ON IDEALS IN BANACH SPACES

T.S.S.R.K. RAO

ABSTRACT. In this paper we study the notion of an *ideal*, which was introduced by Godefroy, Kalton and Saphar in [**7**] and was called "locally one complemented" in [**11**], for injective and projective tensor products of Banach spaces. For a Banach space X and an ideal Y in X, we show that the injective tensor product space $Y \otimes_\varepsilon Z$ is an ideal in $X \otimes_\varepsilon Z$ for any Banach space Z. This as a consequence gives us a way of proving some known results about intersection properties of balls and extensions of operators on injective tensor product spaces in a unified way that does not involve any vector-valued Choquet theory. We also exhibit classes of Banach spaces in which every ideal is the range of a norm one projection.

Introduction. Let Y be a closed subspace of a Banach space X. Y is said to be an *ideal* in X if $Y^\perp$ (the annihilator of Y in the dual space X^* of X) is the kernel of a projection of norm one in X^*. When X is embedded in X^{**} via the map $J : X \to X^{**}$ defined by $J(x)(x^*) = x^*(x)$, the natural projection $\wedge \to \wedge|J(X)$ in X^{***} is of norm one and its kernel is $J(X)^\perp$. Thus X is isometric to an ideal in X^{**}. Sims and Yost have proved in [**21**] that for any separable subspace $Y \subset X$, there exists a separable subspace Z such that $Y \subset Z \subset X$ and Z is an ideal in X. Saab and Saab in the past have studied Banach space properties of injective tensor product spaces and extensions of operators defined on injective tensor product spaces where one of the spaces in the tensor product is an L^1-predual (see [**16**], [**17**], [**18**]). Their method consisted of using vector-valued Choquet theory developed in [**19**]. The authors of [**14**] also use these methods to study intersection properties of balls in injective tensor product spaces. In Section 1 of this paper we first note that the injective tensor product space $X \otimes_\varepsilon Z$ is an ideal in $Y \otimes_\varepsilon Z$ whenever Y is an ideal in X. This allows us to give a new and simpler proof of a result from [**14**] that for an L^1-predual space X and for a space Z having the almost n.k. intersection property, the space $X \otimes_\varepsilon Z$ also has the almost n.k. intersection property. Using the observation that an operator defined on an ideal extends in a norm-preserving way to the entire space (the range however gets enlarged), we extend a result of Saab and Saab ([**18**]) to general injective tensor product spaces.

Turning to projective tensor product spaces, we show that if Y is an ideal in X then for any Z, the projective tensor product space $Y \otimes_\pi Z$ is indeed a subspace of, and is an ideal in, $X \otimes_\pi Z$. Cilia proved in [**1**] that, for the space of Bochner integrable functions, $L^1(\mu, X^{**})$ is isometric to a subspace of $L^1(\mu, X)^{**}$ that contains $J(L^1(\mu, X))$. We extend this to general projective tensor products by showing that if Y has the metric approximation property, then $(Y \otimes_\pi X^{**})$ is isometric to a subspace of $(Y \otimes_\pi X)^{**}$ that contains $J(Y \otimes_\pi X)$. It is known that in general $L^1(\mu, X)$ is not a complemented subspace of *cabv* (μ, X), the space of X-valued countably additive measures of bounded variation that are absolutely continuous with respect to μ (see [**4**]). It follows from our results that $L^1(\mu, X)$ is always an ideal in cabv (μ, X).

In Section 2 we consider examples of spaces in which every ideal is actually the range of a norm one projection. Banach spaces X which are M-ideals (L-summands) in their bidual (under the embedding J) provide a rich class of examples. In the former, it turns out that every ideal is the range of a norm one projection. This provides a new characterization of the space $c_0(\Gamma)$ (functions vanishing at infinity on a discrete space Γ, equipped with the supremum norm) and also has some implications to the M-structure of the space of compact operators. We note that if Y is an ideal in the predual of a von Neumann algebra, then Y is the range of a norm one projection. This section depends heavily on concepts from the M-structure theory that can be found in [**9**].

Most of our notation and terminology is standard and can be found in [**3**] (the only exception being the use of ε and π to denote the injective and projective tensor products). For a Banach space X by X_1, we denote its closed unit ball. All Banach spaces are considered over the real scalar field and are of infinite dimension.

We will be using several times the following characterization of an ideal due to Lima [**11**]. As remarked by him in that paper, the corresponding isomorphic notion was studied by Kalton and Fakhoury (see the reference in [**11**]).

