Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 4, November 2001, pp. 381–397.

Unitary tridiagonalization in $M(4, \mathbb{C})$

VISHWAMBHAR PATI

Stat.-Math. Unit, Indian Statistical Institute, RVCE P.O., Bangalore 560 059, India

MS received 7 April 2001; revised 4 September 2001

Abstract. A question of interest in linear algebra is whether all $n \times n$ complex matrices can be unitarily tridiagonalized. The answer for all $n \neq 4$ (affirmative or negative) has been known for a while, whereas the case $n = 4$ seems to have remained open. In this paper we settle the $n = 4$ case in the affirmative. Some machinery from complex algebraic geometry needs to be used.

Keywords. Unitary tridiagonalization; 4×4 matrices; line bundle; degree; algebraic curve.

1. Main Theorem

Let $V = \mathbb{C}^n$, and $\langle\ ,\ \rangle$ be the usual euclidean hermitian inner product on V. $U(V) = U(n)$ denotes the group of unitary automorphisms of V with respect to $\langle\ ,\ \rangle$. $\{e_i\}_{i=1}^n$ will denote the standard orthonormal basis of V. $A \in M(n, \mathbb{C})$ will always denote an $n \times n$ complex matrix.

A matrix $A = [a_{ij}]$ is said to be *tridiagonal* if $a_{ij} = 0$ for all $1 \leq i, j \leq n$ such that $|i - j| \geq 2$. Then we have:

Theorem 1.1. *For $n \leq 4$, and $A \in M(n, \mathbb{C})$, there exists a unitary $U \in U(n)$ such that UAU^* is tridiagonal.*

Remark 1.2. The case $n = 3$, and counterexamples for $n \geq 6$, are due to Longstaff, [3]. In the paper [1], Fong and Wu construct counterexamples for $n = 5$, and provide a proof in certain special cases for $n = 4$. The article §4 of [1] poses the $n = 4$ case in general as an open question. Our main theorem above answers this question in the affirmative. In passing, we also provide another elementary proof for the $n = 3$ case.

2. Some Lemmas

We need some preliminary lemmas, which we collect in this section. In the sequel, we will also use the letter A to denote the unique linear transformation determined by the matrix $A = [a_{ij}]$ (satisfying $Ae_j = \sum_{i=1}^n a_{ij} e_i$).

Lemma 2.1. *Let $A \in M(n, \mathbb{C})$. For all n, the following are equivalent*:

(i) *There exists a unitary $U \in U(n)$ such that UAU^* is tridiagonal.*

(ii) *There exists a flag (= ascending sequence of* $\mathbb{C}$*-subspaces) of* $V = \mathbb{C}^n$:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that $\dim\, W_i = i,\; AW_i \subset W_{i+1}$ *and* $A^*W_i \subset W_{i+1}$ *for all* $0 \le i \le n-1$.

(iii) *There exists a flag in* V:

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n = V$$

such that $\dim\, W_i = i,\; AW_i \subset W_{i+1}$ *and* $A(W_{i+1}^{\perp}) \subset W_i^{\perp}$ *for all* $0 \le i \le n-1$.

Proof. (i) $\Rightarrow$ (ii). Set $W_i = \mathbb{C}\text{-span}(f_1, f_2, \ldots, f_i)$, where $f_i = U^*e_i$ and e_i is the standard basis of $V = \mathbb{C}^n$. Since the matrix $[b_{ij}] := UAU^*$ is tridiagonal, we have

$$Af_i = b_{i-1,i}\, f_{i-1} + b_{ii}\, f_i + b_{i+1,i}\, f_{i+1}, \quad \text{for} \quad 1 \le i \le n$$

(where b_{ij} is understood to be $= 0$ for $i, j \le 0$ or $\ge n+1$). Thus $AW_i \subset W_{i+1}$. Since $\{f_i\}_{i=1}^n$ is an orthonormal basis for $V = \mathbb{C}^n$, we also have

$$A^* f_i = \overline{b}_{i,i-1}\, f_{i-1} + \overline{b}_{ii}\, f_i + \overline{b}_{i,i+1}\, f_{i+1} \quad 1 \le i \le n$$

which shows $A^*(W_i) \subset W_{i+1}$ for all i as well, and (ii) follows.

(ii) $\Rightarrow$ (iii). $A^*W_i \subset W_{i+1}$ implies $(A^*W_i)^{\perp} \supset W_{i+1}^{\perp}$ for $1 \le i \le n-1$. But since $(A^*W_i)^{\perp} = A^{-1}(W_i^{\perp})$, we have $A(W_{i+1}^{\perp}) \subset W_i^{\perp}$ for $1 \le i \le n-1$ and (iii) follows.

(iii) $\Rightarrow$ (i). Inductively choose an orthonormal basis f_i of $V = \mathbb{C}^n$ so that W_i is the span of $\{f_1, \ldots, f_i\}$. Since $A(W_i) \subset W_{i+1}$, we have

$$Af_i = a_{1i}\, f_1 + a_{2i}\, f_2 + \cdots + a_{i+1,i}\, f_{i+1}. \tag{1}$$

Since $f_i \in (W_{i-1})^{\perp}$, and by hypothesis $A(W_{i-1}^{\perp}) \subset W_{i-2}^{\perp}$, and $W_{i-2}^{\perp} = \mathbb{C}\text{-span}(f_{i-1}, f_i, \ldots, f_n)$, we also have

$$Af_i = a_{i-1,i}\, f_{i-1} + a_{ii}\, f_i + \cdots + a_{ni}\, f_n \tag{2}$$

and by comparing the two equations (1), (2) above, it follows that

$$Af_i = a_{i-1,i}\, f_{i-1} + a_{ii}\, f_i + a_{i+1,i}\, f_{i+1}$$

for all i, and defining the unitary U by $U^*e_i = f_i$ makes UAU^* tridiagonal, so that (i) follows. □

Lemma 2.2. *Let* $n \le 4$. *If there exists a* 2*-dimensional* $\mathbb{C}$*-subspace* W *of* $V = \mathbb{C}^n$ *such that* $AW \subset W$ *and* $A^*W \subset W$*, then* A *is unitarily tridiagonalizable*.

Proof. If $n \le 2$, there is nothing to prove. For $n = 3$ or 4, the hypothesis implies that A maps $W^{\perp}$ onto itself. Then, in an orthonormal basis $\{f_i\}_{i=1}^n$ of V which satisfies $W = \mathbb{C}\text{-span}(f_1, f_2)$ and $W^{\perp} = \mathbb{C}\text{-span}(f_3, \ldots, f_n)$ the matrix of A is in $(1, 2)$ (resp. $(2, 2)$) block-diagonal form for $n = 3$ (resp. $n = 4$), which is clearly tridiagonal. □

Lemma 2.3. *Every matrix* $A \in M(3, \mathbb{C})$ *is unitarily tridiagonalizable*.

Proof. For $A \in M(3, \mathbb{C})$, consider the homogeneous cubic polynomial in $v = (v_1, v_2, v_3)$ given by

$$F(v_1, v_2, v_3) := \det(v, Av, A^*v).$$

Note $v \wedge Av \wedge A^*v = F(v_1, v_2, v_3) e_1 \wedge e_2 \wedge e_3$. By a standard result in dimension theory (see [4], p. 74, Theorem 5) each irreducible component of $V(F) \subset \mathbb{P}^2_{\mathbb{C}}$ is of dimension ≥ 1, and $V(F)$ is non-empty. Choose some $[v_1 : v_2 : v_3] \in V(F)$, and let $v = (v_1, v_2, v_3)$ which is non-zero. Then we have the two cases:

Case 1. v is a common eigenvector for A and A^*. Then the 2-dimensional subspace $W = (\mathbb{C}v)^{\perp}$ is an invariant subspace for both A and A^*, and applying the Lemma 2.2 to W yields the result.

