PROBABILISTIC ASPECTS OF RINGS OF OPERATORS

By

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P R E F A C E

This thesis is submitted to the Indian Statistical Institute in support of the author's application for the degree of Doctor of Philosophy. The thesis embodies the research carried out by the author under the supervision of Dr. J. Sethuraman at the Indian Statistical Institute, Calcutta.

This thesis consists of three chapters. The first chapter deals with convergence in measure in rings (of operators) and allied results, the second, is devoted to a study of stability and mixing in rings, and the last chapter is concerned with some dominated convergence theorems in rings.

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C O N T E N T S

I N T R O D U C T I O N

Rings of Operators, known also as von Neumann Algebras, are rightly regarded as the most distinguished of all Banach Algebras. The study of such rings, initiated by Murray and von Neumann, was sedulously fostered by several mathematicians, notably, Segal, Kadison, Kaplansky, Dye etc., of the American School, Dixmier, Godement etc., of the French School and Umegaki, Nakamura, Takeda, Misonou Turumaru, etc., of the Japanese School.

The year 1953 saw a sensational development in the theory of rings of operators, when, Segal, motivated by certain investigations in quantum mechanics, in harmonic analysis on groups, and his own earlier work on operator algebras, brought out the fundamental paper [7], laying, thereby, the foundations of a non-commutative integration theory. He introduced the concepts of measurable operators affiliated with a gage space, convergence nearly everywhere in a ring, integrable operators and square-integrable operators. He established the completeness of the space of integrable operators and the space of square-integrable operators. He also proved the Radon-Nikodym Theorem for certain classes of linear functionals on arbitrary gage spaces. Thus, Segal's work verily blazed a new trial, in that, it enabled mathematicians to build a theory of rings, which closely parallels measure theory.

The kind of work, initiated by Segal, was ably pursued by Stinespring [8]. He defined convergence in measure in gage spaces, studied the inter-relation between convergence in measure and convergence nearly everywhere, and proved, inter alia, non-commutative versions of Fatou's Lemma and Fubini's Theorem. And, deserving of special mention are his results which state that under some mild restrictions, a continuous function preserves convergence in the L_2-mean, and that, a sequence (T_n) of self-adjoint measurable operators, converges in measure to a self-adjoint measurable operator T, if and only if, for any real continuous function Φ with compact support on the real line, $(\Phi(T_n))$ converges in measure to $\Phi(T)$.

Utilising the powerful analytical machinery set up by Segal and Dixmier, Umegaki evolved the concept of conditional expectation in rings. Combining the results of Segal, Dixmier and Kaplansky, he set out, in a series of interesting papers, [10], [11], to prove the existence and uniqueness of conditional expectation, the existence of M-nets (known, in the commutative case, as martingales), and some of their important properties (which, in the commutative case include martingale convergence theorems).

The concept of conditional expectation opened up many new vistas in the theory of finite rings. Umegaki himself was fully aware of this. In a ring, he introduced, and also made a somewhat extensive study of, probabilistic concepts, such as the Kullback-Leibler Information, the divergence in that information, statistical sufficiency etc., etc., [19], [11].

Drawing freely on the works of Segal, Stinespring and Umegaki, we have, in the present thesis, extended certain standard results in Probability Theory to finite rings.

Throughout this thesis, (except in Section 1 of Chapter III), we shall be concerned exclusively with a probability gage space (i.e.) a gage space (H, α, m), wherein gage of the identity operator is unity (m(I) = 1). This thesis is divided into three chapters.

In the ~~first~~ second section of Chapter I, we prove that convergence in measure is preserved by a continuous functi which is expressible as the sum of a finite number of real, and monotonic continuous functions. We indicate a few applications of this result, including one to operator-entropy. We, then obtain a necessary and sufficient condition for convergence in measure and convergence nearly everywhere to coincide, and also a necessary and sufficient condition for convergence in the L_1-mean to take place. Further, we prove a non-commutative version of Egoroff's Theorem and its converse.

In the ~~second~~ subsequent sections, we introduce the notion of a distribution function of a self-adjoint measurable operator with respect to a state (positive, normal linear funtional) ϕ of α. Let F_n be the distribution function of T_n, and F that of T, with respect to ϕ, $n = 1,2,\ldots$ Let x be any continuity-point of F. We prove that, if (T_n) converges to T in measure, then $F_n(x) \to F(x)$. Let $\phi = m$, and F_n, F and x be as above. Then, we also show that, (T_n) converges to T in measure, if and only if, for each such x, (P_n^x) converges in measure to P^x, where, P_n^x and P^x are the spectral projections of T_n and T respectively, corresponding to $(-\infty, x]$.

A few simple corollaries are also deduced.

The proofs of the above results lean heavily on some theorems of Stinespring [8].

In Chapter II, we define the concepts of 'zero-one,' 'mixing,' and 'strongly mixing.' Let ϕ be any state whose support is central. The main result of this chapter is that a sequence of projections is zero-one with respect to ϕ, if and only if, it is strongly mixing with respect to ϕ. The proof is accomplished by an application of the non-commutative martingale convergence theorem due to Umegaki, and some other lemmas.

Chapter III consists of two sections.

In Section I, we prove two dominated convergence theorems, dropping the assumption that the gage of the identity is finite. One of our theorems is a generalisation of a theorem of Stinespring [8].

In Section II, we prove two dominated convergence theorems, assuming $m(I) = 1$.

As applications, we show that, if R is a bounded self-adjoint operator, and (T_n) a sequence of non-negative square-integrable operators converging in the L_2-mean to an operator T, then, the operator-entropy of T_n tends in the L_1-mean to the operator-entropy of T, and, the information about T_n contained in R(as defined by Umegaki and Nakamura) tends to the information about T contained in R.

CHAPTER I

CONVERGENCE IN MEASURE IN RINGS OF OPERATORS, AND RELATED RESULTS

Section 1: Summary and Preliminaries

Summary: Throughout this thesis, the notation and terminology will be the same as those of [7] and [8]. Let (H, α, m) be a *gage* space in the sense of [7], I the identity operator, and m, a regular gage on α with $m(I) = 1$. By an operator we shall always mean an operator measurable with respect to α in the sense of [7]. Let T be any self-adjoint operator and P_E its spectral projection corresponding to any borel set E. Let ϕ be any faithful state of α((i.e.) a faithful, positive normal linear functional with $\phi(I) = 1$). The measure μ defined by $\mu(E) = \phi(P_E)$ will be called *the probability measure associated with* T (with respect to ϕ). And the point-function F defined at any point x by $F(x) = \mu\{(-\infty, x]\}$ will be called the distribution function of T with respect to ϕ. A point x at which F is continuous is called a continuity-point of F. Clearly a point x is a continuity-point of F, if and only if $\mu(x) = 0$ ((i.e.) the single-point x carries zero μ-measure). When we say simply distribution function of T, we mean its distribution function with respect to the gage m. Following the measure theoretic case, we define that a sequence (T_n) of self-adjoint operators with distribution functions (F_n), converges weakly to a self-adjoint operator T with distribution function F, if, at every continuity-point x of F, $F_n(x) \to F(x)$. The main results of this Chapter are as follows:-

1. Let (T_n) be a sequence of self-adjoint operators converging in measure to a self-adjoint operator T. Let Φ be a real continuous function expressible as the sum of a finite number of real, and monotonic continuous functions. Then it is shown that $(\Phi(T_n))$ converges in measure to $\Phi(T)$. A few applications are given.

2. A non-commutative version of Egoroff's Theorem and its converse is proved. Then a necessary and sufficient condition under which convergence in measure and convergence nearly everywhere coincide is obtained. Also established is a necessary and sufficient condition under which a sequence (T_n) of self-adjoint integrable operators converges in the L_1-mean to a self-adjoint operator T.

3. Let ϕ be any faithful state of α. Let G be the distribution function of a self-adjoint operator T w.r.t. ϕ, and G_n that of a self-adjoint operator T_n, w.r.t. ϕ $(n = 1, 2, \ldots)$. Let now (T_n) converge in measure to T. Then it is proved that at every continuity point x of G, $G_n(x) \to G(x)$.

4. If $(T_n - S_n)$ converges in measure to zero, and (T_n) converges to T weakly, then, (S_n) also converges to T weakly.

5. Corresponding to $(-\infty, x]$, (the infinite interval closed at x), let P_n^x denote the spectral projection of T_n and P^x that of T. Then (T_n) converges in measure to T, if and only if, for every continuity-point x of the distribution function of T, (P_n^x) converges in measure to P^x.

As a corollary, a generalisation of Slutsky's result in the commutative case is obtained. Also established is a necessary and sufficient condition for weak convergence and convergence in measure to be equivalent.

The techniques adopted in proving the above results and in particular, results 3, 4 and 5, are totally different from those of the measure-theoretic case since the latter do not directly extend to the general case.

<u>Preliminaries</u>: Although the notation and terminology of this thesis will be the same as those of the papers [7 and 8] we shall define some concepts and explain some symbols, which, we shall repeatedly deal with.

Let (H, α, m) be the underlying gage space. For any operator A in α, $\| A \|$ will denote the operator-norm of A. For any measurable operator T (not necessarily bounded), $| T |$ will denote $(T^* T)^{1/2}$. The extension of the gage m to the class of all integrable operators will also be denoted by m. The L_1-norm of T, denoted as $\|T\|_1$ is the number $m(|T|)$. The L_2-norm of T, denoted as $\|T\|_2$ is the number $[m(T^*T)]^{1/2}$. For any two projections P and Q in α, $P \vee Q$ and $P \wedge Q$ will denote respectively the lattice-sum and lattice-product of P and Q. A projection and its range will be denoted by the same symbol.

Let A and B be any two operators measurable with respect to α. Throughout this thesis (except Chapter II), $A + B$ and $A.B$ will denote respectively the strong-sum and strong-product of A and B [7].

Let T and T_n be operators measurable with respect to α $(n = 1, 2, \ldots)$.

The sequence (T_n), will be said to converge in measure to T, if, given any $\epsilon > 0$, there exists a se sequence (Q_n) of projections in α, such that $||(T_n - T)Q_n|| < \epsilon$ for all n, and $m(Q_n) \to 1$.

It will be said to converge nearly everywhere to T, if, given any $\epsilon > 0$, there exists a sequence (Q_n) of projections in α, such that $||(T_n - T)Q_n|| < \epsilon$ for all n and $Q_n \uparrow I$.

It will be said to converge to T in the L_p-mean, if $||T_n - T||_p \to 0$, as $n \to \infty$, $p = 1, 2$, and converge to T almost uniformly, if given any $\partial > 0$, one can find a projection P in α with $m(P) \geq 1 - \partial$ and such that $||(T_n - T)P|| \to 0$ as $n \to \infty$.

