Waves on a film of power-law fluid flowing down an inclined plane at moderate Reynolds number

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Abstract

Waves that occur at the surface of a power-law fluid film flowing down an inclined plane are investigated. Using the method of integral relations, an evolution equation is derived for two types of wave equations which are possible under long wave approximation. This equation is valid for moderate Reynolds numbers and reveals the presence of both kinematic and dynamic wave processes which may either act together or singularly dominate the wave field depending on the order of different parameters. It is shown that, at a small flow rate, kinematic waves dominate the flow field and it acquires energy from the mean flow, while, for high flow rate, inertial waves dominate and the energy comes from the kinematic waves. This energy transfer from kinematic waves to inertial waves depends on the power-law index n. Linear stability analysis predicts the contribution of different terms in the wave mechanism. Further, it is found that surface tension plays a double role, for the kinematic wave process, it exerts dissipative effects so that a finite amplitude case may be established, but for the dynamic wave process it yields dispersion. The evolution equation is capable of predicting amplitudes, shapes, and interaction at the finite amplitude level. It is also shown that the results of the interaction may lead either to forward breaking waves or solitary waves with dark soliton depending on the flow rate, Weber number and the angle of inclination with the horizon. Power-law index n plays a vital role in the wave mechanism.

Keywords: Power-law fluid film; Waves on falling film; Stability of power-law fluid film; KdV waves; Forward breaking waves

1. Introduction

Flow of thin liquid film on an inclined plane has drawn the attention of studies since the last five decades due to its various applications in the technological development of modern science. Linear stability of long waves on a layer of viscous fluid flowing down an inclined plane was investigated by Yih (1963), who found the critical Reynolds number by a regular perturbation method. Prior to this study, Benjamin (1957) approximated the eigenfunction in the Orr–Sommerfeld equation governing

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linear stability by a power series in the coordinate normal to the inclined plane and determined the phase velocity of the wave. Using momentum integral method, Kapitza (1948), Shkadov (1967, 1968) and Krylov et al. (1969) performed the analysis to predict the dependencies of the growth rate and the phase velocity of the wave. They have predicted the critical Reynolds number and the characteristic of the fastest growing waves. A detailed account is documented by Alekseenko et al. (1994). Finite amplitude stability of a layer of viscous liquid flowing down an inclined plane were made by Benney (1966), Lin (1969), Gjevik (1970) and Nakaya (1975). It is interesting to note that most of the studies on the flow of thin liquid film on an inclined plane assumed the fluid to be Newtonian. These results of Newtonian fluid cannot describe the rheological behaviour of the non-Newtonian fluid. As most of the fluid used in industry in connection with plastic manufacturing, flow of molten metals/lava, in coating process, movement of biological fluid are basically non-Newtonian. Of course, some studies on linear stability of non-Newtonian liquid film flow down an inclined plane were made by Gupta (1967) considering fluid be second order; by Liu and Mei (1989) a Bingham fluid, Lai (1967) for a Oldroyd-B fluid and Hwang et al. (1994) and Berezin et al. (1998) for power-law model. Using Benney's (1966) approach, Dandapat and Gupta (1978) studied the stability of a falling film of an incompressible second-order fluid with respect to two-dimensional disturbances of small but finite amplitude. They found that in the presence of surface tension, the stability of flow of the falling film is supercritically stable and an initially growing monochromatic wave reaches an equilibrium state of the finite amplitude. Further, they found that the equilibrium amplitude first increases with the elastic parameter M (say) of the fluid, reaches a maximum and then decreases with the increase in M. In a recent study, Dandapat and Gupta (1997) have shown the existence and the role of the solitary wave in the finite amplitude instability of a layer of a second-order fluid flowing down an inclined plane. They observed that the number of solitary waves decreases with the increase in M. Ng and Mei (1994) studied the roll waves on a layer of a mud modelled as a power-law fluid flowing down an inclined plane. They found through linearized instability that the growth rate of unstable disturbances increases monotonically with the wave number, this prevents them from predicting any preferred wavelength for the roll waves. Further, they observed that the existence of long roll waves depends on the power-law index even if the corresponding uniform flow is stable. It is to be pointed out here that Ng and Mei (1994) have neglected the surface tension term in their analysis. It is well known that the wavelength, amplitudes and their relation with the flow rate are of primary importance for the design of process devices. Further, the analysis of the non-linear wave evolution equation is in general very complicated. So to extract more results from it, most of the previous studies on non-linear wave process are based on the assumption of stationary waves which does not change its phase speed and shape during the course of propagation. This study will be based on quasi-stationary wave process to analyse the evolution equation of a power-law film flow on an inclined plane. In this process, the phase speed c is assumed to be approximately constant so that the wave profile in a moving co-ordinate system is deformed slightly during the course of propagation. Alekseenko et al. (1979) have initiated this analysis to study the wave formation in a liquid film flow on a vertical wall. For a better understanding of a physical phenomena on a power-law fluid film flowing down an inclined plane, it is therefore desirable to investigate the types of waves that occur under various parameter regions. One should remember here that the power-law model represents a class of non-Newtonian fluids which do not exhibit any elastic, yield or stress relaxation properties but these fluids show the behaviour of shear-thinning (pseudo-plastic) or shear-thickening (dilatant) properties. Power-law models have been found to be successful in describing the behaviour of colloids and suspensions and a variety of polymeric liquids and low molecular weight biological liquids, in the field of glaciology, blood-rheology and geology. Since this model is different from a Newtonian fluid only in that its viscosity depends on the symmetric part of the velocity gradient so that in a simple shear flow, the viscosity depends on the shear rate. To be more precise such effects are inconsequential, the dominant departure from the Newtonian behaviour being shear-thinning or shear-thickening. Andersson and Irgens (1989) documented a review of power-law fluids which contains the names of different non-Newtonian fluids and their corresponding values of the power-law index n.

2. Mathematical formulation of the problem

Consider a layer of an incompressible fluid obeying the power-law model that flows down an inclined plane of inclination θ with the horizon where x-axis along and z-axis normal to the plane (Fig. 1). The governing equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0, \tag{1}$$

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\rho g\sin\theta+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}, \tag{2}$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial z}-\rho g\cos\theta+\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z}, \tag{3}$$

where the flow is assumed to be two dimensional and the stress tensor τ_{ij} defined by

$$\tau_{ij}=2\mu_n(2D_{kl}D_{kl})^{(n-1)/2}D_{ij} \tag{4}$$

where

$$D_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right] \tag{5}$$

denotes the strain-rate tensor, μ_n is the viscosity coefficient of dimension $[ML^{(-1)}T^{(n-2)}]$ and n is the power-law index which is positive. $n=1$ represents a Newtonian fluid with constant dynamic

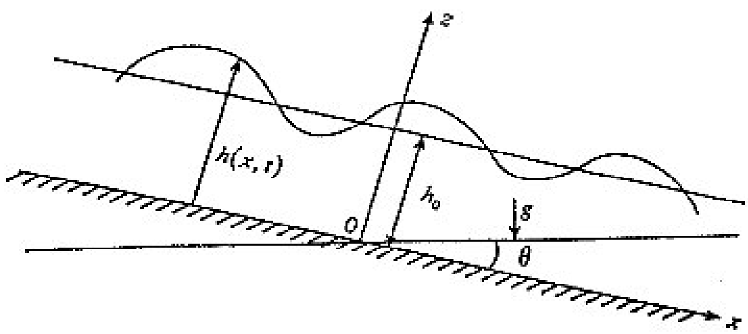

Fig. 1. Sketch of the problem.

coefficient of viscosity μ, while $n<1$ and $n>1$ corresponds to the case of pseudoplastic (shear-thinning) and dilatant (shear-thickening) fluids, respectively. u, v, ρ and p have their usual meaning and g is the acceleration due to gravity.

The boundary conditions are as follows: no-slip on the plane $z=0$

$$u=0, \qquad v=0. \tag{6}$$

At the free surface $z=h(x,t)$, the shear stress vanishes and the normal stress difference just balances that due to surface tension and reads as

$$\tau_{zx}(1-h_x^2)-(\tau_{xx}-\tau_{zz})h_x=0 \tag{7}$$

and

$$p_0-p+(\tau_{xx}h_x^2-2\tau_{zx}h_x+\tau_{zz})(1+h_x^2)^{-1}=\sigma h_{xx}(1+h_x^2)^{-3/2}. \tag{8}$$

Where σ, p_0 and h denote the surface tension, atmospheric pressure and deflection from the mean depth h_0.

