THE RECOVERY OF INTER-BLOCK INFORMATION IN INCOMPLETE BLOCK DESIGNS

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Introduction

Non-orthogonal designs involving blocks with fewer plots than the number of treatments have been developed by Fisher, Yates, Bose, Nair, Kishen and Rao. The designs given by Fisher and Yates are variously called confounded designs, quasi-factorial designs and balanced incomplete block designs. These designs have been generalized into what is known as "partially balanced incomplete block designs" by some of the later workers. For a given number of treatments there is a considerable variety of alternative partially balanced incomplete block designs to choose from. But designs whose blocks separate into replication-groups (which may be called resolvable designs) will be the most suitable for practical experimentation. An exhaustive enumeration of such designs is now under way in the Calcutta Statistical Laboratory.

In the statistical analysis of the partially balanced designs discussed in the earlier papers only the intra-block information was utilised for estimating treatment differences. Recently, Yates showed that the efficiency of non-orthogonal designs could be much increased if block differences are taken into account in estimating treatment differences. He has considered only the balanced incomplete block and the quasi-factorial designs and their counterparts in Latin square arrangements.

In the present paper, Yates' ideas have been extended to cover the more general class of partially balanced incomplete block designs. In fact the method of extracting both intra-block and inter-block information can be applied to any general two-way non-orthogonal data, so that, when incomplete block designs lose some of their combinatorial properties, owing to accidental occurrences of missing plots, there will not be any difficulty in recovering both types of information.

Treatment Differences in Partially Balanced Incomplete Block Designs.

The combinatorial properties of these designs have been fully discussed in earlier papers[1, 2]. There are v treatments, each replicated in r plots distributed over b blocks, each containing k plots. No treatment will occur more than once in a block. With respect to every treatment, the remaining treatments fall in m associate classes of $n_1, n_2, \ldots, n_m$ the members of the i-th class occurring with the initial treatment in λ_i blocks. The λ's need not all be unequal. If $\lambda_1 = \lambda_2 = \ldots = \lambda_m$ we get balanced incomplete block designs. The number of treatments common to the j-th associates of any treatment θ and the k-th associates of a treatment ϕ, where θ and ϕ are themselves i-th associates, is constant and is denoted by p^i_{jk}. The following relations exist.

$$vr = bk \tag{1}$$

$$\sum_{i}^{m} n_i \lambda_i = r(k-1) \tag{2}$$

$$\sum_{k=1}^{m} p^i_{jk} = n_j - 1 \text{ or } n_j, \text{ according as } i = j \text{ or } i \neq j \tag{3}$$

$$n_i\, p^i_{jk} = n_j\, p^j_{ik} = n_k\, p^k_{ij} \tag{4}$$

If by Q_i we denote the sum of the yields of treatment i minus the sum of the r corresponding block means; and by Q'_i the latter sum minus r times the general mean, the intra-block and inter-block estimates of treatment differences will depend on Q_i and Q'_i respectively as shown in earlier papers[6, 7]. Separate inter-block estimates are obviously not obtainable unless $b \geqq v$.

In partially balanced designs b can be less than v and that is one of their main practical advantages. In general, therefore, besides the intra-block estimates based on Q_i we can get only pooled estimates based on both Q_i and Q'_i. In terms of treatment effects: v_i, we have

$$Q_i = r\left(1-\frac{1}{k}\right)v_i - \frac{\lambda_1}{k}\Sigma v_{i1} - \dots - \frac{\lambda_m}{k}\Sigma v_{im} \tag{5}$$

$$Q'_i = r\left(\frac{1}{k}-\frac{1}{v}\right)v_i + \left(\frac{\lambda_1}{k}-\frac{r}{v}\right)\Sigma v_{i1} + \dots + \left(\frac{\lambda_m}{k}-\frac{r}{v}\right)\Sigma v_{im} \tag{6}$$

