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OPTIMALITY OF PARTIAL GEOMETRIC DESIGNS

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We find a sufficient condition on the spectrum of a partial geometric design d^* such that, when d^* satisfies this condition, it is better (with respect to all convex decreasing optimality criteria) than all unequally replicated designs (binary or not) with the same parameters b, v, k as d^*.

Combining this with existing results, we obtain the following results:

(i) For any $q \geq 3$, a linked block design with parameters $b = q^2$, $v = q^2+q$, $k = q^2-1$ is optimal with respect to all convex decreasing optimality criteria in the unrestricted class of all connected designs with the same parameters.

(ii) A large class of strongly regular graph designs are optimal w.r.t. all type 1 optimality criteria in the class of all binary designs (with the given parameters). For instance, all connected singular group divisible (GD) designs with $\lambda_1 = \lambda_2 + 1$ (with one possible exception) and many semiregular GD designs satisfy this optimality property.

Specializing these general ideas to the A-criterion, we find a large class of linked block designs which are A-optimal in the un-restricted class. We find an even larger class of regular partial geometric designs (including, for instance, the complements of a large number of partial geometries) which are A-optimal among all binary designs.

1. Introduction. Experiment is the first step of a decision making process. Thus, in order that an "optimum" decision can be taken, the experiment must also be "optimum."

Consider the problem of comparing v "treatments" on a certain kind of material. In a typical experiment, the treatments are applied to the available experimental units and observations are taken. An appropriate linear model is assumed and 'estimates' of "treatment effects" are obtained from the observations by the least square method.

Now, the "estimates" of course depend on how the treatments are allocated. From a badly designed experiment, some or all of the parametric functions of interest may not be estimable. So, the first requirement for a design is to ensure that all parametric functions of interest are estimable. Usually, for a given experimental set up there are many such designs. Thus the next question is how to choose one from them.

More explicitly, given the linear model and given a linear function of the vector of treatment effects, what design should one choose so that the best linear unbiased estimator (BLUE) of the given linear function obtained from this design is better in some sense than those obtained from other designs?

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In classical inference, a "good" estimator is meant to be an unbiased estimator with small variance, in the univariate case. The Fisher information matrix is considered to be the natural generalisation of variance in the multivariate case. Therefore, given two competing designs d_1 and d_2, we should prefer d_1 to d_2 if the information matrix (commonly known as the C-matrix) C_1 of d_1 is, in some suitable sense, "smaller" than the C-matrix C_2 of d_2. Following this idea, Kiefer introduced in [15] various optimality criteria to measure the "smallness" or otherwise of the C-matrix. Each of these (A-, D- and E-) criteria demands a specific statistical property of the BLUE, and it amounts to the minimisation of a particular convex function of the eigenvalues of the C-matrix.

In [16], Kiefer defined a very large class Φ of optimality criteria containing the earlier defined A-, D- and E-criteria. He termed a design universally optimal if is optimal with respect to each criterion in the class Φ. He also found certain sufficient conditions for universal optimality.

In a block design set up, there are v treatments and b blocks. Each block consists of k experimental units. The units in the same block are homogeneous and those from different blocks are subject to different "block effects." The problem is to find an optimal design among all designs with given parameters b, k and v. Kiefer [16] proved that a BIBD (balanced incomplete block design) is universally optimal (when it exists for the given parameters) over the general class of designs.

Now, certain divisibility conditions on b, k, v are necessary (but not sufficient) for the existence of a BIBD. So, when these conditions are not satisfied, what is the optimal design? This problem is far from completely solved. In [8], Cheng proved the optimality of most balanced group divisible designs (MBGDDs) in the general class with respect to a subclass of Φ.

Apart from these, the existing optimality results are of two types; namely, specific optimality (mostly E-, a few D- and A-) in the general class and general optimality within a restricted class (equireplicate and/or binary). In [14], Jacroux derived some sufficient conditions for proving general optimality in the general class. However, it appears that it is not possible to derive a general result using any of these conditions as one has to verify the appropriate condition for each optimality criterion and each set of parameters separately, often requiring the use of a computer.

In this paper we derive sufficient conditions for proving general optimality (more general than Cheng [8] and Jacroux [14]) as well as specific (A-) optimality for a class of designs described below. These conditions are easy to verify. This is illustrated by the explicit examples of several families of optimal designs given below, some in the general class and others in the binary class.

The partial geometric designs were introduced by Bose, Shrikhande and Singhi in [3] (see Definition 4.5). This class includes all proper linked block designs, all singular and semi-regular group divisible designs, as well as partial geometries and their complements. We prove that whenever the nullity of the concurrence matrix of a partial geometric design d^* is small enough,

d^* is better (with respect to all convex decreasing optimality criteria) than all unequally replicated designs (binary or not) with the same parameters $b,\ k,\ v$ as d^*. As one application, we show that any linked block design with parameters $b = q^2,\ v = q^2 + q,\ k = q^2 - 1$ (i.e., the dual of the complement of an affine plane of order q) is optimal (with respect to all the criteria mentioned above) in the unrestricted class of all connected designs with the same $b,\ k,\ v$, whenever $q \geq 3$. Designs in this series are known to exist whenever q is a power of a prime number.

A partial geometric design which is also a regular graph design is termed a regular partial geometric design (see Definition 4.6). Combining the result of Cheng and Bailey [11] with ours, we obtain a large class of regular partial geometric designs which are type 1 optimal in the class of all connected binary designs. These include many semi- regular group divisible designs with $\lambda_2 = \lambda_1 + 1$ and all connected singular group divisible design with $\lambda_1 = \lambda_2 + 1$, with a possible exception in case $m = 3,\ n = 2$.