Further, the kinematic condition at the free surface $z=h(x,t)$ is

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}=v. \tag{9}$$

The basic velocity $[u(z),0]$ in the steady flow down the plane is

$$u=\frac{(1+2n)}{(1+n)}\bar{u}_0\left[1-\left(1-\frac{z}{h_0}\right)^{(1+n)/n}\right]. \tag{10}$$

To obtain Eq. (10) we have used the no-slip condition $u(0)=0$ and the condition of zero shear stress at the free surface $z=h_0$ which is the undisturbed layer thickness. Here, $\bar{u}_0$ is the depth averaged characteristic velocity defined by

$$\bar{u}_0=\frac{1}{h_0}\int_0^{h_0}u(z)\,\mathrm{d}z=\frac{n}{(1+2n)}\left(\frac{\rho g\sin\theta}{\mu_n}\right)^{1/n}h_0^{(1+n)/n}. \tag{11}$$

This steady flow has the pressure distribution $p(z)$ given by

$$p=p_0+\rho g h_0\left(1-\frac{z}{h_0}\right)\cos\theta. \tag{12}$$

We assume the characteristic longitudinal length scale to be l_0 whose order may be considered the same as the wavelength λ_0 and the mean film thickness h_0 as the length scale in transverse direction. We define the dimensionless quantities as

$$x=l_0x^*, \quad (h,z)=h_0(h^*,z^*), \quad t=\left(\frac{l_0}{\bar{u}_0}\right)t^*, \quad u=\bar{u}_0u^*, \quad v=\left(\frac{h_0}{l_0}\right)\bar{u}_0v^*, \quad p=\rho\bar{u}_0^2p^*,$$

$$(\tau_{xx},\tau_{zz})=\mu_n(\bar{u}_0/h_0)^{n-1}(\bar{u}_0/l_0)(\tau_{xx}^*,\tau_{zz}^*) \quad\text{and}\quad (\tau_{xz},\tau_{zx})=\mu_n(\bar{u}_0/h_0)^n(\tau_{xz}^*,\tau_{zx}^*). \tag{13}$$

Using (13) in Eqs. (1)–(3) and in Eqs. (6)–(9), we arrive after dropping the asterisk at

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0, \tag{14}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\sin\theta}{\varepsilon\mathrm{Fr}}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial\tau_{xx}}{\partial x}+\frac{1}{\varepsilon\mathrm{Re}}\frac{\partial\tau_{xz}}{\partial z}, \tag{15}$$

$$\varepsilon^2\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial z}-\frac{\cos\theta}{\mathrm{Fr}}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial\tau_{zx}}{\partial x}+\frac{\varepsilon}{\mathrm{Re}}\frac{\partial\tau_{zz}}{\partial z}, \tag{16}$$

$$u=0,\quad v=0\quad \text{at } z=0, \tag{17}$$

$$\tau_{zx}\left[1-\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right]-\varepsilon^2(\tau_{xx}-\tau_{zz})\frac{\partial h}{\partial x}=0\quad \text{at } z=h, \tag{18}$$

$$p_0-p+\left[\frac{\varepsilon^3}{\mathrm{Re}}\tau_{xx}\left(\frac{\partial h}{\partial x}\right)^2-\frac{2\varepsilon}{\mathrm{Re}}\tau_{zx}\frac{\partial h}{\partial x}+\frac{\varepsilon}{\mathrm{Re}}\tau_{zz}\right]\left[1+\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2\right]^{-1}$$
$$=\varepsilon^2\mathrm{We}\,\frac{\partial^2 h}{\partial x^2}\left[1+\left(\frac{\partial h}{\partial x}\right)^2\right]^{-3/2}\quad \text{at } z=h, \tag{19}$$

and

$$v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\quad \text{at } z=h. \tag{20}$$

Where Re is the Reynolds number

$$\mathrm{Re}=\frac{\rho\bar{u}_0^{(2-n)}h_0^n}{\mu_n},$$

$\mathrm{We}=\sigma/\rho\bar{u}_0^2h_0$ is the Weber number, $\mathrm{Fr}=\bar{u}_0^2/gh_0$ is the Froude number and $\varepsilon=h_0/l_0\ll 1$ is the aspect ratio for long wavelength approximation. Using the dimensionless form of Eqs. (4) and (5) in Eqs. (14)–(20) under usual boundary layer approximations for long-wave expansions, we arrive at

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}=0, \tag{21}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\sin\theta}{\varepsilon\mathrm{Fr}}+\frac{1}{\varepsilon\mathrm{Re}}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)^n, \tag{22}$$

$$0=-\frac{\partial p}{\partial z}-\frac{\cos\theta}{\mathrm{Fr}}. \tag{23}$$

The boundary conditions are

$$u=0=v\quad \text{at } z=0, \tag{24}$$

$$\frac{\partial u}{\partial z}=0\quad \text{at } z=h(x,t), \tag{25}$$

$$p=p_0-\varepsilon^2\mathrm{We}\frac{\partial^2 h}{\partial x^2}\quad \text{at } z=h(x,t) \tag{26}$$

and the kinematic condition is

$$v=\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\quad \text{at } z=h. \tag{27}$$

Here we assumed that the Weber number We is large enough which is a fact for thin films. The z-momentum equation (23) and normal stress boundary condition (26) are used to eliminate $\partial p/\partial x$ in Eq. (22) and the resulting system reduces to

$$u_x + v_z = 0, \tag{28}$$

$$u_t + uu_x + vu_z = \mathrm{We}\varepsilon^2 h_{xxx} - \mathrm{Fr}^{-1}\cos\theta h_x + (\varepsilon\mathrm{Fr})^{-1}\sin\theta + (\varepsilon\mathrm{Re})^{-1}[(u_z)^n]_z, \tag{29}$$

$$u = 0 = v \quad \text{at } z = 0, \tag{30}$$

$$u_z = 0 \quad \text{at } z = h(x,t), \tag{31}$$

$$h_t + uh_x = v \quad \text{at } z = h(x,t). \tag{32}$$

Here the subscripts denote the derivative of the respective variables with respect to t, x and z.

Integrating Eqs. (28) and (29) with respect to z from 0 to h under the assumption that the velocity profile (10) is valid in a non-transient and non-uniform flow, we get

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{33}$$

$$q_t + \left(\beta\frac{q^2}{h} + \alpha\frac{h^2}{2}\right)_x = \mathrm{We}\varepsilon^2 hh_{xxx} + \frac{1}{\varepsilon\gamma}\left[h - \left(\frac{q}{h^2}\right)^n\right], \tag{34}$$

where the flow rate per unit film width is

$$q = \int_0^h u\,\mathrm{d}z \tag{35}$$

and the shape factor β is defined as

$$\beta = \frac{1}{h\bar{u}_0^2}\int_0^h u^2\,\mathrm{d}z = \frac{2(1+2n)}{(2+3n)}. \tag{36}$$

For shear-thinning fluids, $0 < n \leqslant 1$, the range of β is $1 \leqslant \beta \leqslant \frac{6}{5}$. $\bar{u}$ is defined as the depth-averaged velocity

$$\bar{u} = h^{-1}\int_0^h u\,\mathrm{d}z = \frac{q}{h}. \tag{37}$$

Relations

$$\frac{\sin\theta}{\mathrm{Fr}} = \frac{1}{\mathrm{Re}}\left(\frac{1+2n}{n}\right)^n = \frac{1}{\gamma}\ \text{(say)}$$

and

$$\alpha = \frac{\cos\theta}{\mathrm{Fr}} = \left(\frac{1+2n}{n}\right)^n\frac{\cot\theta}{\mathrm{Re}} = \frac{\cot\theta}{\gamma} \tag{38}$$

have been used in deriving Eq. (34). Using Eq. (37) in Eqs. (33) and (34), we get

$$h_t + (\bar{u}h)_x = 0, \tag{39}$$

$$(\overline{u}h)_t + \left(\beta\overline{u}^2 h + \frac{\alpha h^2}{2}\right)_x = \mathrm{We}\varepsilon^2 h h_{xxx} + \frac{1}{\gamma\varepsilon}\left[h - \left(\frac{\overline{u}}{h}\right)^n\right]. \tag{40}$$