<u>Section 2: Preservation of Convergence in Measure by Continuous Functions</u>

<u>Theorem 1</u>: Let (T_n) be a sequence of operators converging in measure to an operator T. Let R_n^{ϵ} denote the spectral projection of $|T_n - T|$ corresponding to the interval (ϵ, ∞). Then there exists a subsequence (n_k) such that

$$\sum_{k=1}^{\infty} m(R_{n_k}^{\epsilon}) < \infty .$$

Proof: In the case of an arbitrary gage space, it has been proved by Stinespring [8, Page 23], that (T_n) has a subsequence which converges to T nearly everywhere. His methods, together with the finiteness of the gage, yield the desired result.

Theorem 2: Let (T_n) be an arbitrary sequence of self-adjoint operators converging in measure to a self-adjoint operator T. Let Φ be a real-valued and everywhere defined continuous function expressible as the sum of a finite number of monotonic continuous functions. Then $(\Phi(T_n))$ converges to $\Phi(T)$ in measure.

Proof:

Case I: Φ is strictly increasing, continuous and range of Φ is the whole real line.

Proof: Let $\Phi(T_n) = S_n$ and $\Phi(T) = S$. Then S and S_n are self-adjoint. Let 'a' be any continuous function with compact support. Let $a \circ \Phi$ denote the composite map defined thus:- For any real number λ, $(a \circ \Phi)(\lambda) = a(\Phi(\lambda))$. Easy to verify that $(a \circ \Phi)(T_n) = a(\Phi(T_n)) = a(S_n)$. As Φ is strictly increasing and its range is the whole real line, $a \circ \Phi$ is once again a continuous function with compact support.

Since (T_n) converges in measure to T, it follows by Stinespring's Theorem [8 , Page 33, Theorem 5.5], that $((a \circ \Phi)(T_n))$ converges in the L_2-mean to $(a \circ \Phi)(T)$ (i.e.,) $(a(S_n))$ converges in the L_2-mean to $a(S)$. Since 'a' is arbitrary, it follows by the same Theorem that (S_n) converges in measure to S.

__Case 2:__ Φ is continuous, strictly increasing and its range is a bounded interval.

__Proof:__ Let $a(\lambda) = \lambda$. Let $b(\lambda) = a(\lambda) + \Phi(\lambda)$. Then by case 1, $(b(T_n))$ converges in measure to $b(T)$, so that $(b(T_n) - T_n)$ converges in measure to $b(T) - T$ (i.e) $(\Phi(T_n))$ converges in measure to $\Phi(T)$.

__Case 3:__ Φ is strictly increasing and continuous and its range is an unbounded interval with a finite left-hand end point.

__Proof:__ Let $\Phi(0) = k$. Without loss of generality, we can assume $k = 0$, as otherwise we can consider $f(\lambda) = \Phi(\lambda) - k$. Define a new function $g(\lambda)$ thus:-
For $\lambda > 0$, $g(\lambda) = \Phi(\lambda)$. For $\lambda = 0$, and $\lambda < 0$, $g(\lambda)=\lambda$.
Let $b(\lambda) = g(\lambda) + \Phi(\lambda)$. By case 1, $(g(T_n))$ converges in measure to $g(T)$ and $(b(T_n))$ to $b(T)$. Hence

$(\Phi(T_n))$ converges in measure to $\Phi(T)$. The case where the range of Φ is an interval bounded on the right but not bounded on the left can be disposed of similarly. Thus the Theorem has been proved for any strictly increasing continuous function Φ. Similarly one can show that it is true when Φ is decreasing strictly. More generally if Φ is the sum of a finite number of monotonic, continuous functions, then $(\Phi(T_n))$ converges in measure to $\Phi(T)$.

<u>Corollary 1</u>: Let Φ be expressible as the sum of a finite number of monotonic (not necessarily strictly) continuous functions. Then $(\Phi(T_n))$ converges in measure to $\Phi(T)$.

<u>Proof</u>: It suffices to prove the corollary when Φ itself is decreasing and continuous. Let $g(\lambda) = \Phi(\lambda) - \lambda$, so that $\Phi(\lambda) = g(\lambda) + \lambda$. As $g(\lambda)$ is strictly decreasing, $(g(T_n))$ converges in measure to $g(T)$. Hence $(\Phi(T_n))$ converges in measure to $\Phi(T)$.

<u>Applications</u>:

1. Let S be a self-adjoint measurable operator, P the spectral projection of S, corresponding to $[0, \infty)$ and $Q = I - P$. The operator SP will be denoted by S^+ and the operator $-SQ$ by S^-. Let (T_n) converge in

measure to T. Then one can show that (T_n^+) converges in measure to T^+, and $-(T_n^-)$, to T^-.

<u>Proof</u>: Let $g(\lambda) = \lambda$, $\lambda \geq 0$

and $= 0$, $\lambda < 0$.

Then, by the above Theorem, $(g(T_n))$ converges in measure to $g(T)$. But, $g(T_n) = T_n^+$ and $g(T) = T^+$ The other part can be proved similarly.

2. Let (T_n) be an arbitrary sequence of operators (not necessarily self-adjoint), converging in measure to T. Then $(|T_n|)$ converges in measure to $|T|$.

<u>Proof:</u> It is known that if (T_n) converges in measure to T, then $(T_n^* T_n)$ converges in measure to T^*T. Let $g(\cdot)$ be defined thus:- $g(\lambda) = 0$, $\lambda \leq 0$, and $g(\lambda) = +(\lambda)^{1/2}$, for $\lambda \geq 0$. Let $S_n = T_n^* T_n$, and $S = T^*T$. Then $(g(S_n))$ converges in measure to $g(S)$.

3. Let (T_n) be a sequence of non-negative operators. For each n, $- T_n \log T_n$ is called the operator-entropy of T_n, and $m(-T_r \log T_n)$ is called the numerical entropy of T_n. Let (T_n) converge in measure to an operator T. Then T can be shown to be non-negative so that $T \log T$ can be defined. Now the function $\lambda \log \lambda$ is

expressible as the sum of a finite number of monotonic, continuous functions so that by the above Theorem, the operator-entropy of T_n, converges in measure to the operator-entropy of T. Let now (T_n) be uniformly bounded (i.e.), there exists a positive integer k such that $||T_n|| \leq k$ for all n. In this case, there is convergence even in the L_1-mean. Hence $m(-T_n \log T_n) \to m(-T \log T)$. In other words, the entropy function is continuous in bounded sets of α in the topology of convergence in measure.

4. Let g be a strictly monotonic and real function, continuous everywhere on the real line. Let $g(0) = 0$. Also let g be bounded (i.e.) for some positive integer k, $|g(\lambda)| < k$, for all real λ. For any two measurable operators T and S, define $p(T, S) = m(g(|T-S|))$. Then, (T_n) converges in measure to T, if and only if, $p(T_n, T) \to 0$ as $n \to \infty$.

<u>Proof</u>: Firstly let (T_n) converge in measure to T. This implies that $(|T_n - T|)$ converges in measure to zero. Set $S_n = |T_n - T|$. As (S_n) converges in measure to zero, it follows by Theorem 2, that $(g(S_n))$ converges in measure to $g(0) = 0$. Since $|| g(S_n)|| \leq k$, for all n, it even follows that $m(g(S_n)) \to 0$ (i.e.) $p(T_n, T) \to 0$.

Conversely, let $p(T_n, T) \to 0$. This implies that $m(g(S_n)) \to 0$. As $g(S_n)$ is non-negative for each n, it follows that $(g(S_n))$ converges to zero in measure. Now, if possible let S_n not converge in measure to zero. By a result of Stinespring [8], there exists, therefore, some $\epsilon > 0$, such that the sequence $(m(R_n^\epsilon))$ does not converge to zero, where R_n^ϵ is the spectral projection of S_n corresponding to (ϵ, ∞). Hence, there exists, at least one subsequence, which will be denoted by $(m(R_{n_k}^\epsilon))$ and which converges to a strictly positive number L. As g is strictly monotonic, $g(\epsilon) > g(0) = 0$.

$$
\begin{aligned}
\text{Now} \quad 0 < g(\epsilon).L = g(\epsilon). &\operatorname*{Lt}_{n_k \to \infty} m(R_{n_k}^\epsilon) \\
&= \operatorname*{Lt}_{n_k \to \infty} m(g(\epsilon).R_{n_k}^\epsilon) \\
&< \operatorname*{Lt}_{n_k \to \infty} m(g(S_n)) \\
&= 0.
\end{aligned}
$$

Thus $L = 0$. And this contradiction shows that (S_n) converges in measure to zero.

Section 3: Egoroff's Theorem and the Various Modes of Convergence

Theorem 3: A sequence (T_n) converges nearly everywhere to T, if and only if it converges to T almost uniformly.

Remark: This Theorem generalises Egoroff's Theorem[1] and its converse.

Proof: Let (T_n) converge to T nearly everywhere. We shall show that (T_n) converges to T almost uniformly. Without loss of generality, we may take $T = 0$. Let (ϵ_k) be a sequence of positive numbers converging to zero. By the definition of nearly everywhere convergence, corresponding to ϵ_j, $(j = 1,2,\dots)$, there exists a sequence (Q_{jn}) such that $||T_n Q_{jn}|| < \epsilon_j$ for all n, and for a fixed j, $Q_{jn} \uparrow I$ as $n \to \infty$. Given any positive number $\delta > 0$, one can, in view of the latter property, find a projection Q_{1n_1} such that $m(Q_{1n_1}) > 1 - \frac{\delta}{2}$, and similarly a projection Q_{2n_2} such that $m(Q_{2n_2}) > 1 - \frac{\delta}{2^2}$, ... and in general, a projection Q_{jn_j} (from the sequence (Q_{jn})), such that $m(Q_{jn_j}) > 1 - \frac{\delta}{2^j}$. For convenience write $Q_j = Q_{jn_j}$. Let $S = \bigwedge_{j=1}^{\infty} Q_j$. Let $S^{\perp} = I - S$. Then

$m(S^{\perp}) = m(\vee_j Q_j^{\perp}) \leq \sum_j m(Q_j^{\perp}) = \sum_j \frac{\partial}{2^j} = \partial$. Hence $m(S) \geq 1 - \partial$.

Let an arbitrary positive number ϵ be given. Since $\epsilon_k \to 0$ as $k \to \infty$, $\epsilon_k < \epsilon$ for all $k \geq$ some positive integer L. Hence for any $k \geq n_L$, $||T_k S|| \leq ||T_k Q_{L,n_L}||$

$$\leq \epsilon_L < \epsilon. \text{ And } m(S) > 1 - \partial.$$

This being true of any arbitrary $\partial > 0$, it follows that (T_n) converges to 0 almost uniformly.