Theorem (Lima). *Let Y be a closed subspace of X.* TFAE.

1. *Y is an ideal in X.*

2. *$Y^{\perp\perp}$ is the range of a norm one projection in X^{**}.*

3. *If F is a finite dimensional subspace of X and $\varepsilon > 0$, there exists an operator $T : F \to Y$ such that*

$$Ty = y \quad \text{for } y \in F \cap Y \text{ and } \|T\| \leq 1 + \varepsilon.$$

Note that it follows immediately from 3) that if $Y \subset Z \subset X$ and Y is an ideal in X, then Y is an ideal in Z.

Section 1. We first collect some properties of an ideal that we will be needing later in the form of a lemma. Let $Y \subset X$ be an ideal. Let $P : X^* \to X^*$ be a projection of norm one with $\operatorname{Ker} P = Y^\perp$. For any $y^* \in Y^*$ and for any Hahn-Banach extension x^* of y^*, $y^* \to P(x^*)$ is a well-defined linear map and since P is of norm one, by taking a norm-preserving Hahn-Banach extension, we see that Y^* is isometric to $P(X^*)$. We call this the canonical embedding of Y^*.

Lemma 1. *Let X be a Banach space and Y an ideal in X.*

(i) *If under the canonical embedding of Y^* in X^*, Y_1^* is w^*-dense in X_1^*, then X is isometric to a subspace of Y^{**} by an isometry whose restriction to Y is the embedding J on Y. Conversely if X is isometric to a subspace of Y^{**} containing $J(Y)$, then Y is an ideal in X in the above sense.*

(ii) *If $T : Y \to Z$ is a bounded linear operator, then there exists a norm preserving extension $S : X \to Z^{**}$. When Z is a dual space, there exists an extension that takes values in Z.*

Proof. (i) Assume that Y^* is canonically embedded in X^*. Define $\Phi : X \to Y^{**}$ by $\Phi(x) = x|Y^*$. Since Y_1^* is w^*-dense in X_1^*, clearly Φ is an isometry and its restriction to Y is the embedding J. Now ignoring the isometry on X and J, suppose $Y \subset X \subset Y^{**}$. Define $P : X^* \to X^*$ in the following way. For any $x^* \in X^*$, put $y^* = x^*|Y$. Since y^* acts as a continuous linear functional on Y^{**}, put $P(x^*) = y^*|X$. Clearly P is a projection of norm one whose kernel is $Y^\perp$. Since the unit ball of Y^* is a w^*-dense subset of Y_1^{***}, we get the desired conclusion.

(ii) Let $P : X^{**} \to X^{**}$ be a norm one projection with $\operatorname{Range} P = Y^{\perp\perp} = Y^{**}$. Now $T^{**} \circ P \circ J = S : X \to Z^{**}$ is clearly an extension of T and, since $\|P\| = 1$, we have $\|S\| = \|T\|$. When Z is a dual space, one composes S with the canonical norm one projection from the triple dual of a space to its dual to get the desired conclusion.

Remark 1. Let $X = C([0,1])$ and $Y = \{f \in C([0,1]) : f([0,1/2]) = 0\}$. Since $\mu \to \mu|[1/2,1]$ is a norm one projection in $C([0,1])^*$ whose kernel is $Y^{\perp}$, we get that Y is an ideal in X. However, the unit ball of Y^* is not w^*-dense in the unit ball of $C([0,1])^*$. Moreover, since this is an L projection, there is no other projection in X^* of norm one whose kernel is $Y^{\perp}$ (see [**9**, Proposition I.1.2.]. Therefore, the hypothesis of (i) is not satisfied. We will also give an example later on (after Proposition 1) to show that it is essential for X to contain the copy $J(Y)$ of Y for the validity of the statement in (i).

However, there are several naturally occurring examples where the hypothesis of (i) is satisfied (see also (ii) of Theorem 1 below).

Example 1. If Y has the metric approximation property (MAP) then for any Banach space X, it follows from Lemma 1 of [**8**] that the space of compact operators $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$, the space of bounded operators.

Since functionals of the form $x \otimes y^*$ defined by $(x \otimes y^*)(T) = T^*(y^*)(x)$, $x \in X_1$, $y^* \in Y_1^*$, $T \in \mathcal{L}(X,Y)$, in $\mathcal{K}(X,Y)^*$ determine the norm of an operator, and since the projection defined in Lemma 1 of [**8**] is an identity on these objects, it is easy to see that the hypothesis of (i) is satisfied. Clearly any subspace of operators that contains $\mathcal{K}(X,Y)$ satisfies these conditions.