Case 2. v is not a common eigenvector for A and A^*. Say it is not an eigenvector for A (otherwise interchange the roles of A and A^*). Set $W_1 = \mathbb{C}v$, $W_2 = \mathbb{C}\text{-span}(v, Av)$, $W_3 = V = \mathbb{C}^3$. Then $\dim W_i = i$, for $i = 1, 2, 3$, and the fact that $v \wedge Av \wedge A^*v = 0$ shows that $A^* W_1 \subset W_2$. Thus, by (ii) of Lemma 2.1, we are done. □

Note. From now on, $V = \mathbb{C}^4$ and $A \in M(4, \mathbb{C})$.

Lemma 2.4. *If A and A^* have a common eigenvector, then A is unitarily tridiagonalizable*.

Proof. If $v \neq 0$ is a common eigenvector for A and A^*, the 3-dimensional subspace $W = (\mathbb{C}v)^{\perp}$ is invariant under both A and A^*, and unitary tridiagonalization of $A_{|W}$ exists from the $n = 3$ case of Lemma 2.3 by a $U_1 \in U(W) = U(3)$. The unitary $U = 1 \oplus U_1$ is the desired unitary in $U(4)$ tridiagonalizing A. □

Lemma 2.5. *If the main theorem holds for all $A \in S$, where S is any dense* (*in the classical topology*) *subset of $M(4, \mathbb{C})$, then it holds for all $A \in M(4, \mathbb{C})$.*

Proof. This is a consequence of the compactness of the unitary group $U(4)$. Indeed, let T denote the closed subset of tridiagonal (with respect to the standard basis) matrices.

Let $A \in M(4, \mathbb{C})$ be any general element. By the density of S, there exist $A_n \in S$ such that $A_n \to A$. By hypothesis, there are unitaries $U_n \in U(4)$ such that $U_n A_n U_n^* = T_n$, where T_n are tridiagonal. By the compactness of $U(4)$, and by passing to a subsequence if necessary, we may assume that $U_n \to U \in U(4)$. Then $U_n A_n U_n^* \to UAU^*$. That is $T_n \to UAU^*$. Since T is closed, and $T_n \in T$, we have UAU^* is in T, viz., is tridiagonal. □

We shall now construct a suitable dense open subset $S \subset M(4, \mathbb{C})$, and prove tridiagonalizability for a general $A \in S$ in the remainder of this paper. More precisely:

Lemma 2.6. *There is a dense open subset $S \subset M(4, \mathbb{C})$ such that*:

(i) *A is nonsingular for all $A \in S$.*

(ii) *A has distinct eigenvalues for all $A \in S$.*

(iii) *For each* $A \in S$*, the element* $(t_0 I + t_1 A + t_2 A^*) \in M(4, \mathbb{C})$ *has rank* ≥ 3 *for all* $(t_0, t_1, t_2) \neq (0, 0, 0)$ *in* $\mathbb{C}^3$.

Proof. The subset of singular matrices in $M(4, \mathbb{C})$ is the complex algebraic subvariety of complex codimension one defined by $Z_1 = \{A : \det A = 0\}$. Let S_1, (which is just $GL(4, \mathbb{C})$) be its complement. Clearly S_1 is open and dense in the classical topology (in fact, also in the Zariski topology).

A matrix A has distinct eigenvalues iff its characteristic polynomial ϕ_A has distinct roots. This happens iff the discriminant polynomial of ϕ_A, which is a 4th degree homogeneous polynomial $\Delta(A)$ in the entries of A, is not zero. The zero set $Z_2 = V(\Delta)$ is again a codimension-1 subvariety in $M(4, \mathbb{C})$, so its complement $S_2 = (V(\Delta))^c$ is open and dense in both the classical and Zariski topologies.

To enforce (iii), we claim that the set defined by

$$\begin{aligned} Z_3 := \{A \in M(4, \mathbb{C}) :\ & \text{rank}\,(t_0 I + t_1 A + t_2 A^*) \leq 2 \text{ for some } (t_0, t_1, t_2) \\ & \neq (0, 0, 0) \ \text{ in } \ \mathbb{C}^3\} \end{aligned}$$

is a proper *real* algebraic subset of $M(4, \mathbb{C})$. The proof hinges on the fact that three general cubic curves in $\mathbb{P}^2_{\mathbb{C}}$ having a point in common imposes an algebraic condition on their coefficients.

Indeed, saying that rank $(t_0 I + t_1 A + t_2 A^*) \leq 2$ for some $(t_0, t_1, t_2) \neq (0, 0, 0)$ is equivalent to saying that the third exterior power $\bigwedge^3 (t_0 I + t_1 A + t_2 A^*)$ is the zero map, for some $(t_0, t_1, t_2) \neq 0$. This is equivalent to demanding that there exist a $(t_0, t_1, t_2) \neq 0$ such that the determinants of all the 3×3-minors of $(t_0 I + t_1 A + t_2 A^*)$ are zero.

Note that the (determinants of) the (3×3)-minors of $(t_0 I + t_1 A + t_2 A^*)$, denoted as $M_{ij}(A, t)$ (where the ith row and jth column are deleted) are complex valued, complex algebraic and $\mathbb{C}$-homogeneous of degree 3 in $t = (t_0, t_1, t_2)$, with coefficients real algebraic of degree 3 in the variables $(A_{ij}, \bar{A}_{ij})$ (or, equivalently, in $\mathrm{Re}\, A_{ij}$, $\mathrm{Im}\, A_{ij}$), where $A = [A_{ij}]$.

We know that the space of all homogeneous polynomials of degree 3 with complex coefficients in (t_0, t_1, t_2) (up to scaling) is parametrized by the projective space $\mathbb{P}^9_{\mathbb{C}}$ (the Veronese variety, see [4], p. 52). We first consider the complex algebraic variety:

$$X = \{(P, Q, R, [t]) \in \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} : P(t) = Q(t) = R(t) = 0\},$$

where $[t] := [t_0 : t_1 : t_2]$, and (P, Q, R) denotes a triple of homogeneous polynomials. This is just the subset of those $(P, Q, R, [t])$ in the product $\mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ such that the point $[t]$ lies on all three of the plane cubic curves $V(P)$, $V(Q)$, $V(R)$. Since X is defined by multihomogenous degree (1, 1, 1, 3) equations, it is a complex algebraic subvariety of the quadruple product. Its image under the first projection $Y := \pi_1(X) \subset \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}}$ is therefore an algebraic subvariety inside this triple product (see [4], p. 58, Theorem 3). Y is a proper subvariety because, for example, the cubic polynomials $P = t_0^3$, $Q = t_1^3$, $R = t_2^3$ have no common non-zero root.

Denote pairs (i, j) with $1 \leq i, j \leq 4$ by capital letters like I, J, K etc. From the minorial determinants $M_I(A, t)$, we can define various *real algebraic* maps:

$$\begin{aligned} \Theta_{IJK} : M(4, \mathbb{C}) \quad &\to \quad \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \\ A \quad &\mapsto \quad (M_I(A, t), M_J(A, t), M_K(A, t)) \end{aligned}$$

for I, J, K distinct. Clearly, $\bigwedge^3(t_0 I + t_1 A + t_2 A^*) = 0$ for some $t = (t_0, t_1, t_2) \neq (0, 0, 0)$ iff $\Theta_{IJK}(A)$ lies in the complex algebraic subvariety Y of $\mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}} \times \mathbb{P}^9_{\mathbb{C}}$, for all I, J, K distinct. Hence the subset $Z_3 \subset M(4, \mathbb{C})$ defined above is the intersection:

$$Z_3 = \bigcap_{I,J,K} \Theta^{-1}_{IJK}(Y),$$

where I, J, K runs over all distinct triples of pairs (i, j), $1 \le i, j \le 4$.

We claim that Z_3 is a proper real algebraic subset of $M(4, \mathbb{C})$. Clearly, since each $M_I(A, t)$ is real algebraic in the variables $\operatorname{Re} A_{ij}$, $\operatorname{Im} A_{ij}$ the map Θ_{IJK} is real algebraic. Since Y is complex and hence real algebraic, its inverse image $\Theta^{-1}_{IJK}(Y)$, defined by the real algebraic equations obtained upon substitution of the components $M_I(A, t)$, $M_J(A, t)$, $M_K(A, t)$ in the equations that define Y, is also real algebraic. Hence the set Z_3 is a real algebraic subset of $M(4, \mathbb{C})$.