<u>**Converse:**</u> Let (T_n) converge to 0 almost uniformly. We shall show that (T_n) converges to 0 nearly everywhere.

<u>**Proof:**</u> Let ∂ be a positive number less than 1. In view of our assumption there exists a sequence (S_k) of projections such that $m(S_k) > 1 - \frac{\partial}{2^k}$, and for each fixed k, $||T_n S_k|| \to 0$ as $n \to \infty$. Let $R_1 = \bigwedge_{k=1}^{\infty} S_k$, $R_2 = \bigwedge_{k=2}^{\infty} S_k, \ldots$ $\ldots, R_n = \bigwedge_{k=n}^{\infty} S_k, \ldots$ Then $m(R_n^{\perp}) \leq \sum_{i=n}^{\infty} m(S_i^{\perp}) = \sum_{j=n}^{\infty} \frac{\partial}{2^j} = \frac{\partial}{2^{n-1}}$. So $m(R_n) \to 1$. As m is regular, and $R_k < R_{k+1}$, for all k, it follows that $R_n \uparrow I$. Now, for each k, $R_k < S_k$. Since for each fixed k, $||T_n S_k|| \to 0$ as $n \to \infty$, given k, and $\epsilon > 0$, one can find a positive integer N_k such that $||T_n S_k|| < \epsilon$ for all $n \geq N_k$,

($k = 1, 2, 3, \ldots$). Define a sequence (E_k) of projections thus:-

$$E_1 = E_2 = \ldots = E_{N_1 - 1} = 0 \quad \text{(the zero projection)}$$

$$E_{N_1} = E_{N_1+1} = \ldots = E_{N_1+N_2-1} = R_1$$

$$E_{N_1+N_2} = E_{N_1+N_2+1} = \ldots = E_{N_1+N_2+N_3-1} = R_2.$$ And, in general,

$$E_{N_1+N_2+\ldots+N_k} = \ldots = E_{N_1+N_2+\ldots+N_{k+1}-1} = R_k, \ldots, \quad k=1,2,\ldots$$

The sequence $E_n \uparrow I$. For $n < N_1$, $\| T_n E_n \| = 0$ and so is less than ϵ. For $n \geq N_1$, n lies between two numbers $N_1 + \ldots + N_k$ and $N_1 + N_2 + \ldots + N_{k+1} - 1$, so that $\| T_n E_n \| = \| T_n R_k \| < \| T_n S_k \| < \epsilon$ (since $R_k \leq S_k$). Thus for all n, $\| T_n E_n \| < \epsilon$. For any given $\epsilon > 0$, the choice of one such sequence (E_n) being possible. it follows that (T_n) converges nearly everywhere to 0. Hence the Theorem.

In what follows, a projection P in α will be said to be minimal, if for any projection Q in α, $Q \leq P$ implies $Q = P$ or $Q = 0$.

<u>Theorem 4</u>: Convergence in measure and convergence nearly everywhere are equivalent if and only if each projection in α contains a minimal projection.

<u>Proof:</u> Let each projection contain a minimal projection. We shall show that these two notions of convergence coincide. Let $I = P_1 + P_2 + \dots + P_n + \dots$, be some resolution of the identity into a sequence of pairwise orthogonal minimal projections. Let $m(P_i) = \partial_i$, since $\partial_i \geq 0$, and $\sum_{i=1}^{\infty} \partial_i = 1$, without loss of generality, we can assume that $\partial_1 \geq \partial_2 \geq \dots \geq \partial_n \geq \dots$ Let (T_n) be a sequence of operators converging in measure to an operator T. We may take $T = 0$. By the definition of convergence in measure, given ϵ, there exists a sequence (Q_n) of projections such that $m(Q_n) \to 1$ and $\|T_n Q_n\| < \epsilon$ for all n. As $m(Q_n) \to 1$, there exists a positive integer N_1, such that $m(Q_n) > 1 - \frac{\partial_1}{2}$ for all $n \geq N_1$, and similarly a positive integer N_2, such that $m(Q_n) > 1 - \frac{\partial_1}{2} - \frac{\partial_2}{2}$ for all $n \geq N_2$, and in general, a positive integer N_k such that $m(Q_k) \geq 1 - \frac{\partial_1}{2} - \frac{\partial_2}{2} \dots - \frac{\partial_k}{2}$, k ranging over all positive integral values. Obviously for $n \geq N_1$, $Q_n \wedge P_1$ is non-null. But as P_1 is minimal, this implies $P_1 \leq Q_n$ for $n \geq N_1$.

Similarly $P_1 \vee P_2 \leq Q_n$ for $n \geq N_1 + N_2$ etc. Define a sequence (R_n) of projections thus:-

Define a sequence (R_n) of projections thus:-

$R_1 = 0 = \ldots \qquad = R_{N_1 - 1}$

$R_{N_1} = \bar{P}_1 = \ldots \qquad = R_{N_1 + N_2 - 1}$

$R_{N_1 + N_2} = P_1 \vee P_2 \ldots \qquad = R_{N_1 + N_2 + N_3 - 1}$ etc.

As $\sum_{i=1}^{\infty} m(P_i) = 1$, it follows that $R_n \uparrow I$, and $||T_n R_n|| < C$ for all n. Hence convergence in measure implies convergence nearly everywhere. Since in the case of a finite gage space, convergence in measure is always implied by convergence nearly everywhere, it follows that they are equivalent in this case.

Converse: Let convergence in measure and convergence nearly everywhere coincide. We shall show that each projection has to contain a minimal projection. If not, let there exist a non-null projection P, with the following property:- $m(P) = \partial$, and for any β

there exists a projection $R < P$, wi

can show by arguments similar to tho

Measure Theory, that this implies th ... on.

can be expressed as the sum of n pairwise orthogonal projections $P_{1n},\ldots P_{nn}$, such that $m(P_{in}) = \frac{m(P)}{n}$, $i = 1,2,\ldots, n$. Now, clearly the sequence P_{11}, $2P_{21}$, $2P_{22}$, $3P_{31}$, $3P_{32}$, $3P_{33},\ldots$, nP_{n1}, $nP_{n2},\ldots$, nP_{nn}, $\ldots$, converges in measure to 0. If possible, let this converge nearly everywhere to 0. By the non-commutative version of Egoroff's Theorem, one can find a projection S with $m(S) > 1 - m(P)$, and such that $||nP_{ni}\, S|| \to 0$. Let $\epsilon < 1$. Hence there exists a positive integer N such that

$$||nP_{ni}\, S|| < \epsilon \qquad \ldots \qquad \ldots \qquad (1)$$

for all $n \geq N$. $P \wedge S$ is non-null, as $m(S) > 1 - m(P)$. Let x be a unit vector in $P \wedge S$. Then, for any $n \geq N$,

$$1 = ||\, x\, ||^2 = ||Px||^2 = ||P_{n1}\, x||^2 + \ldots + ||P_{nn}\, x||^2.$$

But by (1), $||P_{ni}\, x||^2 < \frac{\epsilon^2}{n^2}$, $i = 1,2,\ldots, n$. Hence

$$1 < \frac{\epsilon^2}{n^2} + \ldots + \frac{\epsilon^2}{n^2}\ (n \text{ times}) = \frac{\epsilon^2}{n} < \epsilon^2 < 1.$$

This contradiction proves the result.

<u>Theorem 5</u>: A sequence (T_n) of self-adjoint integrable operators converges in the L_1-mean to a self-adjoint operator T, if and only if (T_n) converges in measure to T and (T_n) satisfies the following U-continuity condition.

Given any $\epsilon > 0$ there exists a '∂' > 0, such that for any projection P with $m(P) < \partial$, one has $||T_n P||_1 < \epsilon$ for all n.

__Proof:__ That convergence in the L_1-mean implies convergence in measure is clear [%, Page 24, Theorem 3.2]. To show it also implies U-continuity.

Let (T_n) converge to T in the L_1-mean. Each T_n being integrable, it follows that T is integrable. Hence there exists a bounded operator S, such that $||T - S||_1 < \frac{\epsilon}{4}$. Let $||S|| = k$. Let $\partial_1 = \frac{\epsilon}{4k}$. Then, for any projection P with $m(P) < \partial_1$, $||SP||_1 < \frac{\epsilon}{4}$, so that $||TP||_1 < \frac{\epsilon}{2}$. Since $T_n \to T$ in the L_1-mean, there exists a positive integer N such that $||T_n - T||_1 < \frac{\epsilon}{2}$ for $n \geq N$. Hence $||T_n P||_1 < \epsilon$ whenever $m(P) < \partial$. Also, as $T_1, T_2, \ldots, T_{N-1}$ are all integrable and $N-1$ is finite, one can find a positive number ∂_2 such that for any projection Q with $m(Q) < \partial_2$, we have $||T_n P||_1 < \epsilon$ $n = 1, 2, \ldots N-1$. Let $\partial = \min(\partial_1, \partial_2)$. Thus for any projection P, with $m(P) < \partial$, $||T_n P||_1 < \epsilon$ for all positive integral values of n. For any given ϵ, the choice of one such ∂ being possible, the U-continuity follows.

<u>Converse:</u> Let an arbitrary number $\epsilon > 0$ be given. In view of U-continuity, there exists a $\partial > 0$, such that for any projection P, with $m(P) < \partial$, $||T_n P||_1 < \epsilon$, for all n.

<u>Case 1:</u> Each T_n is non-negative. In this case T is also non-negative. Let Q be the spectral projection of T corresponding to $[N, \infty]$. Choose N large enough so that $m(Q) < \partial$. Then by the non-commutative version of Fatou's Lemma [8 , Page 31], $m(TQ) = m(QTQ) \leq$

$$\leq \liminf_{n \to \infty} m(QT_nQ)$$

$$< \epsilon \quad \text{(i. e.)}.$$

$$||TQ||_1 < \epsilon.$$

Let $\partial = \min \left(\frac{\epsilon}{2N}, \partial_1\right)$. For any projection R with $m(R) < \partial$, $||TR||_1 \leq ||T(I - Q)R||_1 + ||TQR||_1 \leq \frac{\epsilon}{2} + \epsilon \leq 2\epsilon$

Since (T_n) converges to T in measure, given $\epsilon > 0$, there exists a sequence (Q_n) of projections such that $||(T_n - T)Q_n|| < \epsilon$ for all n and $m(I - Q_n) \to 0$ as $n \to \infty$ so that for all $n >$ some positive integer N, $m(I - Q_n) < \partial$. Now for any $n > N$,

$$\| T_n - T \|_1 \leq \| (T_n - T) Q_n \|_1 + \| (T_n - T)(I - Q_n) \|_1$$
$$\leq \epsilon + \| T_n (I - Q_n) \|_1 + \| T (I - Q_n) \|_1$$
$$\leq \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Case 2: Let T_n be self-adjoint for each n. In this case, it follows by a result of Stinespring [8], that the limit T is also self-adjoint. By the corollary to Theorem 2, (T_n^+) converges in measure to T^+ and (T_n^-) converges in measure to T^-. Also, if (T_n) satisfies the U-continuity condition, then so do (T_n^+) and (T_n^-). By applying the result of Case 1, to (T_n^+) and (T_n^-) separately, the proof is completed.