It is to be noted here that the momentum integral method has been used by earlier researchers in connection with boundary layer theory Schlichting (1968) and on stability theory starting from Kapitza (1948), Alekseenko et al. (1985), Jurman and McCready (1989) and others for Newtonian fluid and Ng and Mei (1994) for power-law fluids. It is to be noted here that the first term on the right-hand side (r.h.s.) of Eq. (40) is missing in Eq. (3.9) of Ng and Mei (1994) as they have not considered the surface tension term in their analysis. Further, the difference on the second term of r.h.s. is a factor $1/\gamma\varepsilon$. This discrepancy is due to their particular choice of horizontal characteristic length scale l_0. In the following analysis, we have discussed wave evaluations/generations for various parameter regions depending on flow rate, surface tension and angle of inclination. Throughout the analysis, we have assumed that the Weber number is large enough which is a fact for thin film. Moreover, it is to be remembered that surface tension is the fluid property and it does not have any bearing on the length scale and according to our definition $\mathrm{We} = \{(1+2n)/n\}^{n(3n-2)/(2+n)}[Fi\mathrm{Re}^{-(2+3n)}]^{1/(2+n)}(\sin\theta)^{(2-3n)/(2+n)}$, where $Fi = \sigma^{(2+n)}/[\rho^{(2+n)} g^{(3n-2)} \nu_n^4]$ is the Film (Kapitza) number and $\nu_n = \mu_n/\rho$ is the kinematic viscosity. Experimental results as reported in Alekseenko et al. (1994), for vertical thin film with Re $\sim$ 1, $Fi^{1/11} = 9.54$ and 4 are observed for water and maximum viscous solutions, respectively, used in their experiment. These two results lead to We $\simeq$ 5600 and 250, respectively, which are large enough. Further Alekseenko et al. (1985) have shown that the governing equations (21)–(27) are valid for all real fluids within the range of Reynolds number $1 \leqslant \mathrm{Re} \leqslant \varepsilon^{-2}$.

3. Linear stability analysis for the uniform flow

Following standard linear stability analysis, we introduce the perturbed field $h = 1 + H(x,t)$ and $\bar{u} = 1 + U(x,t)$ in Eqs. (39) and (40), where H and U are infinitely small so that their product or higher order terms are neglected to obtain the perturbed equations

$$H_t + (U + H)_x = 0, \tag{41}$$

$$(U + H)_t + (2\beta U + (\alpha + \beta)H)_x = \mathrm{We}\varepsilon^2 H_{xxx} + \frac{1}{\gamma\varepsilon}[(n+1)H - nU]. \tag{42}$$

We assume that the perturbations of the system of equations (41) and (42) are of the form of a travelling wave

$$(H, U) = \mathrm{Re}[(\hat{H}, \hat{U})\exp\{\mathrm{i}(k\hat{x} - \omega\hat{t})\}], \tag{43}$$

where the wave number k is real and $\omega = \omega_r + \mathrm{i}\omega_i$, is the complex frequency, $\hat{x} = x/\varepsilon$ and $\hat{t} = t/\varepsilon$. The dispersive relation

$$\omega^2 - (2\beta k - \mathrm{i}\gamma^{-1} n)\omega - \mathrm{i}\gamma^{-1}(1+2n)k + (\beta - \alpha)k^2 - \mathrm{We}k^4 = 0 \tag{44}$$

is obtained by using Eq. (43) in Eqs. (41) and (42). The solution of Eq. (44) gives

$$\omega^{\pm} = \beta k - \frac{\mathrm{i}n}{2\gamma} \pm (a + \mathrm{i}b)^{1/2}, \tag{45}$$

where

$$a = \mathrm{We}k^4 + [\beta(\beta-1)+\alpha]k^2 - \frac{1}{4}\gamma^{-2}n^2 \quad \text{and} \quad b = (1+2n-\beta n)\gamma^{-1}k. \tag{46}$$

It is clear from the definition of β that b is positive for all n and γ, $k \neq 0$. The separation of the real and imaginary parts of Eq. (45) gives

$$\omega_r^{\pm} = \beta k \pm \left(\frac{1}{2}[a + (a^2+b^2)^{\frac{1}{2}}]\right)^{1/2}$$

and

$$\omega_i^{\pm} = -\frac{n}{2\gamma} \pm \left(\frac{1}{2}[-a + (a^2+b^2)^{\frac{1}{2}}]\right)^{1/2}.$$

It is obvious from above that ω_i^- gives stability while ω_i^+ will ensure stability provided

$$\left(\frac{1}{2}[-a + (a^2+b^2)^{\frac{1}{2}}]\right)^{1/2} < \frac{n}{2\gamma}.$$

In other words, for $k \neq 0$,

$$\alpha + \mathrm{We}k^2 > \frac{1+2n}{n^2}. \tag{47}$$

Using relation (38) for α, stability criterion reduces to

$$\mathrm{Re} < \mathrm{Re}_{\mathrm{linear}} = n\left(\frac{n}{1+2n}\right)^{(1-n)} \cot\theta \left[1 - \frac{\mathrm{We}n^2k^2}{1+2n}\right]^{-1}. \tag{48}$$

This shows that the surface tension renders stability to the flow under long wavelength approximation. This result confirms the earlier findings of the stabilizing role of surface tension. Further, $\mathrm{Re}_{\mathrm{linear}}$ depends on power-law index n. It is clear from Fig. 2 that the Reynolds number $\mathrm{Re}_{\mathrm{linear}}$ increases with the power-law index n. In other words, increase of non-Newtonian character stabilizes the flow. For example, if the flow of Polystyrene at 422 K ($n = 0.4$) is stable, then the flow of 3% Polyisobutylene in decalin ($n = 0.77$) is also stable but the reverse is not true. For large k, the asymptotic amplification rate is

$$\omega_i^+ \sim -\frac{1}{2}n + \frac{b}{2\sqrt{a}} \sim -\frac{1}{2}n \tag{49}$$

ensuring stability while the phase speed is

$$\frac{\omega_r^{\pm}}{k} \sim \pm k\sqrt{\mathrm{We}}. \tag{50}$$

In the neutral state $\omega_i = 0$ gives

$$c_0 = \frac{1+2n}{n}. \tag{51}$$

It is to be noted here that for Newtonian fluid Yih (1963), Benjamin (1957) have found the phase speed equal to 3. It is clear from Fig. 3 that c_0 increases as n decreases.

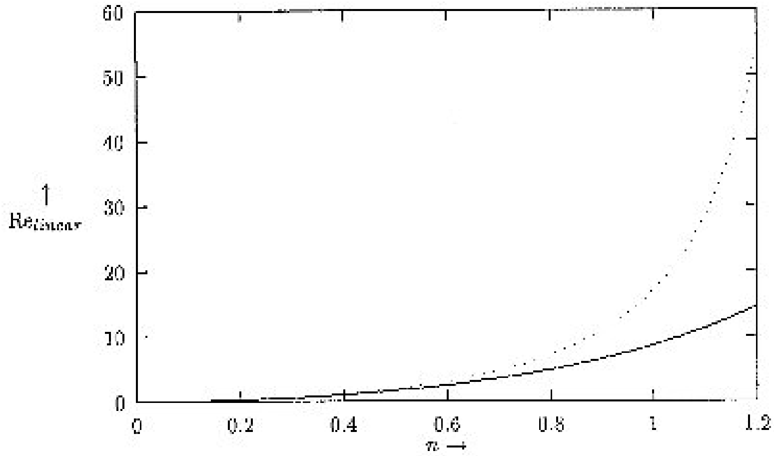

Fig. 2. Effect of surface tension on the variation of $\mathrm{Re}_{\mathrm{linear}}$ with n for fixed wave number $k = 0.05$ and $\theta = 10^{\circ}$. Solid and dotted lines denote We = 400 and 800, respectively.

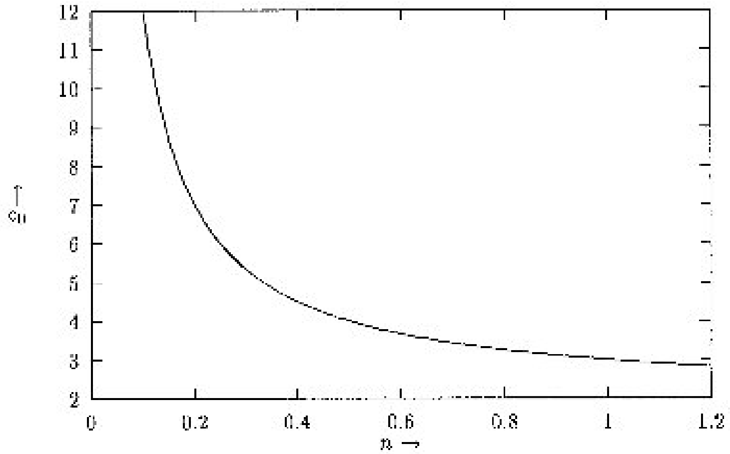

Fig. 3. Variation of the phase velocity c_0 with respect to power-law index n.

