EVALUATION OF POLYNOMIALS IN ONE UNKNOWN AND THEIR REAL ROOTS ON PUNCHED CARD TABULATOR TYPE IBM 421

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SUMMARY. A mechanical procedure of evaluation of polynomials in one unknown and their real roots with the help of the IBM tabulator type 421 is discussed in this paper. The author considers the equispaced values of the independent variable and applies the method of differences in evaluating the polynomials. To locate a real root of a polynomial equation in one unknown the method adopted to find a pair of consecutive values of the unknown between which the polynomial changes its sign. The interval formed by the pair then contains a root.

1. Introduction

We describe the evaluation of polynomials in one unknown and their real roots on punched card tabulator type IBM 421 using the method of differences. For simplicity a polynomial of third degree has been considered although the procedure discussed below is quite general and can be extended to the evaluation of higher degree polynomials and their real roots. The only limitation is the number of counter positions available in the tabulator.

2. Use of differences

Let $f(x) = a_3x^3+a_2x^2+a_1x+a_0$ be a polynomial of the third degree in x, which we are required to tabulate for the equispaced values $x_0, x_1, x_2, \ldots$ of x, where $x_i = x_0+ih$, h being the interval of tabulation. For simplicity, we shall write $y_i = f(x_i)$. The differences of the y_i's are defined as follows :

$$\text{first order differences : } \Delta y_i = y_{i+1}-y_i$$

$$r\text{-th order differences : } \Delta^r y_i = \Delta^{r-1}y_{i+1}-\Delta^{r-1}y_i$$

The third order differences are constant : $\Delta^3 y_i = 6a_3h^3$ and we shall denote this by A_3. The values of the y_i's can then be built up recursively from the relations :

$$y_{i+1} = y_i+\Delta y_i$$

$$\Delta y_{i+1} = \Delta y_i+\Delta^2 y_i$$

$$\Delta^2 y_{i+1} = \Delta^2 y_i+A_3$$

starting with the initial values :

$$A_0 = y_0$$

$$A_1 = \Delta y_0 = b_2x_0^2+b_1x_0+b_0$$

$$A_2 = \Delta^2 y_0 = c_1x_0+c_0$$

where $b_2 = 3a_3h$, $b_1 = 3a_3h^2+2a_2h$, $b_0 = a_3h^3+a_2h^2+a_1h$, $c_1 = 2b_2h$

and $c_0 = b_2h^2+b_1h$.

To locate a real root of the equation $f(x) = 0$, we evaluate the polynomial $f(x)$, for equispaced values of x and if the sign of $f(x)$ changes between x_i and x_{i+1} then the interval (x_i, x_{i+1}) contains a root.

How the IBM Accounting Machine Type 421 can be programmed for evaluation and location of roots using the above logic is described in the following sections.

3. Mechanised procedure of tabulating the values of $f(x)$

Using the intial value x_0 of x the values of $y_0 = f(x_0)$, Δy_0, $\Delta^2 y_0$ and $\Delta^3 y_0$ are calculated for a certain value of h. These values together with x_0, h and n are punched in a card, $(n+1)$ being the number of values of $f(x)$ to be tabulated. This card is then fed through the IBM Accounting Machine Type 421. Counters 8A, 8B, 8C, 8D, 4A, 4B and 4C are instructed to add y_0, Δy_0, $\Delta^2 y_0$ $\Delta^3 y_0$, x_0, n and h respectively. The digit 1 from digit-emitter is also entered into counter 2A. A counter is asked to plus or minus according as the number fed into it is positive or negative. For this purpose an x is punched in the card column containing the unit digit of the negative number. The tabulator is set up to start programming by a card count impulse. Three programme steps are called for and the following chart shows the operations to be carried out during these steps.