Section 4: *Weak Convergence*

Let (X, Γ, P) be a probability space. By a random variable is meant a real, almost everywhere finite-valued Γ-measurable function. With each random variable ξ, one can associate a probability measure μ defined on the borel sets of the line as $\mu(E) = P[\xi^{-1}(E)]$. This μ is known as the probability measure corresponding to ξ. Let $(-\infty, x] = \{y : -\infty < y \leq x\}$. The function F defined by $F(x) = \mu\{(-\infty, x]\}$ is called the distribution function of ξ. F is always right-continuous. A point x at which F is also left-continuous is known as a continuity-point of F. It can easily be seen that x is a continuity-point of F if and only if $\mu(x) = 0$. The set of continuity-points of a distribution function is known to be dense on the real line. A sequence (ξ_n) of random variables with respective distribution functions (F_n) is said to converge weakly to a random variable ξ, with distribution function F, if, at every continuity-point x of F, $F_n(x) \to F(x)$. Let μ_n be the probability measure associated with ξ_n, and μ, the measure associated with ξ, $n = 1,2,\ldots$ Then it is known that weak convergence takes place if and only if for any bounded, continuous function g, $\int_R g d\mu_n \to \int_R g d\mu$, where R is the real line.

Let now T be any self-adjoint operator and P_E its spectral projection corresponding to the borel set E. We shall call the measure μ, defined by $\mu(E) = m(P_E)$, the probability measure associated with T. And the point-function F defined at any point x by $F(x) = \mu\{(-\infty, x]\}$, we shall call the distribution function of T. Following the measure-theoretic case, we define that a sequence (T_n) of self-adjoint operators with distribution functions (F_n) converges weakly to a self-adjoint operator T with distri-

distribution function F, if, at every continuity-point x of F, $F_n(x) \to F(x)$. (As before, if μ_n is the probability measure associated with T_n, $n = 1, 2, \ldots$, and μ, the measure associated with T, then it is known that a necessary and sufficient condition for weak convergence is that $\int_R g\,d\mu_n \to \int_R g\,d\mu$, for any bounded continuous function g, and R denoting the real line).

More generally, let σ be any faithful state of α, with $\sigma(I) = 1$. Let T be any self-adjoint operator. Since σ is completely additive, using σ, one can, as before, associate a distribution function G with T. This G we shall call the distribution function of T with respect to σ. As σ is faithful, it can easily be seen that a point x is a continuity-point of G, if and only if the spectral projection of T, corresponding to the singleton x is the zero projection. When we say simply the distribution function of an operator, we mean its distribution function with respect to the gage m.

In what follows (T_n) will denote a sequence of self-adjoint operators converging in measure to a self-adjoint operator T, F the distribution function of T, and F_n that of T_n, $(n = 1, 2, \ldots)$. We shall now state and prove four theorems.

__Theorem 6.__ Let x be an arbitrary continuity-point of F. Then $F_n(x) \to F(x)$ as $n \to \infty$.

__Theorem 7.__ Let σ be any faithful state. Let G be the distribution function of T with respect to σ, and G_n, that of T_n with respect to σ; $n = 1, 2, \ldots$ Let x be an arbitrary continuity-point of G. Then $G_n(x) \to G(x)$.

__Theorem 8.__ Let (A_n) and (B_n) be two sequences of self-adjoint operators such that $(A_n - B_n)$ converges in measure to zero. Let (A_n) converge weakly to a self-adjoint operator A ((i.e.) if H_n is the distribution of A_n, and H that of A, then $H_n(x) \to H(x)$ at every point x, which is a continuity-point of H). Then (B_n) also converges weakly to A.

__Theorem 9:__ Let x be an arbitrary continuity-point of F. Let, for each n, P_n^x denote the spectral projection of T_n corresponding to the interval $(-\infty, x]$ and P^x that of T corresponding to the same interval. Then, a necessary and sufficient condition for (T_n) to converge in measure to T, is that, corresponding to each continuity-point x of F, (P_n^x) converges in measure to P^x.

<u>Remarks:</u> Theorems 6 and 7 are consequences of Theorem 9, however, they are stated separately, because the proof of Theorem 9 depends on that of Theorem 6. Theorem 7 is more general than Theorem 6, but cannot be proved directly and has only to be deduced from Theorem 9. In proving the above theorems, we shall utilise the following results of Stinespring.

[8, Page 32, Corollary 5.2]. 'A sequence (A_n) of measurable operators converges in measure to a measurable operator A, if and only if, for any $C > 0$, $m(R_n^C) \to 0$ as $n \to \infty$ where R_n^C is the spectral projection of $|A_n - A|$ corresponding to (C, ∞)'.

[8, Page 33, Theorem 5.5]. 'A sequence (T_n) of self-adjoint measurable operators converges in measure to a self-adjoint measurable operator T, if and only if for any continuous function σ with compact support on the real line $\sigma(T_n) \to \sigma(T)$ in the L_2-mean.'

<u>Proof of Theorem 6.</u>

<u>Case 1:</u> Let the T_n's be uniformly bounded (i.e.) there exists a positive integer k such that $||T_n|| < k$ for all n. In this case, the proof is exceedingly simple. Clearly for each n, the spectrum of T_n is contained in

the closed interval $[-k, k]$. Let μ_n be the probability measure associated with T_n. Easy to verify that all the μ_n's vanish[illegible]e the compact set $[-k, k]$. Hence, in this case, to establish weak convergence, suffice to consider continuous functions with compact support. Now, $\int_R g d\mu_n = m(g(T_n))$ and $\int_R g d\mu = m(g(T))$. But by Stinespring's Theorem mentioned above, $(g(T_n))$ converges in the L_2-mean to $g(T)$; and hence, in particular,

$$m(g(T_n)) \rightarrow m(g(T)), \text{ (i.e.) } \int_R g d\mu_n \rightarrow \int_R g d\mu$$

Hence weak convergence in this case.

<u>Case 2:</u> (General Case):- Let (T_n) be an arbitrary sequence of self-adjoint operators converging in measure to a self-adjoint operator T. Let F_n be the distribution function of T_n, and F that of T. Let x be an arbitrary continuity-point of F. To prove $F_n(x) \rightarrow F(x)$. Corresponding to $(-\infty, x]$, let P_n^x be the spectral projection of T_n, and P^x that of T. Then $F_n(x) = m(P_n^x)$ and $F(x) = m(P^x)$.

The proof in the measure-theoretic case, as given in standard text-books on Probability Theory, such as [1], [2, Page 168] does not directly extend to the non-commutative case. The argument given in those books is of the following type:- Let (X, β, P) be a probability space and x an arbitrary point of the basic space X. Let f_n and f be two random variables, c any fixed point on the real line, and $\epsilon > 0$. Let $A = f_n^{-1}\{(-\infty, c]\}$ and $B = f^{-1}\{(c + \epsilon, \infty)\}$. Let $C = X - B$. Then,

$$A = A \cap B + A \cap C \qquad (1)$$

This is the crucial decomposition on which the proof in the measure-theoretic case hinges. In the general case, let P_n be the spectral projection of T_n corresponding to $(-\infty, c]$, P the spectral projection of T corresponding to $(c + \epsilon, \infty)$, and $Q = I - P$. The equality $P_n = P_n \wedge P + P_n \wedge Q$, is not in general valid, since the associative law does not in general hold for the lattice of projections. In other words, the crucial decomposition (1) does not generalise to the case of rings of operators. Hence, we shall furnish a proof which is different from the one in the measure-theoretic case.

Let x be any fixed continuity-point of F. We shall show that $m(P^x)$ is the only limit-point of the bounded sequence $(m(P_n^x))$. (Note that for all n, $0 \leq m(P_n^x) \leq 1$). Let L be a limit-point of $(m(P_n^x))$. Hence there exists a subsequence, which also will be denoted by $(m(P_n^x))$, and which converges to L. If possible, let $L < m(P^x)$. So, for some $\partial > 0$, one can write $L = m(P^x) - 2\partial$. As x is a continuity-point of F, one can find for some $\varepsilon > 0$, a point $x - \varepsilon$ (to the left of x) such that $F(x) - F(x-\varepsilon) < \frac{\partial}{2}$. (Note that if $P^{x-\varepsilon}$ denotes the spectral projection of T corresponding to $(-\infty, x - \varepsilon]$, then $F(x - \varepsilon) = m(P^{x-\varepsilon})$). Hence $L < m(P^{x-\varepsilon})$. Since $m(P_n^x) \to L$, it follows that there exists a positive integer N_1 such that for all $n \geq N_1$, $m(P_n^x) < L + \frac{\partial}{4} < m(P^x) - \partial$. Let $A_n^x = I - P_n^x$. It follows that for

$$n \geq N_1, \quad m(A_n^x \wedge P^{x-\varepsilon}) \geq \partial \qquad (2)$$

Also, as (T_n) converges in measure to T, it follows, by a result mentioned above that, if S_n denotes the spectral projection of $|T_n - T|$ corresponding to the interval $[0, \varepsilon/2)$, then $m(S_n) \to 1$ as $n \to \infty$. So there exists a positive integer N_2 such that $m(S_n) > 1 - \frac{\partial}{2}$ for all $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then, for any $n \geq N$, $K_n = S_n \wedge A_n^x \wedge P^{x-\varepsilon}$ is non-null. Let c_n be any unit vector in K_n. Since c_n belongs to K_n, and so to S_n, one has, $\| \, |T_n - T| \, c_n \| \leq \frac{\partial}{2}$. But as c_n belongs to $P^{x-\varepsilon}$, $(Tc_n, c_n) \leq x - \varepsilon$. And since c_n is in A_n^x, one has

$(T_n c_n, c_n) \geq x$. Thus $\frac{\epsilon}{2} \geq \| \, |T_n - T| \, c_n \| = \| (T_n - T) c_n \| \geq$ $|((T_n - T) c_n, c_n)| = |(T_n c_n, c_n) - (T c_n, c_n)| \geq x - (x - \epsilon) = \epsilon.$

This contradiction shows that the limit point L cannot be less than $m(P^x)$. Similarly by using the fact that F is right-continuous, one can show that the assumption that $L > m(P^x)$ will also lead to a contradiction. Thus $m(P^x)$ is the only limit-point of the bounded sequence $(m(P_n^x))$. Hence $m(P_n^x) \to m(P^x)$ (i.e.) $F_n(x) \to F(x)$. Hence the Theorem.

