Unsteady flow of thin liquid film on a disk under nonuniform rotation

B. S. Dandapat[a)]
Physics and Applied Mathematics Unit, Indian Statistical Institute, 203, B. T. Road, Calcutta-700 035, India

(Received 5 October 1999; accepted 19 March 2001)

Development of thin liquid film under nonuniform rotation has been studied numerically by using finite difference technique under the assumption of planar interface. For impulsive rotation of the disk an anomalous behavior of the rate of film thinning with the variation of the Reynolds number is observed in a different time zone. It is also shown that the azimuthal velocity field develops into the entire depth faster with the smaller impulsive rotation. A physical explanation for the above observation is provided. Further it is found that faster rate of thinning can be obtained if the spinner starts impulsively and then increases its spinning rate continuously.
Physics. [DOI: 10.1063/1.1377615]

I. INTRODUCTION

Free surface flow of a liquid film on a rotating disk is an interesting fluid mechanical problem. A number of forces like friction exerted by the disk, the centrifugal, and Coriolis forces due to rotation, the inertia of the existing fluid on the disk and the shear stress due to air/vapor flow at the free surface act on the system jointly to shape the flow structure. It is interesting to note that the flow of thin liquid film under the action of centrifugal force has not been studied as much as film flow due to the action of gravity. It is well known that the centrifugal force can be controlled to any desired level by regulating the spinning of the disk in a laboratory, on the other hand gravity driven flow has its own limitations. Still for an unknown reason not much attention has been paid in the literature to the former type of flow, although it has enormous technological applications. For example, the thin film produced on a rotating disk is used to promote the absorption of vapor into it. Specifically the absorber unit of a space-based vapor-absorption refrigeration system will use a liquid film thinned by the centrifugal force on a rotating disk to enhance the absorption of the refrigerant vapor into the absorbent. Since a falling film cannot be produced in a microgravity environment, the vapor-absorption cycle is more appropriate than a vapor-compression cycle for a microgravity application. Thin film on a rotating disk can also be used in coating the substrates in micro-electronics industry known as spin coating in the literature. The first theoretical study on hydrodynamical film flow on a rotating disk was made by Emslie *et al.*[1] and predicted the flow pattern as well as the thickness of the film. Later, a number of theoretical as well as experimental studies on the development of thin film on a rotating disk were presented in the pertinent literature (Acrivos *et al.*,[2] Washo,[3] Meyerhofer,[4] Givens and Daughton,[5] Lai,[6] Chen,[7] Jenekhe and Schuldt,[8] Sukanek,[9] Hwang and Ma,[10] and others) by considering different aspects of the process. Higgins[11] first considered the full Navier–Stokes equations and studied the flow problem from initial stage of development through matched asymptotic expansion procedure. Later on Dandapat and Ray[12–14] and Ray and Dandapat[15] extended the thin film development on a rotating disk to study analytically the effect of thermocapillarity and/or magnetic field on the rate of film thinning. In all these studies it is tacitly assumed that the disk is wet so that the classical no-slip boundary condition can be applied at every point along the disk surface. The assumption of wet wall is justified in the light of the experimental observation on advancing contact lines by Schwartz and Tejada[16] and Giradella and Radigan,[17] in which they have indicated the presence of an unseen precursor layer of fluid ahead of the contact line which is only an angstrom thick. Further it has been argued by physical chemists that the presence of a precursor layer is a very real phenomenon arising as a consequence of evaporation from the drop in a small region local to the contact line followed by diffusion and adsorption (De Gennes[18]). Another important class of problems viz., spreading of a liquid drop on a rotating disk in connection with the spin-coating is also studied by Troian *et al.*,[19] Melo *et al.*,[20] Moriarty and Schwartz,[21] and McKinley *et al.*[22] In these studies main interest was on how the contact line moves during the spreading of the drop on the disk and its stability. The present study is based on the former assumption of wet disk and we are interested to know the effects of three different types of nonuniform rotations of the disk for entire time scale. *The disk starts rotation (i) impulsively from rest and then maintains its speed for future time, (ii) starts impulsive rotation from rest and increases its angular speed continuously with time t, and (iii) starts from rest and increases its angular speed smoothly with time to attain a constant finite value as* $t\rightarrow\infty$. In general, the aim of this paper is to study the transient as well as steady behavior of the film development under different types of rotation. In particular, this study will address the question: If the disk starts from rest with impulsive rotation and maintains it for the rest of the period of rotation then *will the film thinning rate be more for the stronger rotational speed from the beginning of the spinning? If not, why?*

It may be pointed out here that other works on the unsteady development of thin film on a rotating disk are by Matsumoto *et al.*[23] and Wang *et al.*[24] Matsumoto *et al.*[23] considered a hemispherical liquid blob on the wetted surface of the disk and studied how this blob spreads at the initial stage of the rotation and the action of different forces like gravitational, Coriolis, centrifugal, etc., on film development. They have correctly mentioned that after 0.01 s from the start of the rotation only gravitational force dominates the film development as centrifugal force has very little effect at this stage. At time 0.03 s after the start of the rotation, the gravitational force is dominated by the centrifugal force. The liquid rotates similar to rigid rotation causing the uniformity of the film thickness at time lapse of 0.07 s from the start. This implies that the centrifugal force controls the film development process. Wang *et al.*[24] studied the fluid flow over an accelerating rotating disk. By using a special type of similarity transformation they were able to reduce the unsteady N–S equations to a set of nonlinear ODE. They further found the existence of nonunique solutions depending on film thickness over the rotating disk for the same value of the unsteady parameter $S \equiv \alpha/\Omega_0$, where both α and Ω_0 are positive with dimension $[T^{-1}]$. These constants are related to the unsteady acceleration of the disk given by $\Omega(t) = \Omega_0(1-\alpha t)^{-1}$. It is to be pointed out here that the work of Wang *et al.*[24] is a parallel study to our case (ii) for continuous increase of the angular speed after the impulsive start of the disk. It should be noted that the objective of the present study differs from that of Wang *et al.*[24] As stated earlier, we are more interested in studying the initial value problem to determine the unsteady development of velocity field, film thickness, etc., after the impulsive start of rotation of the disk and their corresponding changes with time if the angular speed increases continuously. Wang *et al.*[24] instead focused their attention on the effect of the unsteady parameter *S*, on the velocity field and the film thickness.

