Nonlinear Phase Changes in a Deformed Hilbert Space

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In this paper we study nonlinear phase changes of some states with respect to a phase distribution on a deformed Hilbert space.

1. INTRODUCTION

Quantum-mechanical description of phase has a long history, starting with the work of Dirac (1927), who attempted definition of a phase operator with the help of polar decomposition of the annihilation operator in radiation field. Thereafter, Susskind and Glogower (1964), Carruthers and Nieto (1968), Pegg and Barnett (1989), and Shapiro and Shepard (1991) have further studied this topic. Susskind and Glogower modified Dirac's phase operator, though it is a one-sided unitary operator. Nevertheless, their phase operator has been extensively used in quantum optics. Shapiro and Shepard introduced phase measurement statistics through quantum estimation theory (Helstrom, 1976). Pegg and Barnett (1989) carried out a polar decomposition of the annihilation operator in a truncated Hilbert space of dimension $s + 1$, and defined a Hermitian phase operator in this finite-dimensional space. Now, given a state in the finite-dimensional Hilbert space one first computes the expectation value with the restricted state to the $(s + 1)$-dimensional space. It is natural now to take the limit s to infinity and recover a Hermitian phase operator in the full Hilbert space. However, in this limit the PB phase operator does not converge to a Hermitian phase operator, but the distribution does converge to the SG phase distribution. Thus it appears to be computationally advantageous to describe the quantum-mechanical phase via a phase distribution rather than through a phase operator. This view was manifested in the work of Shapiro and Shepard. Agarwal and coworkers (1992) adopted this point of view in investigating the quantum-mechanical phase properties of the nonlinear optical phenomena.

Keeping the ideas of Susskind and Glogower in mind, I (Das, 1999a, 2000) recently, described a phase operator in a deformed Hilbert space and studied phase distribution of Kerr vectors. Here, I shall adopt the viewpoint of Agarwal and coworkers to investigate the phase properties of several states (vectors) in a deformed Hilbert space.

The work is organized as follows. In Section 2, we give a brief description of phase distribution that we would like to associate to a given density operator. In Section 3, we describe a few illustrative examples. In fact, we describe how the phase distribution will look like when we take incoherent vector, coherent vector, coherent phase vector, and Kerr vector in the deformed space. In Section 4, we consider the evolution of the phase distribution associated with a field as it propagates through nonlinear mediums. We shall discuss two well-known Kerr-like phenomena with examples. In Section 5, we observe how the phase distribution changes in the process of photon absorption from a thermal beam and finally we give a conclusion.

2. PHASE DISTRIBUTION IN A DEFORMED HILBERT SPACE

Before we describe Phase distribution in brief, we narrate a few preliminaries and notations.

2.1. Preliminaries and Notations

We consider the set

$$H_q = \left\{ f : f(z) = \sum a_n z^n \text{ where } \sum [n]! |a_n|^2 < \infty \right\},$$

where $[n] = (1 - q^n)/(1 - q)$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we define addition and scalar multiplication as follows:

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$\lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda\, a_n z^n. \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]!}$ belongs to H_q.

Now we define the inner product of two functions $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ belonging to H_q as

$$(f, g) = \sum [n]!\, \bar{a}_n\, b_n. \tag{3}$$

Corresponding norm is given by

$$\|f\|^2 = (f, f) = \sum [n]!|a_n|^2 < \infty.$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space. Hence H_q forms a Hilbert space.

In a recent paper (Das, 1998, 1999b), I have proved that the set $\{z^n/\sqrt{[n]!}, n = 0, 1, 2, 3, \dots\}$ forms a complete orthonormal set. If we consider the following action on H_q,

$$\begin{aligned} T f_n &= \sqrt{[n]} f_{n-1} \\ T^* f_n &= \sqrt{[n+1]} f_{n+1}, \end{aligned} \tag{4}$$

where T and its adjoint T^* are the backward shift and forward shift operators, respectively, on H_q, and $f_n(z) = z^n/\sqrt{[n]}!$. Then we have shown (Das, 1998, 1999b) that the solution of the following eigenvalue equation

$$T f_\alpha = \alpha f_\alpha \tag{5}$$

is given by

$$f_\alpha = e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n. \tag{6}$$

We call f_α a *coherent vector* in H_q.