4. Derivation of the two-wave equation

To study the slightly non-linear waves, let us assume

$$h = 1 + H(x,t), \quad q = 1 + Q(x,t), \quad H, Q \ll 1, \tag{52}$$

where H and Q are dimensionless perturbations of the film thickness and flow rate, respectively. Substituting Eq. (52) into Eqs. (33) and (34) and retaining the terms up to second-order fluctuations, the continuity and momentum equations reduce to

$$H_t + Q_x = 0, \tag{53}$$

$$Q_t + 2\beta Q_x + (\alpha - \beta)H_x - \frac{1}{\gamma\varepsilon}[(2n+1)H - nQ] - \mathrm{We}\varepsilon^2 H_{xxx}$$
$$= \frac{n(1+2n)}{\gamma\varepsilon}H^2 - 2nHQ_t - 2\beta[QQ_x + (2n-1)HQ_x - QH_x]$$
$$- [2\beta(1-n) + \alpha(2n+1)]HH_x + \frac{n(1-n)}{2\gamma\varepsilon}Q^2 + \mathrm{We}\varepsilon^2(2n+1)HH_{xxx}. \quad (54)$$

Eqs. (53) and (54) can be expressed into a single equation for the film height disturbance H by differentiating Eq. (54) with respect to x and eliminating Q and its derivative according to the following procedure described below:
(i) To eliminate the linear derivative of Q use Eq. (53) and for
(ii) the non-linear terms, approximation methods of quasistationary process is to be used. Alekseenko et al. (1985) have used this method for a vertical film. In this method, the basic assumption used is in conformation with the experimental observation that the waves generally evolve in shape rather slowly with the downstream distance. In effect, this procedure limits the ability of the equation to describe the behaviour of very rapidly growing or decaying waves (Jurman and McCready, 1989). Following Alekseenko et al. (1979), we assume the system of coordinate moving with velocity c, which allows the coordinate transformation $(t,x) \to (t, \xi = x - ct)$. It is further assumed that the phase velocity c is approximately constant for quasistationary waves in the interval Δt. Under this transformation, Eq. (53) gives

$$H_t - cH_\xi + Q_\xi = 0. \quad (55)$$

The wave profile in a moving coordinate system is deformed slightly in the quasistationary process, this approximates Eq. (55) as $cH_\xi = Q_\xi$ from which relations are obtained

$$Q = cH, \quad (56)$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}. \quad (57)$$

After using rule (i) and substituting relations (56) and (57), where needed, into Eq. (54), we get

$$(H_t + c_0 H_x) + \frac{\gamma\varepsilon}{n}(\partial_t + c_1\partial_x)(\partial_t + c_2\partial_x)H + \frac{\gamma \mathrm{We}}{n}\varepsilon^3 H_{xxxx}$$
$$= -\left[(1+2n) + \frac{1-n}{2}c^2\right](H^2)_x + \frac{2\gamma\varepsilon}{n}(\beta - n)(HH_t)_t + \frac{4(1-n)}{n}\beta\gamma\varepsilon(HH_x)_t$$
$$+ \frac{\gamma\varepsilon}{n}[2\beta(1-n) + \alpha(1+2n)](HH_x)_x - \frac{1+2n}{n}\mathrm{We}\gamma\varepsilon^3(HH_{xxx})_x, \quad (58)$$

where c_0 as in Eq. (51) and

$$c_{1,2} = \beta \pm \sqrt{\beta^2 - \beta + \alpha}. \quad (59)$$

It should be noted here that weakly non-linear waves are small in curvature, therefore, the contribution from the higher order derivatives of the quadratic terms on the r.h.s. of Eq. (58) are very small and hence may be neglected. Therefore, Eq. (58) consists of a two-wave structure which reveals that two-wave processes occur simultaneously on the thin liquid film. They are according to Whitham

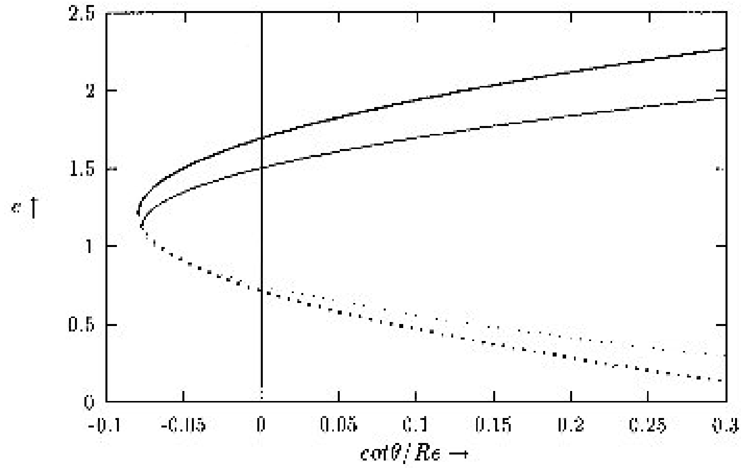

Fig. 4. Variation of the wave velocities $c_{1,2} - vs - \cot\theta/\mathrm{Re}$ with power-law index n. Solid lines correspond to c_1 and dotted lines to c_2. Bold and faint lines for both cases correspond to $n = 1$ and 0.4, respectively.

(1974) (i) *Kinematic waves*: These are the lower order waves with characteristic velocity c_0. These waves are non-dispersive and are expected to be a low frequency disturbance. These waves are responsible for the transport of fluids. (ii) *Dynamic waves*: These are higher order waves with characteristic velocities approximated by c_1 and c_2. These waves are dispersive, their speeds in general depend on fluid inertia, gravity and surface tension. No net transport of the fluid is associated with the motion of these types of waves. On the other hand, these waves may be called inertial waves, since Eq. (58) has appeared due to the inertial term of the Navier–Stokes equation. It can be seen from Fig. 4 that c_1 increases while c_2 decreases with the increase of either $\cot\theta/\mathrm{Re}$ or n.

Inspecting the non-linear wave equation (58), it can be seen that the kinematic waves, associated with the first order terms, dominate the wave field for $\mathrm{Re} \sim 1$, while for large Reynolds number $\mathrm{Re} \sim \frac{1}{\varepsilon^2} \gg 1$, the waves of higher order dominate. In ranges where one of these processes controls the type of waves and its behaviour, hence for easier analysis, Eq. (58) is linearized since analytic solutions are rarely available for the full non-linear system. On the other hand, linear results may be used to infer the corresponding non-linear behaviour of the various waves. To do so, the physical processes of the two distinct wave equations (58) will be reduced following the method of Whitham (1974). In this process of reduction, the dominant wave type is first determined from the relative orders of the parameters of the original wave equation. For Eq. (58), the parameters are considered $\mathrm{Re} \sim 1, \mathrm{We} \sim 1/\varepsilon^2, \cot\theta \leqslant 1$. Under this approximation, Eq. (58) reduces to

$$H_t + c_0 H_x = 0$$

which describes the kinematic wave.

In the next approximation (order of ε) the insensitive time derivative i.e., those that do not correspond to the dominant wave process, are replaced by $\partial/\partial t = -c_0 \partial/\partial x$, where c_0 is the approximate speed of the wave and is used to eliminate the time scale by which the overall equation is governed. To study the effect of different waves, orders of different parameters are considered.

4.1. Case-I: Small flow rate; Kinematic waves, for $\mathrm{Re} \sim 1, \mathrm{We} \sim \varepsilon^{-2}$, $\cot\theta \leqslant 1$

In this limit, the kinematic waves are expected to dominate the wave field. To get the reduction equation as outlined above following Whitham (1974), the time derivatives of higher order terms are replaced by $-c_0\partial/\partial x$ in Eq. (58), the equation then reduces to

$$\begin{aligned} &H_t + c_0 H_x + \frac{\gamma\varepsilon}{n}(c_1 - c_0)(c_2 - c_0)H_{xx} + \frac{\mathrm{We}\gamma\varepsilon^3}{n}H_{xxxx} + [2(1+2n) - c^2(1-n)]HH_x \\ &\quad + \frac{\gamma\varepsilon}{n}[4\beta c_0(1-n) - 2(\beta - n)c_0^2 - 2\beta(1-n) - \alpha(1+2n)](HH_x)_x \\ &\quad + \frac{1+2n}{n}\gamma\mathrm{We}\varepsilon^3(HH_{xxx})_x = 0, \end{aligned} \tag{60}$$

where the non-linear quadratic terms with higher order derivatives in x may have been neglected on the basis of weak non-linearity, but have been retained here for the sake of completeness. It would be better to take a note on the contribution of different terms of Eq. (60), although the following conclusions are valid for the entire range of Re. In general, odd-order spatial derivatives contribute to the celerity of the waves. Due to the presence of viscosity, there exists a mechanism for energy pumping from the mean flow to the perturbations, as a result, the second order derivatives pump energy to the perturbation which results in the instability but the fourth order derivative containing surface tension term introduces dissipative effects resulting in stability. The non-linear HH_x term causes an asymmetric sharpening of the peak to the steeper front and more shallow back as is observed for solitary waves, while the non-linear H_x^2 term contributes to the symmetric growth of the peak. On the other hand, both allow for weak interaction between modes. To obtain a better understanding, linear stability analysis is needed.