In Sec. II, mathematical formulation is done by using the Von-Kármán[25] similarity transformation on the equations of motion and the corresponding boundary conditions. These transformed equations are then expressed in suitable dimensionless form. The resulting set of partial differential equations in one space coordinate and time is solved numerically using finite-difference technique. The method of solution and the theoretical results and discussion on these are presented in Secs. III and IV. Section V contains conclusion on the study. It is to be noted here that the film height gradually decreases with time. This shows the physical domain varies with time. To fix the computational domain into a finite fixed region, a coordinate transformation has been used to solve the problem.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a film of viscous liquid of uniform thickness h_0 on a disk whose radius is large compared with the thickness of the film and rotates with a nonuniform angular velocity $\Omega(t)$. The origin is fixed at the center of the disk and the z-axis pointing vertically upward is the axis of rotation. Let (u,v,w) be the velocity components along (r, θ, z) directions of a cylindrical polar coordinate system. The axisymmetric equations of continuity and the Navier–Stokes equations are written as

$$\partial_r u + \left(\frac{u}{r}\right) + \partial_z w = 0, \tag{1}$$

$$\partial_t u + u\partial_r u - \left(\frac{v^2}{r}\right) + w\partial_z u = -\frac{1}{\rho}\partial_r p + \nu\left[\partial_{rr}u + \partial_r\left(\frac{u}{r}\right) + \partial_{zz}u\right], \tag{2}$$

$$\partial_t v + u\partial_r v + w\partial_z v + \left(\frac{uv}{r}\right) = \nu\left[\partial_{rr}v + \partial_r\left(\frac{v}{r}\right) + \partial_{zz}v\right], \tag{3}$$

$$\partial_t w + u\partial_r w + w\partial_z w = -\frac{1}{\rho}\partial_z p + \nu\left[\partial_{rr}w + \frac{1}{r}\partial_r w + \partial_{zz}w\right]. \tag{4}$$

Here p denotes the pressure. The suffix denotes the derivative with respect to the indicated variables.

Boundary conditions at $z=0$ *and* $z=h(t)$:

At the disk surface $(z=0)$:

No-slip condition gives $u=0$, $v=\Omega r$, $w=0$.

At the free-surface $z=h(t)$:

The dynamic boundary conditions at the free-surface are: (i) the jump in the normal stress across the interface is balanced by surface tension times curvature and (ii) the shear stress vanishes along the interface. Under the assumption of planar interface these conditions are represented by

$$p_a - p + 2\mu\partial_z w = 0, \tag{5}$$

$$\partial_r w + \partial_z u = 0 \quad \text{and} \quad \partial_z v = 0, \tag{6}$$

where p_a is the atmospheric pressure and μ is the dynamic coefficient of viscosity of the fluid. The kinematic condition at the free-surface is

$$\partial_t h = w(r,h,t), \tag{7}$$

which states that the free surface is advected by the fluid motion.

The initial conditions $(t=0)$ are

$$u(r,z,0) = v(r,z,0) = w(r,z,0) = 0, \quad h(0) = h_0, \quad \partial_t h = 0. \tag{8}$$

Introducing the following similarity variables:

$$u(r,z,t) = rf(z,t), v(r,z,t) = rg(z,t), \quad w(r,z,t) = w(z,t), \text{ and } p = -\frac{r^2}{2}\rho\mathcal{A}(z,t) + \mathcal{B}(z,t)\rho \tag{9}$$

in the above system of Eqs. (1)–(4), the following set of equations obtained after equating the different powers of r are

$$\partial_t f + f^2 - g^2 + w\partial_z f = \mathcal{A}(z,t) + \nu\partial_{zz}f, \tag{10}$$

$$\partial_t g + 2fg + w\partial_z g = \nu\partial_{zz}g, \tag{11}$$

$$\partial_t w + \frac{1}{2}\partial_z w^2 - \nu\partial_{zz}w + \frac{\partial_z\mathcal{B}}{\rho} = 0, \tag{12}$$

$$2f+\partial_z w=0, \tag{13}$$

$$\partial_z \mathcal{A}=0. \tag{14}$$

Substituting (9d) for p in Eq. (5) we get

$$\mathcal{A}=0,\ \ \mathcal{B}=2\rho\nu\partial_z w \ \ \text{at}\ \ z=h(t). \tag{15}$$

Equations (14) and (15a) give $\mathcal{A}=0$ and $\mathcal{B}(z,t)$ can be found by integrating Eq. (12) with respect to z, from z to $z=h(t)$, and thus we can evaluate the pressure from (9d). The following dimensionless quantities:

$$\tau=t\Omega_0,\ \ \eta=z/h_0,\ \ H=h/h_0,\ \ F=f/\Omega_0,$$
$$G=g/\Omega_0,\ \ \text{and}\ \ W=w/(h_0\Omega_0), \tag{16}$$

may be introduced into the system of equations and boundary conditions, where Ω_0 and h_0 are the initial angular velocity of the system and the film thickness (at $t=0$), respectively.