2.2. Phase Vectors

To obtain the phase vector, we first consider the Susskind–Glogower type *phase operator* $P = (q^n + T^*T)^{-1/2} T$ and try to find the solution of the following eigenvalue equation:

$$P f_\beta = \beta f_\beta, \tag{7}$$

where

$$f_\beta(z) = \sum_{n=0}^{\infty} a_n\, z^n = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n(z). \tag{8}$$

and we arrive at

$$f_\beta = \sum_{n=0}^{\infty} a_n \sqrt{[n]!} f_n$$

$$= a_0 \sum_{n=0}^{\infty} \beta^n \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\cdots(q^n+[n-1])}{[n]!}}\, f_n.$$

where $\beta = |\beta| e^{i\theta}$ is a complex number.

For details we refer to Das (1999a).

These vectors are normalizable in a strict sense only for $|\beta| < 1$.

Now if we take $a_0 = 1$ and $|\beta| = 1$, we have

$$f_\beta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\cdots(q^n+[n-1])}{[n]!}}\, f_n. \tag{9}$$

Henceforth, we shall denote this vector as

$$f_\theta = \sum_{n=0}^{\infty} e^{in\theta} \sqrt{\frac{(q+[0])(q^2+[1])(q^3+[2])\cdots(q^n+[n-1])}{[n]!}}\, f_n, \tag{10}$$

$0 \le \theta \le 2\pi$ and call f_θ a *phase vector* in H_q.

The phase vectors f_θ are neither normalizable nor orthogonal, but form a complete set and yield the following resolution of the identity:

$$I = \frac{1}{2\pi} \int_X \int_0^{2\pi} d\nu(x,\theta)\, | f_\theta >< f_\theta | \tag{11}$$

where

$$d\nu(x,\theta) = d\mu(x)\, d\theta, \tag{12}$$

which may be proved as follows:

Here we consider the set X consisting of the points $x = 0, 1, 2, \ldots$ and $\mu(x)$ is the measure on X which equals

$$\mu_n \equiv \frac{[n]!}{(q+[0])(q^2+[1])\cdots(q^n+[n-1])}$$

at the point $x = n$, and θ is the Lebesgue measure on the circle.

Define the operator

$$| f_\theta >< f_\theta | : H_q \to H_q \tag{13}$$

by

$$| f_\theta >< f_\theta | f = (f_\theta, f) f_\theta, \tag{14}$$

with $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Now,

$$
\begin{aligned}
&(f_\theta, f) \\
&= \sum_{n=0}^{\infty} [n]! \frac{e^{-in\theta}}{\sqrt{[n]!}} \sqrt{\frac{(q+[0])\,(q^2+[1])\,(q^3+[2])\cdots(q^n+[n-1])}{[n]!}}\, a_n \\
&= \sum_{n=0}^{\infty} e^{-in\theta} \sqrt{(q+[0])\,(q^2+[1])\,(q^3+[2])\cdots(q^n+[n-1])}\, a_n. \qquad (15)
\end{aligned}
$$

Then,

$$
\begin{aligned}
(f_\theta, f) f_\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n e^{i(m-n)\theta} &\sqrt{\frac{(q+[0])(q^2+[1])\cdots(q^m+[m-1])}{[m]!}} \\
&\times \sqrt{(q+[0])(q^2+[1])\cdots(q^n+[n-1])}\, f_m. \qquad (16)
\end{aligned}
$$

Using

$$
\int_0^{2\pi} d\theta\; e^{i(m-n)\theta} = 2\pi\,\delta_{mn}, \qquad (17)
$$

we have

$$
\begin{aligned}
&\frac{1}{2\pi} \int_X \int_0^{2\pi} d\nu(x,\theta)\, | f_\theta >< f_\theta | f \\
&= \int_X d\mu(x) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n\, f_m \sqrt{\frac{(q+[0])(q^2+[1])\cdots(q^m+[m-1])}{[m]!}} \\
&\quad \times \sqrt{(q+[0])(q^2+[1])\cdots(q^n+[n-1])}\, \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta \\
&= \sum_{n=0}^{\infty} a_n f_n \int_X \frac{(q+[0])(q^2+[1])\cdots(q^n+[n-1])}{\sqrt{[n]!}}\, d\mu(x) \qquad (18) \\
&= \sum_{n=0}^{\infty} a_n f_n \frac{(q+[0])(q^2+[1])\cdots(q^n+[n-1])}{\sqrt{[n]!}} \\
&\quad \times \frac{[n]!}{(q+[0])(q^2+[1])\cdots(q^n+[n-1])} \\
&= \sum_{n=0}^{\infty} \sqrt{[n]!}\, a_n f_n \\
&= f.
\end{aligned}
$$

Thus, (11) follows.