If σ and σ' are the standard errors per plot for intra-block and inter-block comparisons, the pooled estimating equations for v_i can be shown to depend on $wQ_i + w'Q'_i$ where $1/w = \sigma^2$ and $1/w' = \sigma'^2$. Then we have

$$k[w\, Q_i + w'\, Q'_i] = [r(k-1)\, w + r\left(1-\frac{k}{v}\right)w']\, v_i + [-\lambda_1\, w + \left(\lambda_1 - \frac{rk}{v}\right)w']\, \Sigma v_{i1} + \dots$$

$$+ [-\lambda_m\, w + \left(\lambda_m - \frac{rk}{v}\right)w']\, \Sigma v_{im} \tag{7·0}$$

$$k[w\, \Sigma Q_{i1} + w'\, \Sigma Q'_{i1}] = [-n_1\lambda_1\, w + n_1\left(\lambda_1 - \frac{rk}{v}\right)w']\, v_i + [rk\left(w - \frac{n_1}{v}w'\right)$$

$$-(r + \Sigma\lambda_k\, p^1_{1k})\,(w-w')]\, \Sigma v_{i1} + \left[-\Sigma\lambda_k\, p^2_{1k}(w-w') - \frac{krn_1}{v}\, w'\right]\Sigma v_{i2} + \dots \tag{7·1}$$

$$k[w\Sigma Q_{im} + w'\, \Sigma Q'_{im}] = [-n_m\lambda_m\, w + n_m\left(\lambda_m - \frac{rk}{v}\right)w']\, v_i + [-\Sigma\lambda_k\, p^1_{mk}(w-w') - \frac{rkn_m}{v}\, w']\Sigma v_{i1}$$

$$+ \dots + [rk(w - \frac{n_m}{v}\, w') - (r + \sum_{1}^{m}\lambda_k\, p^m_{mk})\,(w-w')]\Sigma v_{im} \tag{7·m}$$

The left and right hand sides of the $(m+1)$ equations (7.0) to (7.m) separately add up to 0. They do not therefore supply unique estimates of v_i. But the sum of the coefficients of the v's in any equation is also 0. We can therefore reduce the unknowns in the equations to $\Sigma v_{i1} - n_1 v_i, \Sigma v_{i2} - n_2 v_i, \dots \Sigma v_{im} - n_m v_i$. These m unknowns may be uniquely solved from the last m equations (7.1) to (7.m), for sake of symmetry. In fact, if the v equations of the form (7.0) were written down, a set of $v-1$ differences between say v_1 and the others, like $v_1 - v_2$, $v_1 - v_3$, .. $v_1 - v_v$ could be uniquely estimated from $v-1$ of the equations. From these differences all possible differences between pairs of treatments could be obtained. As v is usually large, the solution of a set of $v-1$ linear equations will be rather laborious. It will be convenient, therefore, to solve $v_1, \Sigma v_{1_1}, \ldots, \Sigma v_{1_m}$ from $m+1$ independent equations. The missing independent equation is usually provided by the arbitrary condition $\sum_{1}^{v} v_i = 0$. Of course, whatever the form of this arbitrary constraint, estimates of $v_i - v_j$ will be unaffected. The solution for v_1 will now be

$$v_1/_h = \frac{\begin{vmatrix} 0 & 1 & \ldots & 1 \\ w\Sigma Q_{11} + w'\Sigma Q'_{11} & rk\left(w - \frac{n_1 w'}{v}\right) - (r + \Sigma\lambda_k\, p^1_{1k})(w-w') & \ldots & rk\left(-\frac{n_1 w'}{v}\right) - (\Sigma\lambda_k\, p^m_{1k})(w-w') \\ w\Sigma Q_{1m} + w'\Sigma Q'_{1m} & rk\left(-\frac{n_m w'}{v}\right) - (\Sigma\lambda_k\, p^1_{mk})(w-w') & \ldots & rk\left(w - \frac{n_m w'}{v}\right) - (r + \Sigma\lambda_k\, p^m_{mk})(w-w') \end{vmatrix}}{\begin{vmatrix} rk\,w - (r + \Sigma\lambda_k\, p^1_{1k} - n_1\lambda_1)(w-w') & \ldots & -(\Sigma\lambda_k\, p^m_{1k} - n_1\lambda_1)(w-w') \\ -(\Sigma\lambda_k\, p^1_{mk} - n_m\lambda_m)(w-w') & \ldots & rk\,w - (r + \Sigma\lambda_k\, p^m_{mk} - n_m\lambda_m)(w-w') \end{vmatrix}} \qquad (8)$$

By putting $w' = 0$ in (8) we get the intra-block estimate of v_1 and by putting $w=0$ we get the inter-block estimate, if one exists. By putting $w = w'$, we get $v_i = (Q_i + Q'_i)/r$ = treatment mean—general mean, which will be called the unadjusted estimate of v_i.