Specializing these general ideas to the case of A-optimality, we show that if δ is a BIBD with parameters $b_0,\ v_0,\ r_0,\ k_0,\ \lambda_0$, where $v_0 \geq (k_0^2 + 4)/2$ (this holds, e.g., whenever $\lambda_0 \leq 2$), then the dual d^* of the complement of δ is A-optimal in the un-restricted class. We find an even larger class of regular partial geometric designs (including, for instance, the complements of many partial geometries) which are A-optimal amongst all binary designs.

To prove these results (which go a long way beyond existing optimality results) we exploit Tomic's characterization of majorization and apply Schur's theorem on majorization in a highly non-trivial manner. However, it may be noted that majorization (in our sense) was already used by Bagchi and Shah in [1] as a tool in optimality theory. (One of the referees has kindly informed us that applications of majorization to design theory occured even earlier in the 1979 Ph.D. dissertation of Magda [17].)

2. Majorization technique.

NOTATION 2.1. For any vector $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$, $(x_{(1)}, x_{(2)}, \dots, x_{(n)})$ will denote the increasing reordering of x, that is, it is the vector, obtained from x by permuting co-ordinate positions, such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$.

DEFINITION 2.1 (cf. [18]). For $x, y \in \mathbb{R}^n$, x is said to be weakly majorized from above by y (in symbols, $x \prec^w y$) if

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \qquad k = 1, 2, \dots, n. \tag{2.1}$$

Clearly $\prec^w$ is a pre-order on $\mathbb{R}^n$. That is, it is a reflexive and transitive binary relation.

We begin with two useful results on majorization. Their trivial proofs are omitted.

LEMMA 2.1. *If $x_i \geq y_i$ for $1 \leq i \leq n$ then $x \prec^w y$.*

LEMMA 2.2. *If $x,\ y \in \mathbb{R}^n$ are vectors with $x_n = y_n$ and $\hat{x},\ \hat{y} \in \mathbb{R}^{n-1}$ are the vectors obtained from $x,\ y$ by deleting their (equal) last components, then $x \prec^w y$ if and only if $\hat{x} \prec^w \hat{y}$.*

We shall also need the following two theorems. See pages 109 and 218 of Marshall and Olkin [18] for their proofs.

THEOREM 2.1 (Tomic). *$x \prec^w y$ if and only if*

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

for every convex decreasing function $g : \mathbb{R} \to \mathbb{R}$.

NOTATION 2.2. The vector of diagonal elements and the vector of eigenvalues of a real symmetric matrix A will be denoted by $\delta(A)$ and $\mu(A)$ respectively. While the ordering of the components of $\delta(A)$ will be the one naturally induced by the matrix (just read its diagonal entries from top left to bottom right), those of $\mu(A)$ will be taken to be in the increasing order.

THEOREM 2.2 (Schur). *For every real symmetric matrix A we have $\delta(A) \prec^w \mu(A)$.*

COROLLARY 2.1. *For any $n \times n$ real symmetric matrix A and any convex function $g : \mathbb{R} \to \mathbb{R}$, we have*

$$\sum_{i=1}^{n} g(\delta_i(A)) \leq \sum_{i=1}^{n} g(\mu_i(A))$$

PROOF. The proof is contained in the third proof of Schur's theorem given in [18]. $\square$

3. Optimality criteria. All designs in this paper are connected block designs with constant block size. We shall use the following more or less standard notation.

NOTATION 3.1. (i) $\mathscr{D} = \mathscr{D}_{b,k,v}$ denotes the class of all connected block designs with v treatments and b blocks of size k each.

(ii) $\mathscr{D}^B_{b,k,v}$ denotes the class of binary designs in $\mathscr{D}_{b,k,v}$.

(iii) $r := bk/v$ is assumed to be an integer. $\mathscr{D}_{b,k,v,r}$ denotes the class of all equireplicate designs in $\mathscr{D}_{b,k,v}$ Naturally, the constant replication number of the designs in this subclass equals r.

(iv) $\mathscr{D}^B_{b,k,v,r}$ denotes the class of all equireplicate and binary designs in $\mathscr{D}_{b,k,v}$.

(v) The replication number of the ith treatment in a design $d \in \mathscr{D}$ will be denoted by $r_i = r_i(d)$ $(1 \leq i \leq v)$. $R(d)$ will denote the diagonal matrix diag $(r_1(d), r_2(d), \ldots, r_v(d))$.

(vi) For a design $d \in \mathscr{D}$, $S(d)$ will denote the concurrence matrix $S(d) = N(d)N(d)^T$, where $N(d)$ is the usual $(v \times b)$ treatment-block incidence matrix of d. $C(d)$ will denote the information matrix of $d : C(d) := R(d) - \frac{1}{k}S(d)$.

We shall drop d from the notations in (v) and (vi) when there is no scope of confusion as to which design is meant.

Motivated by Theorem 2.1, we introduce the following.

DEFINITION 3.1. A design $d_1 \in \mathscr{D}$ is said to be better than another design $d_2 \in \mathscr{D}$ in the sense of majorization (in short M-better) if (in terms of Notation 2.2)

$$\mu(C(d_1)) \prec^w \mu(C(d_2)).$$

$d^* \in \mathscr{D}_{b,k,v}$ is said to be optimal in the sense of majorization in a subclass of $\mathscr{D}_{b,k,v}$ (or, in short, d^* is M-optimal in this subclass) if it is M-better than every member of this subclass.