Example 2. Let K be any compact set and let $WC(K,X)$ denote the space of X-valued functions on K that are continuous when X has the weak topology, equipped with the supremum norm. Let $C(K,X)$ denote the space of norm continuous functions.

We note that for any $f \in WC(K,X)$, there corresponds a weakly compact operator $T : X^* \to C(K)$ defined by $T(x^*)(k) = x^*(f(k))$.

This association is a linear isometry and, when f is continuous with respect to norm, we get a compact operator. Since $C(K)$ has the metric approximation property, following the arguments given during the proof of Lemma 1 in [**8**], it is easy to construct a norm one projection $P : WC(K,X)^* \to WC(K,X)^*$ such that $\operatorname{Ker} P = C(K,X)^\perp$. Again, since functionals of the form $\delta(k) \otimes x^*$, $k \in K$, $x^* \in X_1^*$, defined by $(\delta(k) \otimes x^*)(f) = x^*(f(k))$ in $C(K,X)^*$ determine the norm of a function in $WC(K,X)$, it follows that the hypothesis of (i) is satisfied. See [**2**] for another construction of such a projection using properties of weakly compact subsets of a Banach space.

Example 3. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. In the class of Bochner integrable functions, $L^\infty(\mu, X^*)$ as a subspace of $L^1(\mu, X)^*$ satisfies the hypothesis of (i). To see this, we can use the identification of $L^1(\mu, X)^*$ as $\mathcal{L}(L^1(\mu), X^*)$ (see [**3**]); under this identification the space of representable operators gets mapped onto $L^\infty(\mu, X^*)$. Since $L^\infty(\mu)$ has the MAP, using the results from [**8**], this time with respect to the domain space, one gets the conclusion following the line of reasoning given in Example 1.

It is well known in tensor product theory that for Banach spaces X and Y, the projective tensor product space $X \otimes_\pi Y$ is a subspace of $X \otimes_\pi Y^{**}$ (see [**3**]). Cilia proved in [**1**] that for any finite measure space $(\Omega, \mathcal{A}, \mu)$, the space of Bochner integrable functions $L^1(\mu, Y^{**})$ is isometric to a subspace of $L^1(\mu, Y)^{**}$. Our first theorem is an extension of both of these results.

Theorem 1. (i) *Let X and Z be Banach spaces and Y an ideal in Z; then $X \otimes_\pi Y$ is a subspace of $X \otimes_\pi Z$ and is an ideal.*

(ii) *If X has the* MAP, *then for any Banach space Y, $X \otimes_\pi Y^{**}$ is isometric to a subspace of $(X \otimes_\pi Y)^{**}$ that contains $J(X \otimes_\pi Y)$.*

Proof. (i) Consider the identity embedding of $X \otimes_\pi Y$ in $X \otimes_\pi Z$. For $u \in X \otimes_\pi Y$, choose $T \in (X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*)$ such that $\|u\| = T(u)$, $\|T\| = 1$. Since Y^* canonically embeds in Z^*, we have $T \in \mathcal{L}(X, Z^*) = (X \otimes_\pi Z)^*$ and $\|T\| = 1$. Therefore the identity embedding is an isometry. If P is a norm one projection in Z^* with

Ker $P = Y^\perp$, then the mapping $T \to P \circ T$ is a projection of norm one in $(X \otimes_\pi Z)^*$ whose kernel is $(X \otimes_\pi Y)^\perp$. Hence $X \otimes_\pi Y$ is an ideal in $X \otimes_\pi Z$.

(ii) Suppose X has the metric approximation property. From the first part we know that $X \otimes_\pi Y$ is an ideal in $X \otimes_\pi Y^{**}$. We shall now verify that the hypothesis of (i) of Lemma 1 is satisfied here, and then the conclusion follows.

With the usual identifications

$$(X \otimes_\pi Y^{**})^* = \mathcal{L}(X, Y^{***}) = \mathcal{L}(X, Y^*) \oplus (X \otimes_\pi Y)^\perp$$

since X has the MAP, it follows from Corollary 3.3 of [**20**] that the unit ball of $\mathcal{L}(X, Y^*)$ is w^*-dense in the unit ball of $\mathcal{L}(X, Y^{***})$ (since we are working on a dual unit ball, weak*-operator-dense implies w^*-dense).