Accuracy of Estimates of Treatment Differences

The sampling variance of a treatment effect, say, v_i, as estimated by (8) and the covariance between two effects, v_i and v_j are given by C_{ii} and C_{ij} respectively which are the solutions of v_i and v_j when $wQ_i + w'Q'_i$ is replaced by $1 - 1/v$ and all other $(wQ + w'Q')$'s by $-1/v$ in (7). The variance of the difference between two treatment effects is $2(C_{11} - C_{j1})$ and is derived easily, for two s-th associates, by replacing the $(s+1)$-th element of the first column of the upper determinant of (8) by -2 and all other elements in this column by 0. The mean variance for all the $v(v-1)/2$ treatment differences is $2C_{11}/(1-1/v)$, which involves both w and w'. By putting $w' = 0$ in it, we get the mean variance for intra-block comparisons and can find out the gain in accuracy achieved by recovering the inter-block information. If $\sigma = \sigma'$, $C_{11} = \sigma^2(1/r - 1/vr)$.

If the partially balanced incomplete block design is resolvable (as, for instance, quasi-factorial and confounded designs) with n blocks per replication, so that $v = nk$, $b = nr$, and if the lay-out was arranged in the field in compact areas for each separate replication, the design could be analysed as an ordinary randomized block experiment. The variance of each treatment difference should be taken as same, namely, $2/r$ times error variance within blocks of v plots and can be expressed in terms of σ and σ'. Thus the $(v-1)$ degrees of freedom within a single replication splits into a set of $n-1$ degrees of freedom with variance σ'^2 and a second set of $n(k-1)$ degrees of freedom with variance σ^2, giving an average plot variance of $[(v-k)\sigma'^2 + v(k-1)\sigma^2]/k(v-1)$. The mean variance of treatment differences is $2/r$ times this average plot variance. The efficiency of the weighted estimates of treatment differences of resolvable partially balanced designs is therefore $[(\sigma^2 - \sigma'^2/v) + (\sigma'^2 - \sigma^2)/k]/rC_{11}$ which is greater than or equal to unity according as σ and σ' are unequal or not. But σ cannot be greater than σ' from *a priori* consideration

ESTIMATION OF σ AND σ'

If there had been no real treatment differences, the analysis of variance of all the plot yields will resolve into a 'between block' variance with $b-1$ degrees of freedom and a 'within block' variance with $b(k-1)$ degrees of freedom. The expectations of these variances are, by definition, σ'^2 and σ^2 respectively. If the field is completely random*, so that, yields of pairs of plots are uncorrelated, whatever be the distance between them, then $\sigma' = \sigma$. Agricultural fields are seldom random in this sense. On the other hand pairs of plots are positively correlated with decreasing intensity as the distance between them increases. The actual values of this correlation will depend on a number of other factors also. σ' will be a function of σ, k and the various correlations between each of the $k(k-1)/2$ pairs of plots. When correlations are positive, σ' is greater than σ, so that the plots of different blocks have different expectations. Let us denote by x_{ij} the yield of the i-th plot of the j-th block. Let $E(x_{ij}) = b_j$ or $x_{ij} = b_j + e_{ij}$. Let $V(e_{ij}) = A$ and $V(b_j) = B$. Then $\sigma^2 = A$ and $\sigma'^2 = kB + A$.

When real treatment differences are suspected, $E(x_{ij}) = v_i + b_j$ when the i-th plot is the plot occupied by the i-th treatment. Differences among the v's stand for treatment differences within blocks. Let $V(v_i) = C$.

Calculating $Q_{i.}$ and $Q_{.j}$ as indicated in an earlier paper[4], we have the analysis of variance given in Table (1).