To see that Z_3 is a *proper* subset of $M(4, \mathbb{C})$, we simply consider the matrix (defined with respect to the standard orthonormal basis $\{e_i\}_{i=1}^4$ of $\mathbb{C}^4$):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $t = (t_0, t_1, t_2) \neq 0$, we see that

$$t_0 I + t_1 A + t_2 A^* = \begin{bmatrix} t_0 & t_1 & 0 & 0 \\ t_2 & t_0 & t_1 & 0 \\ 0 & t_2 & t_0 & t_1 \\ 0 & 0 & t_2 & t_0 \end{bmatrix}.$$

For the above matrix the minorial determinant $M_{41}(A, t) = t_1^3$, whereas $M_{14}(A, t) = t_2^3$. The only common zeros to these two minorial determinants are points $[t_0 : 0 : 0]$. Setting $t_1 = t_2 = 0$ in the matrix above gives $M_{ii}(A, t) = t_0^3$ for $1 \le i \le 4$. Thus t_0 must also be 0 for all the minorial determinants to vanish. Hence the matrix A above lies outside the real algebraic set Z_3.

It is well-known that a proper real algebraic subset in euclidean space cannot have a non-empty interior. Thus the complement Z_3^c is dense and open in the classical and real-Zariski topologies. Take $S_3 = Z_3^c$.

Finally, set

$$S := S_1 \cap S_2 \cap S_3 = \left(\bigcup_{i=1}^{3} Z_i \right)^c$$

which is also open and dense in the classical topology in $M(4, \mathbb{C})$. Hence the lemma. □

Remark 2.7. One should note here that for *each* matrix $A \in M(4, \mathbb{C})$, there will be at least a curve of points $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbb{C}}$ (defined by the vanishing of $\det(t_0 I + t_1 A + t_2 A^*)$), on which $(t_0 I + t_1 A + t_2 A^*)$ is singular. Similarly for each A there is at least a curve of points on which the trace $\operatorname{tr}\left(\bigwedge^3(t_0 I + t_1 A + t_2 A^*)\right)$ vanishes, and so a non-empty (and generally a finite) set on which *both* these polynomials vanish, by dimension theory ([4], Theorem 5, p. 74). Thus for *each* $A \in M(4, \mathbb{C})$, there is at least a non-empty finite set of points $[t]$ such that $(t_0 I + t_1 A + t_2 A^*)$ has 0 as a repeated eigenvalue. For example, for the matrix A constructed at the end of the previous lemma, we see that the matrix $(t_0 I + t_1 A + t_2 A^*)$ is strictly upper-triangular and thus has 0 as an eigenvalue of multiplicity 4 for all $(0, t_1, 0) \neq 0$, but nevertheless has rank 3 for all $(t_0, t_1, t_2) \neq (0, 0, 0)$.

Indeed, as (iii) of the lemma above shows, for A in the open dense subset S, the kernel $\ker(t_0 I + t_1 A + t_2 A^*)$ is at most 1-dimensional *for all* $[t] = [t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbf{C}}$.

3. The varieties $\boldsymbol{C}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{D}$

Notation 3.1. In the light of Lemmas 2.5 and 2.6 above, we shall henceforth assume $A \in S$. As is easily verified, this implies $A^* \in S$ as well. We will also henceforth assume, in view of Lemma 2.4 above, that *A and A^* have no common eigenvectors*. (For example, this rules out A being normal, in which case we know that the main result for A is true by the spectral theorem.) Also, in view of Lemma 2.2, we shall assume that A and A^* do not have a common 2-dimensional invariant subspace.

In $\mathbb{P}^3_{\mathbf{C}}$, the complex projective space of $V = \mathbb{C}^4$, we denote the equivalence class of $v \in V \setminus 0$ by $[v]$. For a $[v] \in \mathbb{P}^3_{\mathbf{C}}$, we define $W([v])$ (or simply $W(v)$ when no confusion is likely) by

$$W([v]) := \mathbb{C}\text{-span}(v, Av, A^*v).$$

Since we are assuming that A and A^* have no common eigenvectors, we have $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3_{\mathbf{C}}$.

Denote the four distinct points in $\mathbb{P}^3_{\mathbf{C}}$ representing the four linearly independent eigenvectors of A (resp. A^*) by E (resp. E^*). By our assumption above, $E \cap E^* = \phi$.

Lemma 3.2. *Let* $A \in M(4, \mathbb{C})$ *be as in* 3.1 *above. Then the closed subset*:

$$C = \{[v] \in \mathbb{P}^3_{\mathbf{C}} : v \wedge Av \wedge A^*v = 0\}$$

is a closed projective variety. This variety C is precisely the subset of $[v] \in \mathbb{P}^3_{\mathbf{C}}$ *for which the dimension* $\dim W([v]) = \dim(\mathbb{C}\text{-span}\{v, Av, A^*v\})$ *is exactly* 2.

Proof. That C is a closed projective variety is clear from the fact that it is defined as the set of common zeros of all the four (3×3)-minorial determinants of the (3×4)-matrix

$$\Lambda := \begin{bmatrix} v \\ Av \\ A^*v \end{bmatrix}$$

(which are all degree-3 homogeneous polynomials in the components of v with respect to some basis). Also C is nonempty since it contains $E \cup E^*$.

Also, since A and A^* are nonsingular by the assumptions in 3.1, the wedge product $v \wedge Av \wedge A^*v$ of the three non-zero vectors v, Av, A^*v vanishes precisely when the space $W([v]) = \mathbb{C}\text{-span}(v, Av, A^*v)$ is of dimension ≤ 2. Since by 3.1, A, A^* have no common eigenvectors, the dimension $\dim W([v]) \geq 2$ for all $[v] \in \mathbb{P}^3_{\mathbf{C}}$, so C is precisely the locus of $[v] \in \mathbb{P}^3_{\mathbf{C}}$ for which the space $W([v])$ is 2-dimensional. $\square$

Now we shall show that for A as in 3.1, the variety C defined above is of pure dimension one. For this, we need to define some more associated algebraic varieties and regular maps.

DEFINITION 3.3

Let us define the bilinear map:

$$\begin{aligned} B : \mathbb{C}^4 \times \mathbb{C}^3 &\to \mathbb{C}^4 \\ (v, t_0, t_1, t_2) &\mapsto B(v, t) := (t_0 I + t_1 A + t_2 A^*)v. \end{aligned}$$

We then have the linear maps $B(v, -) : \mathbb{C}^3 \to \mathbb{C}^4$ for $v \in \mathbb{C}^4$ and $B(-, t) : \mathbb{C}^4 \to \mathbb{C}^3$ for $t \in \mathbb{C}^3$.

Note that the image Im $B(v, -)$ is the span of $\{v, Av, A^*v\}$, which was defined to be $W(v)$. For a fixed t, denote the kernel

$$K(t) := \ker(B(-, t) : \mathbb{C}^4 \to \mathbb{C}^4).$$

Denoting $[t_0 : t_1 : t_2]$ by $[t]$ and $[v_1 : v_2 : v_3 : v_4]$ by $[v]$ for brevity, we define

$$\Gamma := \{([v], [t]) \in \mathbb{P}^3_{\mathrm{C}} \times \mathbb{P}^2_{\mathrm{C}} : B(v, t) = 0\}.$$

Finally, define the variety D by

$$D \subset \mathbb{P}^2_{\mathrm{C}} := \{[t] \in \mathbb{P}^2_{\mathrm{C}} : \ \det\, B(-, t) = \det\,(t_0 I + t_1 A + t_2 A^*) = 0\}.$$

Let

$$\pi_1 : \mathbb{P}^3_{\mathrm{C}} \times \mathbb{P}^2_{\mathrm{C}} \to \mathbb{P}^3_{\mathrm{C}}, \quad \pi_2 : \mathbb{P}^3_{\mathrm{C}} \times \mathbb{P}^2_{\mathrm{C}} \to \mathbb{P}^2_{\mathrm{C}}$$

denote the two projections.