Corollary 1. *For any measure space $(\Omega, \mathcal{A}, \mu)$, $L^1(\mu, X^{**})$ is isometric to a subspace of $L^1(\mu, X)^{**}$ that contains $J(L^1(\mu, X))$.*

Proof. Note that $L^1(\mu, X^{**}) = L^1(\mu) \otimes_\pi X^{**}$ and $L^1(\mu)$ has the MAP.

Remark 2. For a finite measure space $(\Omega, \mathcal{A}, \mu)$ and Banach space X, let *cabv* (μ, X) denote the space of X-valued countably additive measures on $\mathcal{A}$ of bounded variation that are absolutely continuous with respect to μ. The noncomplementability of $L^1(\mu, X)$ in *cabv* (μ, X) has recently attracted considerable attention (see [**4**] and references listed there). It follows from the proof of the theorem and the remarks preceding it in [**12**] that *cabv* $(\mu, X) \subset L^1(\mu, X)^{**}$ and contains $J(L^1(\mu, X))$. Therefore $L^1(\mu, X)$ is always an ideal in *cabv* (μ, X).

We next consider ideals in injective tensor product spaces. One of the motivations here is to show that some results of Saab and Saab [**17**], [**18**] that have been proved for L^1-predual spaces, using methods of vector-valued Choquet theory, can be done easily (and in a more general way) under this scheme. We also obtain a new proof of the main result in [**14**].

We first state a result that characterizes L^1-predual spaces. One part of this, $3 \Rightarrow 1$, has been remarked by Lima in [**11**] and the other parts can be proved using standard facts from L^1-predual theory, see [**10**].

Proposition 1. *For any Banach space X,* TFAE.

(i) *If Y is such that X is isometric to a subspace of Y, then X is an ideal in Y.*

(ii) *X is isometric to an ideal in some $C(K)$ space, (for a compact set K).*

(iii) *X is an L^1-predual space.*

We first use this to give the example promised earlier.

Example 4. There is a standard way of embedding c_0 into any $C(K)$ space for an infinite compact set K (see [**10**, p. 112]). Let $K = [0,1]$. By the above proposition we have that such a copy of c_0 is an ideal in $C([0,1])$. On the other hand, $C([0,1])$ being a separable space is isometric to a subspace of the bidual of c_0, namely l^∞. However, if the unit ball of the range of a norm one projection P with $\operatorname{Ker} P = c_0^\perp$ (under any embedding) is dense in $C([0,1])^*_1$, then using Lemma 1 one can conclude that $C([0,1])$ is isometric to a subspace of l^∞ containing $J(c_0)$. Since c_0 is an M-ideal in l^∞ under the embedding J, we get that c_0 is isometric to an M-ideal of $C([0,1])$. Since any M-ideal in $C([0,1])$ is of the form $\{f \in C([0,1]) : f(E) = 0\}$ for some closed set $E \subset [0,1]$ (see [**9**, Chapter 1]), we get a contradiction.

Lemma 2. *Let X and Z be Banach spaces, and let Y be an ideal in Z. Then the injective tensor product $Y \otimes_\varepsilon X$ is an ideal in $Z \otimes_\varepsilon X$.*

Proof. Let $P : Z^* \to Z^*$ be a norm one projection with $\operatorname{Ker} P = Y^\perp$.

For any $\Phi \in (Z \otimes_\varepsilon X)^*$ and $x \in X$, define $\Phi_x : Z \to \mathbf{R}$ by $\Phi_x(z) = \Phi(z \otimes x)$.

Define $Q : (Z \otimes_\varepsilon X)^* \to (Z \otimes_\varepsilon X)^*$ by $Q(\Phi)(\sum_{i=1}^n z_i \otimes x_i) = \sum_{i=1}^n P(\Phi_{x_i})(z_i)$.

It is easy to verify that Q is a projection of norm one.

If $\Phi \in (Y \otimes_\varepsilon X)^\perp$, then $\Phi_x(y) = 0$ for all $y \in Y$ so that $P(\Phi_x) = 0$ and hence $Q(\Phi) = 0$. On the other hand, if $Q(\Phi) = 0$, then for any $x \in X$ and $y \in Y$, since $(\Phi_x - P(\Phi_x)) \in Y^\perp$, we get $\Phi(y \otimes x) = 0$. Hence, $(Y \otimes_\varepsilon X)^\perp = \operatorname{Ker} Q$.

Therefore, $Y \otimes_\varepsilon X$ is an ideal in $Z \otimes_\varepsilon X$.