We use the vectors f_θ to associate, to a given density operator ρ, a phase distribution as follows:

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}\sum_{m,n=0}^{\infty}\sqrt{\frac{(q+[0])\cdots(q^m+[m-1])}{[m]!}} \\ &\quad\times\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}e^{i(n-m)}(f_m, \rho f_n) \end{aligned} \tag{19}$$

The $P(\theta)$ as defined in (19) is positive (owing to the positivity of ρ) and is normalized

$$\int_X \int_0^{2\pi} P(\theta)\, d\nu(x,\theta) = 1, \tag{20}$$

where

$$d\nu(x,\theta) = d\mu(x)\, d\theta \tag{21}$$

for,

$$\begin{aligned} \int_X \int_0^{2\pi} P(\theta)\, d\nu(x,\theta) &= \int_X d\mu(x) \sum_{m,n=0}^{\infty}\sqrt{\frac{(q+[0])\cdots(q^m+[m-1])}{[m]!}} \\ &\quad\times\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}} \\ &\quad\times\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta(f_m, \rho f_n) \\ &= \int_X d\mu(x) \sum_{n=0}^{\infty}\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}(f_n, \rho f_n) \\ &= \sum_{n=0}^{\infty}(f_n, \rho f_n) \\ &= 1. \end{aligned} \tag{22}$$

In particular, the *phase distribution* over the window $0 \le \theta \le 2\pi$ for any vector

f is then defined by

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, |f><f|f_\theta) \\ &= \frac{1}{2\pi}|(f_\theta, f)|^2. \end{aligned} \tag{23}$$

3. EXAMPLES

We now consider some specific vectors in the Hilbert space H_q and compute their corresponding phase ditributions.

3.1. Incoherent Vectors

For the incoherent vectors we take the density operator to be

$$\rho = \sum_{n=0}^{\infty} p_n |f_n><f_n|, \tag{24}$$

with

$$p_n \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} p_n = 1.$$

Now we calculate the phase distribution $P(\theta)$ as

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n (f_\theta, |f_n><f_n|f_\theta) \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n |(f_\theta, f_n)|^2 \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty} p_n \frac{(q+[0])\cdots(q^n+[n-1])}{[n]!} \end{aligned} \tag{25}$$

3.2. Coherent Vectors

For the coherent vectors (Das, 1998)

$$f_\alpha = e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n. \tag{26}$$

We take the density operator to be

$$\rho = |f_\alpha><f_\alpha|, \quad \alpha = |\alpha| e^{i\theta_0}, \tag{27}$$

and calculate the phase distribution $P(\theta)$ as

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}(f_\theta, |f_\alpha >< f_\alpha| f_\theta) \\ &= \frac{1}{2\pi}|(f_\theta, f_\alpha)|^2 \\ &= \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} e^{in(\theta_0-\theta)}\frac{|\alpha|^n}{\sqrt{[n]!}}e_q(|\alpha|^2)^{-1/2}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\right|^2 \end{aligned} \tag{28}$$

3.3. Coherent Phase Vectors

For a coherent phase vector (Das, 1999a)

$$f_\beta = \Phi(|\beta|^2)^{-1/2}\sum_{n=0}^{\infty}\beta^n\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}f_n, \tag{29}$$

with $|\beta| < 1$ and

$$\Phi(|\beta|^2) = \sum_{n=0}^{\infty}|\beta|^{2n}\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}. \tag{30}$$

We take the density operator to be

$$\rho = |f_\beta >< f_\beta|, \tag{31}$$

and calculate the phase distribution $P(\theta)$ as

$$\begin{aligned} P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}(f_\theta, |f_\beta >< f_\beta| f_\theta) \\ &= \frac{1}{2\pi}|(f_\theta, f_\beta)|^2 \\ &= \frac{1}{2\pi}\left|\Phi(|\beta|^2)^{-1/2}\sum_{n=0}^{\infty}\beta^n e^{-in\theta}\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}\right|^2 \\ &= \frac{1}{2\pi}\frac{|\Phi(e^{-i\theta}\beta)|^2}{\Phi(|\beta|^2)}. \end{aligned} \tag{32}$$