Lemma 3.4. *We have the following facts*:

(i) $\pi_1(\Gamma) = C$, *and* $\pi_2(\Gamma) = D$.

(ii) $\pi_1 : \Gamma \to C$ *is* 1-1, *and the map* g *defined by*

$$g := \pi_2 \circ \pi_1^{-1} : C \to D$$

is a regular map so that Γ *is the graph of* g *and isomorphic as a variety to* C.

(iii) $D \subset \mathbb{P}^2_{\mathrm{C}}$ *is a plane curve, of pure dimension one. The map* $\pi_2 : \Gamma \to D$ *is* 1-1, *and the map* $\pi_1 \circ \pi_2^{-1} : D \to C$ *is the regular inverse of the regular map* g *defined above in* (ii). *Again* Γ *is also the graph of this regular inverse* g^{-1}, *and* D *and* Γ *are isomorphic as varieties. In particular,* C *and* D *are isomorphic as varieties, and thus* C *is a curve in* $\mathbb{P}^3_{\mathrm{C}}$ *of pure dimension one*.

(iv) *Inside* $\mathbb{P}^3_{\mathrm{C}} \times \mathbb{P}^2_{\mathrm{C}}$, *each irreducible component of the intersection of the four divisors* $D_i := (B_i(v, t) = 0)$ *for* $i = 1, 2, 3, 4$ (*where* $B_i(v, t)$ *is the* i*-th component of* $B(v, t)$ *with respect to a fixed basis of* $\mathbb{C}^4$) *occurs with multiplicity* 1. (*Note that* Γ *is set-theoretically the intersection of these four divisors, by definition*).

Proof. It is clear that $\pi_1(\Gamma) = C$, because $B(v,t) = t_0 v + t_1 A v + t_2 A^* v = 0$ for some $[t_0 : t_1 : t_2] \in \mathbb{P}^2_{\mathbb{C}}$ iff dim $W(v) \leq 2$, and since A and A^* have no common eigenvectors, this means dim $W(v) = 2$. That is, $[v] \in C$.

Clearly $[t] \in \pi_2(\Gamma)$ iff there exists a $[v] \in \mathbb{P}^3_{\mathbb{C}}$ such that $B(v,t) = 0$. That is, iff dim ker $B(-,t) \geq 1$, that is, iff

$$G(t_0, t_1, t_2) := \det\ B(-,t) = 0.$$

Thus $D = \pi_2(\Gamma)$ and is defined by a single degree 4 homogeneous polynomial G inside $\mathbb{P}^2_{\mathbb{C}}$. It is a curve of pure dimension 1 in $\mathbb{P}^2_{\mathbb{C}}$ by standard dimension theory (see [4], p. 74, Theorem 5) because, for example $[1:0:0] \notin D$ so $D \neq \mathbb{P}^2_{\mathbb{C}}$. So $\pi_2(\Gamma) = D$, and this proves (i).

To see (ii), for a given $[v] \in C$, we claim there is exactly one $[t]$ such that $([v],[t]) \in \Gamma$. Note that $([v],[t]) \in \Gamma$ iff the linear map:

$$\begin{aligned} B(v,-) : \mathbb{C}^3 &\to \mathbb{C}^4 \\ t &\mapsto (t_0 I + t_1 A + t_2 A^*) v \end{aligned}$$

has a non-trivial kernel containing the line $\mathbb{C}t$. That is, dim Im $B(v,-) \leq 2$. But the image Im $B(v,-) = W(v)$, which is of dimension 2 for all $v \in C$ by our assumptions. Thus its kernel must be exactly one dimensional, defined by ker $B(v,-) = \mathbb{C}t$. Thus $([v],[t])$ is the unique point in Γ lying in $\pi_1^{-1}[v]$, viz. for each $[v] \in C$, the vertical line $[v] \times \mathbb{P}^2_{\mathbb{C}}$ intersects Γ in a single point, call it $([v], g[v])$. So $\pi_1 : \Gamma \to C$ is 1-1, and Γ is the graph of a map $g : C \to D$. Since $g([v]) = \pi_2 \pi_1^{-1}([v])$ for $[v] \in C$, and Γ is algebraic, g is a regular map. This proves (ii).

To see (iii), note that for $[t] \in D$, by definition, the dimension dim ker $B(-,t) \geq 1$. By the fact that $A \in S$, and (iii) of Lemma 2.6, we know that dim ker $B(-,t) \leq 1$ for all $[t] \in \mathbb{P}^2_{\mathbb{C}}$. Thus, denoting $K(t) :=$ ker $B(-,t)$ for $[t] \in D$, we have

$$\dim K(t) = 1 \quad \text{for all} \quad t \in D. \tag{3}$$

Hence we see that the unique projective line $[v]$ corresponding to $\mathbb{C}v = K(t)$ yields the unique element of C, such that $([v],[t]) \in \Gamma$. Thus $\pi_2 : \Gamma \to D$ is 1-1, and the regular map $\pi_1 \circ \pi_2^{-1} : D \to C$ is the regular inverse to the map g of (ii) above. Γ is thus also the graph of g^{-1} and, in particular, is isomorphic to D. Since g is an isomorphism of curves, and D is of pure dimension 1, it follows that C is of pure dimension one. This proves (iii).

To see (iv), we need some more notation.

Note that $D \subset \mathbb{P}^2_{\mathbb{C}} \setminus \{[1;0;0]\}$, (because there exists no $[v] \in \mathbb{P}^3_{\mathbb{C}}$ such that $I.v = 0$!). Thus there is a regular map:

$$\begin{aligned} \theta : D &\to \mathbb{P}^1_{\mathbb{C}} \\ [t_0 : t_1 : t_2] &\mapsto [t_1 : t_2]. \end{aligned} \tag{4}$$

Let $\Delta(t_1, t_2)$ be the discriminant polynomial of the characteristic polynomial $\phi_{t_1 A + t_2 A^*}$ of $t_1 A + t_2 A^*$. Clearly $\Delta(t_1, t_2)$ is a homogeneous polynomial of degree 4 in (t_1, t_2), and it is not the zero polynomial because, for example, $\Delta(1,0) \neq 0$, for $\Delta(1,0)$ is the discriminant of ϕ_A, which has distinct roots (=the distinct eigenvalues of A) by the assumptions 3.1 on A. Let $\Sigma \subset \mathbb{P}^1_{\mathbb{C}}$ be the zero locus of Δ, which is a finite set of points. Note that the fibre $\theta^{-1}([1:\mu])$ consists of all $[t:1:\mu] \in D$ such that $-t$ is an eigenvalue of $A + \mu A^*$, which are at most four in number. Similarly the fibres $\theta^{-1}([\lambda : 1])$ are also finite. Thus the subset of D defined by

$$F := \theta^{-1}(\Sigma)$$

is a finite subset of D. F is precisely the set of points $[t] = [t_0 : t_1 : t_2]$ such that $B(-, t) = (t_0 I + t_1 A + t_2 A^*)$ has 0 as a repeated eigenvalue.

Since $\pi_2 : \Gamma \to D$ is 1-1, the inverse image:

$$F_1 = \pi_2^{-1}(F) \subset \Gamma$$

is a finite subset of Γ.

We will now prove that for each irreducible component Γ_α of Γ, and each point $x = ([a], [b])$ in $\Gamma_\alpha \setminus F_1$, the four equations $\{B_i(v, t) = 0\}_{i=1}^4$ are the generators of the ideal of the variety Γ_α in an affine neighbourhood of x, where $B_i(v, t)$ are the components of $B(v, t)$ with respect to a fixed basis of $\mathbb{C}^4$. Since F_1 is a finite set, this will prove (iv), because the multiplicity of Γ_α in the intersection cycle of the four divisors $D_i = (B_i(v, t) = 0)$ in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ is determined by generic points on Γ_α, for example all points of $\Gamma_\alpha \setminus F_1$. We will prove this by showing that for $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$, the four divisors $(B_i(v, t) = 0)$ intersect transversely at x.