In the first part of the next theorem, for the sake of completeness we give the proof due to Emmanuele [**5**] of a result remarked by Saab and Saab in [**17**]. The second part extends Proposition 2 of [**18**] to a general Banach space.

Theorem 2. (i) *Let X be an infinite dimensional L^1 predual space. If $X \otimes_\varepsilon Y$ contains a complemented copy of l^1, then Y contains a complemented copy of l^1.*

(ii) *Suppose X is an ideal in Y. For any Banach space Z_1, Z_2, any bounded linear operator $T : X \otimes_\varepsilon Z_1 \to Z_2$ has a norm preserving extension $S : Y \otimes_\varepsilon Z_1 \to Z_2^{**}$.*

Proof. (i) Let K be any compact space such that $X \subset C(K)$. By Proposition 1, X is an ideal in $C(K)$. Therefore by Lemma 2, $X \otimes_\varepsilon Y$ is an ideal in $C(K) \otimes_\varepsilon Y = C(K, Y)$. Therefore $C(K, Y)^*$ contains an isometric copy of $(X \otimes_\varepsilon Y)^*$ canonically.

Hence if l^1 is complemented in $X \otimes_\varepsilon Y$, then $(X \otimes_\varepsilon Y)^*$, and hence $C(K, Y)^*$, contain a copy of c_0. It now follows from the proof of Theorem 1 in [**16**] that Y^* contains a copy of c_0. Therefore Y contains a complemented copy of l^1.

(ii) Again we have that $X \otimes_\varepsilon Z_1$ is an ideal in $Y \otimes_\varepsilon Z_1$; therefore by part (ii) of Lemma 1, we get the desired norm-preserving extension.

Remark 3. Under some special conditions on X one can make the extension in (ii) to take values in Z_2. If X is such that every unconditionally converging operator on X is weakly compact, then one can imitate the arguments given during the proof of (i) in Theorem 3 in [**18**] to prove that if T is unconditionally converging, then S takes values in Z_2. Also if K is any compact space (not necessarily the dual unit ball) such that X is an ideal in $C(K)$, then since the domain of S is $C(K, Z_1)$, one can exploit the knowledge of operators defined on this domain.

Let K be a compact Choquet simplex and let $A(K, X)$ denote the space of X-valued affine continuous functions, equipped with the supremum norm. Using the identification of the space $A(K,X)$ with the space of w^*-weak continuous compact operators from $A(K)$ to X, we have observed in [**13**] that $A(K,X)$ is in the canonical way, isometric to $A(K) \otimes_\varepsilon X$. We use this in the next corollary to give a proof of Corollary 1 in [**16**] that does not use any vector-valued Choquet theory.

Corollary 2. *Let K be a compact Choquet simplex. If $A(K,X)$ contains a complemented copy of l^1, then X contains a complemented copy of l^1.*

Proof. Since K is a Choquet simplex, $A(K)$ is an L^1-predual space. Also $A(K,X) = A(K) \otimes_\varepsilon X$. Therefore the conclusion follows from part i) of the above theorem.

Remark 4. It can be verified that when K is a Choquet simplex, $A(K)$ as a subspace of $C(K)$ satisfies the hypothesis of (i) in Lemma 1.

The authors of [**14**] have studied intersection properties of balls in injective tensor product spaces using the machinery developed by Saab in [**19**]. We next present an easy proof of (i) $\Rightarrow$ (ii) in Theorem 3.1 that depends only on the ideas developed here.

First we need some definitions due to Lima (see [**14**] and the references listed there).

A Banach space X has the almost n.k. intersection property (a.n.k.I.P for short) if, for every family $\{B(x_j,\gamma_j)\}_{j=1}^n$ of n closed balls in X such that any k of them intersect, we have $\cap_{j=1}^n B(x_j,\gamma_j+\varepsilon) \neq \phi$ for all $\varepsilon > 0$.

Corollary 3. *Let X be an L^1-predual space. If E is any Banach space with the almost* n.k.I.P, *then $X \otimes_\varepsilon E$ also has the* a.n.k.I.P.

Proof. Let K be any compact space such that X is an ideal in $C(K)$. It follows from Lemma 1 of [**14**] (see also the remark following the proof of that lemma that suggests an easier proof) that $C(K,E)$ has the a.n.k.I.P.