CYCLE CHART

step	operations
1	(i) read out counter 8B, plus of counter 8A
	(ii) read out counter 4C, plus of counter 4A
2	(i) read counter 8C, plus of counter 8B
	(ii) read out and print counters 4A and 8A
	(iii) read out counter 2A, minus of counter 4B
3	(i) read out counter 8D, plus of counter 8C
	(ii) reset all the counters (conditional)
	(iii) programme stop (conditional)

Thus at the end of the first cycle the value of the argument x and the corresponding value of $f(x)$ are formed in counters 4A and 8A respectively. In the second step first differences are formed in counter 8B and at the same time $f(x)$ and the corresponding x are printed. During this step 1 is subtracted from the content of counter 4B. As soon as counter 4B shows a negative balance a selector is picked up through its All Step pick up by a Counter Controlled All Cycles Exit Minus pulse. The selector will be picked up immediately and will remain transferred till the end of the next card feed cycle. The first operation of the third step is carried out until the selector is transferred. This operation will result in the formation of second differences in counter 8C. The second and third operations of this step are performed only when the selector is transferred. Thus the above programmes are repeated n times printing the values of $f(x_i)$ and x_i where $x_i = x_0 + ih$, $i = 1, 2, \ldots, n$ before the counter 4B becomes negative and the selector is transferred when the programme stops and all the counters are reset to zero. The initial values $f(x_0)$ and x_0 may be listed from the card itself.

Since in a standard 421 tabulator ten Programme Steps Counters are available, seven idle cycles will take place and hence some time will be lost between the third and the first step during the repetition of the above programmes. This difficulty can be overcome by instructing the operations in the i-th ($i = 1, 2, 3$) step to be carried out in both $(3+i)$-th and $(6+i)$-th steps. This is made possible by connecting the i-th, $(3+i)$-th and $(6+i)$-th Programme Steps Counter hubs together to one Programme Steps Out hub, the corresponding Programme Steps In hub being wired from All Cycles hub.

The total time required for tabulating $f(x_0), f(x_0+h), \ldots, f(x_0+nh)$ being approximately equal to $n/50$ minutes, this mechanical procedure becomes more worthwhile and time saving in comparison to manual computation as n becomes very large. Moreover, this method can be used in evaluating any regular function of one unknown after approximating it by a polynomial of suitable degree.

4. Mechanised procedure of solving the equation $f(x) = 0$

The above method with some changes in programming can be used for finding the real root of the equation $f(x) = 0$. With a starting value x_0 of x and suitable value of h, the values of $f(x_0)$, $\Delta f(x_0)$, $\Delta^2 f(x_0)$ and $\Delta^3 f(x_0)$ are calculated. Then following the same procedure as discussed in Section 3, these calculated values together with x_0 and h are taken into seven counters as shown below.

input values	$f(x_0)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	x_0	h
counter alloted	8A	8B	8C	8D	4A and 4B	4C

Following programme steps are then initiated by a card count impulse.

step	operations
1	(i) read out counter 8B, plus of counter 8A
	(ii) read out counter 4C, plus of counter 4B
2	(i) read out counter 8C, plus of counter 8B
3	(i) read out counter 8D, plus of counter 8C
	(ii) read out counter 4C, plus of counter 4A

All the three steps are repeated until the content of counter 8A changes its sign which becomes positive or negative according as the function is increasing or decreasing. To obviate this difficulty the input values are added in the counters in such a manner that irrespective of the nature of the function, the value of $f(x)$ changes its sign always from negative to positive. This is accomplished by inverting the signs of the input values of a decreasing function. As soon as counter 8A shows a positive balance an inpulse is available from the corresponding Counter Controlled All Cycles Exit Plus hub which is used to pick up a selector through its All Step pick up hub. Through the transfer side of this selector the following operations instead of the above operations are then performed in the second and third steps.

step	operations
2	(i) read out and print counters 4A and 4B
3	(i) reset all the counters (ii) programme stop

Thus counters 4A and 4B give the values x_i and x_{i+1} respectively of x such that $f(x)$ changes its sign between x_i and x_{i+1}.

5. Comments

The interval of tabulation h can be taken as small as possible depending, of course, on the time needed for the entire process. Taking into consideration the fact that the maximum time required for evaluating x upto n decimal places with $h = \frac{1}{10^n}$ being approximately equal to $\frac{10^n}{50}$ minutes it will be more convenient to find x upto two decimal places as the first approximation. Then starting with the first approximate value of x and repeating the same process the second approximation upto four decimal places can be obtained. Following this iterative procedure we can locate the root upto the desired decimal places. Even if we assume that about five minutes will be spent in computing the input values and in transferring them on a card, the solution of the equation upto six decimal places can be obtained in about twenty minutes.

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