So let Γ_α be some irreducible component of Γ, with $x = ([a], [b]) \in \Gamma_\alpha \setminus F_1$.

Fix an $a \in \mathbb{C}^4$ representing $[a] \in C_\alpha := \pi_1(\Gamma_\alpha)$, and also fix $b \in \mathbb{C}^3$ representing $[b] = g([a]) \in g(C_\alpha)$. Also fix a 3-dimensional linear complement $V_1 := T_{[a]}(\mathbb{P}^3_{\mathbb{C}}) \subset \mathbb{C}^4$ to a and similarly, fix a 2-dimensional linear complement $V_2 = T_{[b]}(\mathbb{P}^2_{\mathbb{C}}) \subset \mathbb{C}^3$ to b. (The notation comes from the fact that $T_{[v]}(\mathbb{P}^n_{\mathbb{C}}) \simeq \mathbb{C}^{n+1}/\mathbb{C}v$, which we are identifying non-canonically with these respective complements V_i.) These complements also provide local coordinates in the respective projective spaces as follows. Set coordinate charts ϕ around $[a] \in \mathbb{P}^3_{\mathbb{C}}$ by $[v] = \phi(u) := [a+u]$, and ψ around $[b] \in \mathbb{P}^2_{\mathbb{C}}$ by $[t] = \psi(s) := [b+s]$, where $u \in V_1 \simeq \mathbb{C}^3$, and $s \in V_2 \simeq \mathbb{C}^2$. The images $\phi(V_1)$ and $\psi(V_2)$ are affine neighbourhoods of $[a]$ and $[b]$ respectively. These charts are like 'stereographic projection' onto the tangent space and depend on the initial choice of a (resp. b) representing $[a]$ (resp. $[b]$), and are *not* the standard coordinate systems on projective space, but more convenient for our purposes.

Then the local affine representation of $B(v, t)$ on the affine open $V_1 \times V_2 = \mathbb{C}^3 \times \mathbb{C}^2$, which we denote by β, is given by

$$\beta(u, s) := B(a + u, b + s).$$

Note that ker $B(a, -) = \mathbb{C}b$, where $[b] = g([a])$, so that $B(a, -)$ passes to the quotient as an isomorphism:

$$B(a, -) : V_2 \xrightarrow{\sim} W(a), \tag{5}$$

where $W(a)$ is 2-dimensional.

Similarly, since $B(-, b)$ has one dimensional kernel $\mathbb{C}a = K(b) \subset \mathbb{C}^4$, by (3) above, we also have the other isomorphism:

$$B(-, b) : V_1 \xrightarrow{\sim} \mathrm{Im}\ B(-, b), \tag{6}$$

where Im $B(-, b)$ is 3-dimensional, therefore.

Now one can easily calculate the derivative $D\beta(0,0)$ of β at $(u,s) = (0,0)$. Let $(X,Y) \in V_1 \times V_2$. Then, by bilinearity of B, we have

$$\begin{aligned}\beta(X,Y) - \beta(0,0) &= B(a+X, b+Y) - B(a,b)\\ &= B(X,b) + B(a,Y) + B(X,Y).\end{aligned}$$

Now since $B(X,Y)$ is quadratic, it follows that

$$\begin{aligned}D\beta(0,0) : V_1 \times V_2 &\to \mathbb{C}^4\\ (X,Y) &\mapsto B(X,b) + B(a,Y).\end{aligned} \tag{7}$$

By eqs (5) and (6) above, we see that the image of $D\beta(0,0)$ is precisely $\operatorname{Im} B(-,b) + W(a)$.

Claim. For $([a],[b]) \in \Gamma_\alpha \setminus F_1$, the space $\operatorname{Im} B(-,b) + W(a)$ is all of $\mathbb{C}^4$.

Proof of Claim. Denote $T := B(-,b)$ for brevity. Clearly $a \in W(a)$ by definition of $W(a)$. Also, $a \in \ker T = K(b)$. We claim that a is not in the image of T. For, if $a \in \operatorname{Im} T$, we would have $a = Tw$ for some $w \notin K(b) = \ker T$ and $w \neq 0$. In fact w is not a multiple of a since $Tw = a \neq 0$ whereas $a \in \ker T$. Thus we would have $T^2 w = 0$, and completing $f_1 = a = Tw$, $f_2 = w$ to a basis $\{f_i\}_{i=1}^4$ of $\mathbb{C}^4$, the matrix of T with respect to this basis would be of the form:

$$\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

Thus $T = B(-,b)$ would have 0 as a repeated eigenvalue. But we have stipulated that $([a],[b]) \notin F_1$, so that $[b] \notin F$, and hence $B(-,b)$ does not have 0 as a repeated eigenvalue. Hence the non-zero vector $a \in W(a)$ is not in $\operatorname{Im} T$. Since $\operatorname{Im} T$ is 3-dimensional, we have $\mathbb{C}^4 = \operatorname{Im} T + W(a)$, and this proves the claim. □

In conclusion, all the points of $\Gamma_\alpha \setminus F_1$ are in fact smooth points of Γ_α, and the local equations for Γ_α in a small neighbourhood of such a point are precisely the four equations $\beta_i(u,s) = 0$, $1 \le i \le 4$. This proves (iv), and the lemma. □

4. Some algebraic bundles

We construct an algebraic line bundle with a (regular) global section over C. By showing that this line bundle has positive degree, we will conclude that the section has zeroes in C. Any zero of this section will yield a flag of the kind required by Lemma 2.1. One of the technical complications is that none of the bundles we define below are allowed to use the hermitian metric on V, orthogonal complements, orthonormal bases etc., because we wish to remain in the $\mathbb{C}$-algebraic category. As a general reference for this section and the next, the reader may consult [2].

DEFINITION 4.1

For $0 \neq v \in V = \mathbb{C}^4$, we will denote the point $[v] \in \mathbb{P}^3_{\mathbb{C}}$ by v, whenever no confusion is likely, to simplify notation. We have already denoted the vector subspace $\mathbb{C}\text{-span}(v, Av, A^*v) \subset \mathbb{C}^4$ as $W(v)$. Further define $W_3(v) := W(v) + AW(v)$, and $\widetilde{W}_3(v) := W(v) + A^*W(v)$. Clearly both $W_3(v)$ and $\widetilde{W}_3(v)$ contain $W(v)$.

Since A and A^* have no common eigenvectors, we have $\dim\, W(v) \geq 2$ for all $v \in \mathbb{P}^3_{\mathbb{C}}$, and $\dim\, W(v) = 2$ for all $v \in C$, because of the defining equation $v \wedge Av \wedge A^*v = 0$ of C. Also, since $\dim\, W(v) = 2 = \dim\, AW(v)$ for $v \in C$, and since $0 \neq Av \in W(v) \cap AW(v)$, we have $\dim\, W_3(v) \leq 3$ for all $v \in C$. Similarly $\dim\, \widetilde{W}_3(v) \leq 3$ for all $v \in C$.

If there exists a $v \in C$ such that $\dim\, W_3(v) = 2$, then we are done. For, in this case $W_3(v)$ must equal $W(v)$ since it contains $W(v)$. Then the dimension $\dim\, \widetilde{W}_3(v) = 2$ or $= 3$. If it is 2, $W(v)$ will be a 2-dimensional invariant space for both A and A^*, and the main theorem will follow by Lemma 2.2. If $\dim\, \widetilde{W}_3(v) = 3$, then the flag:

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = \widetilde{W}_3(v) \subset W_4 = V$$

satisfies the requirements of (ii) in Lemma 2.1, and we are done. Similarly, if there exists a $v \in C$ with $\dim\, \widetilde{W}_3(v) = 2$, we are again done. Hence we may assume that:

$$\dim\, W_3(v) = \dim\, \widetilde{W}_3(v) = 3 \quad \text{for all} \quad v \in C. \tag{8}$$

In the light of the above, we have the following:

Remark 4.2. We are reduced to the situation where the following condition holds: For each $v \in C$, $\dim\, W(v) = 2$, $\dim\, W_3(v) = \dim\, \widetilde{W}_3(v) = 3$.