TABLE (1). ANALYSIS OF VARIANCE OF INCOMPLETE BLOCK DESIGNS

	D. F.	Observed Sum of Squares	Expected Value of Sum of Squares
Blocks	$b-1$	$\sum\limits_{1}^{b} b_j Q_{.j} = \sum\limits_{1}^{b} (b_j - b_1) Q_{.j}$	$v(r-1)B + (b-1)A$
Treatments	$v-1$	$\sum\limits_{1}^{v} v_i Q_{i.} = \sum\limits_{1}^{v} (v_i - v_1) Q_{i.}$	$b(k-1)C + (v-1)A$
Error	$rv-b-v+1$	$\sum\sum x^2_{ij} - \sum v_i Q_{i.} - \frac{1}{k}\sum\limits_{1}^{b} T^2_{.j}$	$(rv-b-v+1)A$
Total	$rv-1$	$\sum\sum x^2_{ij} - G^2/rv$	$r(v-1)C + k(b-1)B + (rv-1)A$

Each sum of squares in the above Table has to be calculated separately.

We note that no independent estimates of σ and σ' can be obtained from Table (1). Unbiased estimates of them or of w and w' can, however, be obtained by equating the observed error and block variances, M and M' with their respective expectations. Thus

$$w = \frac{1}{M} \text{ and } w' = \frac{v(r-1)}{k(b-1)M' - (v-k)M}$$

When the design is resolvable ($v = nk$, $b = nr$), the $b-1$ degrees of freedom for blocks of Table (1) splits into a set of $(r-1)$ degrees of freedom for replications and $(b-r)$ degrees of freedom for blocks within replications. For a plot belonging to i-th treatment in the j-th block of the p-th replication, we now have $E(x_{pij}) = v_i + b_{jp} + r_p$. Let $B = V(b_{jp})$, $D = V(r_p)$. Then we have the analysis of variance for the block sum of squares as in Table (2).

* A full discussion on Random and Non-random fields will be found in a paper on 'Large Scale Sample Surveys' by P. C. Mahalanobis which will shortly be published in *Philosophic Transactions of the Royal Society*, London.

TABLE (2). ANALYSIS OF BLOCK VARIANCE OF RESOLVABLE DESIGNS

	D. F.	Observed sum of squares	Expected Value of sum of squares
Replications	$r-1$	$\sum_{1}^{r} R_u^2/v - G^2/rv$	$(r-1)(vD+A)$
Blocks within replications	$b-r$	†	$(v-k)(r-1)B+(b-r)A$
Blocks	$b-1$	$\sum_{1}^{b} b_j Q_{\cdot j} = \sum_{1}^{b} (b_j - b_0) Q_{\cdot j}$	$v(r-1)D+(v-k)(r-1)B+(b-1)A$

† Obtained by subtraction.

The treatment and error sums of squares for resolvable designs will be the same as in Table (1). If M'' is the observed variance for blocks within replications, estimate of w' is

$$w' = \frac{(v-k)(r-1)}{k(b-r)M'' - (v-k)M} = \frac{r-1}{rM''-M}$$

and estimate of $w = \frac{1}{M}$, as before.

Whenever M' or M'' happens to be less than M, we shall assume that $B = 0$ or $w = w'$, and use the unadjusted treatment means for estimating treatment differences.

If the resolvable designs are treated as ordinary randomized blocks, the analysis of variance will be as in Table (3).

TABLE (3). ANALYSIS OF RESOLVABLE DESIGNS AS ORDINARY RANDOMIZED BLOCKS

	D. F.	Observed sum of squares	Expected Value of sum of squares
Replications	$r-1$	$\sum R_u^2/v - G^2/rv$	$(r-1)(vD+A)$
Treatments	$v-1$	$\sum V_i^2/r - G^2/rv$	$(v-1)(rC+A)+(v-k)B$
Error	$(v-1)(r-1)$	†	$(v-k)(r-1)B+(v-1)(r-1)A$
Total	$rv-1$	$\sum\sum x_{ij}^2 - G^2/rv$	$v(r-1)D+r(v-1)C+k(b-r)B+(rv-1)A$

† Obtained by subtraction.

The treatment differences will now be obtained from the unadjusted treatment means. For the null-hypothesis that treatment differences are not present, in which case $C=0$, expectation of the treatment and error variances of Table (3) have the same value, namely, $[(v-k)/(v-1)]B + A$. The analysis of Table (3), therefore, provides a valid estimate of the average error variance of treatment differences. If the design is actually laid out in the field in compact areas for each replication, the analysis of variance of Table (1) becomes valid for testing intra-block estimates of treatment differences. The inter-block estimates (if they exist) can also be similarly tested. For resolvable designs the inter-block estimates can exist only if $b \geq v+r-1$. An analysis of variance for the unadjusted block sum of squares as shown in Table (4) gives a perfectly valid test for inter-block estimates of treatment differences.

