A NOTE ON THE DISTRIBUTION OF THE SUM OF CHI-SQUARES

By A. BHATTACHARYYA
Statistical Laboratory, Calcutta

Mr. B. C. Bhattacharyya (1943) has considerd the distribution of the sum of two chi-squares, each of which is the sum of squares of the same number of independent normal variates with zero mean and the same standard deviation, but the two sets differing in their standard deviations. He has shown that the distribution involves some Bessel functions considered by Mckay. We shall here briefly indicate how the same distribution can be expressed as a convergent series in Laguerre's polynomials. We shall consider the more general case where the two chi-squares are not independently distributed.

2. Let $(x_1, y_1;\ x_2, y_2; \ldots x_n, y_n)$ be n pairs of independent correlated normal variates with zero mean and s.d.'s σ_1 and σ_2; also let the correlation coefficient be ρ. Let us define χ^2_1 as the sum of squares of the x's and χ^2_2 that of the y's. The case when the two χ^2 are independent is obtained when $\rho = 0$. We shall consider the distribution of $\chi^2_1 + \chi^2_2$. The characteristic function of $x^2_1 + y^2_1$ is

$$\phi(t) = \{1 - 2it(\sigma^2_1 + \sigma^2_2) - 4(1 - \rho^2)\sigma^2_1 \sigma^2_2 t^2\}^{-\frac{1}{2}}$$

and hence the characteristic function of $\chi^2_1 + \chi^2_2$ is $(\phi(t))^n$. From the form it takes it is obvious that the same Bessel function form is valid in this case also. Similarly it is easily seen that the distribution of $\chi^2_1 - \chi^2_2$ follows the other Bessel function form (Bhattacharyya, 1943).

The characteristic function of $u = (\chi^2_1 + \chi^2_2)/(\sigma^2_1 + \sigma^2_2)$ is easily found to be

$$\psi(t) = [1 - 2it - 4t^2 \sigma^2_1 \sigma^2_2 (1 - \rho^2)(\sigma^2_1 + \sigma^2_2)^{-2}]^{-n/2} = [(1 - it)^2 + at^2]^{-n/2} \qquad .. \quad (2{\cdot}1)$$

where $a = 1 - 4\sigma^2_1 \sigma^2_2 (1 - \rho^2)(\sigma^2_1 + \sigma^2_2)^{-2}$. As $2\sigma_1 \sigma_2 < \sigma^2_1 + \sigma^2_2$, and $\rho^2 < 1$, we must have $|a| < 1$ and so $|at^2(1 - it)^{-2}| = |a| t^2 (1 + t^2)^{-1} < 1$, for all values of t. So, from (2.1) we get

$$\psi(t) = (1 - it)^{-n} [1 + at^2(1 - it)^{-2}]^{-n/2} = \sum_{r=0}^{\infty} \binom{-n/2}{r} a^r t^{2r} (1 - it)^{-n-2r} \qquad .. \quad (2{\cdot}2)$$

The infinite series is uniformly convergent for all values of t. Now, differentiating the relation

$$\int_{-\infty}^{\infty} (1 - it)^{-n-2r} \, \text{Exp.}(-itu) dt/2\pi = e^{-u} u^{n+2r-1}/\Gamma(n + 2r)$$

r-times with respect to u, we get

$$\int_{-\infty}^{\infty} (1 - it)^{-n-2r} (-it)^{2r} \, \text{Exp.}(-itu) dt/2\pi = \left(\frac{d}{du}\right)^{2r} e^{-u} u^{n+2r-1} / \Gamma(n + 2r)$$

$$= \Gamma(n) e^{-u} u^{n-1} L_{2r}(u, n)/\Gamma(n + 2r) \qquad .. \quad (2{\cdot}3)$$

where $L_{2r}(u, n)$ is a Laguerre's polynomial. Hence applying Fourier's integral theorem to $\psi(t)$ and using (2.3) we get the distribution function of u as

$$\Gamma(n) e^{-u} u^{n-1} \sum_{r=0}^{\infty} \binom{-n/2}{r} a^r L_{2r}(u, n)/\Gamma(2r + n) \qquad .. \quad (2{\cdot}4)$$

The above series is convergent for all values of u. Proceeding in a similar way it could be shown, under some restrictions, that the distribution of the sum of three or more χ^2's can be expressed in a series of Laguerre's polynomials.

3. The non-null distribution of χ^2 can be expressed, under certain conditions, in a series of Laguerro's polynomials. Let x_k, $(k = 1, 2, \ldots n)$ be n independent normal variates with zero mean and standard deviation σ_k. Then the distribution of $\chi^2 = \sum_{k=1}^{n} x^2_k$ can be regarded as the non-null distribution. There is another type of non-nullity, namely when the normal variates have the same standard deviation, but different means. This has already been considered by Wicksell (1933). The classical D^2-statistic follows the same distribution (Bose, 1930). We shall consider the other type of non-null case. The characteristic function of χ^2 is obviously

$$\prod_{k=1}^{n} (1 - 2it\,\sigma_k^2)^{-\frac{1}{2}} \qquad \ldots \quad (3\cdot1)$$

If $n\sigma^2 = \sum_{k=1}^{n} \sigma^2_k$ and $\varepsilon_k = \sigma^2_k - \sigma^2$ then the above characteristic function can be written as

$$\prod_{k=1}^{n} (1 - 2it\,\sigma^2 - 2i\,\varepsilon_k t)^{-\frac{1}{2}} = (1 - 2it\,\sigma^2)^{-n/2} \prod_{k=1}^{n} \{1 - 2it\,\varepsilon_k/(1 - 2it\,\sigma^2)\}^{-\frac{1}{2}}$$

The characteristic function of χ^2/σ^2 is

$$(1 - 2it)^{-n/2} \prod_{k=1}^{n} \{1 - 2it\,\varepsilon_k/\sigma^2(1 - 2it)\}^{-\frac{1}{2}} \qquad \ldots \quad (3\cdot2)$$

If now we suppose that $\varepsilon^2_k < \sigma^4$ for all values of k, then each of the expressions $\{1 - 2it\,\varepsilon_k/\sigma^2(1 - 2it)\}^{-\frac{1}{2}}$ can be expanded in a uniformly convergent series, which then can be multiplied together giving a uniformly convergent infinite series for (3.2) of the form

$$\sum_{r=0}^{\infty} c_r (it)^r (1 - 2it)^{-r-n/2} \qquad \ldots \quad (3\cdot3)$$

where $c_0 = 1$, $c_1 = 0$, $c_2 = \Sigma\,\varepsilon^2_k/\sigma^4$, $c_3 = 4\Sigma\,\varepsilon^3_k/3\sigma^6$ etc. Applying, as in section 2, Fourier's integral theorem and remembering (2.3) we get for the distribution function of $x = \chi^2/\sigma^2$ an expression of the form

$$\Gamma(n/2)\, e^{-x/2}\, x^{n-1} \sum_{r=0}^{\infty} d_r\, L_r(x, n)/\Gamma(n + r)$$

where of course d_r depends on c_r or ultimately on ε_r.

There are many other non-null forms of χ^2 which we cannot adequately consider in this short note. It is hoped to consider these in detail at some future date, when other interesting properties of χ^2 will be considered.

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Paper received : 29 September, 1943.