Now our main task is to prove that there exists a $v \in C$ such that the two 3-dimensional subspaces $W_3(v)$ and $\widetilde{W}_3(v)$ are the *same*. In that event, the flag

$$0 = W_0 \subset W_1 = \mathbb{C}v \subset W_2 = W(v) \subset W_3 = W(v) + AW(v) = W(v) + A^*W(v) \subset W_4 = V$$

will meet the requirements of (ii) of the Lemma 2.1. The remainder of this discussion is aimed at proving this.

DEFINITION 4.3

Denote the trivial rank 4 algebraic bundle on $\mathbb{P}^3_{\mathbb{C}}$ by $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, with fibre $V = \mathbb{C}^4$ at each point (following standard algebraic geometry notation). Similarly, $\mathcal{O}^4_C$ is the trivial bundle on C. In $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, there is the tautological line-subbundle $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$, whose fibre at v is $\mathbb{C}v$. Its restriction to the curve C is denoted as $\mathcal{W}_1 := \mathcal{O}_C(-1)$.

There are also the line subbundles $A\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$ (respectively $A^*\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$) of $\mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$, whose fibre at v is Av (respectively A^*v). Both are isomorphic to $\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1)$ (via the global linear automorphisms A (resp. A^*) of V). Similarly, their restrictions $A\mathcal{O}_C(-1)$, $A^*\mathcal{O}_C(-1)$, both isomorphic to $\mathcal{O}_C(-1)$. *Note that throughout what follows, bundle isomorphism over any variety X will mean algebraic isomorphism, i.e. isomorphism of the corresponding sheaves of algebraic sections as $\mathcal{O}_X$-modules.*

Denote the rank 2 algebraic bundle with fibre $W(v) \subset V$ at $v \in C$ as $\mathcal{W}_2$. It is an algebraic sub-bundle of $\mathcal{O}_C^4$, for its sheaf of sections is the restriction of the subsheaf

$$\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) + A\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) + A^*\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(-1) \subset \mathcal{O}^4_{\mathbb{P}^3_{\mathbb{C}}}$$

to the curve C, which is precisely the subvariety of $\mathbb{P}^3_{\mathbb{C}}$ on which the sheaf above is locally free of rank 2 (=rank 2 algebraic bundle).

Denote the rank 3 algebraic sub-bundle of $\mathcal{O}_C^4$ with fibre $W_3(v) = W(v) + AW(v)$ (respectively $\widetilde{W}_3(v) = W(v) + A^*W(v)$) by $\mathcal{W}_3$ (respectively $\widetilde{\mathcal{W}}_3$). Both $\mathcal{W}_3$ and $\widetilde{\mathcal{W}}_3$ are of rank 3 on C because of Remark 4.2 above, and both contain $\mathcal{W}_2$ as a sub-bundle. We denote the line bundles $\bigwedge^2 \mathcal{W}_2$ by $\mathcal{L}_2$, and $\bigwedge^3 \mathcal{W}_3$ (resp. $\bigwedge^3 \widetilde{W}_3$) by $\mathcal{L}_3$ (resp. $\widetilde{\mathcal{L}}_3$). Then $\mathcal{L}_2$ is a line sub-bundle of $\bigwedge^2 \mathcal{O}_C^4$, and $\mathcal{L}_3$, $\widetilde{\mathcal{L}}_3$ are line sub-bundles of $\bigwedge^3 \mathcal{O}_C^4$.

Finally, for X any variety, with a bundle $\mathcal{E}$ on X which is a sub-bundle of a trivial bundle $\mathcal{O}_X^m$, the *annihilator* of $\mathcal{E}$ is defined as

$$\mathrm{Ann}\mathcal{E} = \{\phi \in \hom_X(\mathcal{O}_X^m, \mathcal{O}_X) : \phi(\mathcal{E}) = 0\}.$$

Clearly, by taking $\hom_X(-\,, \mathcal{O}_X)$ of the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_X^m \to \mathcal{O}_X^m/\mathcal{E} \to 0,$$

the bundle

$$\mathrm{Ann}\mathcal{E} \simeq \hom_X(\mathcal{O}_X^m/\mathcal{E}, \mathcal{O}_X) = (\mathcal{O}_X^m/\mathcal{E})^*,$$

where $*$ always denotes the (complex) dual bundle.

Lemma 4.4. *Denote the bundle $\mathcal{W}_3/\mathcal{W}_2$ (resp. $\widetilde{\mathcal{W}}_3/\mathcal{W}_2$) by Λ (resp. $\widetilde{\Lambda}$). Then we have the following identities of bundles on C*:

(i)

$$\begin{aligned}
0 &\to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0\\
0 &\to \mathcal{W}_2 \to \widetilde{\mathcal{W}}_3 \to \widetilde{\Lambda} \to 0\\
0 &\to \mathcal{L}_3 \xrightarrow{i} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\pi} \Lambda^* \to 0\\
0 &\to \widetilde{\mathcal{L}}_3 \xrightarrow{\tilde{i}} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\tilde{\pi}} \widetilde{\Lambda}^* \to 0,
\end{aligned}$$

(ii)

$$\mathcal{L}_3 \simeq \mathcal{L}_2 \otimes \Lambda \quad \text{and} \quad \widetilde{\mathcal{L}}_3 \simeq \mathcal{L}_2 \otimes \widetilde{\Lambda},$$

(iii)

$$\bigwedge^2 \mathrm{Ann}\mathcal{W}_2 \simeq \bigwedge^2 \mathcal{W}_2,$$

(iv)

$$\Lambda \simeq \widetilde{\Lambda},$$

(v)

$$\mathcal{L}_2 \simeq \Lambda \otimes \mathcal{O}_C(-1) \simeq \widetilde{\Lambda} \otimes \mathcal{O}_C(-1),$$

(vi)

$$\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^* \otimes \widetilde{\Lambda}^{*2} \simeq \mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2).$$

Proof. From the definition of Λ, we have the exact sequence:

$$0 \to \mathcal{W}_2 \to \mathcal{W}_3 \to \Lambda \to 0$$

from which it follows that:

$$0 \to \Lambda \to \mathcal{O}_C^4/\mathcal{W}_2 \to \mathcal{O}_C^4/\mathcal{W}_3 \to 0$$

is exact. Taking $\hom_C(-, \mathcal{O}_C)$ of this exact sequence yields the exact sequence:

$$0 \to \mathrm{Ann}\mathcal{W}_3 \to \mathrm{Ann}\mathcal{W}_2 \to \Lambda^* \to 0.$$

Now, via the canonical isomorphism $\bigwedge^3 V \to V^*$ which arises from the non-degenerate pairing

$$\bigwedge^3 V \otimes V \to \bigwedge^4 V \simeq \mathbb{C},$$

it is clear that $\mathrm{Ann}\mathcal{W}_3 \simeq \bigwedge^3 \mathcal{W}_3 = \mathcal{L}_3$.

Thus the first and third exact sequences of (i) follow. The proofs of the second and fourth are similar. From the first exact sequence in (i), it follows that $\bigwedge^3 \mathcal{W}_3 \simeq \bigwedge^2 \mathcal{W}_2 \otimes \Lambda$. This implies the first identity of (ii). Similarly the second exact sequence of (i) implies the other identity of (ii).

Since for every line bundle γ, $\gamma \otimes \gamma^*$ is trivial, we get from the first identity of (ii) that $\mathcal{L}_2 \simeq \mathcal{L}_3 \otimes \Lambda^*$. From third exact sequence in (i) it follows that $\bigwedge^2 \mathrm{Ann}\mathcal{W}_2 \simeq \mathcal{L}_3 \otimes \Lambda^*$, and this implies (iii).