DEFINITION 3.2 (cf. [8]). Let M be a number larger than the eigenvalues of $C(d)$ for all $d \in \mathscr{D}$. Then, a thrice differentiable function $f : (0, M) \to \mathbb{R}$ is said to a (generalized) optimality criterion of type 1 (respectively, type 2) if (i) $f(0+) = \infty$, (ii) $f' < 0$, (iii) $f'' > 0$, (iv) $f''' < 0$ (respectively, $f''' > 0$). If f is such a function, then define $\Psi_f : \mathscr{D} \to \mathbb{R}$ by $\Psi_f(d) = \sum_{i=2}^{v} f(\mu_i(d))$, $d \in \mathscr{D}$. [Here, in keeping with Notation 2.2, $\mu_i(d)$ is the ith smallest eigenvalue of $C(d)$.] We say that the design d_1 is better than the design d_2 with respect to the criterion f (in short f-better) if $\Psi_f(d_1) \leq \Psi_f(d_2)$. A design d^* is said to be type 1 (respectively type 2) optimal in a subclass of $\mathscr{D}$ if it is f-better than all the designs in this subclass for all type 1 (respectively type 2) optimality criteria f.

REMARK 3.1. Any convex decreasing function on a compact interval of reals can easily be extended to a convex decreasing function on the entire real line. If $f : (0, M) \to \mathbb{R}$ is a type 1 (or type 2) optimality criterion, then taking a number ε in the interval $(0, M)$ which is smaller than all the non-zero eigenvalues of $C(d)$ for all designs d in $\mathscr{D}$ and applying this observation to the restriction of f to $[\varepsilon, M - \varepsilon]$, one sees that if d^* is M-better than d then d^* is better than d with respect to all the criteria of type 1 as well as of type 2. In particular, M-optimality implies optimality with respect to all the usual criteria, including E-, A- and D-optimality.

REMARK 3.2. In Bagchi and Shah [1], certain row-column designs are shown to be M-optimal in the class of all equireplicate row-column designs.

Let us recall that by the dual of a design d with b blocks and v treatments is meant the design $\overline{d}$ with v blocks and b treatments, such that, for $1 \leq$

$i \leq b$, $1 \leq j \leq v$, the ith treatment appears in the jth block of $\overline{d}$ the same number of times as the jth treatment of d appears in the ith block of d. Thus the treatment-block incidence matrices of d and $\overline{d}$ are the transposes of each other. In particular when d has constant block size k and constant replication number r then we have $C(\overline{d}) = kI - r^{-1}N(d)^T N(d)$. This, in view of the fact that the positive eigenvalues of $N^T N$ are the same as those of NN^T, yields the following:

For $d_1, d_2 \in \mathscr{D}_{b,k,v,r}$, d_1 is M-better than d_2 iff $\overline{d_1}$ is M-better than $\overline{d_2}$.

Hence we have the following result.

THEOREM 3.1. *If d^* is equireplicate and M-optimal in $\mathscr{D}_{b,k,v,r}$, then its dual $\overline{d^*}$ is M-optimal in $\mathscr{D}_{v,r,b,k}$.*

Universally optimal designs are clearly M-optimal. In particular, any BIBD (balanced incomplete block design) is M-optimal. Since an LBD (linked block design) is by definition the dual of a BIBD, an immediate consequence of the preceding theorem is:

COROLLARY 3.1. *Any linked block design d^* is M-optimal in the class of all equi-replicate designs with the same parameters b, k, v as d^*.*

REMARK 3.3. The optimality (within the class of equi-replicate designs) of the duals of optimal designs have been noted by many authors, such as Sinha [22], Shah et al [21], Cheng [9], Jacroux [12], to name a few. Various optimality properties of an LBD in $\mathscr{D}_{b,k,v,r}$ have also been noted. However, Corollary 3.1 describes the optimality of the LBDs in in the equi-replicate class in what seems to be the utmost generality. In the unrestricted class $\mathscr{D}_{b,k,v}$, the LBDs are known to be E- (Cheng [9], Jacroux [12]) and D-optimal (Cheng [10], Pohl [20]). In this paper, we show that a large class of LBDs are also A-optimal in $\mathscr{D}_{b,k,v}$ (see Theorem 6.4). We also find one infinite series of LBD's which are majorization optimal in $\mathscr{D}_{b,k,v}$ (see Theorem 6.1).

4. Partial geometric designs. We begin by recalling some standard definitions.

DEFINITION 4.1. A connected binary equi-replicate design is called a regular graph design (RGD) if the off-diagonal entries of its concurrence matrix take just two distinct values and these two values differ by 1, that is, if there are constants λ_1 and λ_2 such that $\lambda_1 = \lambda_2 \pm 1$ and any two distinct treatments occur together in λ_1 or λ_2 blocks.

DEFINITION 4.2. If d^* is an RGD then the underlying graph $G(d^*)$ of d^* is defined to be the (undirected, loopless) graph whose vertices are the treatments of d^* and two distinct vertices are adjacent if and only if the corresponding treatments occur together in λ_1 blocks of d^*.