The final set of dimensionless equations become

$$\mathrm{Re}[\partial_\tau F+F^2-G^2+W\partial_\eta F]=\partial_{\eta\eta}F, \tag{17}$$

$$\mathrm{Re}[\partial_\tau G+2\,FG+W\partial_\eta G]=\partial_{\eta\eta}G, \tag{18}$$

$$2F+\partial_\eta W=0. \tag{19}$$

The dimensionless boundary conditions are

$$\begin{aligned} &F(\tau,0)=0,\ \ G(\tau,0)=\Omega(\tau)/\Omega_0,\ \ W(\tau,0)=0,\\ &\partial_\eta F(\tau,H)=0,\ \ \partial_\eta G(\tau,H)=0,\ \ \partial_\tau H=W(\tau,H). \end{aligned} \tag{20}$$

The initial conditions of the problem are

$$\begin{aligned} &F(0,\eta)=G(0,\eta)=W(0,\eta)=0,\\ &H(0)=1,\quad \partial_\tau H(0)=0, \end{aligned} \tag{21}$$

where the Reynolds number $\mathrm{Re}=\Omega_0 h_0^2/\nu$. In the present study the rotation $\Omega(\tau)$ has been chosen for three different forms viz., (*i*) *the disk starts with impulsive acceleration and maintains this constant angular speed thereafter*, (*ii*) *the disk starts impulsively and its angular speed increases continuously with time*, and (*iii*) *the disk starts its rotation from rest and its angular speed increases continuously to a finite value as* $\tau\to\infty$.

III. METHOD OF SOLUTION

The above coupled nonlinear system of Eqs. (17)–(21) with boundary conditions can be solved by the finite-difference technique. It is to be noted that the conventional finite-difference method cannot be used in this problem as the film height is continuously decreasing with time. For this reason one has to transform the time-dependent physical domain to a fixed computational domain [0, 1] such that the film thickness will always remain in fixed computational domain for all times. Further it is well known that fine grid distribution is needed for large velocity gradients near the disk surface when Reynolds number is large. It should be pointed out here that the said transformation will be useful for the fine as well as uniform grid distribution. Following Roberts,[26] we choose the transformation as below:

$$\xi(\tau)=1-a_1\ln\left(\frac{a_2H(\tau)-\eta}{b_2H(\tau)+\eta}\right),\ \ 1<c<\infty. \tag{22}$$

Here $a_1=[\ln(a_2/b_2)]^{-1}$, where $a_2=(c+1)$ and $b_2=(c-1)$. The parameter c controls the grid spacing in the physical domain. Small values of c cluster grid points at the disk surface where as large values make the grid spacing uniform throughout the liquid film.

The Crank–Nicholson scheme is used to solve the transformed nonlinear system of Eqs. (17)–(21) after approximating the nonlinear terms according to the Newton's linearization technique (Fletcher[27]). Computation is carried out in each time level on the following linear tridiagonal system of algebraic equations:

$$PG_{j-1}^{n+1}+QG_j^{n+1}+RG_{j+1}^{n+1}=(S_1)_j^n, \tag{23}$$

$$PF_{j-1}^{n+1}+QF_j^{n+1}+RF_{j+1}^{n+1}=(S_2)_j^n. \tag{24}$$

Here n and j denote the time level and spatial level of discretization. The quantities $P,Q,R,(S_1)_j^n$, and $(S_2)_j^n$ are defined as

$$P=\frac{B-A}{4\,\delta\xi}-\frac{C}{2\,\delta\xi^2},$$

$$Q=\frac{1}{\delta\tau}+2\,F_j^n+\frac{C}{\delta\xi^2},$$

$$R=\frac{A-B}{4\,\delta\xi}-\frac{C}{2\,\delta\xi^2},$$

$$(S_1)_j^n=G_j^n\left(\frac{1}{\delta\tau}-\frac{C}{\delta\xi^2}\right)+G_{j-1}^n\left(\frac{A-B}{4\,\delta\xi}+\frac{C}{2\,\delta\xi^2}\right)+G_{j+1}^n\left(\frac{B-A}{4\,\delta\xi}+\frac{C}{2\,\delta\xi^2}\right),$$

$$(S_2)_j^n=F_j^n\left[\frac{1}{\delta\tau}+F_j^n-\frac{C}{\delta\xi^2}\right]+(G_j^n)^2+F_{j-1}^n\left[\frac{A-B}{4\,\delta\xi}+\frac{C}{2\,\delta\xi^2}\right]+F_{j+1}^n\left[\frac{B-A}{4\,\delta\xi}+\frac{C}{2\,\delta\xi^2}\right],$$

$$A=\frac{a_1(a_2+b_2)(H^nW_j^n-\xi_jH_\tau^n)}{(a_2H^n-\xi_j)(b_2H^n+\xi_j)},$$

$$B=\mathrm{Re}\left[\frac{a_1(a_2+b_2)H^n\{(b_2-a_2)H^n+2\,\xi_j\}}{(a_2H^n-\xi_j)^2(b_2H^n+\xi_j)^2}\right],$$

$$C=\left[\frac{a_1(a_2+b_2)H^n}{(a_2H^n-\xi_j)(b_2H^n+\xi_j)}\right]^2.$$

At each and every time level G_j^{n+1} and F_j^{n+1} are computed from (23) to (24) and then the axial velocity W_j^{n+1} is obtained from the finite-difference representation of the continuity equation by using the values of F_j^{n+1} at that time level. The iteration process is continued until it achieves the desired level of film height.