4.1.1. Linear stability analysis for the kinematic waves

In this section, we shall examine the linear response for a sinusoidal perturbation of the film by assuming a perturbation of the form

$$H = \delta \exp[\mathrm{i}(k\tilde{x} - \omega\tilde{t})], \tag{61}$$

where $\omega(=\omega_r + \mathrm{i}\omega_i)$ is the complex wave speed and the amplitude δ is real. We shall first use the transformation $x = \varepsilon\tilde{x}$ and $t = \varepsilon\tilde{t}$ in Eq. (60) and then use Eq. (61) on the linearized part which gives

$$-\mathrm{i}\omega + \mathrm{i}kc_0 - \frac{\gamma k^2}{n}[c_0^2 - 2\beta c_0 + \beta - \alpha] + \frac{\gamma\mathrm{We}}{n}k^4 = 0.$$

Equating the real and imaginary parts, we get

$$\omega_r = c_0 k$$

and

$$\omega_i = \frac{\gamma k^2}{n}(c_0^2 - 2\beta c_0 + \beta - \alpha) - \frac{\gamma\mathrm{We}}{n}k^4. \tag{62}$$

Hence the phase velocity

$$c = \frac{\omega_r}{k} = c_0 = \frac{2n+1}{n}$$

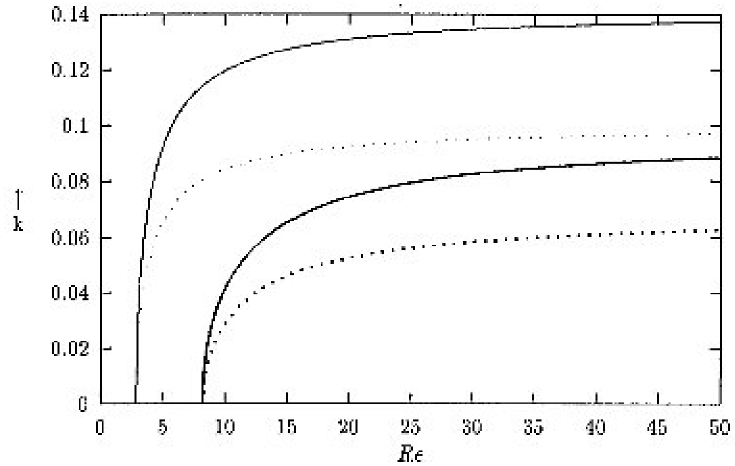

Fig. 5. Variation of the neutral curve and line of maximum growth with power-law index n. Solid lines and $k=0$ line denote neutral curve and dotted lines indicate the line of maximum growth. Bold and faint lines for both cases correspond to $n=0.85$ and 0.5, respectively.

is obtained. This shows that the phase velocity is independent of the wave number k, implying non-dispersive waves. But, ω_i is different from zero containing two summands which appeared due to second and fourth derivatives present in Eq. (60). It is clear that the second term which is related to surface tension is always negative and results in the attenuation of perturbations implying dissipation. On the other hand, the first term may have any sign. When $\alpha > (c_0^2 - 2\beta c_0 + \beta)$, then this term also contributes to dissipation. But for $k \neq 0$, if

$$\alpha < (c_0^2 - 2\beta c_0 + \beta) - \mathrm{We}k^2,$$

then the flow becomes unstable. In other words, if

$$\mathrm{Re} > \mathrm{Re}_{\mathrm{linear}} = n\left(\frac{n}{1+2n}\right)^{1-n} \cot\theta \left[1 - \frac{\mathrm{We}n^2k^2}{1+2n}\right]^{-1}, \tag{63}$$

then the perturbation grows resulting in instability. It should be pointed out here for We $=0$, that this result was obtained by Ng and Mei (1994) and it agrees with our earlier result equation (48) for stability criterion. This shows that the second derivative yields the energy pumping into the perturbation causing the instability while the fourth derivative term describes the dissipation effects.

For neutral perturbations ($\omega_i = 0$) gives two relations

$$k = 0, \tag{64a}$$

$$k_N = \left[\left(\frac{1+2n}{n^2} - \alpha\right)\frac{1}{\mathrm{We}}\right]^{\frac{1}{2}}. \tag{64b}$$

This shows two branches of the neutral curve and the flow instability takes place between them. They are depicted in Fig. 5 for We $=400$, $\theta = 5^\circ$ with different values of n. The minimum Re, at which instability sets in may be denoted as critical Reynolds number Re_c for wave formation and

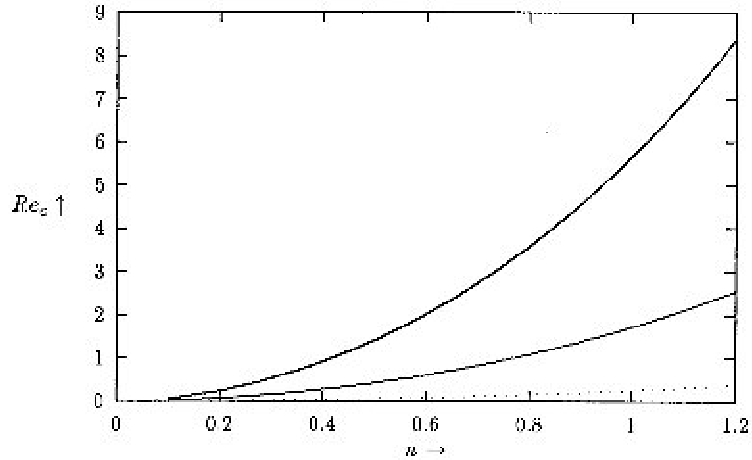

Fig. 6. Variation of critical Reynolds number $\mathrm{Re_c}$ with power-law index n at different angles of inclination θ. Bold, slim and dotted lines are for $\theta = 10^\circ$, 30° and 75°, respectively.

obtained from Eq. (63) as

$$\mathrm{Re_c} = n\left(\frac{n}{1+2n}\right)^{1-n}\cot\theta. \tag{65}$$

It is clear from Fig. 5 that at this value of $\mathrm{Re_c}$, two neutral curves bifurcate. The wave number of the waves with maximum growth is obtained from the relation $\mathrm{d}\omega_i/\mathrm{d}k = 0$, gives

$$k_m = \left[\left(\frac{1+2n}{n^2} - \alpha\right)\frac{1}{2\mathrm{We}}\right]^{1/2} = \frac{k_N}{\sqrt{2}}, \tag{66}$$

where k_N is given by formula (64b). Further, it is shown in Fig. 6 that a flow which is stable for Newtonian fluid may become unstable for shear-thinning ($n < 1$) fluid or a stable shear-thickening ($n > 1$) fluid flow loses its stability if the fluid is replaced by Newtonian fluid.

4.2. Case-II: High flow rate; Dynamic or inertial waves

This section will deal with the waves of higher order that dominate in the range of large Reynolds numbers $\mathrm{Re} \sim 1/\varepsilon^2 \gg 1$. In this range, dynamic or inertial waves have a controlling position over the kinematic waves. Different limiting cases are considered depending on the relative orders of magnitude of the parameter We and the angle of inclination θ.

4.2.1. Case-(i): $\mathrm{Re} \sim \frac{1}{\varepsilon^2}$, $\mathrm{We} \sim \frac{1}{\varepsilon}$, $\cot\theta \leqslant 1$

Under this limit, Eq. (58) will be controlled by the dynamic or inertial wave field. Retaining the linear terms of the order of $1/\varepsilon$ as the first basic approximation, Eq. (58) reduces to

$$\left(\frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x}\right)H = 0. \tag{67}$$

The equivalent forms of the above equation are

$$\left(\frac{\partial}{\partial t}+c_1\frac{\partial}{\partial x}\right)H=0, \tag{68}$$

$$\left(\frac{\partial}{\partial t}+c_2\frac{\partial}{\partial x}\right)H=0 \tag{69}$$

describing the propagation of the travelling waves in the mean flow direction with velocities c_1 and c_2, given in Eq. (59) above. It is clear from Eqs. (68), (69) and (59) that the first and second wave propagates faster and slower, respectively, than the mean flow. The factorization of the classical wave equation results in two waves moving in opposite directions with the same velocity. The same result may be obtained if systems (68) and (69) are transformed through the system of coordinates moving with velocity $\frac{(c_1+c_2)}{2}$ and it results in

$$\frac{\partial H}{\partial t}+\frac{(c_1-c_2)}{2}\frac{\partial H}{\partial \xi}=0,$$

$$\frac{\partial H}{\partial t}-\frac{(c_1-c_2)}{2}\frac{\partial H}{\partial \xi}=0,$$

where $\xi=x-(c_1+c_2)t/2$.