Notice that if $d^* \in \mathscr{D}_{b,k,v,r}$ is an RGD, then the graph $G(d^*)$ is regular of degree $a := \pm(r(k-1) - \lambda_2(v-1))$ (i.e., each vertex is adjacent to a vertices)—hence the name RGD. Also, the usual $0-1$ adjacency matrix $A(d^*)$ of the graph is related to the concurrence matrix $S(d^*)$ of d^* by

$$S(d^*) = (r - \lambda_2)I + \lambda_2 J \pm A(d^*) \tag{4.1}$$

where the plus or minus sign is chosen according as $\lambda_1 - \lambda_2$ is $+1$ or -1. As usual, I and J denotes the identity and all-one matrix, respectively, of order v.

DEFINITION 4.3. Recall that a strongly regular graph with parameters (n, a, c, d) is a regular graph of degree a on n vertices such that given any two distinct vertices of the graph, the number of their common neighbors is c or d according as the two vertices are adjacent or not. Equivalently (see, e.g., Cameron and van Lint [7]), it is a regular graph whose adjacency matrix has at most three distinct eigenvalues.

DEFINITION 4.4 (cf. [11]). A design is called a strongly regular graph design (SRGD) if it is an RGD whose underlying graph is strongly regular. In view of Equation 4.1 and Definition 4.3, an SRGD may also be defined as an RGD whose concurrence matrix (or, equivalently, information matrix) has at most three different eigenvalues.

DEFINITION 4.5. The original (combinatorial) definition of partial geometric designs (PGD) may be found in Bose, Shrikhande and Singhi [3]. In view of the main theorem in Bose, Bridges and Shrikhande [4], a PGD may equivalently be defined as a binary equi-replicate connected design whose concurrence matrix is a singular matrix with at most three distinct eigenvalues.

REMARK 4.1. The C-matrix of a PGD have exactly one strictly positive eigenvalue other than r.

Next we introduce:

DEFINITION 4.6. A design d^* will be called a regular partial geometric design (RPGD) if it is a regular graph design which is also a partial geometric design. In consequence of the preceding definitions, an RPGD is just an SRGD with a singular concurrence matrix.

REMARK 4.2. There is an ambiguity in our definition of the underlying graph of an RGD: interchanging λ_1 and λ_2 replaces the graph by its complement. This ambiguity could be removed by requiring in the definition of an RGD that we have $\lambda_1 = \lambda_2 + 1$. But this would conflict with the conventional notation for the parameters of group divisible designs. So we prefer to live with the ambiguity.

EXAMPLES OF PGD'S AND RPGD'S. (i) The dual of any PGD is either a PGD or a BIBD. (However, the dual of an RPGD need not be an RPGD.)

(ii) The complement of any PGD is a PGD and the complement of any RPGD is an RPGD. (Recall that the complement d^+ of a binary design d has the same treatments and blocks as d but a treatment x is incident in d^+ with a block β if and only if x is non-incident in d with β.)

(iii) The proper Linked Block designs (i.e., duals of non-symmetric BIBDs) are PGDs. In general, these are not RPGDs.

(iv) The Singular Group Divisible designs are PGDs. These are RPGDs provided $\lambda_1 = \lambda_2 + 1$.

(v) The semi-regular Group Divisible designs are PGDs. These are RPGDs provided $\lambda_2 = \lambda_1 + 1$.

(vi) The partial geometries and their complements are RPGDs.

REMARK 4.3. Bose introduced the notion of partial geometries in [2]. A partial geometry with parameters $(r,\ k,\ t)$ [in short, a $pg(r, k, t)$] is nothing but an RPGD with replication number r, block size k, $\lambda_1 = 1,\ \lambda_2 = 0$, such that, given any block B and treatment x outside B, there are exactly t treatments y in B such that x and y are on a common block. The dual of a $pg(r, k, t)$ is a pg(k, r, t).

SOME COMMON EXAMPLES OF PARTIAL GEOMETRIES. (i) The BIBDs with $\lambda = 1$ (these are the partial geometries with $t = k$) and their duals $(t = r)$.

(ii) The semi-regular Group Divisible Designs with $\lambda_1 = 0,\ \lambda_2 = 1$ are partial geometries $(t = k - 1)$; their duals $(t = r - 1)$ are also known as Bruck nets (see [5]) and as lattice designs - these are known to be abundantly available because of their interpretation as sets of mutually orthogonal latin squares.

(iii) The partial geometries with $t = 1$ are also known as generalized quadrangles (GQs). Many families of GQs are known; [19].

Any partial geometry not covered by these examples [i.e., with $1 < t < \min(r-1, k-1)$] is called a proper partial geometry. Proper partial geometries are surveyed in [6].

It can be checked easily that the number of treatments of a $pg(r, k, t)$ is given by

$$v = k(1 + (r-1)(k-1)/t) \tag{4.2}$$

and the spectrum of its concurrence matrix is

$$o^g (r + k - t - 1)^{v-g-1} (rk)^1,$$

where

$$v - g - 1 = k(k-1)r(r-1)/(t(r + k - t - 1)). \tag{4.3}$$

5. Optimality results. Let us first prove a simple but useful lemma (recall Notation 2.2).

LEMMA 5.1. *Let d be any unequally replicated design in $\mathscr{D}_{b,k,v}$. Suppose $r := bk/v$ is an integer. Let $\hat{d}$ be a binary design in $\mathscr{D}_{b,k,v}$ having replication numbers* $r-1$, r and $r+1$ *with respective multiplicities* 1, $v-2$ and 1. *Then*