<u>Corollary 1</u>: Let x_1 and x_2 be two continuity-points of F. $(x_1 < x_2)$. Let R_n be the spectral projection of T_n corresponding to the interval $(x_1, x_2]$, (open at x_1 and closed at x_2), and R the spectral projection of T corresponding to the same interval. Then $m(R_n) \to m(R)$.

The proof is easy and is omitted.

Theorem 7 cannot be proved in the same way as above, since unlike a gage, a state may not be subadditive (i.e.) for any two arbitrary projections P and Q, the inequality $\sigma(P \vee Q) \leq \sigma(P) + \sigma(Q)$ is not in general valid. We shall deduce Theorem 7 from Theorem 9 after proving the latter.

Proof of Theorem 8:- Theorem 8 is not a consequence of Theorem 6, since the notion of weak convergence is not additive even in the commutative case. However, a proof can be given along the following lines:- Let F_n be the distribution function of A_n, G_n, the distribution function of B_n, $(n = 1,2,\ldots)$ and F that of A. Let x be any continuity-point of F. Corresponding to the interval $(-\infty, x]$, let P_n be the spectral projection of A_n; Q_n that of B_n, and P that of A. By assumption $m(P_n) \to m(P)$. Let L be a limit-point of the sequence $(m(Q_n))$. As such there exists a subsequence, which also will be denoted by $(m(Q_n))$, and which converges to L. If possible, let $L < m(P)$. So, for some $\partial > 0$, $L = m(P) - 2\partial$. As in the argument of Theorem 6, one can find a number $\epsilon > 0$, such that $x-\epsilon$ is a continuity point of F, and that if R is the spectral projection of A corresponding to $(-\infty, x-\epsilon]$, then $m(R) > m(P) - \frac{\partial}{2}$, so that $L < m(R) - \partial$. As $m(Q_n) \to L$, one can find a positive integer M, such that for all $n \geq M$, $m(Q_n) < L + \partial < m(R) - \frac{\partial}{2}$. For each n, let R_n be the spectral projection of A_n corresponding to $(-\infty, x-\epsilon]$. Since $x-\epsilon$ is a continuity-point of F, and (A_n) converges weakly to A, it follows that $m(R_n) \to m(R)$. Hence, one can find a positive integer N_1, such that

$m(Q_n) < m(R_n) - \frac{\delta}{4}$, for all $n \geq N_1$. Let S_n denote the spectral projection of $|A_n - B_n|$ corresponding to the interval $[0, \varepsilon/2)$. Since $(A_n - B_n)$ converges in measure to zero, $m(S_n) \to 1$ and so is $\geq 1 - \frac{\delta}{8}$ for all $n \geq$ some positiv integer N_2. Let $N = \max(N_1, N_2)$. Then for any $n \geq N$, $S_n \wedge Q_n^{\perp} \wedge R_n$ is non-null. Let c_n be any unit vector in $S_n \wedge Q_n^{\perp} \wedge R_n$. Then as c_n is in S_n, $\| \, |A_n - B_n| c_n \| \leq \frac{\varepsilon}{2}$. But as c_n is in $Q_n^{\perp}$, $(B_n c_n, c_n) \geq x$. And as c_n is R_n, $(A_n c_n, c_n) < x - \varepsilon$. Thus $\frac{\varepsilon}{2} > \| (A_n - B_n) c_n \| \geq |(A_n c_n, c_n) - (B_n c_n, c_n)| = |(x - \varepsilon) - x| = \varepsilon$. This contradiction shows that L cannot be less than $m(P)$. Similarly one can show that L cannot be greater than $m(P)$. Thus $m(P)$ is the only limit-point of the bounded sequence $(m(Q_n))$, which proves the Theorem.

As a consequence of the Theorem, we obtain the following corollary, which, in the commutative case, has been proved by Slutsky.

<u>Corollary 2:</u> Let for each n, F_n be the distribution function of a self-adjoint operator S_n, and F that of a self-adjoint operator S. Let (S_n) converge to S weakly, and let (D_n) be another sequence of self-adjoint operations converging in measure to cI (c some real number, and I

the identity operator). Let G_n be the distribution function of $S_n + D_n$ and G that of $S + cI$. Let x be any continuity-point of F. Then $G_n(x+c) \to G(x+c)$.

The proof of this corollary rests on the following proposition:-

'Let N be any self-adjoint operator with distribution function H. Let c and I be as in the above Theorem. Let J be the distribution function of $N+cI$. Then, for any point y on the real line,

$$H(y) = J(y+c)'.$$

The proof is as follows:- For any borel set E on the line, let $E+c$ denote the set of points $z+c$, where z is any point of E. Then one can verify that the spectral projection of N corresponding to E, is the same as the spectral projection of $N+cI$, corresponding to $E+c$.

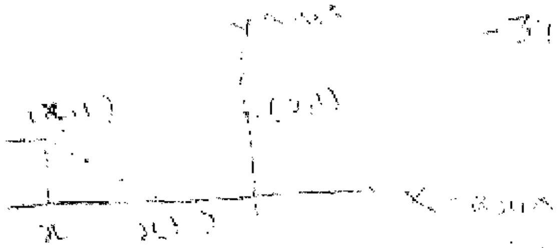
Proof of the Corollary: Set $A_n = S_n+D_n-cI$, $B_n=S_n$. Then $(A_n - B_n)$ converges in measure to zero. Also (B_n) converges weakly to S. Hence by Theorem 8, (A_n) converges weakly to S. Let H_n be the distribution function of A_n and G_n that of A_n+cI. Then $H_n(x) \to F(x)$ where x is any continuity-point of F. By the above proposition

$H_n(x) = G_n(x+c)$ and $F(x) = G(x+c)$. Hence $G_n(x+c) \to G(x+c)$ which proves the corollary.

Proof of Theorem 9:

Let (T_n) converge in measure to T. Let x be any continuity-point of the distribution function F of T. We shall now show that $\{P_n^x\}$ converges in measure to P^x. Let $\psi(\cdot)$ be the Characterestic Function of the set $(-\infty, x]$. Then $P_n^x = \psi(T_n)$ and $P^x = \psi(T)$. We shall prove that $m(|\psi(T_n) - \psi(T)|) \to 0$ which will imply that $\{\psi(T_n)\}$ converges to $\psi(T)$ in the L_1-mean and hence in measure. Let an arbitrary positive number ϵ be given. The continuity-points of F being dense on the line, one can find a sufficiently small δ, $(\delta > 0)$ such that $x+\delta$ is a continuity-point of F and $F(x+\delta) - F(x) < \epsilon$ ... (1). Let R^x denote the spectral projection of T, corresponding to $(x, x+\delta]$. As x and $x+\delta$ are both continuity-points of F, it follows by Corollary 1 that $m(R_n^x) \to m(R^x)$. Hence one can find a positive integer N_1 such that $|m(R_n^x)-m(R^x)| < \epsilon$ for $n \geq N_1$ (i.e.) $m(R_n^x) < 2\epsilon$ for $n \geq N_1$.

Now let σ be the continuous function defined thus:-



For all $\lambda < x$. $\sigma(\lambda) = 1$. For any $\lambda \geq x + \partial$, $\sigma(\lambda) = 0$ In the open interval $(x, x+\partial)$, the graph of σ is the straight line joining the two points $(x, 1)$ and $(x+\partial, 0)$. Clearly, for any λ, $|\sigma(\lambda) - \phi(\lambda)| < 1$, so that $||\sigma(T) - \phi(T)|| \leq 1$. Note that $\sigma(T) - \phi(T)$ is ≥ 0, so that $m(|\sigma(T) - \phi(T)|) = m(\sigma(T) - \phi(T)) = m((\sigma(T) - \phi(T))R^x) \leq ||\sigma(T) - \phi(T)||.m(R^x) < \epsilon$. Similarly for $n > N_1$, $m(|\sigma(T_n) - \phi(T_n)| = m(\sigma(T_n) - \phi(T_n)) = m((\sigma(T_n) - \phi(T_n)).R_n^x) \leq ||\sigma(T_n) - \phi(T_n)||.m(R_n^x) \leq m(R_n^x) < 2\epsilon$. As σ is continuous and decreasing, it follows that $(\sigma(T_n))$ converges in measure to $\sigma(T)$. Since this sequence is uniformly bounded, (bounded in norm by 1), convergence in measure implies convergence in the L_1-mean. So, given ϵ, there exists a positive integer N_2, such that $m(|\sigma(T_n) - \sigma(T)|) < \epsilon$ for $n \geq N_2$. Let $N = N_1 + N_2$. Now for any $n \geq N$, one has

$$m(|\phi(T) - \phi_n(T)|) \leq m(|\phi(T) - \sigma(T)|) + m(|\sigma(T) - \sigma_n(T)|)$$

$$m(|\sigma_n(T) - \phi_n(T)|) \leq \epsilon + \epsilon + 2\epsilon = 4\epsilon \text{ (i.e..}$$

$$|\phi_n(T) - \ldots| - \phi(T)|) \to 0. \text{ (i.e. } (P_n^x) \text{ converges to}$$

in the L_1-mean and hence in measure. And x being any arbitrary continuity-point of F, the desired result follows.

<u>Corollary 3:</u> Let (T_n) converge in measure to T. Let F be the distribution function of T. Let x_1 and x_2 $(x_1 < x_2)$ be any two continuity-points of F. Corresponding to the interval $(x_1, x_2]$, let R_n be the spectral projection of T_n and R that of T. Then (R_n) converges in measure (and because of uniform boundedness, converges in the L_2-mean) to R.

This corollary is an immediate consequence of what has been proved above.

<u>Converse:</u> For any continuity-point x of F, let (P_n^x) denote the spectral projection of T_n and P^x that of T, corresponding to $(-\infty, x]$. Let (P_n^x) converge in measure to P^x. We shall now show that for any continuous function σ with compact support on the real line, $(\sigma(T_n))$ converges in the L_2-mean to $\sigma(T)$, and this will imply by Stinespring's Theorem [8 , Theorem 5] that (T_n) converges in measure to T.

Let σ have compact support $[-k, k]$. Hence $\sigma(-k) = 0 = \sigma(k)$. As σ is uniformly continuous in $[-k, k]$, given any $\epsilon > 0$, one can find a positive number δ, such that for any two points x and y in $[-k, k]$, one has $|\sigma(x) - \sigma(y)| < \epsilon$, whenever $|x-y| < 2\delta$. Choose a point x_1 such that (1) $x_1 > -k$ (2) the distance between x_1 and $-k$

...eds $\frac{\partial}{2}$ but is less than ∂ and (3) x_1 is a conti-...y-point of F. Choose now successively points $x_3, \dots, x_N$ such that (4) $x_2 < x_3 < \dots < x_{N-1} < k$ $x_N > k$ (5) $\partial \geq |x_{i+1} - x_i| \geq \frac{\partial}{2}$ and (6) each x_i ... a continuity point of F, $i = 2,3,\dots, N$. The choice ... such a finite sequence $x_1, x_2, \dots, x_N$ is possible as continuity-points of F are dense on the real line.