3.4. Kerr Vectors

For a Kerr vector (Das, 1999a)

$$\begin{aligned}\phi_\alpha^{\mathrm{K}} &= e_q^{\frac{i}{2}\gamma N(N-1)} f_\alpha \\ &= \sum_{n=0}^{\infty} k_n f_n,\end{aligned} \tag{33}$$

where

$$k_n = e_q(|\alpha|^2)^{-1/2} \frac{\alpha^n}{\sqrt{[n]!}} e_q^{\frac{i}{2}\gamma[n]([n]-1)}. \tag{34}$$

We take the density operator to be

$$\rho = |\phi_\alpha^{\mathrm{K}} >< \phi_\alpha^{\mathrm{K}}|, \tag{35}$$

and calculate the phase distribution $P(\theta)$ as

$$\begin{aligned}P(\theta) &= \frac{1}{2\pi}(f_\theta, \rho f_\theta) \\ &= \frac{1}{2\pi}\left(f_\theta, |\phi_\alpha^{\mathrm{K}} >< \phi_\alpha^{\mathrm{K}}| f_\theta\right) \\ &= \frac{1}{2\pi}\left|\left(f_\theta, \phi_\alpha^{\mathrm{K}}\right)\right|^2 \\ &= \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} e^{-in\theta} k_n \frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}\right|^2.\end{aligned} \tag{36}$$

4. PROPAGATION THROUGH NONLINEAR MEDIUMS

Here, we consider the evolution of the phase distribution associated with a field as it propagates through nonlinear mediums. We shall discuss two well-known Kerr-like phenomena that fall in this category.

4.1. The first dynamic evolution of the density operator for our consideration is given by

$$\rho(t) = e_q(-i\gamma T^{*2}T^2 t)\rho(0)e_q(i\gamma T^{*2}T^2 t) \tag{37}$$

where γ is the Kerr constant of the medium. The time evolution of the corresponding phase distribution is given by

$$P(\theta, t) = \frac{1}{2\pi}(f_\theta, \rho(t) f_\theta). \tag{38}$$

4.1.1. For an initial incoherent vector

$$\rho(0) = |f_n >< f_n|, \tag{39}$$

$P(\theta, t)$ is given by

$$\begin{aligned} P(\theta,\beta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t)|f_n >< f_n|e_q(i\gamma T^{*2}T^2 t) f_\theta\big) \\ &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t)\big(f_n, e_q(i\gamma T^{*2}T^2 t) f_\theta\big) f_n\big) \\ &= \frac{1}{2\pi}\big(f_n, e_q(i\gamma T^{*2}T^2 t) f_\theta\big)\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t) f_n\big) \\ &= \frac{1}{2\pi}\left|\big(f_n, e_q(i\gamma T^{*2}T^2 t) f_\theta\big)\right|^2 \\ &= \frac{1}{2\pi}\left|e^{in\theta}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\, e_q(i\gamma t[n][n-1])\right|^2 \\ &= \frac{1}{2\pi}\left|\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\, e_q(i\beta[n][n-1])\right|^2, \end{aligned} \tag{40}$$

where $\beta = \gamma t$.

4.1.2. For an initial Coherent vector

$$\rho(0) = |f_\alpha >< f_\alpha|, \quad \alpha = |\alpha| e^{i\theta_0} \tag{41}$$

$P(\theta, t)$ is given by

$$\begin{aligned} P(\theta,\beta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t)|f_\alpha >< f_\alpha|e_q(i\gamma T^{*2}T^2 t) f_\theta\big) \\ &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t)\big(f_\alpha, e_q(i\gamma T^{*2}T^2 t) f_\theta\big) f_\alpha\big) \\ &= \frac{1}{2\pi}\big(f_\alpha, e_q(i\gamma T^{*2}T^2 t) f_\theta\big)\big(f_\theta, e_q(-i\gamma T^{*2}T^2 t) f_\alpha\big) \\ &= \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} e_q(|\alpha|^2)^{-1/2}\frac{|\alpha|^n}{\sqrt{[n]!}} e^{in(\theta-\theta_0)}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\right. \\ &\quad \left.\times\, e_q(i\beta[n][n-1])\right|^2, \end{aligned} \tag{42}$$

where $\beta = \gamma t$.