Following the procedure described above, the time derivatives in Eq. (58) are replaced by the relation $\partial/\partial t=-c_1\partial/\partial x$, except for, naturally, the operator $\partial/\partial t+c_1\partial/\partial x$. The time scale c_1 is chosen because it corresponds to the wave in the direction of shear and should be the primary disturbance. The resulting equation after integrating once with respect to x yields

$$\begin{aligned}H_t+c_1H_x-\frac{n}{\gamma\varepsilon}\frac{c_0-c_1}{c_1-c_2}H-\frac{\mathrm{We}\varepsilon^2}{c_1-c_2}H_{xxx}-\left[(1+2n)+\frac{1-n}{2}\right]\frac{n}{\gamma\varepsilon(c_1-c_2)}H^2\\+\left[\frac{(1+2n)\alpha+2(1-n)\beta(1-2c_1)+2(\beta-n)c_1^2}{c_1-c_2}\right]HH_x-\frac{1+2n}{c_1-c_2}\mathrm{We}\varepsilon^2HH_{xxx}=A(t).\end{aligned} \tag{70}$$

Assuming the amplitude of thickness perturbation $\delta\sim\varepsilon$ and keeping the terms up to $O(\varepsilon)$, for periodic stationary waves $(A(t)=0)$ one obtains

$$\begin{aligned}H_t+c_1H_x-\frac{n(c_0-c_1)}{\gamma\varepsilon(c_1-c_2)}H\\+\frac{(1+2n)\alpha+2\beta(1-n)(1-2c_1)+2(\beta-n)c_1^2}{(c_1-c_2)}HH_x-\frac{\varepsilon^2\mathrm{We}}{(c_1-c_2)}H_{xxx}=0.\end{aligned} \tag{71}$$

It can be shown that the third term on the left-hand side is always negative for $\mathrm{Re}\sim 1/\varepsilon^2$ and $\cot\theta\leqslant 1$.

4.2.2. Linear stability analysis for the dynamic/inertial waves

In this section, we are interested in understanding the contribution of separate terms to the formation of the wave process (71). To achieve this goal, linear stability analysis is being performed by assuming the perturbation of the form

$$H=\delta\exp[\mathrm{i}(kx-\omega t)], \tag{72}$$

where $\omega(=\omega_r+\mathrm{i}\omega_i)$ is the complex wave speed and the amplitude δ is real. Use of Eq. (72) in Eq. (71) gives

$$-\mathrm{i}\omega+\mathrm{i}kc_1+\frac{n}{\gamma\varepsilon}\frac{c_0-c_1}{c_1-c_2}+\mathrm{i}k^3\frac{\mathrm{We}\varepsilon^2}{c_1-c_2}=0.$$

By equating the real and imaginary parts, we get

$$\omega_r=c_1k+\frac{\mathrm{We}\varepsilon^2}{c_1-c_2}k^3, \tag{73}$$

$$\omega_i=\frac{n}{\gamma\varepsilon}\frac{c_0-c_1}{c_1-c_2} \tag{74}$$

and the phase speed

$$c=c_1+\frac{\mathrm{We}\varepsilon^2}{c_1-c_2}k^2. \tag{75}$$

It is clear from Eq. (75) that the surface tension yields dispersion in this case. The third term in Eq. (71) comes from the contribution of the kinematic waves and it leads to ω_i (Eq. (74)) which is always positive, imparting instability to the film flow. Thus, it can be concluded that this term whose appearance is due to the kinematic waves is responsible for a low frequency energy pumping resulting in instability to film flow at high Reynolds number. Under this approximation

$$c_1=\beta+\sqrt{\frac{2n(2n+1)}{(2+3n)^2}}+O(\varepsilon^2),\quad c_2=\beta-\sqrt{\frac{2n(2n+1)}{(2+3n)^2}}+O(\varepsilon^2)$$

make the above observation of energy transfer always possible while kinematic waves interact with the dynamic waves. Hence for instability, one obtained from Eq. (74), as

$$\mathrm{Re}>n\left(\frac{n}{1+2n}\right)^{1-n}\cot\theta. \tag{76}$$

A general comment on the wave process described by Eq. (58) can be noted as follows. The lower order waves (kinematic waves) obtain energy from the mean flow through the wave mechanism of higher order and control the process with small Reynolds number. On the other hand, higher order waves (dynamic waves) dominate the mechanism with high Reynolds number and obtain energy from the kinematic wave process. The surface tension plays a double role. For the first case, it exerts dissipative effects or in other words it stabilizes the flow, so that a finite-amplitude case may be established, but for the second case it yields dispersion.

4.2.3. Case-(ii) $\mathrm{Re}\sim 1/\varepsilon^2$, $\mathrm{We}\sim 1/\varepsilon^2$, $\cot\theta\leqslant 1$

At this order of approximations, Eq. (58) will reduce to the form

$$(\partial_t+c_1\partial_x)(\partial_t+c_2\partial_x)H+\varepsilon^2\mathrm{We}H_{xxxx}=0. \tag{77}$$

In deriving Eq. (77), it was assumed that the amplitude of the thickness perturbation $\delta\sim\varepsilon$. To obtain the dispersion relation, thickness perturbation is assumed as

$$H=\delta\exp[\mathrm{i}(k\tilde{x}-\omega\tilde{t})],$$

where $\omega(=\omega_r+\mathrm{i}\omega_i)$ is the complex wave speed, the amplitude δ is real and transformed to $x=\varepsilon\tilde{x}$ and $t=\varepsilon\tilde{t}$. The dispersion relation becomes

$$-\omega^2+2\beta k\omega-(\beta-\alpha)k^2+\mathrm{We}k^4=0.$$

By equating real and imaginary parts, we get

$$\omega_i=0,$$

$$\omega_r=\beta k\pm k\sqrt{\beta^2-\beta+\alpha+\mathrm{We}k^2}$$

and the phase speed

$$c=\beta\pm\sqrt{\beta^2-\beta+\alpha+\mathrm{We}k^2}. \tag{78}$$

It is thus at this order of approximations of the parameters that the exchange of stability takes place and the wave is dispersive.

4.3. *Case-III: Moderate flow rate:* $\mathrm{Re}\sim 1/\varepsilon$, $\mathrm{We}\sim 1/\varepsilon^2$, $\cot\theta\leqslant 1$

At this approximation, the linearized form of Eq. (58) reduces to

$$(\partial_t+c_0\partial_x)H+\frac{\varepsilon\gamma}{n}(\partial_t+c_1\partial_x)(\partial_t+c_2\partial_x)H+\frac{\varepsilon^3\gamma\mathrm{We}}{n}H_{xxxx}=0. \tag{79}$$

To study the stability of this film flow on the basis of the two-wave equation (79), introduce the time varying perturbations of the film height

$$H=\delta\exp[\mathrm{i}k(\tilde{x}-c\tilde{t})+\lambda\tilde{t}]. \tag{80}$$

Here, $\tilde{x}=x/\varepsilon$, $\tilde{t}=t/\varepsilon$, k is the real wave number, c is the real part of the phase velocity and λ is an increment (the imaginary part of the frequency). Using Eq. (80) in Eq. (79), we get, after equating real and imaginary parts of the dispersion relation,

$$\lambda\mathrm{Re}=-\frac{n}{2}\left(\frac{1+2n}{n}\right)^n\frac{c-c_0}{c-\beta}, \tag{81}$$

$$\lambda+\frac{\gamma}{n}\lambda^2-\frac{\gamma}{n}[c^2-2\beta c+\beta-\alpha]k^2+\frac{\gamma}{n}\mathrm{We}k^4=0. \tag{82}$$

Elimination of λ from Eq. (82) by using Eq. (81) gives a quadratic relation for $(k\mathrm{Re})^2$ as

$$(k\mathrm{Re})^4-\frac{\mathrm{Re}^2}{\mathrm{We}}[(c-c_1)(c-c_2)](k\mathrm{Re})^2$$
$$-\frac{\mathrm{Re}^2}{\mathrm{We}}\left(\frac{n}{2}\right)^2\left(\frac{1+2n}{n}\right)^{2n}\left(\frac{c-c_0}{c-\beta}\right)\left(\frac{c+c_0-2\beta}{c-\beta}\right)=0. \tag{83}$$