Define a new function $\phi(\lambda)$ thus:-

$$\phi(\lambda) = 0 \quad \text{for } \lambda \leq x_1$$

$$\phi(\lambda) = \sigma(x_1), \quad x_1 < \lambda \leq x_2$$

$$\phi(\lambda) = \sigma(x_2), \quad x_2 < \lambda \leq x_3$$

$$\phi(\lambda) = \sigma(x_{N-1}), \quad x_{N-1} < \lambda \leq x_N$$

$\phi(\lambda) = 0$, $\lambda > x_N$. Clearly for any λ,

$$|\phi(\lambda) - \sigma(\lambda)| \leq C.$$

Corresponding to $(x_{i-1}, x_i]$, let R_n^i denote the ...ectral projection of T_n and R^i that of T. $(i = 2, \dots N)$. ~~corollary 3~~ assumption, (R_n^i) converges in measure to R^i, $i = 1, 2, \dots N)$, so that $\phi(T_k) = \sum_{i=2}^{N} \sigma(\lambda_{i-1}) R_k^i$ converges ... measure to $\sum_{i=2}^{n} \sigma(\lambda_{i-1}) R^i = \phi(T)$. Easy to verify that ...$\phi(\lambda) - \sigma(\lambda)| < \epsilon$ for all λ, so that $\|\phi(T_n) - \sigma(T_n)\| < \epsilon$

or all n. Since $((\phi)(T_n))$ converges in measure to $(\phi)(T)$ and is uniformly bounded in norm, it follows that $(\phi)(T_n) \to (\phi)(T)$ in the L_2-mean; (i.e.) $\|(\phi)(T_n) - (\phi)(T)\|_2 \to 0$ as $n \to \infty$. As such, there exists a positive integer N_1 such that $\|(\phi)(T_n)-(\phi)(T)\|_2 < \epsilon$ or all $n \geq N_1$. Again, for each n, $\|(\phi)(T_n)-\sigma(T_n)\|_2 <$ $\|(\phi)(T_n) - \sigma(T_n)\|.m(I) = \epsilon . 1 = \epsilon$. Hence, for any $n \geq N_1$,

$$\|\sigma(T_n)-\sigma(T)\|_2 \leq \|\sigma(T_n) -(\phi)(T)\|_2 + \|(\phi)(T)-(\phi)(T_n)\|_2 + \|(\phi)(T_n) - \sigma(T_n)\|_2 < \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Hence the Theorem.

We shall now deduce Theorem 7 from Theorem 9 . Let x be any continuity-point of the distribution-function of T. Then by Theorem 1 , (P_n^x) converges in measure and because of uniform boundedness, converges in the L_2-mean, to P^x. (i.e.) $m((P_n^x - P^x)^*(P_n^x - P^x)) \to 0$. Since σ is absolutely continuous with respect to m, it follows that $\sigma((P_n^x - P^x)^*(P_n^x - P^x)) \to 0$. And this implies $\sigma(P_n^x-P^x) \to 0$, or $\sigma(P_n^x) \to \sigma(P^x)$, which completes the proof of Theorem

We conclude this chapter with a necessary and sufficient condition for weak convergence and convergence in measure to coincide.

eorem 10: Let (T_n) be an arbitrary sequence of self-joint operators converging weakly to a self-adjoint erator T. Then (T_n) converges in measure to T, if and ly if $T = cI$ for some real constant c.

oof:

Firstly we shall show that the given condition is ufficient. Let $T = cI$, for some real c. We shall prove at (T_n) converges in measure to T. Let F_n be the istribution function of T_n and F thatof T. Easy to see hat $F(x) = 0$, if $x < c$, and $F(x) = 1$ if $x \geq c$. Also y point y $(\neq c)$ is a continuity-point of F. Correspon-ing to $(-\infty, y]$, let P_n^y denote the spectral projection f T_n and P^y that of T, $(n = 1,2,\ldots)$. When $y < c$, $^y = 0$, and when $y \geq c$, $P^y = I$. In view of weak convergence, $(P_n^y) \rightarrow m(P^y) = 0 = P^y$ whenever $y < c$. (i.e.. (P_n^y) con-erges in measure to $0 = P^y$, whenever $y < c$. Whenever $> c$, $m(P^y) = 1$, so that, $m(P_n^y) \rightarrow m(P^y) = 1$, which implies that (P_n^y) converges in measure to I. Thus, for any arbitrary continuity-point y of F, (P_n^y) converges in measure to (P^y). ence by Theorem 9 , (T_n) converges in measure to T.

nverse: We shall give an example of a sequence of ojections converging weakly to a projection, but still, ere is no convergence in measure.

Let α be a continuous finite factor and m the faithful normal trace on α with $m(I) = 1$. Let P, Q, R, and S be four mutually orthogonal projections in α, with $m(P) = \frac{1}{4} = m(Q) = m(R) = m(S)$. Let (A_n) be a decreasing sequence of projections contained in Q, such that $m(A_n) \to 0$. Let $R_n = P_n + A_n$. Then $m(R_n) \to \frac{1}{4} = m(S)$ and $m(R_n^{\perp}) \to \frac{3}{4} = m(S^{\perp})$. Let F be the distribution function of S and F_n that of R_n. All the involved operators being projections the spectrum of each one of them is concentrated at the two points 0 and 1. Thus, for any x with $x < 0$, $F_n(x) = 0 = F(x)$. For $0 \leq x < 1$, $F_n(x) = m(R_n^{\perp}) \to \frac{3}{4} = m(S^{\perp})$ and $F(x) = m(S^{\perp})$. For $x \geq 1$, $F_n(x) = 1 = F(x)$. And any point x other than 0 and 1 is a continuity-point of F. Hence for any x (in fact for all x), $F_n(x) \to F(x)$. But (R_n) cannot converge in measure to S. For, if it does, then $(R_n.S)$ will converge in measure to $S.S. = S$. But $R_n.S = 0$ for all n, while $S \neq 0$. Since $S \neq c.I$ for any c, this counter-example completes the proof.

CHAPTER II

STABILITY AND MIXING IN RINGS OF OPERATORS

Section 1.

Summary: For a sequence of events in a probability space, the concept of stability was introduced by Renyi in [6] and those of Mixing and Strongly Mixing were introduced by Sucheston in [9].

Let $\{S_n\}$ be a sequence of elements in the unit sphere of α. In this chapter, we introduce the notion of stability of $\{S_n\}$ with respect to a state on α. We then prove in Theorem 1 that any sequence of projections in α contains a subsequence which is stable with respect to any state of α, and that, if $\{S_n\}$ is stable with respect to any faithful, normal trace on α, then it is stable with respect to any state of α. As an application we show that if $\{S_n\}$ is stable with respect to a state and (ψ) any state absolutely continuous with respect to ϕ, then the sequence is stable with respect to (ψ) also (i.e. stability is invariant under absolute continuity.

Let $\{P_n\}$ be any sequence of projections in α. For P_n we introduce the concepts of 'zero-one', 'Mixing',

and 'Strongly Mixing' with respect to a state of α. As another application of Theorem 1, we show that both these concepts are invariant under absolute continuity. Using the above results and the non-commutative Martingale Convergence Theorem due to Umegaki, we prove the most important result of this chapter (Theorem 4) which runs thus:- Let ϕ be any state of α, whose support is central (in particular, let ϕ be a faithful state). Then a sequence (P_n) is zero-one with respect to ϕ if and only if it is strongly mixing with respect to ϕ.

Notation and terminology: In what follows U will denote the unit sphere of α. A sequence $\{S_n\}$ in U is said to be stable with respect to a state ϕ of α, if, for any projection Q in α, the sequences $\{\phi(S_n Q)\}$ and $\{\phi(QS_n)\}$ both converge. And, if there exists an operator T such that $\phi(S_n Q) \to \phi(TQ)$ and $\phi(QS_n) \to \phi(QT)$, then T is called the local density of S_n with respect to ϕ.

The terms 'strongly dense domain', 'essentially measurable operator', and 'strong-product of measurable operators' all have the same meaning as in [7]. For any two measurable operators A and B, AB and $A \times B$ will denote respectively the ordinary product and strong product of A and B.

A sequence P_n of projections in α is said to mixing with density γ, $(0, < \gamma < 1)$ with respect to a te ϕ, if $\phi(P_n) \to \gamma$, and for any projection Q in α, t$_\infty$ $(\phi(P_n Q) - \phi(P_n) \cdot \phi(Q)) = 0$. Let β_n be the ring rated by the sequence $\{P_{n+r}\}$, $r = 0, 1, 2, \ldots$ Let $\bigcap_{n=1}^{\infty} \beta_n$. The sequence $\{P_n\}$ is said to be zero-one with ect to ϕ, if, for any projection R in β, $\phi(R) = 0$ or A sequence $\{Q_n\}$ of projections is said o follow $_n\}$, if, for every n, Q_n belongs to β_n. The sequence $P_n\}$ is said to be strongly mixing, if every sequence lowing $\{P_n\}$ is mixing. Clearly, strongly mixing lies mixing. For, every sequence trivially follows elf.

All the results which we are going to prove in Section will appear in [4].

tion 2: Some Theorems on Stability and Mixing.

Now, we shall state and prove

orem 1: Any sequence $\{S_n\}$ in U, contains a subsence, which also we shall denote by $\{S_n\}$, with the lowing property:- 'There exists a unique operator T in α, such that for any state ϕ of α, $\{S_n\}$ is stable with respect to ϕ, and has local density T with respect to ϕ. In fact, a sequence $\{S_n\}$ stable with respect to m, and having local density T with respect to m, is stable with respect to any arbitrary state ϕ, and has local density T with respect to ϕ'.

Before proving the Theorem, we shall state three lemmas, but shall not prove them, as the proofs are quite elementary.

Lemma 1: The sequence $\{S_n\}$ is stable with respect to m, if and only if for any integrable operator A, the sequence $\{m(AS_n)\}$ converges.

Lemma 2: If $\{S_n\}$ is stable with respect to m, then there exists a unique element T in α, such that for any integrable operator A, $m(AT) = \lim_{n \to \infty} m(AS_n)$.

Lemma 3: Any sequence in U contains a subsequence which is stable with respect to m. Lemma 3, is an immediate consequence of lemmas 1 and 2 and elementary properties of abstract Hilbert Spaces.