4.1.3. For an initial coherent phase vector (Das, 1999a)

$$f_\beta = \Phi(|\beta|^2)^{-1/2}\sum_{n=0}^{\infty}\beta^n\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\,f_n, \tag{43}$$

with $|\beta| < 1$ and

$$\Phi(|\beta|^2) = \sum_{n=0}^{\infty}|\beta|^{2n}\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}, \tag{44}$$

we take the initial density operator to be

$$\rho(0) = |f_\beta><f_\beta|. \tag{45}$$

Then $P(\theta, t)$ is given by

$$\begin{aligned}
P(\theta,\delta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2t)|f_\beta><f_\beta|e_q(i\gamma T^{*2}T^2t)f_\theta\big)\\
&= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma T^{*2}T^2t)\big(f_\beta, e_q(i\gamma T^{*2}T^2t)f_\theta\big)f_\beta\big)\\
&= \frac{1}{2\pi}\big(f_\beta, e_q(i\gamma T^{*2}T^2t)f_\theta\big)\big(f_\theta, e_q(-i\gamma T^{*2}T^2t)f_\beta\big)\\
&= \frac{1}{2\pi}\left|\sum_{n=0}^{\infty}\Phi(|\beta|^2)^{-1/2}\bar{\beta}^n e^{in\theta}\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}\right.\\
&\qquad\left.\times\, e_q(i\delta[n][n-1])\right|^2,
\end{aligned} \tag{46}$$

where $\delta = \gamma t$.

4.2. The second dynamic evolution of the density operator for our consideration is given by

$$\rho(t) = e_q\big(-i\gamma(T^*T)^2t\big)\rho(0)e_q\big(i\gamma(T^*T)^2t\big) \tag{47}$$

where γ is the Kerr constant of the medium. The time evolution of the corresponding phase distribution is given by,

$$P(\theta, t) = \frac{1}{2\pi}(f_\theta, \rho(t)f_\theta) \tag{48}$$

4.2.1. For an initial incoherent vector

$$\rho(0) = |f_n><f_n| \tag{49}$$

$P(\theta, t)$ is given by

$$\begin{aligned} P(\theta,\beta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2t)|f_n>< f_n|e_q(i\gamma(T^*T)^2t)f_\theta\big) \\ &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2t)\big(f_n, e_q(i\gamma(T^*T)^2t)f_\theta\big)f_n\big) \\ &= \frac{1}{2\pi}\big(f_n, e_q(i\gamma(T^*T)^2t)f_\theta\big)\big(f_\theta, e_q(-i\gamma(T^*T)^2t)f_n\big) \\ &= \frac{1}{2\pi}\left|\big(f_n, e_q(i\gamma(T^*T)^2t)f_\theta\big)\right|^2 \\ &= \frac{1}{2\pi}\left|e^{in\theta}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}e_q(i\gamma t[n]^2)\right|^2 \\ &= \frac{1}{2\pi}\left|\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}e_q(i\beta[n]^2)\right|^2 \end{aligned} \tag{50}$$

where $\beta = \gamma t$.

4.2.2. For an initial Coherent vector

$$\rho(0) = |f_\alpha>< f_\alpha|, \quad \alpha = |\alpha|e^{i\theta_0} \tag{51}$$

$P(\theta, t)$ is given by

$$\begin{aligned} P(\theta,\beta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2t)|f_\alpha>< f_\alpha|e_q(i\gamma(T^*T)^2t)f_\theta\big) \\ &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2t)\big(f_\alpha, e_q(i\gamma(T^*T)^2t)f_\theta\big)f_\alpha\big) \\ &= \frac{1}{2\pi}\big(f_\alpha, e_q(i\gamma(T^*T)^2t)f_\theta\big)\big(f_\theta, e_q(-i\gamma(T^*T)^2t)f_\alpha\big) \\ &= \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} e_q(|\alpha|^2)^{-1/2}\frac{|\alpha|^n}{\sqrt{[n]!}}e^{in(\theta-\theta_0)}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\right. \\ &\quad \left.\times\, e_q(i\beta[n]^2)\right|^2 \end{aligned} \tag{52}$$

where $\beta = \gamma t$.