The solution of Eq. (83) reduces to

$$(\mathrm{Re}k)^2=\frac{\mathrm{Re}^2}{2\mathrm{We}}(c-c_1)(c-c_2)$$
$$\left[1\pm\sqrt{1+\frac{\mathrm{We}}{\mathrm{Re}^2}n^2\left(\frac{1+2n}{n}\right)^{2n}\frac{(c-c_0)(c+c_0-2\beta)}{(c-c_1)^2(c-c_2)^2(c-\beta)^2}}\right]. \tag{84}$$

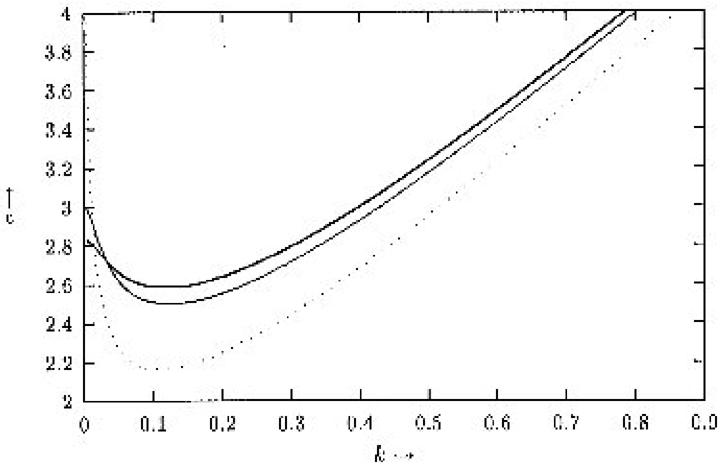

Fig. 7. Variation of dispersion curve for different values of n. Bold, slim and dotted lines correspond to $n=1.2$, 1 and 0.4 respectively.

From relations (81) and (84), we can find that on the neutral curve ($\lambda=0$), the phase velocity $c=c_0$, which gives

$$k=0,$$

$$k=\left[\left(\frac{1+2n}{n}-\alpha\right)\Big/\mathrm{We}\right]^{1/2}. \tag{85}$$

It should be pointed out here that relation (85) was obtained earlier in connection with the kinematic waves (case-I, Eq. (64)). It is further clear that the perturbations will decay so long as $c>c_0$. But for the growth of perturbations, the phase velocity c must lie in $\beta<c<c_0$. It is evident from Fig. 7 that dispersion curves for growing waves $c<c_0$ have points of local velocity minimum for which growth rate λ becomes maximum. Further, λ depends on the power-law index n of the fluid.

So far, we have discussed when the angle of inclination is large enough or near to vertical. It may be of interest to study the cases for small θ or when inclination is near to horizontal.

4.4. *Case-IV: At moderate flow rate with very small angle of inclination with the horizon:* $\mathrm{Re}\sim\frac{1}{\varepsilon}$, $\mathrm{We}\sim\frac{1}{\varepsilon^3}$, $\cot\theta\sim\frac{1}{\varepsilon^3}$

In this approximation, the wave speed $c_{1,2}$ given in Eq. (59) will be modified as

$$c_{1,2}\sim\beta\pm N,$$

where

$$N=\left(\frac{1+2n}{n}\right)^{n/2}\sqrt{\frac{\cot\theta}{\mathrm{Re}}}\sim O(1/\varepsilon), \tag{86}$$

is the dimensionless velocity of gravitational waves on the surface of a thin layer. Since the characteristic time of the system depends on the wave velocity, so the time derivative will have the order of $1/\varepsilon$ as N has. This is due to the fact that the time is made dimensionless through the film velocity. In this relation, Eq. (58) may be approximated by considering the order of magnitude of each term as

$$
\begin{aligned}
&(\partial_t + c_0\partial_x)H + \frac{\varepsilon\gamma}{n}[(\partial_t + N\partial_x) + \beta\partial_x][(\partial_t - N\partial_x) + \beta\partial_x]H + \frac{\varepsilon^3\gamma \mathrm{We}}{n}H_{xxxx} + \left[(1+2n) + \frac{1-n}{2}c^2\right](H^2)_x \\
&(1/\varepsilon \quad 1) \qquad 1[(1/\varepsilon \quad 1/\varepsilon) \quad 1 \quad][1/\varepsilon \quad 1/\varepsilon \qquad 1)] \qquad\qquad 1/\varepsilon \qquad [1 \qquad\qquad\qquad] \quad \delta \\
&\qquad - \frac{4(1-n)}{n}\beta\gamma\varepsilon(HH_x)_t - \frac{2\varepsilon\gamma}{n}(\beta - n)(HH_t)_t - \frac{\varepsilon\gamma}{n}[2\beta(1-n) + \alpha(1+2n)](HH_x)_x \\
&\qquad\qquad 1 \quad (1/\varepsilon) \quad \delta \qquad\qquad 1 \qquad 1/\varepsilon^2 \quad \delta \qquad 1\,[\qquad 1 \qquad 1/\varepsilon^2 \qquad] \quad \delta \\
&\qquad\qquad\qquad + \left(\frac{1+2n}{n}\right)\mathrm{We}\gamma\varepsilon^3(HH_{xxx})_x = 0. \\
&\qquad\qquad\qquad\qquad\qquad 1/\varepsilon \qquad \delta
\end{aligned}
\tag{87}
$$

Assuming the amplitude of perturbations $\delta \sim \varepsilon$ and keeping the terms of order $1/\varepsilon^2$, Eq. (87) reduces as the factorized equation for gravitational waves on shallow water

$$(\partial_t + N\partial_x)(\partial_t - N\partial_x)H = 0. \tag{88}$$

Following the procedure as outlined above, the time derivative in Eq. (87) are replaced by the relation $\partial/\partial t = -N\partial/\partial x$ and keeping the terms up to the order $1/\varepsilon$, one derives a non-linear equation

$$H_t + NH_x + \frac{n}{2\gamma\varepsilon}H + \frac{N}{2}(1+2\beta)HH_x - \frac{\varepsilon^2 \mathrm{We}}{2N}H_{xxx} + \beta H_x = 0. \tag{89}$$

If the system of coordinate is transformed for a moving coordinate with velocity β then the Eq. (89) reduces to

$$H_t + NH_\xi + \frac{n}{2\gamma\varepsilon}H + \frac{N}{2}(1+2\beta)HH_\xi - \frac{\varepsilon^2 \mathrm{We}}{2N}H_{\xi\xi\xi} = 0. \tag{90}$$

It can be noted here that Eq. (90) represents for dynamic (gravitational) waves on a slightly inclined film and having analogous structure to Eq. (71) for dynamic (inertial) waves on a near to vertical film. The basic difference between two Eqs. (71) and (90) is the sign in front of the third term on l.h.s which produces energy pumping for the inertial waves and attenuation for gravitational waves.

4.5. Case-V: At high flow rate with very small angle of inclination with the horizon

One may be tempted to know what type of wave may occur during high flow rate with very small angle of inclination when Weber number changes.

4.5.1. Moderately high Weber number: $\mathrm{Re} \sim 1/\varepsilon^2$, $\mathrm{We} \sim 1/\varepsilon^2$, $\cot\theta \sim 1/\varepsilon^4$

Although the values of Reynolds number Re and $\cot\theta$ are changed, but as stated earlier, $N \sim O(1/\varepsilon) \gg 1$ is maintained. Therefore, in the first approximation one obtains from Eq. (87) the gravitational waves described by the same factorized equation (88). But in the next approximation along with $\delta \sim \varepsilon$, we obtained

$$H_t + NH_x + \frac{N}{2}(1+2\beta)HH_x + \beta H_x = 0, \tag{91}$$

instead of Eq. (89). Using

$$(N+\beta)+\frac{N(1+2\beta)}{2}H=\zeta(x,t)$$

in Eq. (91), we get

$$\zeta_t+\zeta\zeta_x=0. \tag{92}$$

It is interesting to note that Eq. (92) represents a forward breaking wave. Propagation of surface wave speed in Eq. (92) is $c(\zeta)=\zeta$. Since $\mathrm{d}c/\mathrm{d}\zeta>0$, the velocity of propagation will be faster for higher values of ζ. This fact may cause steepening of the surface profile in the course of time.