TABLE (4). VARIANCE RATIO TEST FOR INTER-BLOCK ESTIMATES OF TREAMENT DIFFERENCES FOR RESOLVABLE DESIGNS

	D. F.	Observed Sum of Squares	Expected Value of Sum of Squares
Treatments	$v-1$	$\Sigma v'_i\, Q'_i$	$(b-r)\, C + (v-1)\, (kB+A)$
Error (inter-block)	$b-v-r+1$	†	$(b-v-r+1)\, (kB+A)$
Blocks within replications	$b-r$	$\sum^{b} T^2_{.j}/k - \sum^{r} R^2_s/v$	$(b-r)\, (C+kB+A)$

† Obtained by subtraction.

Values of v'_i can be obtained from equations (7) by putting $w = 0$. The inter-block variance with $b-v-r+1$ degrees of freedom gives a direct estimate of σ', independent of the estimated intra-block variance, M.

When the design is not resolvable, inter-block estimates may exist if $b > v$. The analysis of variance of Table (5) provides a valid test.

TABLE (5). VARIANCE RATIO TEST FOR INTER-BLOCK ESTIMATES OF TREATMENT DIFFERENCES FOR NON-RESOLVABLE DESIGNS

	D. F.	Observed Sum of Squares	Expected Value of Sum of Squares
Treatments	$v-1$	$\Sigma v'_i\, Q'_i$	$(b-r)\, C + (v-1)\, (kB+A)$
Error (inter-block)	$b-v$	†	$(b-v)\, (kB+A)$
Blocks	$b-1$	$\Sigma T^2_{.j}/k - \dfrac{G^2}{vr}$	$(b-r)\, C + (b-1)\, (kB+A)$

† Obtained by subtraction.

Non-resolvable designs cannot be analysed as ordinary randomized blocks, as the replications cannot be separated in the field. Only possibility is to regard them as completely randomised experiments. The analysis of variance will then be as in Table (6).

TABLE (6). ANALYSIS OF NON-RESOLVABLE DESIGNS AS COMPLETELY RANDOMIZED ARRANGEMENTS

	D. F.	Observed Sum of Squares	Exp. Value of Sum of Squares
Treatments	$v-1$	$\sum^{v} T^2_{i.}/r - G^2/vr$	$(v-1)\, (rC+A) + (v-k)B$
Within treatments	$v(r-1)$		$v(r-1)\, (B+A)$
Total	$vr-1$	$\Sigma\Sigma x^2_{ij} - G^2/vr$	$r(v-1)C + k(b-1)B + (vr-1)A$

Obtained by subtraction.

In the null-hypothesis, $C = 0$, but the expectations of the treatment and error variances do not become equal (unless $k = 1$) and so no valid estimate of the average variance of unadjusted treatment differences is possible. No exact test of significance is available for comparing either the adjusted estimates given by (8) or the unadjusted ones.

INCOMPLETE BLOCK DESIGNS WITH BLOCKS OF UNEQUAL SIZE

The designs considered above have same block size, namely, k plots. But the quasi-factorial design given by Yates, when $v = pq$, has blocks of p and q plots. Again the designs developed by Kishen has, in general, m different block sizes $k_1, k_2, \ldots, k_m$.

For recovering both intra-block and inter-block informatio in these cases, we have to consider m different intra-block plot variances $A_1, A_2, \ldots A_m$ and m different block variances $B_1, B_2, \ldots B_m$ so that $1/w_i = A_i$, $1/w'_i = k_i B_i + A_i$ $(i = 1, 2, \ldots m)$.

With the minimum number of replications usually used for these designs it is not possible to estimate the A's and B's, in every case. But we may, as a first approximation, assume that $\frac{1}{w_i} = A_i = A$ and $1/w'_i = k_i B + A$, which will be justified to a great extent if $k_1, k_2, \ldots k_m$ are nearly equal to one another.

Incomplete Block Designs with Constant Block Size but having several Missing Plots.

Here the number of plots in the various blocks will be $k_1, k_2, \ldots$ but the known plots of any block will be scattered over a space originally occupied by a compact block of k plots. We will therefore be perfectly justified in assuming that $A_i = A$ and $B_i = B$.