$$\delta(C(\hat{d})) \prec^w \delta(C(d)).$$

PROOF. Let the replication numbers of d be $r_1 \le r_2 \le \cdots \le r_v$. Since the ith diagonal entry of C_d is $\le r_i(k-1)/k$ (with equality when d is binary), by Lemma 2.1, it is enough to show that (in terms of Notation 3.1) $\delta(R(\hat{d})) \prec^w \delta(R(d))$. Now put $u_i = r_i - r$, $1 \le i \le v$. Thus $u_1 \le u_2 \le \cdots \le u_v$ and $\sum_{i=1}^{v} u_i = 0$. Also, not all the $u's$ are zero as d is unequally replicated. Hence $\sum_{i=1}^{j} u_i < 0$, for $1 \le j \le v-1$. Since the $u_i's$ are integers, it follows that $\sum_{i=1}^{j} u_i \le -1$, that is, $\sum_{i=1}^{j} r_i \le r - 1 + (j-1)r$ for $1 \le j \le v-1$. $\square$

The next lemma (which is perhaps of some independent interest) is crucial to the proof of the main results (Theorems 5.1, 6.1 and 6.3) of this article.

LEMMA 5.2. *If $d^* \in \mathscr{D}_{b,k,v,r}$ is a partial geometric design (regular or not) such that the nullity g of the concurrence matrix of d^* satisfies*

$$g \le \frac{(v-1)(k-1)}{r(v-k)} \tag{5.1}$$

then d^* is M-better than every unequally replicated design (binary or non-binary) in $\mathscr{D}_{b,k,v}$.

PROOF. Let μ^* denote the nonzero eigenvalue of $C(d^*)$ other than r. (See Remark 4.1.) Clearly, μ^* satisfies

$$\mu^* = \frac{r(v - g - \frac{v}{k})}{v - g - 1}. \tag{5.2}$$

Let a be the real number given by

$$a = \max\left(\frac{\mu^*}{v}, \frac{r}{k} - \frac{k-1}{kg}\right).$$

For any design $d \in \mathscr{D}_{b,k,v}$, define the $v \times v$ matrix $E(d)$ by

$$E(d) = C(d) + aJ.$$

(Here J is the $v \times v$ matrix all whose entries are $= 1$.) Thus the eigenvalues of $E(d)$ are the non-zero eigenvalues of $C(d)$ together with av. In particular, the eigenvalues of $E(d^*)$ are μ^*, av and r with respective multiplicities $v-g-1$, 1 and g. From the formula (5.2), it is clear that $r \ge \mu^*$. Also, our hypothesis (5.1) on g implies that $r \ge (\frac{r}{k} - \frac{k-1}{kg})v$. Therefore, the eigenvalues of $E(d^*)$ are ordered as follows:

$$\mu^* \le av \le r.$$

Let $\hat{d}$ be a binary design in $\mathscr{D}_{b,k,v}$ with replication numbers as in the statement of Lemma 5.1 (such a design may be constructed by a little perturbation of d^*). An immediate consequence of the definition of E_d is that Lemma 5.1 is equivalent to the statement

$$\delta(E(\hat{d})) \prec^w \delta(E(d)). \tag{5.3}$$

Again, two applications of Lemma 2.2 on $\mu(E(d^*))$ and $\mu(E(d))$ imply that $\mu(C(d^*)) \prec^w \mu(C(d))$ holds if and only if $\mu(E(d^*)) \prec^w \mu(E(d))$. Thus, to complete this proof, it suffices to prove the following:

$$\mu(E(d^*)) \prec^w \delta(E(\hat{d})). \tag{5.4}$$

This is because the relation (5.4), together with Lemma 5.1 and Theorem 2.2, would imply the following string of majorizations:

$$\mu(E(d^*)) \prec^w \delta(E(\hat{d})) \prec^w \delta(E(d)) \prec^w \mu(E(d)). \tag{5.5}$$

Since $\prec^w$ is transitive, the result would follow.

We therefore concentrate on proving (5.4). Let δ_i, $i = 1, 2, 3$ denote the distinct entries of $\delta(E(\hat{d}))$. Thus,

$$\delta_i = (r + i - 2)t + a, \qquad i = 1, 2, 3, \tag{5.6}$$

where

$$t = 1 - k^{-1}.$$

Since $\operatorname{trace}(E(\hat{d})) = \operatorname{trace}(E(d^*))$, the vectors $\delta(E(\hat{d}))$ and $\mu(E(d^*))$ have the same total sum. Therefore, to establish the claim (5.4), we only need to show that

$$j\mu^* \geq \delta_1 + (j - 1)\delta_2 \qquad \text{for } 1 \leq j \leq v - g - 1 \tag{5.7}$$

and

$$(v - g - 1)\mu^* + av + jr \geq \delta_1 + (v - g - 1 + j)\delta_2 \tag{5.8}$$

for $0 \leq j \leq g - 1$. Now, it can be seen that if (5.7) holds for $j = v - g - 1$, then it would hold for each j in the range $1 \leq j \leq v - g - 1$; similarly, it is enough to prove (5.8) for $j = 0$. Thus, we are to prove the folowing inequalities:

$$(v - g - 1)\mu^* \geq (v - g - 1)\delta_2 - t, \tag{5.9}$$

and

$$(v - g - 1)\mu^* + av \geq (v - g)\delta_2 - t. \tag{5.10}$$

In the case $a = \frac{\mu^*}{v} \geq \frac{r}{k} - \frac{k-1}{kg}$, LHS minus RHS of (5.10) equals

$$g\left(\frac{\mu^*}{v} - \left(\frac{r}{k} - \frac{k-1}{kg}\right)\right) \geq 0,$$

so that (5.10) holds. Also, the formula for μ^* implies that $\mu^* \leq tr + \mu^*/v$, that is, $av \leq \delta_2$. Therefore (5.9) follows from (5.10) in this case. Similarly, in case $a = \frac{r}{k} - \frac{k-1}{kg}$, we find that the inequality (5.10) actually holds with equality, while LHS minus RHS of (5.9) equals $\frac{(v-k)r}{gk}(\frac{(v-1)(k-1)}{r(v-k)} - g)$ which is ≥ 0 by our hypothesis on g, so that (5.9) also holds. Thus, in either case, we have proved $\mu(E(d^*)) \prec^w \delta(E(\hat{d}))$. □