To see (iv), note that

$$\Lambda \simeq \frac{\mathcal{W}_2 + A\mathcal{W}_2}{\mathcal{W}_2} \simeq \frac{A\mathcal{W}_2}{A\mathcal{W}_2 \cap \mathcal{W}_2}.$$

The automorphism A^{-1} of V makes the last bundle on the right isomorphic to the line bundle $\mathcal{W}_2/(\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2)$ (note all these operations are happening inside the rank 4 trivial bundle $\mathcal{O}_C^4$). Similarly, $\widetilde{\Lambda}$ is isomorphic (via the global isomorphism A^{*-1} of V) to the line bundle $\mathcal{W}_2/(\mathcal{W}_2 \cap A^{*-1}\mathcal{W}_2)$. But for each $v \in C$, $W(v) \cap A^{-1}W(v) = \mathbb{C}v = W(v) \cap A^{*-1}W(v)$, from which it follows that the line sub-bundles $\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2$ and $\mathcal{W}_2 \cap A^{*-1}\mathcal{W}_2$ of $\mathcal{W}_2$ are the same ($= \mathcal{W}_1 \simeq \mathcal{O}_C(-1)$). Thus $\Lambda \simeq \widetilde{\Lambda}$, proving (iv).

To see (v), we need another exact sequence. For each $v \in C$, we noted in the proof of (iv) above that $\mathbb{C}v = W(v) \cap A^{-1}W(v)$. Thus the sequence of bundles:

$$0 \to \mathcal{O}_C(-1) \to \mathcal{W}_2 \to \frac{\mathcal{W}_2}{\mathcal{W}_2 \cap A^{-1}\mathcal{W}_2} \to 0$$

is exact. But, as we noted in the proof of (iv) above, the bundle on the right is isomorphic to Λ, so that

$$0 \to \mathcal{O}_C(-1) \to \mathcal{W}_2 \to \Lambda \to 0$$

is exact. Hence $\mathcal{L}_2 = \bigwedge^2 \mathcal{W}_2 \simeq \Lambda \otimes \mathcal{O}_C(-1)$. The other identity follows from (iv), thus proving (v).

To see (vi) note that we have by (ii) $\mathcal{L}_3^* \simeq \mathcal{L}_2^* \otimes \Lambda^*$. Thus

$$\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_3^* \otimes \widetilde{\Lambda}^* \simeq \mathcal{L}_2^* \otimes \Lambda^* \otimes \widetilde{\Lambda}^*.$$

However, since by (iv), $\Lambda \simeq \widetilde{\Lambda}$, we have $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^* \otimes \Lambda^{*2}$. Now, substituting $\Lambda^* = \mathcal{L}_2^* \otimes \mathcal{O}_C(-1)$ from (v), we have the rest of (vi). Hence the lemma. $\square$

We need one more bundle identity:

Lemma 4.5. *There is a bundle isomorphism*:

$$\mathcal{L}_2 \simeq \mathcal{O}_C(-2) \otimes g^*\mathcal{O}_D(1).$$

Proof. When $[t] = [t_0 : t_1 : t_2] = g([v])$, we saw in (5) that the linear map $B(v, -) : \mathbb{C}^3 \to \mathbb{C}^4$ acquires a 1-dimensional kernel, which is precisely the line $\mathbb{C}t$, which is the fibre of $\mathcal{O}_D(-1)$ at $[t]$. The image of $B(v, -)$ was the 2-dimensional span $W(v)$ of v, Av, A^*v, as noted there. Thus for $v \in C$, $B(-, -)$ induces a canonical isomorphism of vector spaces:

$$\mathcal{O}_C(-1)_v \otimes \left(\mathbb{C}^3/\mathcal{O}_D(-1)\right)_{g(v)} \to W(v) = \mathcal{W}_{2,v}$$

which, being defined by the global map $B(-, -)$, gives an isomorphism of bundles:

$$\mathcal{O}_C(-1) \otimes g^* \left(\mathcal{O}_D^3/\mathcal{O}_D(-1)\right) \simeq \mathcal{W}_2.$$

From the short exact sequence:

$$0 \to \mathcal{O}_D(-1) \to \mathcal{O}_D^3 \to \mathcal{O}_D^3/\mathcal{O}_D(-1) \to 0,$$

it follows that $\bigwedge^2(\mathcal{O}_D^3/\mathcal{O}_D(-1)) \simeq \mathcal{O}_D(1)$. Thus:

$$\begin{aligned} \mathcal{L}_2 &= \bigwedge^2 \mathcal{W}_2 \simeq \mathcal{O}_C(-2) \otimes g^* \left(\bigwedge^2 (\mathcal{O}_D^3/\mathcal{O}_D(-1))\right) \\ &\simeq \mathcal{O}_C(-2) \otimes g^*\mathcal{O}_D(1). \end{aligned}$$

This proves the lemma. $\square$

5. Degree computations

In this section, we compute the degrees of the various line bundles introduced in the previous section.

DEFNITION 5.1

Note that an *irreducible* complex projective curve C, as a topological space, is a canonically oriented pseudomanifold of real dimension 2, and has a canonical generator $\mu_C \in H_2(C, \mathbb{Z}) = \mathbb{Z}$. Indeed, it is the image $\pi_*\mu_{\tilde{C}}$, where $\pi : \widetilde{C} \to C$ is the normalization map, and $\mu_{\tilde{C}} \in H_2(\widetilde{C}, \mathbb{Z}) = \mathbb{Z}$ is the canonical orientation class for the smooth connected compact complex manifold $\widetilde{C}$, where $\pi_* : H_2(\widetilde{C}, \mathbb{Z}) \to H_2(C, \mathbb{Z})$ is an isomorphism for elementary topological reasons.

If $C = \cup_{\alpha=1}^{r} C_\alpha$ is a projective curve of pure dimension 1, with the curves C_α as irreducible components, then since the intersections $C_\alpha \cap C_\beta$ are finite sets of points (or empty), $H_2(C, \mathbb{Z}) = \oplus_\alpha H_2(C_\alpha, \mathbb{Z})$. Letting μ_α denote the canonical orientation classes of C_α as above, there is a *unique class* $\mu_C = \sum_\alpha \mu_\alpha \in H_2(C, \mathbb{Z})$. Thinking of C as an oriented 2-pseudomanifold, μ_C is just the sum of all the oriented 2-simplices of C.

If $\mathcal{F}$ is a complex line bundle on C, it has a first Chern class $c_1(\mathcal{F}) \in H^2(X, \mathbb{Z})$, and the *degree* of $\mathcal{F}$ is defined by

$$\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_C \rangle \in \mathbb{Z}.$$

It is known that a complex line bundle on a pseudomanifold is topologically trivial iff its first Chern class is zero. In particular, if an algebraic line bundle on a projective variety has non-zero degree, then it is topologically (and hence algebraically) non-trivial.

Finally, if $i : C \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ is an (algebraic) embedding of a curve in some projective space, we define the degree of the bundle $\mathcal{O}_C(1) = i^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$ as the *degree of the curve C* (in $\mathbb{P}^n_{\mathbb{C}}$). We note that $[C] := i_*(\mu_C) \in H_2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ is called the *fundamental class* of C in $\mathbb{P}^n_{\mathbb{C}}$, and by definition $\deg C = \langle c_1(\mathcal{O}_C(1)), \mu_C \rangle = \left\langle c_1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)), [C] \right\rangle$. Geometrically, one intersects C with a generic hyperplane, which intersects C away from its singular locus in a finite set of points, and then counts these points of intersection with their multiplicity.

More generally, a complex projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$ of complex dimension m has a unique orientation class $\mu_X \in H_{2m}(X, \mathbb{Z})$. Its image in $H_{2m}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z})$ is denoted $[X]$, and the degree $\deg X$ of X is defined as $\left\langle (c_1(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)))^m, [X] \right\rangle$. It is known that if $X = V(F)$ for a homogeneous polynomial F of degree d, then $\deg X = d$.

We need the following remark later on.

Remark 5.2. If $f : C \to D$ is a regular isomorphism of complex projective curves C and D, both of pure dimension 1, and if $\mathcal{F}$ is a complex line bundle on D, then $\deg f^*\mathcal{F} = \deg \mathcal{F}$. This is because $f_*(\mu_C) = \mu_D$, so that

$$\deg \mathcal{F} = \langle c_1(\mathcal{F}), \mu_D \rangle = \langle c_1(\mathcal{F}), f_*\mu_C \rangle = \left\langle f^* c_1(\mathcal{F}), \mu_C \right\rangle = \left\langle c_1(f^*\mathcal{F}), \mu_C \right\rangle = \deg f^*\mathcal{F}.$$

Now we can compute the degrees of all the line bundles introduced.