Computation is carried out on 50 nodes in vertical direction with $c=10^4$ (equivalent to a uniform grid in the physical domain). This gives the uniform grid distribution in the physical domain as well as computational domain. Increase

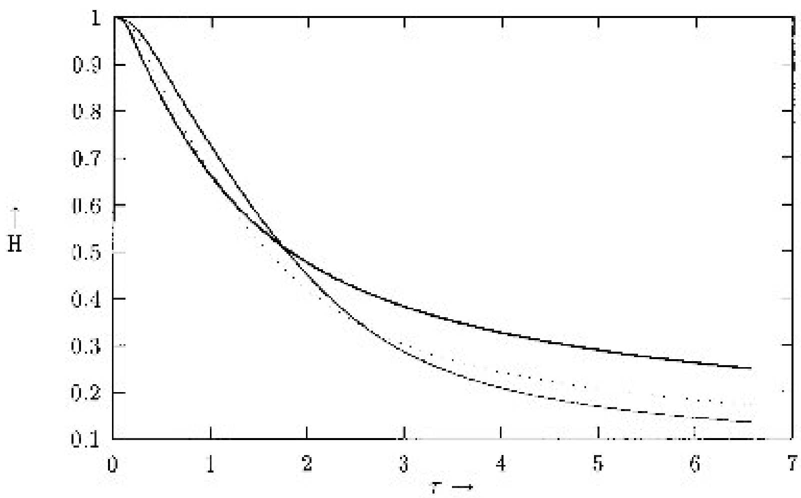

FIG. 1. Film thickness vs time. Thick, dotted, and thin lines represent for Reynolds numbers 2, 5, and 10, respectively.

of Reynolds number needs the increase of grid points in the vertical direction, under this situation one has to decrease c drastically, depending upon the rotational speed of the disk. The time step has been calculated by

$$\delta\tau \leqslant 0.25 \times \delta\xi^2. \quad (25)$$

This comes from the CFL condition of numerical stability. The domain of $\delta\tau$ is chosen smaller than the stability domain for linear equation due to the coupled nonlinear system. For much smaller Reynolds number flow the time-step has to be decreased further subject to condition (25).

IV. RESULTS AND DISCUSSION

The results of the numerical study on the effect of three different types of nonuniform rotation on the development of thin liquid film on a spinning disk for various Reynolds numbers are presented as three different cases of studies:

(i) *Impulsive acceleration*. In this section, it is assumed that the disk is impulsively accelerated from rest to an angular velocity Ω_0 and maintains this speed for the entire period. For Reynolds numbers 2, 5, and 10, respectively, Fig. 1 presents the variation of film thinning for different Re with respect to time. It is clear from the figure that the thinning rate increases with the decrease of the Reynolds number during the time interval $0<\tau<\tau_c$ while film thinning increases with the increase of the Reynolds number for the interval $\tau_c<\tau$. This anomalous behavior of the variation of the film height in different time interval can be explained by considering the two time zones $\tau<\tau_c$ and $\tau>\tau_c$. Here τ_c denotes a point on the time scale. This point represents that time, when the total amount of fluid flowing out at a certain radial distance in two Reynolds number become equal. It is obvious that the point τ_c changes as the corresponding Reynolds number changes. In the first zone ($\tau<\tau_c$), film thinning increases with the decrease of the rotational speed of the disk. This seems to be inconsistent with one's common knowledge that film thins faster with the faster rotation of the disk. A close scrutiny of the physical process reveals that the azimuthal component of velocity develops along the entire depth of the fluid faster with slower rotation of the disk. This is due to the fact that the Reynolds number is the ratio of the centrifugal to the viscous forces. Decrease of the Reynolds number implies that the viscous force dominates over the centrifugal force. As a result, viscous action builds up azimuthal velocity field faster throughout the depth of the fluid for smaller Reynolds number. This result is found in Figs. 2, 3, and 4 which are

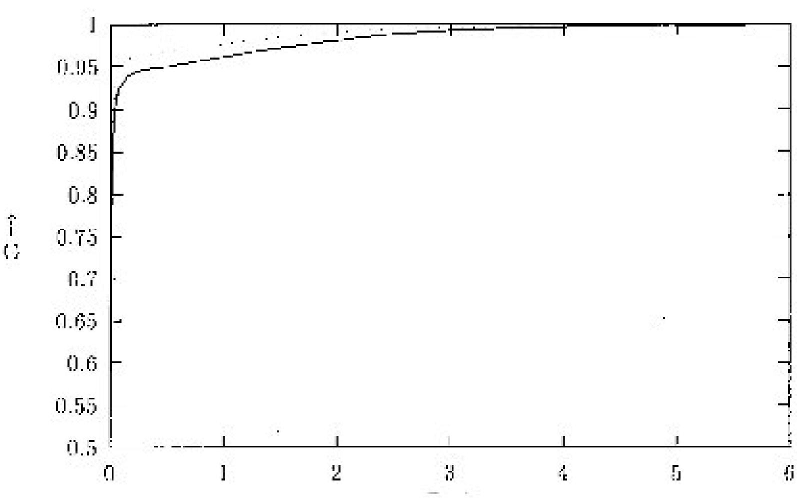

FIG. 2. Variation of azimuthal velocity G with time τ at $\eta=0.1$ H, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