Combining Lemma 5.2 with Theorem 3.1, we get the following result.

THEOREM 5.1. *If* $d^* \in \mathscr{D}_{b,k,v,r}$ *is a partial geometric design* (*regular or not*) *such that*:

(i) *its dual is M-optimal in* $\mathscr{D}_{v,r,b,k}$ *and*
(ii) *it satisfies the condition* (5.1) *then* d^* *is M-optimal in* $\mathscr{D}_{b,k,v}$.

REMARK 5.1. The most important feature of Theorem 5.1 is its simplicity: one can check the condition (5.1) very easily. Of course, not too many designs satisfy this condition, but that is expected since a large class of optimality criteria is involved (considerably larger than the classes considered in [8] and [14]). In this context we may recall that in [8] Cheng has shown that an MBGDD of type 1 is optimal w.r.t. all type 1 criteria while an MBGDD of type 2 is optimal w.r.t all type 2 criteria; thus except for the BIBDs, no single design was known to be optimal w.r.t. type 1 as well as type 2 criteria in the general class. But the M-optimality criterion includes all type 1 and type 2 criteria and many more. Interestingly, we have found an infinite series of designs optimal w.r.t. such a large class of optimality criteria (see Theorem 6.1).

REMARK 5.2. In the above theorem the phrase "M-optimality" can be replaced by "type-1 optimality" or any other general (or specific) optimality criterion, provided the underlying optimality functions are convex and decreasing. However, for a smaller class of optimality criteria, a condition more relaxed than (5.1) should suffice and that would yield many more optimal designs; and we shall next do precisely that for A-optimality.

The analogue of Lemma 5.2 for the A-criterion is the following. The A-optimality results (Theorems 5.2, 6.4, 6.5 and 6.6) of this paper depend on the following lemma.

LEMMA 5.3. *Let d^* be a partial geometric design (regular or not) in $\mathscr{D}_{b,k,r,v}$ such that the nullity g of its concurrence matrix satisfies*

$$2(v-1)\left(v-\frac{v}{k}-g\right) \geq (g+2)(b-r)^2. \tag{5.11}$$

Then d^* is A-better than any unequally replicated design (binary or not) in $\mathscr{D}_{b,k,v}$.

PROOF. Let d be an unequally replicated design with the given parameters. Since $f(x) = 1/x$ is a convex decreasing function on $(0, \infty)$, by a similar argument as in Lemma 5.2, and in view of Theorem 2.1, it is enough to prove the following A-analogue of the condition (5.4): $\mu(E(d^*))$ is better than $\delta(E(\hat{d}))$ in the sense of A-criterion for some choice of $a > 0$. More precisely, we have to prove that

$$(\delta_1)^{-1} + (v-2)(\delta_2)^{-1} + (\delta_3)^{-1} \geq (v-g-1)(\mu^*)^{-1} + (av)^{-1} + gr^{-1}, \tag{5.12}$$

for some $a > 0$. Here the δ_i's are as given in (5.6).

Let us choose (see Remark 5.3 below)

$$a = r(k-1)/k(v-1).$$

With this choice of a, we substitute the values of δ_i from (5.6) to find that LHS minus RHS of (5.12) simplifies to the following expression ϕ:

$$\phi := 2(tv-g) - g((1-t)v-1)^2\{v^2r^2/(v-1)^2 - 1\}/(v-1)$$

where $t = 1 - 1/k$. Using the relations $(v/(v-1))^2 g/(v-1) \leq (g+2)/(v-1)$ and $bk = rv$, we get $\phi > 2(tv-g) - (g+2)(b-r)^2/(v-1)$. Therefore, for the conclusion of the lemma, it suffices to require this lower bound on ϕ to be non-negative. □

REMARK 5.3. In the proof of the above lemma we have chosen a particular value of a. The natural question that arises in this context is: "can we improve upon the theorem by choosing a different value for a?" Toward an answer, we note the following. From (5.12), we see that in order to improve upon the theorem, we have to choose a so as to maximize

$$\psi = (\delta_1)^{-1} + (v-2)(\delta_2)^{-1} + (\delta_3)^{-1} - (av)^{-1}.$$

But ψ is increasing for $a \leq \delta_1/v$ and is decreasing for $a \geq \delta_2/v$, so that the best value of a lies between δ_1/v and δ_2/v. However, pinpointing the best value of a seems to be rather difficult (recall that δ_i's involve a) and since the range $\delta_2 - \delta_1$ is small, it is unlikely to yield many more optimal designs. We therefore decided to be content with the choice $a = \delta_2/v$ which amounts to the choice made above.

