2(\theta) = t_{\mu(2)}(\theta) = a_1\theta_3(s_2 - s_1) = -3a_1$ and $t_3(\theta) = t_{\mu(1)}(\theta) = -a_1\theta_2 s_1 = -6a_1$. Observe that $\sum_{j=1}^{3} t_j(\theta) = \sum_{k=1}^{3} t_{\mu(k)}(\theta) = 0$. Given s and true type vector θ, we consider all possible deviations by unit 1 from its true type θ_1 and argue that the benefits to unit 1, from all these deviations, are non-positive. Define $B(\bar{\theta}_1, \theta_1; \theta_2, \theta_3)$, to be the benefit derived by unit 1 by deviating from its true type θ_1 to $\bar{\theta}_1$, given that the other two units have announced (θ_2, θ_3). Therefore, $B(\bar{\theta}_1, \theta_1; \theta_2, \theta_3) = U_1(\sigma^*(\bar{\theta}_1, \theta_2, \theta_3), t_1(\bar{\theta}_1, \theta_2, \theta_3); \theta_1) - U_1(\sigma^*(\theta), t_1(\theta); \theta_1)$. Consider a deviation by unit 1 from θ_1 to any type $\theta_1' < 3$. Note that, under this deviation, $\mathrm{u}_1 = \frac{\theta_1'}{3} < 1 < \mathrm{u}_2 = 1 < \mathrm{u}_3 = 3$. Therefore, $\sigma^*(\theta_1', \theta_2, \theta_3) = (\sigma_1^*(\theta_1', \theta_2, \theta_3) = 3, \sigma_2^*(\theta_1', \theta_2, \theta_3) = 2, \sigma_3^*(\theta_1', \theta_2, \theta_3) = 1) = \sigma^*(\theta)$ and hence $t_1(\theta_1', \theta_2, \theta_3) = t_1(\theta)$. Thus, $B(\theta_1', \theta_1; \theta_2, \theta_3) = 0$. Consider, the deviation by unit 1 from θ_1 to any type $\theta_1^\alpha \in [3, 9)$. Note that, under this deviation, $\mathrm{u}_2 = 1 \leq \mathrm{u}_1 = \frac{\theta_1^\alpha}{3} < \mathrm{u}_3 = 3$. Therefore, $\sigma^*(\theta_1^\alpha, \theta_2, \theta_3) = (\sigma_1^*(\theta_1^\alpha, \theta_2, \theta_3) = 2, \sigma_2^*(\theta_1^\alpha, \theta_2, \theta_3) = 3, \sigma_3^*(\theta_1^\alpha, \theta_2, \theta_3) = 1) \neq \sigma^*(\theta)$ and hence $t_1(\theta_1^\alpha, \theta_2, \theta_3) = a_1\theta_3(s_1 - s_2) = 3a_1 \neq t_1(\theta) = 9a_1$. Thus, $B(\theta_1^\alpha, \theta_1; \theta_2, \theta_3) = a_1\theta_1 s_2 + t_1(\theta_1^\alpha, \theta_2, \theta_3) - t_1(\theta) = 2a_1 + 3a_1 - 9a_1 = -4a_1 < 0$. Finally, consider the deviation $\theta_1^\beta \geq 9$. Note that, under this deviation, $\mathrm{u}_2 = 1 < \mathrm{u}_3 = 3 \leq \mathrm{u}_1 = \frac{\theta_1^\beta}{3}$. Therefore, $\sigma^*(\theta_1^\beta, \theta_2, \theta_3) = (\sigma_1^*(\theta_1^\beta, \theta_2, \theta_3) = 1, \sigma_2^*(\theta_1^\beta, \theta_2, \theta_3) = 3, \sigma_3^*(\theta_1^\beta, \theta_2, \theta_3) = 2) \neq \sigma^*(\theta)$ and hence $t_1(\theta_1^\beta, \theta_2, \theta_3) = -a_1\theta_3 s_2 = -6a_1 \neq t_1(\theta) = 9a_1$. Thus, $B(\theta_1^\beta, \theta_1; \theta_2, \theta_3) = a_1\theta_1(s_2 + s_3) + t_1(\theta_1^\beta, \theta_2, \theta_3) - t_1(\theta) = 3a_1 - 6a_1 - 9a_1 = -12a_1 < 0$. Therefore, for unit 1 with type θ_1, $B(\bar{\theta}_1, \theta_1; \theta_2, \theta_3) \leq 0$, for all $\bar{\theta}_1 \neq \theta_1$. By applying similar arguments we can check that neither unit 2 nor unit 3 can benefit by deviating from their true types. Therefore, the transfer scheme is both budget balancing and incentive compatible.

4 Conclusion

We can make a comparative study of a linear cost sequencing problem with that of the classic incentive problem of non-excludable public goods where, like the sequencing problem, the set of decisions is finite. In the public goods problem the decision is whether or not to produce the public good. The public goods problem is not first best implementable because the budget balancedness condition cannot be satisfied in all states of the world. The reason for budget imbalance is the externality that an individual can impose on the remaining set of individuals. Here, an individual, by changing his announcement can change the decision of all other individuals (see Green and Laffont 1979). While for the linear cost sequencing problem, the externality that can be imposed by a unit on the remaining set of units is more 'subtle' and is captured by the independence property. If a unit is allotted a position k in the queue in some state, then by changing its cost parameter the unit can either change the cost of the units serviced before it (that is its predecessor set) or the cost of the units serviced after it (that is its successor set). The unit cannot simultaneously affect both the predecessor and the successor sets. This sort of externality, which is certainly less severe than the externality in public goods problem, is one of the crucial requirement for first best implementability of sequencing problems.

The analysis presented in this paper achieves something more. In this paper we have proved that the type of externality that is present in the linear cost sequencing problem is also present in a sequencing problem where the cost function is exponential. In addition to this type of externality we need a nice combinatorial structure of the cost function. This additional need makes linear cost sequencing problems the unique class of first best implementable sequencing problems.

References

Clarke, E.H. (1971) Multi-part pricing of public goods. *Public Choice* 11: 17–33

Curiel, I., Pederzoli, G., Tijs, S. (1989) Sequencing games. *European Journal of Operational Research* 40: 344–351

Dolan, R. (1978) Incentive mechanisms for priority queueing problems. *The Bell Journal of Economics* 9: 421–436

Green, J., Laffont, J.J. (1977) Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica* 45: 427–438

Green, J., Laffont, J.J. (1979). *Incentives in Public Decision Making*. North Holland Publication, Amsterdam

Groves, T., (1973) Incentives in Teams. *Econometrica* 41: 617–631

Groves, T., Ledyard, J.O. (1977) Some Limitations of Demand Revealing Processes. *Public Choice* 29: 107–124

Holmström, B. (1979) Groves' Schemes on Restricted Domains. *Econometrica* 47: 1137–1144

Hurwicz, L. (1975) On the Existence of Allocation Systems Whose Manipulative Nash Equilibria are Pareto Optimal. mimeo, University of Minnesota

Hurwicz, L., Walker, M. (1990) On the generic non-optimality of dominant strategy allocation mechanisms: A general theorem that includes pure exchange economies. *Econometrica* 58: 683–704

Suijs, J. (1996) On incentive compatibility and budget balancedness in public decision making. *Economic Design* 2: 193–209

Walker, M. (1980) On the non-existence of dominant strategy mechanisms for making optimal public decisions. *Econometrica* 48: 1521–1540