<u>Proof of the Theorem:</u>

By lemma 3, $\{S_n\}$ has a stable sequence which also will be denoted by $\{S_n\}$. Let an arbitrary state ϕ of α be given. By the non-commutative version of the Radon-Nikodym Theorem, there exists an integrable operator R, such that for any B in α, $\phi(B) = m(RB)$. Hence, for any projection Q in α, $\phi(S_n Q) = m(RS_n Q) = m((S_n Q) \times R)$. Now the two measurable operators $(S_n Q) \times R$ and $S_n \times (Q \times R)$ both agree with the essentially measurable operator $S_n(QR)$ on the domain of R, which is strongly dense. Hence by a lemma of Segal (7 , Corollary 5.1, Page 413), they both are identical with the closure of $S_n QR$ and hence are themselves identical. So $\phi(S_n Q) = m((S_n Q) \times R) = m(S_n \times (Q \times R))$. Hence

$$\operatorname*{Lt}_{n \to \infty} m(S_n \times (Q \times R)) = m(T \times (Q \times R)) = m(RTQ) = \phi(TQ).$$

Similarly one can show that $\phi(QS_n) \to \phi(QT)$ as $n \to \infty$. Thus S_n is stable with respect to ϕ. Hence the Theorem.

<u>Theorem 2:</u> Let ϕ and ψ be any two states of α. Let $\{S_n\}$ be stable with respect to ϕ, and have local density T with respect to ϕ. If ψ is absolutely continuous with respect to ϕ, then $\{S_n\}$ is stable with respect to ψ, and has local density T with respect to

(ψ). In other words, stability and local density are invariant under absolute continuity.

This Theorem will not follow by a straight forward application of the Random-Nikodym Theorem, since the latter theorem has a simple form, only when either ϕ, or both ϕ and (ψ) are traces. However, we shall prove the Theorem by applying Theorem 1.

Proof:

Case 1: ϕ is faithful. By Theorem 1, $\{S_n\}$ has a subsequence which also we shall denote by $\{S_n\}$ which is stable with respect to m, and has local density, say D with respect to m. Also, by Theorem 1, $\{S_n\}$ had local density D with respect to ϕ. Hence for any projection Q, $\phi(TQ) = \operatorname{Lt}_{n \to \infty} \phi(S_n' Q) = \phi(DQ)$, (i.e. $\phi((T-D)Q) = 0$. As Q is arbitrary, and ϕ is faithful, this implies that T = D. Hence the original sequence $\{S_n\}$ is stable with respect to m, and has local density T with respect to m. Since (ψ) is absolutely continuous with respect to m, the desired result follows from Theorem 1.

Case 2: Let C be the maximal null projection in α of ϕ. Set P = I - C. Then P is the support of ϕ. Let $\frac{1}{L} = 1 + m(C)$. For any A in α, let $\phi_1(A) = L(\phi(A)+m(CA))$.

Then ϕ_1 is also a state on α, which is faithful. For any B in α, $\phi(S_n B) = \frac{1}{L}\cdot\phi_1(PS_n B)$. Let $A_n = PS_n$. Since for any projection Q, $\phi(S_n Q) \to \phi(TQ)$, it follows that $\phi_1(A_n Q) \to \phi_1(TQ)$. By case 1, $\{A_n\}$ is stable with respect to (ψ), and has local density T with respect to (ψ). But, for any projection Q, $(\psi)((A_n - S_n)Q) = 0$. Hence $\{S_n\}$ is stable with respect to (ψ) and has local density T with respect to (ψ).

<u>Theorem 3</u>: The properties of mixing (and density) and strongly mixing are invariant under absolute continuity.

<u>Proof:</u> Here again, the proof is not straightforward. For, if $\{P_n\}$ is mixing with density γ with respect to a state ϕ, and (ψ) another state absolutely continuous with respect to ϕ, it is not at all obvious that $(\psi)(P_n) \to \gamma$, and (P_n) is mixing with respect to (ψ). We shall state a lemma first.

<u>Lemma 4:</u> A sequence $\{P_n\}$ of projections is mixing with respect to a state ϕ if and only if $\{P_n\}$ is stable with respect to ϕ, and its local density with respect to ϕ is a scalar multiple of the identity operator. This non-negative scalar, will, in fact, be the density of the mixing sequence.

Proof of the lemma is elementary and hence omitted.

Now, to prove the Theorem. Let $\{P_n\}$ be mixing (with density γ) with respect to a state ϕ. Let (ψ) be absolutely continuous with respect to ϕ. By lemma 4 and Theorem 2, it follows that $\{P_n\}$ is stable with respect to (ψ) and has local density T with respect to (ψ). Applying lemma 4 again, it follows that mixing is invariant under absolute continuity. Similarly it can be shown that strongly mixing is invariant under absolute continuity.

We now state and prove the most important result of this chapter:-

<u>Theorem 4:</u> Let ϕ be any state of α, whose support is central ((i.e.) a projection belonging to the centre of α). Let $\{P_n\}$ be any arbitrary sequence of projections in α, with $\phi(P_n) \to \gamma$, $(0 < \gamma < 1)$. Then $\{P_n\}$ is zero-one with respect to ϕ, if and only if it is strongly mixing with respect to ϕ.

<u>Proof:</u> Let R be the support of ϕ. Let $m(R) = \frac{1}{k}$. For any T in α, set $r(T) = k.m(RT)$. As R is central, it follows that r is a normal trace on α with $r(I)=1=r(R)$. Since r and ϕ are absolutely continuous with respect to

each other (both having the same support) and since by Theorem 3, the properties of strongly mixing and zero-one are invariant under absolute continuity, it suffices to prove the Theorem for 'r' in place of ϕ. Firstly, we shall state some definitions and prove some lemmas.

For any ring β, $(\beta \subset \alpha)$, let $R\beta = \{RS : S \text{ in } \beta\}$. R being central, easy to check that $R\beta$ is also a ring acting on the range of the projection R. Let us denote the conditional expectation of any element T (in α) with respect to β under the trace r by $E_r(T|\beta)$. However, as r is not faithful, this has to be defined in an indirect way.

For any T in α, RT is an element of $R\alpha$. Since r is a faithful normal trace on $R\alpha$, the conditional expectation (in the sense of Umegaki [10]) of RT with respect to $R\beta$ (under the trace r) is well-defined and is unique. Let it be denoted by $\tilde{S}$ i.e. $E_r(RT|R\beta) = \tilde{S}$. And by the definition of $R\beta$, there exists at least one (there may exist more than one) element S in β with $RS = \tilde{S}$. This S is not unique. However, if S_1 is another element of β with the above property, then easy to verify that for any projection Q in β, $r((S-S_1)Q) = 0$ i.e. S is unique to within the support of r. We write $E_r(T|\beta) = S$. We shall now justify our definition of conditional expectation:-

Firstly, note that for any A in $R\beta$,

$$r((RT)A) = r(\tilde{S}\, A).$$

Now, for any K in β,

$$\begin{aligned} r(SK) &= r(RSK) \\ &= r((RS).(RK)) \\ &= r(\tilde{S}\, RK) \\ &= r((RT).(RK)) \\ &= r(RTK) \\ &= r(TK). \end{aligned}$$

Hence we are justified in writing

$$E_r\,(T|\beta) = S$$

<u>Lemma 5:</u> If (P_n) is a sequence of projections with $r(P_n) \to \gamma$ $(0 < \gamma < 1)$, then the following conditions are equivalent:-

1. The sequence (P_n) is mixing with respect to r with density γ.
2. Let α_n be the ring generated by the single projection P_n. For each projection M in α, $\{E_r(M|\alpha_n)\}$ converges in measure to $r(M)I$.

Proof:

Firstly we shall show condition 1, implies condition 2. As $r(P_n) \to \gamma$, without loss of generality, one can assume that $0 < r(P_n) < 1$ for all n. Easy to see that one version of conditional expectation can be taken to be

$$\frac{r(MP_n)}{r(P_n)} \cdot P_n + \frac{r(MP_n^{\perp})}{r(P_n^{\perp})} \cdot P_n^{\perp},$$

$P_n^{\perp} = I - P_n$ (In fact, in this case, conditional expectation is even unique). Then condition 1 is equivalent to saying that

$$\frac{r(M\ P_n)}{r(P_n)} \to r(M),$$

whence it also follows that

$$\frac{r(MP_n^{\perp})}{r(P_n^{\perp})} \to r(M).$$

So, given any $\epsilon > 0$, one can find a positive integer N such that for all

$$n \geq N, \quad \left|\frac{r(MP_n)}{r(P_n)} - r(M)\right| < \epsilon/2 \quad \text{and} \quad \left|\frac{r(MP_n^{\perp})}{r(P_n^{\perp})} - r(M)\right| < \epsilon/2.$$

Let $\{R_n\}$ be a sequence of projections defined thus:- $R_1 = 0 = R_2 = 0 = \dots = R_N$, $R_{N+K} = I$, $K = 1, 2, \dots$

Then $||(E_r(M|\alpha_n) - r(M)I)R_m|| < \epsilon$ for all n. As ϵ is arbitrary, it follows that $\{E_r(M|\alpha_n)\}$ converges in measure to $r(M)I$. Thus condition 1 implies condition 2.

For the converse, let

$$a_n = \frac{r(MP_n)}{r(P_n)} - r(M), \text{ and}$$

$$b_n = \frac{r(MP_n^{\perp})}{r(P_n^{\perp})} - r(M). \text{ Then}$$

$E_r(M|\alpha_n) - r(M)I = a_nP_n + b_nP_n^{\perp}$. We shall now show that given any $\epsilon > 0$, one can find a positive integer N such that for any $n \geq N$, $|a_n| < \epsilon$, and $|b_n| < \epsilon$, so that the sequences $\{a_n\}$ and $\{b_n\}$ both converge to zero. Let $S_n = a_nP_n + b_nP_n^{\perp}$. Condition 2 implies $\{S_n\}$ converges in measure to zero. So there exists a sequence $\{R_n\}$ of projections with $||S_nR_n|| < \epsilon$ for all n, and $r(R_n) \to 1$. By assumption, $r(P_n) \to \gamma$, and $r(P_n^{\perp}) \to 1 - \gamma$. Let $\rho = \min(\gamma, 1 - \gamma)$. We can assume $\epsilon < \rho$. Hence one can find a positive integer N_1 such that for all $n \geq N_1$, $r(P_n) > \rho - \frac{\epsilon}{2}$ and $r(P_n^{\perp}) > \rho - \frac{\epsilon}{2}$. Similarly, one can find an integer N_2 such that for all $n \geq N_2$, $r(R_n) > 1-9-\epsilon/4$. Let $N = \max(N_1, N_2)$. Then for any $n \geq N$, $P_n \wedge R_n$ and $P_n \wedge R_n^{\perp}$ are non-null. Let x_n be any unit vector in $P_n \wedge R_n$. $||S_nR_n|| < \epsilon$ implies $||S_nR_nx_n|| = ||S_nx_n|| = ||a_nx_n|| = |a_n| < \epsilon$. Similarly for any $n \geq N$, $|b_n| < \epsilon$. Hence the sequences $\{a_n\}$ and $\{b_n\}$ both converge to zero i.e.., $(r(MP_n) - r(M)r(P_n)) \to 0$ as $n \to \infty$, which shows that $\{P_n\}$ is mixing. Thus condition 2 implies condition 1.