Combining the appropriate restriction of Theorem 3.1 with the above lemma we get:

THEOREM 5.2. *If* $d^* \in \mathscr{D}_{b,k,v,r}$ *is a partial geometric design* (*regular or not*) *such that*:

(i) *its dual is A-optimal in* $\mathscr{D}_{v,r,b,k}$ *and*
(ii) *it satisfies the condition* (5.11) *then* d^* *is A-optimal in* $\mathscr{D}_{b,k,v}$.

REMARK 5.4. In [13], Jacroux has given sufficient conditions for A-efficiency. He generalized these ideas in [14] to derive some sufficient conditions for Type I optimality as well as for various specific optimality criteria. It would be interesting to see how the conditions given here compares with those conditions. We see that among these results of Jacroux, only Theorem 3.10 of [14] is relevant for this comparison. We now compare it with the theorems above.

For the sake of ease in readability we restate Theorem 3.10 of [14] using our notation (see Notation 3.1). We have assumed $k \geq 3$.

THEOREM 5.3 (Jacroux). *Suppose* $\bar{d}$ *is an equireplicate, binary and A-optimal design in* $\mathscr{D}_{v,r,b,k}$. *Let* d^* *denote the dual of* $\bar{d}$ *and let* z_i, $i = 1, 2, \ldots, v-1$, *denote the eigenvalues of the C-matrix of* d^*. *Further, let*

$$m_1 = rst - 2s/k, \;\; m_4 = rst + (2/k)((v-1)(v-2))^{-1}$$

and

$$A_1 = m_1^{-1} + (v-2)m_4^{-1},$$

where $s = v/(v-1)$ *and* $t = (k-1)/k$. *If*

$$\sum_{i=1}^{v-1}(z_i)^{-1} < \min(A_1,\;A_2) \tag{5.13}$$

then d^* *is A-optimal in* $\mathscr{D}_{b,k,v}$.

(Here A_2 involves quantities defined elsewhere in the same paper and it does not concern the statement we prove below. We, therefore omit the expression for A_2.)

We now show the following:

OBSERVATION. If d^* in Jacroux's theorem is also a PGD, then Theorem 5.2 is stronger than Jacroux's Theorem 3.10 provided $r \leq k-3$ or $r = k-2$, $v \geq 7$.

SKETCH OF PROOF. It is enough to show that under the given conditions, inequality (5.12) holds (with the choice of a used in the proof of Lemma 5.3) whenever (5.13) holds. Now, if d^* is a PGD, then

$$\sum_{i=1}^{v-1}(z_i)^{-1} = (v-g-1)(\mu^*)^{-1} + gr^{-1}.$$

Therefore (5.12) may be rewritten as

$$\sum_{i=1}^{v-1}(z_i)^{-1} \leq (\delta_1)^{-1} + (v-3)(\delta_2)^{-1} + (\delta_3)^{-1} = A_B \quad \text{say.}$$

Thus it is enough to show that under the given condition, $A_B \geq A_1$. This can be checked by simple but tedious calculations. □

6. Applications.

6.1. *M-optimality. The linked block designs.*

THEOREM 6.1. *For $q \geq 3$, any linked block design with parameters $b = q^2,\ v = q^2+q,\ k = q^2-1$ (i.e., the complement of the dual of any affine plane of order q) is M-optimal within the unrestricted class $\mathscr{D}_{b,k,v}$.*

PROOF. The nullity g of the concurrence matrix of the LBD considered here is $g = q$. That this satisfies (5.1) for every $q \geq 3$ is easy to check. Thus, the result follows from Theorem 5.1. □

REMARK 6.1. It is well known that any proper LBD satisfies $k \geq r+1$ with equality if and only if the LBD is the dual of an affine plane. Using this fact, it is easy to see that the designs listed in Theorem 6.1 are the only LBD's to which Theorem 5.1 applies. Notice that the LBDs in this theorem are actually semi-regular group divisible designs.

We shall now search for general optimality of other RPGDs. Towards this, we shall make use of the general optimality result of Cheng and Bailey [11], which, in view of the spectral characterization of PGD's, may be reformulated as follows:

THEOREM 6.2 (Cheng and Bailey [11]). *If $d^* \in \mathscr{D}_{b,k,v}$ is a regular partial geometric design then d^* is optimal in the class of all binary equi- replicate members of $\mathscr{D}_{b,k,v}$ with respect to all type 1 optimality criteria. In particular, d^* is A-, D- and E- optimal in this class.*

This result, combined with Lemmas 5.2 and 5.3, yields the following.

COROLLARY 6.1. *Let g denote the nullity g of the concurrence matrix of a regular partial geometric design $d^* \in \mathscr{D}_{b,k,v}$.*

(a) *If g satisfies the inequality* (5.1), *then d^* is optimal with respect to all type* 1 *criteria, and*
(b) *if g satisfies* (5.11) *then it is A-optimal in the class $\mathscr{D}^B_{b,k,v}$ of all binary designs.*

6.2. *The group divisible RPGDs.* In the following, as usual, m and n will denote the number of groups and group size, respectively, of group divisible designs.

THEOREM 6.3. *The following GD designs are Type* 1 *optimal among all binary designs in* $\mathscr{D}_{b,k,v}$.

(a) *Any semi-reguler group divisible design with* $\lambda_2 = \lambda_1 + 1$. *satisfying* $v - k < m + n - 2$.

(b) *Every connected singular group divisible design with* $\lambda_2 = \lambda_1 - 1$, *except possibly the case of three groups of size two each.*

PROOF. (a) The parameters of a semi-regular GDD with $\lambda_2 = \lambda_1 + 1$ satisfy $v = mn$ and $r(v-k) = v(n-1)$. Further, here $g = m - 1$. Substituting these in (5.1), we get the given condition.