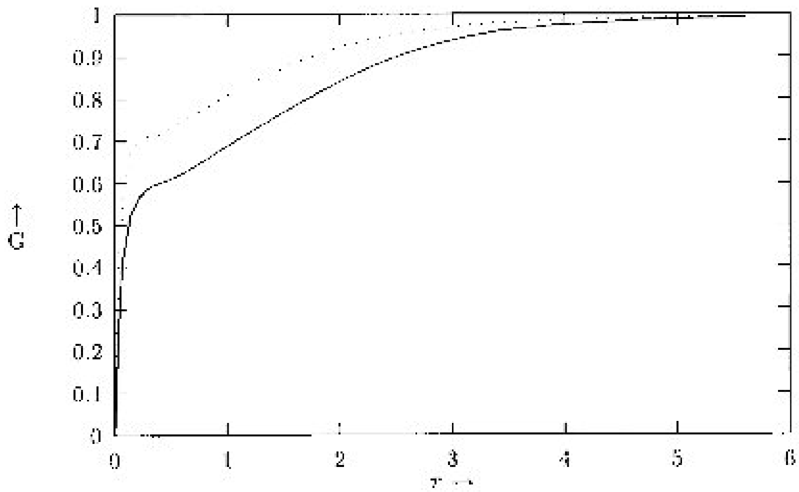

FIG. 3. Variation of azimuthal velocity G with time τ at $\eta = 0.5$ H, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

plotted at different depths of the fluid for Reynolds numbers 5 and 10. The faster development of the azimuthal velocity field introduces the corresponding radial velocity faster for smaller Reynolds number only. Now, for $\tau \ll \tau_c$ one may expect that the azimuthal velocity has developed on the maximum portion of the depth of the fluid for smaller Re. Consequently, for $\tau \ll \tau_c$, the radial velocity develops over a greater depth for the Reynolds number Re_1 than the corresponding depth for Reynolds number $Re_2 > Re_1$. Figures 5, 6, 7, and 8 confirm the above view. As a result, the net amount of fluid flowing out at a certain radius is more for Re_1 than that at the same radius for Re_2. This explains why the film thins faster with smaller Re in time interval $0 < \tau < \tau_c$. In the second zone for $\tau > \tau_c$, azimuthal velocity field has developed into the larger portion of the film depth for both large and small Reynolds numbers and the corresponding centrifugal force is large for larger Re. As a result, fluid moves out of the disk quickly when its rotational speed is high.

The rate of liquid depleted from the disk at a fixed radial distance from the center with time for different values of Reynolds numbers is represented in Fig. 9. It is evident from the figure that in a very small span of time fluid depletion rate is high for small Re compare with the large Re as explained earlier. But for both large and small Re, Q attains its maximum in a short interval of time and then decreases with time and ultimately attains an asymptotic values at large time. Further it is clear from the figure that Q takes larger time to attain this asymptotic value if Re is decreased. This is due to the fact that the decrease of Re means the decrease of the rotational speed Ω_0. This implies that less amount of centrifugal force acts on the fluid to draw it out at a fixed radial distance.

Here, $Q(\tau) = \int_0^H 2\pi r_0 u \, d\eta$ denotes the rate of liquid depleted in time τ at a fixed radial distance $r = r_0$ (π^{-1}, say). It is observed from the numerical computations that the trend of velocity components are more or less same for the other cases of rotations, viz. *(ii) starts impulsively and then increases continuously with time* $\Omega(\tau) = \Omega_0(1 - \delta\tau)^{-1}$, $(\delta\tau) < 1$, *and (iii) smoothly increases from rest with time and attains a finite maximum value at large time* $\Omega(\tau) = \Omega_0(1 - \exp(-\sigma\tau))$. Here both δ and σ are positive constants.

It is observed that the variations of the film thickness is markedly different for three cases of nonuniform rotations and shown in Fig. 10 for $\delta = 0.1$ and $\sigma = 0.1$ with $Re = 2$. It is clear from the figure that the continuously increasing rotation of type (ii) makes the film thinner very fast. It is expected that at large time the thickness of the film in case (iii) con-

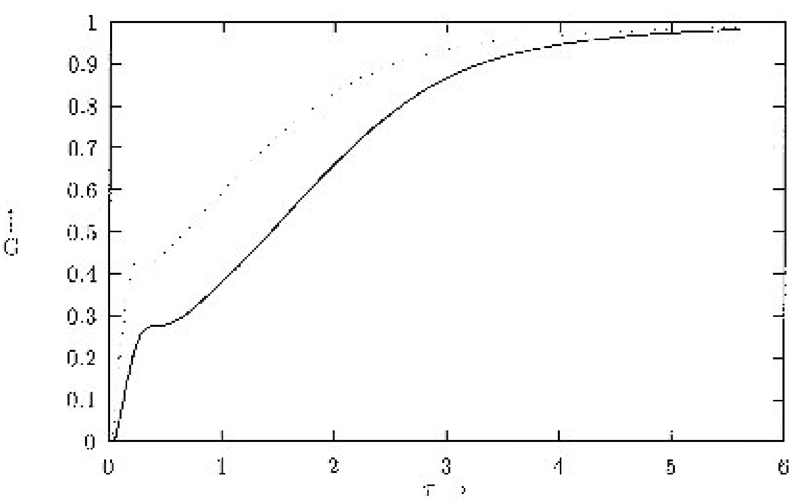

FIG. 4. Variation of azimuthal velocity G with time τ at $\eta = 1.0$ H, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

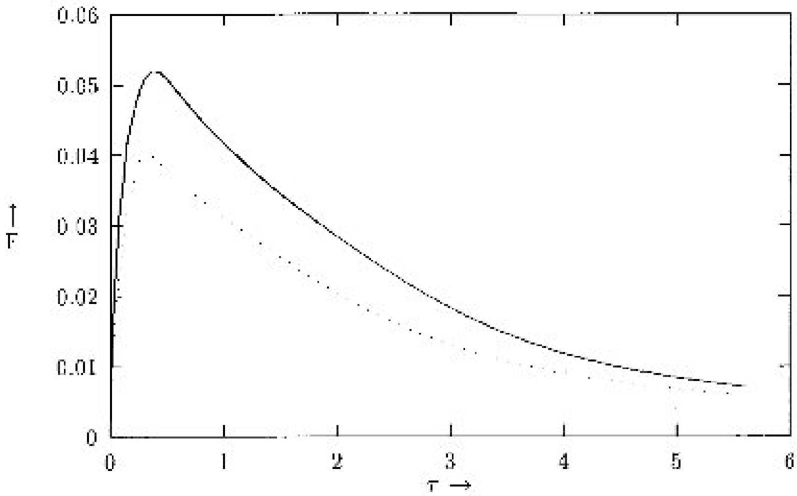

FIG. 5. Variation of radial velocity F with time τ at $\eta = 0.1$ H, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