As before, $X \otimes_\varepsilon E$ is an ideal in $C(K,E)$. To conclude the result now we only need to observe that, if a Banach space Z has the a.n.k.I.P and $Y \subset Z$ is an ideal, then Y has the a.n.k.I.P. To see this, let $\{B(y_j, \gamma_j)\}_{j=1}^n$ be n closed balls in Y such that any k of them intersect. Let $\varepsilon > 0$. Choose $\delta = \min_{1 \le j \le n}\{\varepsilon/(2\gamma_j + \varepsilon)\}$. Now consider n closed balls in Z with centers at y_j and radius γ_j. Clearly any k of them still intersect. Since Z has the a.n.k.I.P, there exists z in Z such that

$$\|z - y_j\| \le \gamma_j + \frac{\varepsilon}{2} \quad \text{for all } j.$$

Now let F be the finite dimensional subspace of Z spanned by $\{z, y_1, \dots, y_n\}$. Since Y is an ideal in Z, by the result of Lima quoted in the introduction, we have a linear map $T : F \to Y$ such that

$$\|T\| \le 1 + \delta \quad \text{and} \quad Ty = y \quad \text{for } y \in F \cap Y.$$

Put $y = T(z)$. Now

$$\begin{aligned}\|y_j - T(z)\| &= \|T(y_j) - T(z)\| \\ &\le (1+\delta)\Big(\gamma_j + \frac{\varepsilon}{2}\Big) \\ &\le \gamma_j + \varepsilon.\end{aligned}$$

Therefore, Y has the a.n.k.I.P.

Section 2. In this section we consider the question, "when is (every) an ideal, the range of a norm one projection?" For a discrete space Γ, let $c_0(\Gamma)$ denote the space of functions on Γ vanishing at infinity, equipped with the supremum norm. We give a new characterization of $c_0(\Gamma)$ in terms of ideals. Several of the results require concepts from M-structure theory which can be found in [**9**].

Suppose Y is a Banach space that is isometric to the range of a norm one projection in some dual space. This implies that Y is the range of a norm one projection, say Q, in Y^{**} (under the embedding J). Therefore, if Y is an ideal in X, with P as a norm one projection in X^{**} whose range is $Y^{\perp\perp} = Y^{**}$, then $Q \circ P|X$ is a norm one projection from X onto Y. In particular, if a dual space Y is an ideal in a space X, then Y is actually the range of a norm one projection in X. Therefore in a reflexive Banach space X, every ideal is the range of a norm one projection in that space. Also note that if every closed subspace of a Banach space X is an ideal, then we have in particular every finite dimensional subspace is the range of a norm one projection, and it then follows from a result of Kakutani (see [**10**]) that X is isometric to a Hilbert space.

Two natural generalizations of reflexivity that have been well studied recently are the notions of X being an M-ideal in X^{**} and that of X being an L-summand in its bidual. See Chapters 3 and 4 of [**9**] for a detailed account of these properties. Our first result extends the above remark to spaces X that are M-ideals in X^{**} and has some interesting consequences in M-structure of operator spaces.

Proposition 2. *If X is an M-ideal in X^{**}, then every ideal in X is the range of a norm one projection.*

Proof. Let $Y \subset X$ be an ideal and P a norm one projection in X^* with $\operatorname{Ker} P = Y^{\perp}$.

Since X^* is an L-summand in its bidual, it follows from Proposition 1.5 in [**9**, p. 161] that the range of P is again an L-summand in its bidual. Since X is an M-embedded space, applying Proposition 1.10 in [**9**, p. 164], we conclude that range of P is a w^*-closed subspace of X^*. Since the kernel and the range of P are w^*-closed, by duality we conclude that Y is the range of a norm one projection.

Corollary 4. *Let Y be an ideal in X.*

(i) *If $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$, then $\mathcal{K}(Y)$ is an M-ideal in $\mathcal{L}(Y)$.*

(ii) *For a dual space Z^*, if $\mathcal{K}(X, Z^*)$ is an M-ideal in $\mathcal{L}(X, Z^*)$, then $\mathcal{K}(Y, Z^*)$ is an M-ideal in $\mathcal{L}(Y, Z^*)$.*

Proof. (i) Suppose $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$. By Proposition 4.4 in [**9**, p. 291], it follows that X is an M-ideal in its bidual. Therefore, Y is the range of a norm one projection in X. Hence, by Proposition 4.2 in [**9**, p. 290], we get that $\mathcal{K}(Y)$ is an M-ideal in $\mathcal{L}(Y)$.

(ii) From part (ii) of Lemma 1, we get that operators in $\mathcal{K}(Y, Z^*)$ $(\mathcal{L}(Y, Z^*))$ have norm preserving extensions in $\mathcal{K}(X, Z^*)(\mathcal{L}(X, Z^*))$. Therefore the conclusion can be derived from the "3-ball property" characterization of M-ideals and the hypothesis.