Lemma 5.3. *The degrees of the various line bundles above are as follows*:

(i) $\deg \mathcal{O}_C(1) = \deg C = 6$

(ii) $\deg \mathcal{O}_D(1) = \deg D = 4$

(iii) $\deg \mathcal{L}_2^* = 8$

(iv) $\deg \hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) = \deg \left(\mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2)\right) = 12.$

Proof. We denote the image of orientation class μ_Γ of the curve Γ (see Definition 3.3 for the definition of Γ) in $H_2(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ by $[\Gamma]$. By the part (iv) of Lemma 3.4, we have that the homology class $[\Gamma]$ is the same as the homology class of the intersection cycle defined by the four divisors $D_i := (B_i(v,t) = 0)$ inside $H_2(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$. By the generalized Bezout theorem in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, the homology class of the last-mentioned intersection cycle is the homology class Poincaré-dual to the cup product

$$d := d_1 \cup d_2 \cup d_3 \cup d_4,$$

where d_i is the first Chern class of the the line bundle L_i corresponding to D_i, for $i = 1, 2, 3, 4$ (see [4], p. 237, Ex. 2).

Since each $B_i(v,t)$ is separately linear in v, t, the line bundle defined by the divisor D_i is the bundle $\pi_1^* \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$, where π_1, π_2 are the projections to $\mathbb{P}^3_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ respectively. If we denote the hyperplane classes which are the generators of the cohomologies $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Z})$ and $H^2(\mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ by x and y respectively, we have

$$d_i = c_1(L_i) = \pi_1^*(x) + \pi_2^*(y).$$

Then we have, from the cohomology ring structures of $\mathbb{P}^3_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ that $x \cup x \cup x \cup x = y \cup y \cup y = 0$. Hence the cohomology class in $H^8(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$ given by the cup-product of d_i is

$$d := d_1 \cup d_2 \cup d_3 \cup d_4 = (\pi_1^*(x) + \pi_2^*(y))^4 = 4\pi_1^*(x^3)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2),$$

where $x^3 = x \cup x \cup x \ldots$ etc. By part (ii) of Lemma 3.4, the map $\pi_1 : \Gamma \to C$ is an isomorphism, so applying the Remark 5.2 to it, we have

$$\begin{aligned} \deg \mathcal{O}_C(1) &= \deg \pi_1^* \mathcal{O}_C(1) \\ &= \left\langle c_1(\pi_1^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1)), [\Gamma] \right\rangle \\ &= \left\langle c_1(\pi_1^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}(1)) \cup d, [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\ &= \left\langle \pi_1^*(x) \cup \left(4\pi_1^*(x^3)\pi_2^*(y) + 6\pi_1^*(x^2)\pi_2^*(y^2) \right), [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\ &= \left\langle 6\pi_1^*(x^3) \cup \pi_2^*(y^2), [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \right\rangle \\ &= 6, \end{aligned} \tag{9}$$

where we have used the Poincaré duality cap-product relation $[\Gamma] = [\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}] \cap d$ mentioned above, and that $\pi_1^*(x^3) \cup \pi_2^*(y^2)$ is the generator of $H^{10}(\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}, \mathbb{Z})$, so evaluates to 1 on the orientation class $[\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}]$, and $x^4 = 0$. This proves (i).

The proof of (ii) is similar, we just replace C by D, and π_1 by π_2, and $\pi_1^*(x)$ by $\pi_2^*(y)$ in the equalities of (9) above, and get 4 (as one should expect, since D is defined by a degree 4 homogeneous polynomial in $\mathbb{P}^2_{\mathbb{C}}$). This proves (ii).

For (iii), we use the identity of Lemma 4.5 that $\mathcal{L}_2 = \mathcal{O}_C(-2) \otimes g^* \mathcal{O}_D(1)$, and the Remark 5.2 applied to the isomorphism of curves $g : C \to D$ (part (iii) of Lemma 3.4) to conclude that $\deg \mathcal{L}_2 = \deg D - 2\deg C = 4 - 12 = -8$, by (i) and (ii) above, so that $\deg \mathcal{L}_2^* = 8$.

For (iv), we have by (vi) of Lemma 4.4 that $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*) \simeq \mathcal{L}_2^{*3} \otimes \mathcal{O}_C(-2)$, so that its degree is $3\deg \mathcal{L}_2^* - 2\deg C = 24 - 12 = 12$ by (i) and (iii) above.

This proves the lemma. □

From (iv) of the lemma above, we have the following.

COROLLARY 5.4

The line bundle $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*)$ *is a non-trivial line bundle*.

6. Proof of the main theorem

Proof of Theorem 1.1. By the third and fourth exact sequences in (i) of Lemma 4.4, we have a bundle morphism s of line bundles on C defined as the composite:

$$\mathrm{Ann}\mathcal{W}_3 = \mathcal{L}_3 \xrightarrow{i} \mathrm{Ann}\mathcal{W}_2 \xrightarrow{\widetilde{\pi}} \widetilde{\Lambda}^* = \mathrm{Ann}\mathcal{W}_2/\mathrm{Ann}\widetilde{\mathcal{W}}_3$$

which vanishes at $v \in C$ if and only if the fibre $\mathrm{Ann}\mathcal{W}_{3,v}$ is equal to the fibre $\mathrm{Ann}\widetilde{\mathcal{W}}_{3,v}$ inside $\mathrm{Ann}\mathcal{W}_{2,v}$. At such a point $v \in C$, we will have $\mathrm{Ann}\mathcal{W}_{3,v} = \mathrm{Ann}\widetilde{\mathcal{W}}_{3,v}$, so that $W_3(v) = \mathcal{W}_{3,v} = W(v) + AW(v) = \widetilde{\mathcal{W}}_{3,v} = W(v) + A^*W(v) = \widetilde{W}_3(v)$.

Now, this morphism s is a global section of the bundle $\hom_C(\mathcal{L}_3, \widetilde{\Lambda}^*)$, which is not a trivial bundle by Corollary 5.4 of the last section. Thus there does exist a $v \in C$, satisfying $s(v) = 0$, and consequently the flag

$$\begin{aligned} 0 \subset W_1 := \mathcal{W}_{1,v} \quad &= \quad \mathbb{C}v \subset W_2 := \mathcal{W}_{2,v} = W(v) = \mathbb{C}\text{-span}\{v, Av, A^*v\} \\ \subset W_3 \quad &:= \quad W_3(v) = W(v) + AW(v) = W(v) + A^*W(v) \\ &= \quad \widetilde{W}_3(v) \subset W_4 = V = \mathbb{C}^4 \end{aligned}$$

satisfies the requirements of (ii) of Lemma 2.1, (as noted after Remark 4.2) and the main theorem 1.1 follows. □

Remark 6.1. Note that since $\dim\ C = 1$, the set of points $v \in C$ such that $s(v) = 0$, where s is the section above, will be a finite set. Then the set of flags that satisfy (ii) of Lemma 2.1 which tridiagonalize A of the kind considered above (viz. A satisfying the assumptions of 3.1), will only be finitely many (at most 12 in number!).

Acknowledgments

The author is grateful to Bhaskar Bagchi for posing the problem, and to B V Rajarama Bhat and J Holbrook for pointing the relevant literature. The author is also deeply grateful to the referee, whose valuable comments have led to the elimination of grave errors, and a substantial streamlining of this paper.

References

[1] Fong C K and Wu P Y, Band Diagonal Operators, *Linear Algebra Appl.* **248** (1996) 195–204
[2] Hartshorne R, Algebraic Geometry, Springer GTM 52 (1977)
[3] Longstaff W E, On tridiagonalisation of matrices, *Linear Algebra Appl.* **109** (1988) 153–163
[4] Shafarevich I R, Basic Algebraic Geometry, 2nd Edition (Springer Verlag) (1994) vol. 1