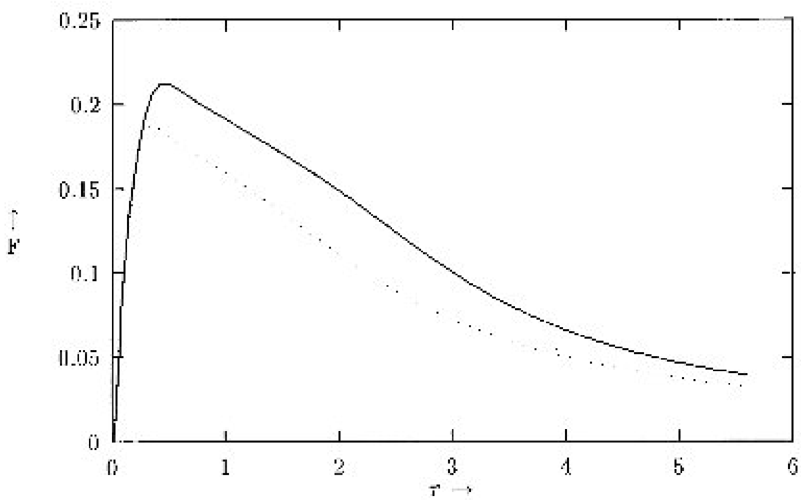

FIG. 6. Variation of radial velocity F with time τ at $\eta = 0.4$ H thick and dotted lines for Reynolds numbers 10 and 5, respectively.

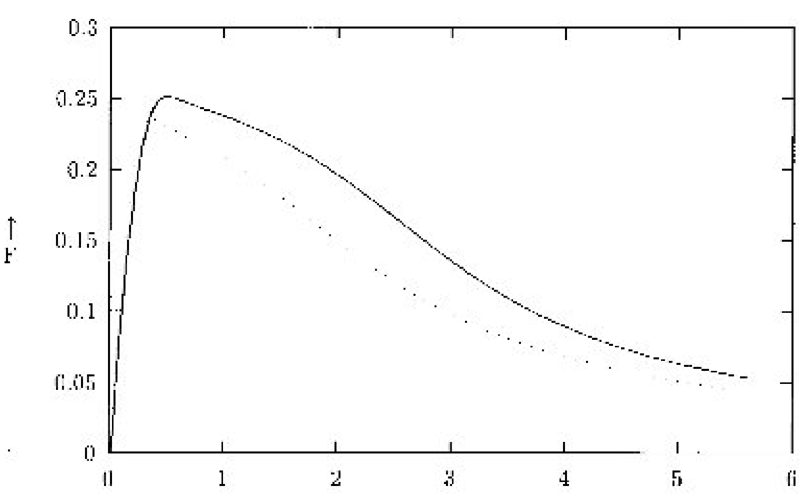

FIG. 7. Variation of radial velocity F with time τ at $\eta = 0.5$ H, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

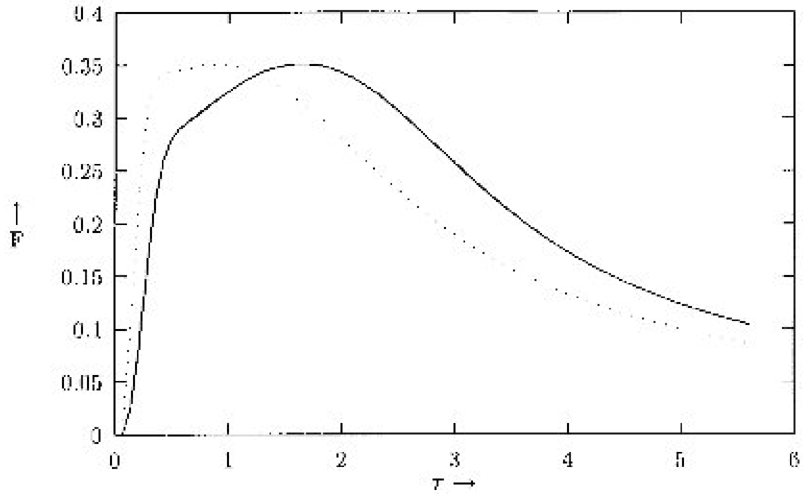

FIG. 8. Variation of radial velocity F with time τ at $\eta = 1.0\,H$, thick and dotted lines for Reynolds numbers 10 and 5, respectively.

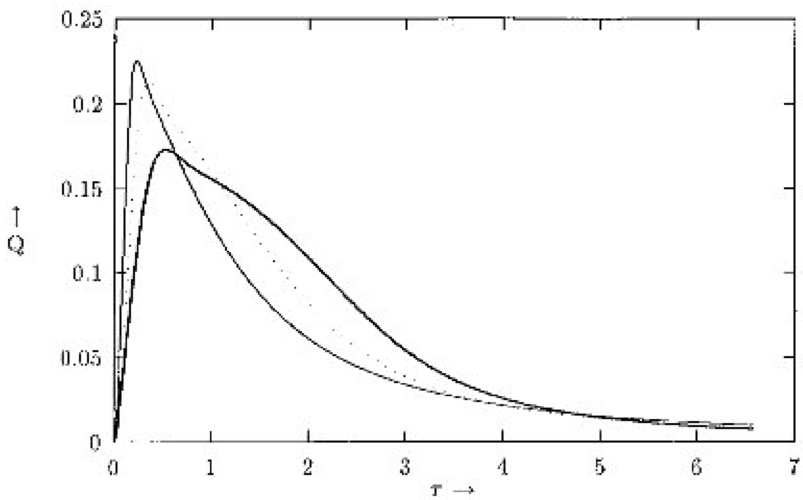

FIG. 9. Depletion rate Q vs time τ. Thick, dotted, and thin lines correspond to the Reynolds numbers 10, 5, and 2, respectively.