The data when arranged in a two-way form for blocks and treatments may be dealt with as the general non-orthogonal data with n_{ij} ($=1$ or 0 in this case) observations in the cell for i-th treatment and j-th block (see Nair). Without loss of generality we may assume that none of the blocks or treatments is totally missing, so that $i = 1$ to v, $j = 1$ to b.

The intra-block information for treatment differences come from the quantities $x_{ij} - \bar{x}_{.j}$ and the inter-block information from $\bar{x}_{.j} - \bar{x}_{..}$, where $\bar{x}_{.j}$ is the mean of the jth block which has $n_{.j}$ plots. $E(x_{ij} - \bar{x}_{.j})$ is free from block parameters, whereas in $E(\bar{x}_{.j} - \bar{x}_{..})$ they are present, but will be ignored for the present purpose. Thus

$$E(x_{ij} - \bar{x}_{.j}) = v_i - \frac{1}{n_{.j}} \sum_{i=1}^{v} n_{ij} v_i = (v_i - v_1) - \frac{1}{n_{.j}} \sum_{i=2}^{v} n_{ij}(v_i - v_1) \tag{9}$$

$$E(\bar{x}_{.j} - \bar{x}_{..}) = \sum_{i=1}^{v} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right) v_i = \sum_{i=2}^{v} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right) (v_i - v_1) \tag{10}$$

For proportional cell frequencies, that is, where $n_{ij} = \frac{n_{i.} \times n_{.j}}{n_{..}}$ it will be noticed that $E(\bar{x}_{.j} - \bar{x}_{..})$ does not involve the v's. The data will then be called 'orthogonal' and no inter-block information can be extracted from such data.

To get pooled intra-block and inter-block estimates of the $v-1$ treatment differences $(v_i - v_1)$ $(i = 2, \ldots v)$ we minimise the following weighted sum of squares with respect to $v_i - v_1$.

$$\frac{1}{A} \sum_{i=1}^{v} \sum_{j=1}^{b} \left[x_{ij} - \bar{x}_{.j} - (v_i - v_1) + \sum_{i=2}^{v} \frac{n_{ij}}{n_{.j}} (v_i - v_1) \right]^2 + \sum_{j=1}^{b} \frac{1}{B + A/n_{.j}} \left[\bar{x}_{.j} - \bar{x}_{..} - \sum_{i=2}^{v} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right) (v_i - v_1) \right]^2 \tag{11}$$

The normal equations are

$$\frac{1}{A} \sum_{j=1}^{b} n_{ij}(x_{ij} - \bar{x}_{.j}) + \sum_{j=1}^{b} \frac{n_{.j}}{A + B n_{.j}} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right) (\bar{x}_{.j} - \bar{x}_{..})$$
$$= \left[\frac{1}{A} \left(n_{i.} - \sum_{j=1}^{b} \frac{n^2_{ij}}{n_{.j}} \right) + \sum_{j=1}^{b} \frac{n_{.j}}{A + B n_{.j}} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right)^2 \right] (v_i - v_1)$$
$$+ \sum_{i' \neq i} \left[\frac{1}{A} \left(- \sum_{j=1}^{b} \frac{n_{ij} \; n_{i'j}}{n_{.j}} \right) + \sum_{j=1}^{b} \frac{n_{.j}}{A + B n_{.j}} \left(\frac{n_{ij}}{n_{.j}} - \frac{n_{i.}}{n_{..}} \right) \left(\frac{n_{i'j}}{n_{.j}} - \frac{n_{i'.}}{n_{..}} \right) \right] (v_{i'} - v_1) \tag{12}$$

When there are no missing plots, $n_{.j} = k$, $n_{i.} = r$ in (9) to (12).

SUMMARY

In agricultural field experiments, it has been found that, unless the field is very heterogeneous, the adoption of incomplete block designs leads to inefficient estimates of treatment differences, if the whole of the apparent block differences is eliminated from such estimates. When the fertility differences between blocks are not big, the efficiency of the estimates of treatment differences can be improved by pooling the intra-block and inter-block information. The process of doing this has been worked out in this paper for partially balanced incomplete block designs, of which the quasi-factorial and confounded designs are special cases. The situations when some plots of these designs are missing has also been discussed.

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