An important example of a Banach space that is an M-ideal in its bidual is the space $c_0(\Gamma)$ for a discrete set Γ. Before stating a characterization of this space, we need a lemma that can be proved by modifying the arguments given during the proof of Proposition 2.6 in [**9**, p. 119].

Lemma 3. *Let X be a Banach space. Every M-ideal in X is an M-summand if and only if X is isometric to the c_0-direct sum $\oplus X_i$ for some family $\{X_i\}$ of Banach spaces where, for each i, X_i has no nontrivial M-ideals.*

Proof. Let $\{X_i\}$ be a family of Banach spaces such that X_i has no nontrivial M-ideals for each i. Put $Y = \oplus X_i$. Since any M-ideal in a space is left invariant by M-projections, using coordinate projections in Y it is easy to see that every M-ideal in Y is an M-summand.

For the converse part we only indicate the modification required. Following the notation in [**9**], for an extreme point p of X_1^*, by our assumption the L-summand complementary to N_p is w^*-closed and, for the same reason, if $N_p \neq N_q$, then $N_p \cap N_q = \{0\}$. Put $X_i = (M_p)_\perp$ and I the quotient of the extreme points of X_1^* after identifying p and q for which $N_p = N_q$. X_i clearly has no nontrivial M-ideals and arguments identical to the ones given during the proof of Proposition 2.6 in [**9**], giving the desired conclusion.

Our next proposition extends Proposition III.2.7 in [**9**].

Proposition 3. *Let X be a Banach space. X is isometric to $c_0(\Gamma)$ for some discrete space Γ if and only if every ideal in X is the range of a norm one projection and*

$$A = \{f \in \partial_e X_1^* : \text{ line}\{f\}, \text{ is an } L\text{-summand}\}$$

(here line$\{f\}$ *stands for the one-dimensional space spanned by $\{f\}$ and $\partial_e X_1^*$ stands for the set of extreme points) is w^*-dense in $\partial_e X_1^*$.*

Proof. Since $(c_0(\Gamma))^* = l^1(\Gamma)$, the "only if" part follows immediately from Proposition 2. Conversely, since any M-ideal M in X by definition is an ideal, we have from the hypothesis that M is the range of a norm one projection. It now follows from Corollary I.1.3 in [**9**] that M is an M-summand. Let Γ now be A quotiented by the same equivalence relation. By hypothesis again, $N_p = \text{line}\{p\}$, $p \in A$. Since A is dense in $\partial_e X_1^*$ and M_p is of codimension one, we conclude that X is isometric to $c_0(\Gamma)$.

Remark 5. If X is a Banach space such that X^* is isometric to an L^1-space, then for any $f \in \partial_e X_1^*$, line$\{f\}$ is an L-summand. Thus our result extends Proposition III.2.7 in [**9**]. We do not know an answer to the following question which is in a sense a noncommutative version of the above proposition.

Question. What are those C^* algebras in which every ideal is the range of a projection of norm one?

Remark 6. The range of a projection of norm one in a C^* algebra is called a JB^* triple (see [**9**, p. 256]). See [**9**, Proposition III.2.9] for the structure of a C^* algebra that is an M-ideal in its bidual.

It follows from Proposition 1 in Section 1 that c_0 is an ideal in every Banach space that contains it; also, as noted before, it is an M-ideal in its bidual. Using this analogy, we next give an abstract version of Proposition II.2.120 in [**9**].

Proposition 4. *Let X be an M-ideal in its bidual. Suppose $X \subset Y$ as an ideal. Then there is a renorming of Y which agrees with the original norm on X, and X is an M-ideal in Y under this new norm.*

Proof. Since X is an ideal in Y, there exists a projection $P^* : Y^{**} \to X^{\perp\perp}$ of norm one. As in the proof of Proposition II.2.10 in [**9**], we can renorm Y to make P^* an M-projection. This norm clearly agrees with the original norm on X. Since X is an M-ideal in $X^{\perp\perp} = X^{**}$, we get that it is an M-ideal in Y with respect to the new norm.

Let X be an L-summand in its bidual and let $Y \subset X$ be an ideal. If

Y is again an L-summand in its bidual, then since Y in particular is the range of a norm one projection in a dual space we have that Y is the range of a norm one projection in X. We do not know if such a Y is always an L-summand in its bidual. This, however, is true for an important class of Banach spaces that are L-summands in their biduals.