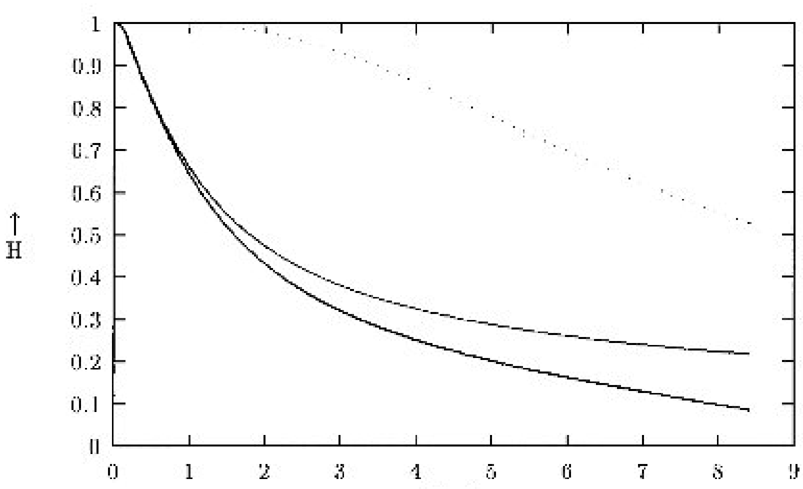

FIG. 10. Variation of height H with time τ for three different forms of rotations. Thin, thick, and dotted lines are correspond to rotation type (i), (ii), and (iii), respectively.

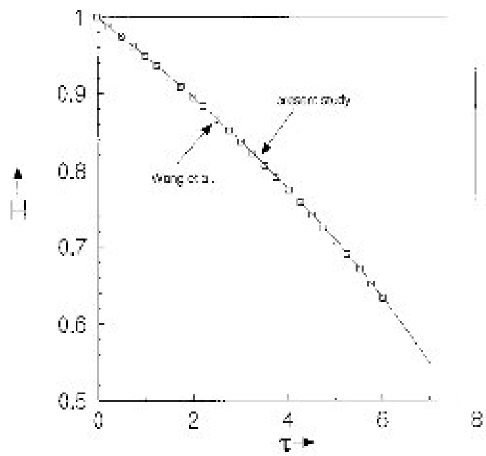

FIG. 11. Comparison on the variation of height H with time τ for similarity approach adopted by Wang *et al.* (rectangles) and the present analysis (solid lines).

verges to that of case (i). It should be pointed out here again that Wang *et al.*[24] have also considered the same kind of unsteadiness as the present case (ii), but solved by similarity approach (SA). They reduced the N–S equation to a set of nonlinear ODE by using a special type of similarity variable η defined in their Eq. (6), which combines time and z. It is therefore possible to compare the results of our case (ii) with that of Wang *et al.*[24] To do that, let us start with their Eq. (11), which can be expressed in dimensionless form using Eq. (16) of the present analysis as

$$H(\tau)=\mathrm{Re}^{-0.5}(1-S\tau)^{0.5}\beta. \tag{26}$$

Here, $H(\tau)$, Re, and τ have the same meaning and expression used in the present study. $S(\equiv\alpha/\Omega_0)$ and β are the dimensionless parameters used by Wang *et al.*[24] $\delta=0.1$ in case (ii) of the present paper is equivalent to $S=0.1$ in (26). Now it is possible to pick out corresponding β from Fig. 1 of Wang *et al.*[24] for $S=0.1$ or by using their Eq. (27), which is very accurate for small S-values and gives $\beta=0.2739$. $\tau=0$ in (26) above gives $H(0)=\mathrm{Re}^{-0.5}\beta$. But according to the definition, $H(0)$ should be equal to 1. Therefore Eq. (26) reveals that the analysis of Wang *et al.*[24] implicitly implies that

$$\mathrm{Re}=\beta^2. \tag{27}$$

This shows that their analysis is valid only for one particular Re for each β-value obtained from their Fig. 1 which relates S with β. For the present purpose of comparison between two studies we have for $S=0.1$, $\beta=0.2739$, and $\mathrm{Re}=0.075\,021$. Figure 11 depicts the results obtain from our finite-difference scheme for case (ii) of N–S equations and that obtained by Wang *et al.*[24] through similarity approach. It is clear from the graph that both the results agree very well.

V. CONCLUSION

To obtain the faster thinning spinner should run impulsively at the beginning and the rate of spinning should increase with time until the desired thinness is obtained. By increasing the spinning rate one may avoid hard skinning which is a common phenomenon in spin-coating. Of course this result needs experimental confirmation. It should be clearly borne in mind that the disk is fully wet and covered by the fluid so that the moving contact line case does not arise. In other types of problems related to the contact line one may observe fingering instabilities on the rim of the moving drop (Troian *et al.*[18] and Melo *et al.*[19]). It should be noted that in the present case the film is very thin and across the depth viscous effect is very prominent to regularize any kind of disturbance due to the deformed interface. Further it should be pointed out here that the underlying theory for film flow is based on the assumption that the free surface remains flat during the entire spinning period, so that, a self-similar solution exists for the velocity fields. In this connection it should be mentioned here that Matsumoto *et al.*[24] have studied the spreading of a hemispherical liquid drop on a spinning disk and shown that the film thins uniformly after a very short time [0.106 s] from the start of the rotation. It is therefore expected that the initial flat deposition of the liquid film will thin uniformly much faster than the hemispherical drop. Therefore, one may conjecture from the present study that the removal of the r dependence for the physical variables takes place in a very short span of time. In other words, the self-similar velocity field may be expected to exist within a very short interval of time from the start of the rotation. The case of the nonuniform free surface will be considered in a future work. Further we would like to put a few remarks regarding the strengths and weaknesses of the two different approaches, viz., the similarity approach (SA) and finite difference (FD) methods. In order to achieve similarity solutions one has to compromise with the generality of the problem although ODEs can be solved more accurately than PDEs. On the other hand (FD) retains its versatility while solving the PDEs.