4.2.3. For an initial coherent phase vector (Das, 1999a)

$$f_\beta = \Phi(|\beta|^2)^{-1/2}\sum_{n=0}^{\infty}\beta^n\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}f_n \tag{53}$$

with $|\beta| < 1$ and

$$\Phi(|\beta|^2) = \sum_{n=0}^{\infty} |\beta|^{2n} \frac{(q + [0]) \cdots (q^n + [n-1])}{[n]!}. \tag{54}$$

We take the initial density operator to be

$$\rho(0) = |f_\beta >< f_\beta|. \tag{55}$$

Then $P(\theta, t)$ is given by

$$\begin{aligned} P(\theta, \delta) &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2 t)|f_\beta >< f_\beta|e_q(i\gamma(T^*T)^2 t) f_\theta\big) \\ &= \frac{1}{2\pi}\big(f_\theta, e_q(-i\gamma(T^*T)^2 t)\big(f_\beta, e_q(i\gamma(T^*T)^2 t) f_\theta\big) f_\beta\big) \\ &= \frac{1}{2\pi}\big(f_\beta, e_q(i\gamma(T^*T)^2 t) f_\theta\big)\big(f_\theta, e_q(-i\gamma(T^*T)^2 t) f_\beta\big) \\ &= \frac{1}{2\pi}\left| \sum_{n=0}^{\infty} \Phi(|\beta|^2)^{-1/2} \bar{\beta}^n e^{in\theta} \frac{(q + [0]) \cdots (q^n + [n-1])}{[n]!} \right. \\ &\quad \left. \times\, e_q(i\delta[n]^2) \right|^2 \end{aligned} \tag{56}$$

where $\delta = \gamma t$.

5. PROCESS OF PHOTON ABSORPTION FROM A THERMAL BEAM

We next consider the phenomenon of photon absorption from a thermal beam (Agarwal, 1992). The density operator associated with the process can be written as

$$\rho = cT^{*s} \rho_0 T^s \tag{57}$$

where c is a normalization constant.

If we take the input field as a coherent vector, then the density operators for the input and the absorbed field are

$$\rho_{\text{in}} = |f_\alpha >< f_\alpha|, \quad \alpha = |\alpha| e^{i\theta_0} \tag{58}$$

and

$$\rho_{\text{out}} = cT^{*s}|f_\alpha >< f_\alpha|T^s, \quad s > 0. \tag{59}$$

Having obtained the density operator for the output field, we can now calculate the corresponding phase distribution. The phase distribution $P_{\text{in}}(\theta)$ corresponding to ρ_{in} has already been calculated in Section 3.2. The phase distribution $P_{\text{out}}(\theta)$ for the absorbed field is given by

$$\begin{aligned}
P_{\text{out}}(\theta) &= \frac{1}{2\pi}(f_\theta, \rho_{\text{out}} f_\theta) \\
&= \frac{1}{2\pi}\left(f_\theta, cT^{*s}|f_\alpha >< f_\alpha|T^s f_\theta\right) \\
&= \frac{c}{2\pi}\left(f_\theta, T^{*s}\left(f_\alpha, T^s f_\theta\right) f_\alpha\right) \\
&= \frac{c}{2\pi}\left|\left(f_\alpha, T^s f_\theta\right)\right|^2 \qquad (60) \\
&= \frac{c}{2\pi}\left|\sum_{n=0}^{\infty} e_q(|\alpha|^2)^{-1/2}\frac{|\alpha|^n}{\sqrt{[n]!}}e^{in(\theta-\theta_0)}\sqrt{\frac{(q+[0])\cdots(q^n+[n-1])}{[n]!}}\right. \\
&\quad \left.\times\sqrt{[n][n-1]\cdots[n-s+1]}\right|^2
\end{aligned}$$

6. CONCLUSION

In conclusion, we have shown how the phase distribution associated with the field evolves in various nonlinear processes. Specifically, we observed how phase distribution evolves when it propagates through Kerr-like mediums and when it undergoes the process of photon absorption from a thermal beam. In all the cases, we have defined phase distribution with the help of quasiprobabilities associated with the fields, and phase operator coming out of Susskind-Glogower type decomposition of annihilation operator has been used.

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