Proposition 5. *Let X be the predual of a von Neumann algebra. If $Y \subset X$ is an ideal, then Y is the range of a norm one projection in X.*

Proof. Let P be a norm one projection in X^* with $\mathrm{Ker}\, P = Y^\perp$. Since X^* is a von Neumann algebra, it follows that $\mathrm{Range}\, P$ has a unique predual and the predual is a Banach space that is an L-summand in its bidual (see the discussion in [**9**, p. 225]). Therefore Y is an L-summand in its bidual and hence is the range of a norm one projection in X.

As a consequence of this proposition, we give another proof of a result of Haagerup from [**6**].

Corollary 5. *Let X be the predual of a von Neumann algebra. Let $Y \subset X$ be a separable subspace. There exists a separable 1-complemented subspace Z such that $Y \subset Z \subset X$.*

Proof. It follows from the results of Sims and Yost [**21**] that there is a separable subspace Z with $Y \subset Z \subset X$, and Z is an ideal in X. The conclusion thus follows from the above proposition.

Acknowledgments. The final revision on this paper was done during the author's stay at the University of Missouri-Columbia. The author is grateful to Professors E. Saab, P. Saab and N.J. Kalton for their hospitality.

REFERENCES

1. R. Cilia, *A remark on the Dunford-Pettis property in $L_1(\mu, X)$*, Proc. Amer. Math. Soc. **120** (1994), 183–184.

2. J.A. de Reyna, et al., *Some observations about the space of weakly continuous functions from a compact space into a Banach space*, Quaestiones Math. **15** (1992), 415–425.

3. J. Diestel and J.J. Uhl, *Vector measures*, American Math. Society, Providence, 1977.

4. L. Drewnowski and G. Emmanuele, *The problem of complementability for some spaces of vector measures of bounded variation with values in Banach spaces containing copies of* c_0, Studia Math. **104** (1993), 111–123.

5. G. Emmanuele, *Addendum to "Remarks on weak compactness of operators defined on certain injective tensor products*, Proc. Amer. Math. Soc. **118** (1993), 335.

6. N. Ghoussoub, et al., *Some topological and geometrical structures in Banach spaces*, Memoirs Amer. Math. Soc. **378** Providence, RI, 1987.

7. G. Godefroy, N.J. Kalton and P. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59.

8. J. Johnson, *Remarks on Banach spaces of compact operators*, J. Funct. Anal. **32** (1979), 304–311.

9. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math. **1547**, Springer-Verlag, Berlin, 1993.

10. H.E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, Berlin, 1974.

11. Å. Lima, *The metric approximation property, norm-one projections and intersection property of balls*, Israel J. Math. **84** (1993), 451–475.

12. T.S.S.R.K. Rao, $L^1(\mu, X)$ *as a complemented subspace of its bidual*, Proc. Indian Acad. Sci. (Math. Sci.) **104** (1994), 421–424.

13. ———, *M structure and the space* $A(K, E)$, Rend. Mat. Appl. (7) **15** (1995), 153–160.

14. T.S.S.R.K. Rao, A.K. Roy and K. Sundaresan, *Intersection properties of balls in tensor products of some Banach spaces*, Math. Scand. **65** (1989), 103–118.

15. W. Ruess, *Duality and geometry of spaces of compact operators*, Proc. Paderborn Conf. Funct. Anal., Survey and Recent Results III (K. Bierstedt and B. Fuchssteiner, eds.), North-Holland Math. Stud. **90** (1984), 59–78.

16. E. Saab and P. Saab, *A stability property of a class of Banach spaces not containing a complemented copy of* l_1, Proc. Amer. Math. Soc. **84** (1982), 44–46.

17. ———, *On unconditionally converging and weakly precompact operators*, Illinois J. Math. **35** (1991), 522–531.

18. ———, *Extensions of some classes of operators and applications*, Rocky Mountain J. Math. **23** (1993), 319–337.

19. P. Saab, *Integral representations by boundary vector measure*, Canad. Math. Bull. **25** (1982), 164–168.

20. M. Sharir, *Extremal structure in operator spaces*, Trans. Amer. Math. Soc. **186** (1975), 91–111.

21. B. Sims and D. Yost, *Linear Hahn-Banach extension operators*, Proc. Edinburgh Math. Soc. (2) **32** (1989), 53–57.

Indian Statistical Institute, R.V. College Post, Bangalore 560059, India
E-mail address: TSS@isibang.ac.in