ACKNOWLEDGMENTS

The author acknowledges the grant from the Council of Scientific and Industrial Research (CSIR) (Scheme No. 03/0778/95 EMR-II), New Delhi, India to carry out this research and expresses his sincere thanks to Professor A. S. Gupta, IIT, Kharagpur, Professor H. I. Andersson, Trondheim, Norway, and the referees for their constructive suggestions and comments to improve the text.

[1] A. C. Emslie, F. D. Bonner, and L. G. Peck, ''Flow of a viscous liquid on a rotating disk,'' J. Appl. Phys. **29**, 858 (1958).
[2] B. D. Washo, ''Rheology and modeling of the spin coating process,'' IBM J. Res. Dev. **21**, 481 (1977).
[3] A. Acrivos, M. J. Shah, and F. F. Petersen, ''On the flow of a non-Newtonian liquid on a rotating disk,'' J. Appl. Phys. **31**, 963 (1960).
[4] D. Meyerhofer, ''Characteristics of resist films produced by spinning,'' J. Appl. Phys. **49**, 3993 (1978).
[5] F. L. Givens and W. L. Daughton, ''On the uniformity of thin film: A new technique applied to polyamides,'' J. Electrochem. Soc. **126**, 269 (1979).
[6] J. H. Lai, ''An investigation of spin coating of electron resists,'' Polymer Eng. Sci. **19**, 1117 (1979).
[7] B. T. Chen, ''Investigation of the solvent-evaporation effect on spin coating of thin films,'' Polymer Eng. Sci. **23**, 399 (1983).
[8] S. A. Jenekhe and S. A. Schuldt, "Coating flow of non-Newtonian fluids on a flat rotating disk," Ind. Eng. Chem. Fundam. **23**, 432 (1984).
[9] P. C. Sukanek, ''Spin-coating,'' J. Imaging Technol. **11**, 184 (1985).

[10] J. H. Hwang and F. Ma, ''On the flow of a thin liquid film over a rough rotating disk,'' J. Appl. Phys. **66**, 388 (1989).
[11] B. G. Higgins, ''The effects of inertia and interfacial shear on film flow on a rotating disk,'' Phys. Fluids **29**, 3522 (1986).
[12] B. S. Dandapat and P. C. Ray, ''Film cooling on a rotating disk,'' Int. J. Non-Linear Mech. **25**, 569 (1990).
[13] B. S. Dandapat and P. C. Ray, ''The effect of thermocapillarity on the flow of a thin liquid film on a rotating disk,'' J. Phys. D **27**, 2041 (1994).
[14] B. S. Dandapat and P. C. Ray, ''Effect of thermocapillarity on the production of a conducting thin film in the presence of a transverse magnetic field,'' Z. Angew. Math. Mech. **78**, 635 (1998).
[15] P. C. Ray and B. S. Dandapat, ''Flow of thin liquid film on a rotating disk in the presence of a transverse magnetic field,'' Q. J. Mech. Appl. Math **47**, 297 (1994).
[16] A. M. Schwartz and S. B. Tejada, ''Studies of dynamic angles on solids,'' J. Colloid Interface Sci. **38**, 359 (1972).
[17] H. Giradella and W. Radigan, ''Electrical resistivity changes in spreading liquid films,'' J. Colloid Interface Sci. **51**, 522 (1975).
[18] P. G. De Gennes, ''Wetting: Statics and dynamics,'' Rev. Mod. Phys. **57**, 827 (1985).
[19] S. M. Troian, E. Herbolzheimer, S. A. Safran, and J. F. Joanny, ''Fingering instabilities of driven spreading films,'' Europhys. Lett. **10**, 25 (1989).
[20] F. Melo, J. F. Joanny, and S. Fauve, ''Fingering instability of spinning drops,'' Phys. Rev. Lett. **63**, 1958 (1989).
[21] J. A. Moriarty, L. W. Schwartz, and E. O. Tuck, ''Unsteady spreading of thin liquid films with small surface tension,'' Phys. Fluids A **3**, 733 (1991).
[22] I. S. McKinley, S. K. Wilson, and B. R. Duffy, ''Spin coating and air jet blowing of thin viscous drops,'' Phys. Fluids **11**, 30 (1999).
[23] Y. Matsumoto, T. Ohara, I. Teruya, and H. Ohashi, ''Liquid film formation on a rotating disk,'' JSME Int. J., Ser. II **32**, 52 (1989).
[24] C. Y. Wang, L. T. Watson, and K. A. Alexander, ''Spinning of a liquid film from an accelerating disk,'' IMA J. Appl. Math. **46**, 201 (1991).
[25] T. von Kármán, ''Uber laminare and turbulente reibung,'' Z. Angew. Math. Mech. **1**, 233 (1921).
[26] G. O. Roberts, *Lecture Notes in Physics* (Springer, New York, 1971), Vol. 8, p. 171.
[27] C. A. J. Fletcher, *Computational Techniques for Fluid Dynamics* (Springer-Verlag, New York, 1988), Vol. II.