(b) For a singular group divisible design with $\lambda_2 = \lambda_1 - 1$, $v = mn$, $r = m - 1$, $g = k = (m-1)n$. Connectedness forces $m \geq 3$. So, the M-optimality condition (5.1) becomes

$$(m-1)^2 n^2 \leq (mn-1)(mn-n-1)$$

which holds whenever $m \geq 3$ and $n \geq 3$. Hence the result. □

REMARK 6.2. The condition in (a) requires $v < 2k$. Since the complement of an SRGDD is also an SRGDD with the same $\lambda_2 - \lambda_1$, the semi-regular GD designs satisfying the above condition are plentiful. For instance, duals of many Bruck nets (i.e., lattice designs; see the second example after Remark 4.3) are included.

REMARK 6.3. The series of LBDs shown to be M-optimal (in Theorem 6.1) are also semi-regular GD designs with $\lambda_2 = \lambda_1 + 1 (= 1)$. So this is a case where the binary restriction of Theorem 6.3 can be removed. It seems safe to conjecture that all the designs named in Theorem 6.3 are actually M-optimal in the unrestricted class.

REMARK 6.4. Excepting the designs named in Theorem 6.1, no other partial geometry nor its complement satisfies (5.1) so that it is not clear whether any other partial geometry or its complement is M-optimal. However, the complements of many partial geometries are A-optimal, as we shall presently see.

6.3. *A-optimality. The linked block designs.*

THEOREM 6.4. *The dual of the complement of any BIBD*$(b_0, v_0, r_0, k_0, \lambda_0)$ *satisfying* $v_0 \geq (k_0^2 + 4)/2$ *is A-optimal among all designs in the un-restricted class* $\mathscr{D}_{b,k,v}$.

PROOF. Here $v = b_0,\ b = v_0,\ r = v_0 - k_0,\ k = b_0 - r_0,\ g = b_0 - v_0$. Now we note that the BIBD satisfying the given condition must have $v_0 \geq 2k_0$. Hence the LBD under consideration has $v/k \leq 2$. Hence the LHS of (5.11) is $\geq 2(v-1)(v-g-2) = 2(v-1)(b-2)$. Again, $r((1-t)v-1) = b-r$ and (since $v_0 \geq 3$), $g+2 \leq v-1$. Therefore the RHS of (5.11) is $\leq (v-1)(b-r)^2$. So, (5.11) would follow if we had $2(b-2) \geq (b-r)^2$, that is, if $2(v_0-2) \geq k_0^2$. But this is what we assumed. □

REMARK 6.5. Any non-symmetric BIBD($b_0,\ v_0,\ r_0,\ k_0,\ \lambda_0$ with $\lambda_0 \leq 2$ satisfies the hypothesis of the above theorem, besides many others.

REMARK 6.6. It may be noted that the A-optimality of some of the LBDs mentioned in Theorem 6.4 may also be proved by Theorem 3.10 of Jacroux [14]. But to verify that one has to go through tedious calculations for every set of parameters often requiring the use of a computer. Also, there are many examples of LBDs proved A-optimal in Theorem 6.4 which fail to satisfy the conditions of Theorem 3.10 of [14]. Here are some examples: (i) $v = 72, b = 28, k = 54$, (ii) $v = 77, b = 22, k = 56$ and (iii) $v = 110, b = 45, k = 88$.

6.4. *The group divisible designs.*

THEOREM 6.5. *Any semiregular group divisible designs with $\lambda_2 = \lambda_1 + 1$ satisfying $(v/k)^2/2 \leq \min(m,n) - 1$ is A-optimal among all binary designs in $\mathscr{D}_{b,k,v}$.*

PROOF. The relation betwen the parameters of a semi-regular GD design as stated in the proof of Theorem 6.3 reduces the condition (5.11) to

$$(m+1)(n-1)^2\left(\frac{v}{k}\right)^2 + 2(mn-1)\left(\frac{v}{k}\right) - 2(mn-1)(mn-m+1) \leq 0.$$

This is a quadratic inequality in $\frac{v}{k}$. So we need only ensure that, under our hypothesis, $\frac{v}{k}$ lies between the two roots. Since the smaller root is negative, this will follow once we verify that the larger root squared is $\geq 2\min(m,n) - 1$. This verification is easy. □

6.5. *Partial geometries.*

THEOREM 6.6. *Whenever $2(k_0 - 2) \geq r_0$, the complement d^* of any partial geometry pg($r_0,\ k_0,\ t_0$) is A-optimal among all binary designs in $\mathscr{D}_{b,k,v}$.*

PROOF. Put $u = v/k$. Since d^* is the complement of a pg(r_0, k_0, t_0), we have $u = v/(v - k_0)$, so that from (4.2) we get

$$u - 1 = t_0/((r_0-1)(k_0-1)) \leq 1/(k_0-1). \tag{6.1}$$

Further, $r(u-1) = r_0$

Using these relations we see that (LHS minus RHS)/$(v-1)$ of (5.11) is greater than or equal to

$$2(v-g-1)-2/(k_0-1)-r_0^2(g+2)/(v-1). \tag{6.2}$$

But from (4.3) we deduce that

$$v-g-1 \geq r_0(k_0-1).$$

It is now immediate that (6.2) is greater than whenever $2(k_0-2) \geq r_0$. □

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