4.5.2. High Weber number: $\mathrm{Re}\sim 1/\varepsilon^2$, $We\sim 1/\varepsilon^3$, $\cot\theta\sim 1/\varepsilon^4$

Following the procedure as outlined above, one may obtain the non-linear equation up to ε order terms as

$$H_t+(N+\beta)H_x+\frac{N}{2}(1+2\beta)HH_x-\frac{\varepsilon^2\mathrm{We}}{2N}H_{xxx}=0. \tag{93}$$

Setting

$$(N+\beta)+\frac{N(1+2\beta)}{2}H=\zeta(x,t) \tag{94}$$

in Eq. (94) one gets

$$\zeta_t+\zeta\zeta_x-\frac{\varepsilon^2\mathrm{We}}{2N}\zeta_{xxx}=0. \tag{95}$$

It is to be noted here that Eq. (95) is a standard KdV equation with a negative sign in front of the dispersion term indicating the existence of dark soliton.

5. Results and discussion

Fig. 2 shows the variation of Reynolds number Re with power-law index n for fixed Weber number and angle of inclination. It is clear that Re increases with n indicating that, slightly non-Newtonian (small values of $n<1$) fluids are more unstable than the highly non-Newtonian (large values of n) fluids. Further, the surface tension plays the role of a stabilizing agent. Fig. 3 depicts the variation of phase speed c_0 with power-law index n. It is evident that phase speed decreases as the non-Newtonian grade of the fluid increases. Using long wavelength approximation, Eq. (58) is derived from Eqs. (33) and (34). It should be emphasized here that Eq. (58) is capable of describing the behaviour of any set of flow rates with non-Newtonian flow properties where the dominant waves are predominantly two dimensional and weakly non-linear. Fig. 4 describes the variation of the velocities of dynamic waves with $\cot\theta/\mathrm{Re}$ for different values of the power-law index n. It is clear that the non-Newtonian character of the fluid has a profound influence on these wave velocities. Fig. 5 represents the graphs of Eqs. (64) and (66) for different values of n. It further indicates the critical Reynolds number where $k=0$ and the solid line touches the Re-axis. It also depicts the line of maximum growth and clearly demonstrates that slightly non-Newtonian fluids are more unstable then highly non-Newtonian fluids. Fig. 6 shows the effects of the angle of inclination on the variation of critical Reynolds number Re_c

with power-law index n. It is clear from the figure that the inclination hastens the instability to the flow. These results confirms the earlier findings on the stability of thin film on an inclined plane for Newtonian or other types of non-Newtonian fluids. Fig. 7 represents the variation of dispersion curve for different values of n. It shows that the phase speed initially decreases with increase of the wave number k, attains a minimum and then increases with k. Further, it is clear from the figure that the least value of c can be obtained only with the least value of n

6. Conclusion

In this section, we shall summarize some of the results of this study. We have analysed the waves that occur at the surface of a thin power-law fluid film flowing down an inclined plane. To do this, we have derived an evolution equation representing two waves equations under long wave approximations. Based on the different ranges of the physical parameters, it is shown that different types of waves are possible on the surface of the film. They are

1. Kinematic waves, $\mathrm{Re} \sim 1$, $\mathrm{We} \sim 1/\varepsilon^2$, $|\cot\theta| \leqslant 1$;

$$\left(\frac{\partial}{\partial t} + c_0\frac{\partial}{\partial x}\right) H = 0.$$

2. Inertial waves, $\mathrm{Re} \sim 1/\varepsilon^2$, $\mathrm{We} \sim 1/\varepsilon, |\cot\theta| \leqslant 1$;

$$\left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + M\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} - M\frac{\partial}{\partial x}\right) H = 0,$$

where $M = \sqrt{2n(1+2n)/(2+3n)^2}$.

3. Gravitational waves, $\mathrm{Re} \sim 1/\varepsilon$, $\mathrm{We} \sim 1/\varepsilon^3, \cot\theta \sim 1/\varepsilon^3$;

$$\left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - N\frac{\partial}{\partial x}\right) H = 0,$$

where $N = \sqrt{\{(1+2n)/n\}\cot\theta/\mathrm{Re}}$. The last two types of waves are known as dynamic waves.

At the next approximation in terms of the aspect ratio ε for a dominating type of waves, we have shown that the results of the interaction with other types of waves are either the exchange of energy or dispersion effects. For example, at a small flow rate, kinematic waves dominate the flow field and it acquires energy from the mean flow, while, for high flow rate, inertial waves dominate and the energy comes from the kinematic waves. In both the cases, surface tension plays a double role, for kinematic wave process, it exerts dissipative effects so that a finite amplitude case may be established, but for a dynamic wave process, it yields dispersion. Further depending on the flow rate, values of Weber number and the angle of inclination, we have shown that the results of the interaction may lead either to forward breaking waves or solitary waves with dark soliton. It should be pointed out here that power-law index n plays a vital role in the wave mechanism. We therefore, summarize on the basis of the above analysis that the waves that occur on the surface of the film of power-law fluid down an inclined plane under long wave approximation is a result of non-linear interaction between kinematic, inertial and gravitational waves.

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References

Alekseenko, S.V., Nakoryakov, V.E., Pokusaev, B.G., 1979. Wave formation at liquid film flow on a vertical wall. Zhurn. Prikl. Mekh. Tekh. Fiz. 6, 77–87.

Alekseenko, S.V., Nakoryakov, V.E., Pokusaev, B.G., 1985. Wave formation on a vertical falling liquid film. A.I.Ch.E. J. 31, 1446–1460.

Alekseenko, S.V., Nakoryakov, V.E., Pokusaev, B.G., 1994. Wave flow of liquid films. Begell House, New York.

Andersson, H.I., Irgens, F., 1989. Film flow of power-law fluids. In: Cheremisinoff, N.P. (Ed.), Encyclopedia of Fluid Mechanics, Vol. 9. Gulf Publishing Company, Houston, TX.

Benjamin, T.B., 1957. Wave formation in laminar flow down an inclined plane. J. Fluid Mech. 2, 554–574.

Benney, D.J., 1966. Long waves on liquid films. J. Math. Phy. 45, 150–155.

Berezin, A.Yu., Hutter, K., Spodareva, L.A., 1998. Stability analysis of gravity driven shear flow with free surface for power-law fluids. Arch. Appl. Mech. 68, 169–178.

Dandapat, B.S., Gupta, A.S., 1978. Long waves on a layer of a visco-elastic fluid down an inclined plane. Rheol. Acta. 17, 492–499.

Dandapat, B.S., Gupta, A.S., 1997. Solitary waves on the surface of a visco-elastic fluid running down an inclined plane. Rheol. Acta. 36, 135–143.

Gjevik, B., 1970. Occurrence of finite amplitude surface waves on a falling liquid films. Phys. Fluid. 13, 1918–1925.

Gupta, A.S., 1967. Stability of a visco-elastic liquid film flowing down an inclined plane. J. Fluid Mech. 28, 17–28.

Hwang, C.C., Chen, J.L., Wang, J.S., Lin, J.S., 1994. Linear stability of power law liquid film flows down an inclined plane. J. Phys. D. Appl. Phys. 27, 2297–2301.

Jurman, L.A., McCready, M.J., 1989. Study of waves on thin liquid films sheared by turbulent gas flows. Phys. Fluids A 1, 522–536.

Kapitza, P.L., 1948. Wave flow of thin viscous liquid film. Zh. Exper. Teor. Fiz. 18, 3–28.

Krylov, V.S., Vorotylin, V.P., Levich, V.G., 1969. On theory of wave motion of thin liquid film. Teor. Osnovy Khim. Tekhn 3, 499–507.

Lai, W., 1967. Stability of elastico-viscous liquid film flowing down an inclined plane. Phys. Fluids 10, 844–847.

Lin, S.P., 1969. Finite amplitude stability of a parallel flow with a free surface. J. Fluid Mech. 36, 113–126.

Liu, K.F., Mei, C.C., 1989. Slow spreading of a sheet of Bingham fluid on an inclined plane. J. Fluid Mech. 207, 505–529.

Nakaya, C., 1975. Long waves on thin fluid layer flowing down an inclined plane. Phys. Fluids 18, 1407–1412.

Ng, C.O., Mei, C.C., 1994. Roll waves on a shallow layer of mud modelled as a power-law fluid. J. Fluid Mech. 263, 151–183.

Schlichting, H., 1968. Boundary layer theory, 6th Edition. McGraw-Hill, New York.

Shkadov, V.Ya., 1967. Wave regimes of thin layer flow of viscous fluid under the gravity effect. Izv. AN SSR. Mekh. Zhidk Gaza 1, 43–51.

Whitham, G.B., 1974. Linear and Nonlinear Waves. Wiley, New York.

Yih, C.S., 1963. Stability of liquid flow down an inclined plane. Phys. Fluids 6, 321–334.