Hence the lemma.

Lemma 6: Let λ be a ring with a faithful normal trace n. Let T be a self-adjoint operator in λ. Let W and V be rings with $W \subset V \subset \lambda$. Then

$$n(|E(T|W)|) \leq n(|E(T|V)|)$$

Proof: Let $S = E(T|W)$. S is self-adjoint, since T is so. Let P denote the spectral projection of S, corresponding to the non-negative real axis (including the origin). Let $Q = I-P$. Then $|S| = SP - SQ$. Let $K = E(T|V)$. Now

$$\begin{aligned} n(|S|) &= n(SP) + n(-SQ) \\ &= n(KP) + n(-KQ) \\ &\leq n(|K|P) + n(|K|Q) \\ &= n(|K|). \end{aligned}$$

Hence the lemma.

Lemma 7: Let T be any element in α, and β and G be rings with $G \subset \beta \subset \alpha$. Then $r(|E(T|G)|) \leq r(|E(T|\beta)|)$.

Proof:

Firstly we shall prove a simple proposition.

'For any S in β, $R|S| = |RS|$'. Both these operators being non-negative, suffice to show their squares are identical

$$(R|S|)^2 = R|S|^2 = (S^*S)R$$

$= (S^*R)(RS) = (RS)^*(RS) = |RS|^2$. Hence, in particular $r(|S|) = r(R|S|) = r(|RS|)$. Hence the proposition.

Now, for the lemma,

$$r(|E(T|G)|) = r(|RE(T|G)|) = r(|E(RT|RG)| \text{ and}$$
$$r(|E(T|\beta)|) = r(|E(RT|R\beta)|).$$

Since r is faithful regarded as a trace on $R\alpha$, the inequality

$$r(|E(RT|RG)| \leq r(|E(RT|R\beta)|)$$

now follows from lemma 6.

Lemma 8: If a sequence $\{P_n\}$ of projections is zero-one with respect to r, then it is mixing with respect to r.

Proof:

Firstly note that for any T in α, $||T||_2=||RT||_2$. For, $||T||_2^2 = r(T^*T) = r(RT^*T) = r((RT)^*(RT)) = (||RT||_2)^2$. Let M be any projection in α. Let F_k be the ring generated by $\{P_{k+r}, r = 0, 1, 2, \ldots\}$ Let RF_k be the ring, consisting of elements RS, where S belongs to F_k. As RF_k is a subalgebra of $R\alpha$ and r is a faithful trace on $R\alpha$, the Martingale convergence Theorem of Umagaki [11] applies. By that Theorem, $\{E_r(RM|RF_k)\}$ converges in the L_2-mean. Hence $\{E_r(M|F_k)\}$ converges in the L_2-mean. And since this sequence is uniformly bounded within the support of r, $(||RE(M|F_k)||) \leq 1$ for all k), convergence in the L_2-mean, convergence in measure, and convergence in the L_1-mean, are all equivalent. Hence $(E_r(M|F_k))$ converges in the L_1-mean also. Let the limit be denoted by N. As $\{F_k\}$ is a decreasing sequence of rings, N is measurable with respect to each F_k, and being bounded, belongs to F_k for each k and hence to $F = \bigcap_{k=1}^{\infty} F_k$. For any projection P in F, $r(E_r(M|F_k)P) = r(MP)$. In view of convergence in the L_2-mean,

$$r(NP) = \lim_{k \to \infty} r(E_r(M|F_k)P) = r(MP)$$

hus $N = E_r(M|F)$. And $E_r(M|F)$ is a scalar multiple of the identity (to within the support of r) if and only if F is trivial i.e. for any projection Q in F, $r(Q) = 0$ or 1. Hence if a sequence $\{P_n\}$ satisfies the conditions of the lemma (i.e. if $\{P_n\}$ is zero-one with respect to r), then $N = r(M)I$ and $E_r(M-r(M)I|F_n) \to 0$ in the L_1-mean. Hence by lemma 7, $E_r(M|F_n) \to r(M)I$ in the L_1-mean, and hence in measure. Now, an application of lemma 5 yields the desired result.

Proof of the Theorem:

Since $\{P_n\}$ is zero-one with respect to r, it is mixing with respect to r by lemma 8. Let $\{Q_n\}$ be any sequence of projections following $\{P_n\}$. Then obviously $\{Q_n\}$ is also zero-one with respect to r, and hence mixing with respect to r. Thus $\{P_n\}$ is strongly mixing with respect to r. For the converse, let $\{P_n\}$ be strongly mixing with respect to r. Let F and F_n be as in lemma 8, and P any projection in F. Then the sequence all whose elements are P, follows the sequence $\{P_n\}$ and is hence mixing, so that $r(P.P) - r(P).r(P) = 0$, i.e. $r(P) = (r(P))^2$. (i.e. $r(P) = 0$ or 1.

Hence the Theorem.

CHAPTER III

SOME DOMINATED CONVERGENCE THEOREMS IN RINGS OF OPERATORS

0. Summary: Let (H, α, m) be a regular gage space, where $m(I)$ is not assumed to be finite. Let $\{A_n\}$ and $\{B_n\}$ denote a pair of sequences of measurable operators.

(*) A_n converges grossly to a measurable operator A.

(**) Each B_n is non-negative, B_n integrable for each n, B_n converges grossly to B, B is integrable, and $m(B_n) \to m(B)$.

In Section 1, the following two theorems are proved.

Theorem 1: If the pair $\{A_n\}$ $\{B_n\}$ given by (*) and (**) respectively, satisfy $|A_n - A| \leq B_n$ for each n, then $A_n \to A$ in the L_1-mean.

Theorem 2: Let each A_n given by (*) be self-adjoint. Suppose there exist operators B and B_n satisfying (**) such that $|A_n| \leq B_n$, $n = 1,2,\ldots$ Then the limit operator A of the sequence A_n is self-adjoint. Also A is integrable and $m(A_n) \to m(A)$. In Section 2, we prove the following two dominated convergence theorems assuming $m(I)=1$.

Let $\{T_n\}$ denote an arbitrary sequence of operators converging in measure to an operator T, and $\{S_n\}$ a sequence of non-negative integrable operators, converging in measure to an integrable operator S, and such that $m(S_n) \to m(S)$. For any A, let $\mathrm{Re}\ A = \frac{A+A^*}{2}$ and $\mathrm{Im}\, A = \frac{A-A^*}{2i}$.

Theorem: If, corresponding to each n, $-S_n \leq \mathrm{Re}\ T_n \leq S_n$, and $-S_n \leq \mathrm{Im}\ T_n \leq S_n$, then $T_n \to T$ in the L_1-mean.

Theorem: If, corresponding to each n, $(\mathrm{Re}\ T_n)^2 \leq S_n$, and $(\mathrm{Im}\, T_n)^2 \leq S_n$, then $T_n \to T$ in the L_2-mean.

In the case of a finite gage space, these results are stronger than the corresponding results of Stinespring.

Let now, $\{A_n\}$ be a sequence of non-negative square-integrable operators converging in the L_2-mean to A. Let R be any bounded self-adjoint operator in α, and β the ring generated by R. Then, as applications of the above theorems, we prove that

1. The operator entropy of A_n $(= -A_n \log A_n) \to$ the operator entropy of A $(= -A \log A)$ in the L_1-mean.
2. The information about A_n contained in R $(= m(A_n \log A_n) - m(E(A_n|\beta) \log E(A_n|\beta)))$ tends to the information about A contained in R $(= m(A \log A) - m(E(A|\beta) \log E(A|\beta)))$.

Section [illegible] Some Dominated Convergence Theorems in a Ring of Operators

In his fundamental paper [8], Stinespring proved several dominated convergence theorems for operators measurable with respect to a gage space. In this part, we state and prove some dominated convergence theorems, one of which is a generalisation of a theorem of Stinespring.

Our definition of gross convergence will be the same as that of Stinespring [8 , Page 27]. It is known [8, Page 27], that convergence in measure always implies gross convergence.

Let $\{A_n\}$ be a sequence of operators converging grossly to an operator A. Suppose there exists an integrable operator B such that $|A_n - A| \leq B$, $n = 1, 2, \ldots$ From a dominated convergence theorem of Stinespring [8, Theorem 4.6], it follows that $A_n \rightarrow A$ in the L_1-mean and in particular $m(A_n) \rightarrow m(A)$.

However, there are cases, which are not covered by this Theorem. We give below an example of a sequence $\{A_n\}$ of non-negative, integrable operators converging grossly to zero (the zero operator) and $m(A_n) \rightarrow m(0) = 0$, but there does not exist any integrable operator T with $A_n \leq T$ for all n. Let α be a continuous finite factor and m, the standard faithful normal trace on α with $m(I) = 1$. Let P_1 be a projection with $m(P_1) = \frac{1}{2}$. Let P_2 be a projection contained in $I - P_1$ with $m(P_2) = \frac{1}{2^2}$, ... and in general P_n a projection contained in $I-P_1 - \ldots - P_{n-1}$ with $m(P_n) = \frac{1}{2^n}$. Let $A_n = \frac{2^n}{n} \cdot P_n$. Each A_n is integrable, and (A_n) converges in measure and hence grossly to zero and $m(A_n) = \frac{1}{n} \to 0 = m(0)$. If possible let there exist a non-negative operator T such that $A_n \leq T$ for all n. Then easy to see that $m(T) \geq m(TP_1) + \ldots + m(TP_n) \geq \sum_{k=1}^{n} \frac{1}{k}$ for all n, which shows that T cannot be integrable. In order to cover such exceptional cases also, we shall state and prove the following theorem and show how it applies to the above example.

In what follows, (H, α, m) will denote a regular gage space and $\{A_n\}$ and $\{B_n\}$ will stand for a pair of sequences of measurable operators.

(*) $\{A_n\}$ converges grossly to a measurable operator A ($A_n \to A$, grossly say).

(**) Each B_n is non-negative ($B_n \geq 0$, say), B_n integrable for each n, $B_n \to B$ grossly, B is integrable and $m(B_n) \to m(B)$.

We shall discuss some modes of numerical convergence of $\{A_n\}$, where each A_n is dominated by B_n in various types.

<u>Theorem 1:</u> If the pair (A_n), (B_n) given by (*) and (**) respectively, satisfy $|A_n - A| \leq B_n$, $n = 1, 2, \ldots$, then $A_n \to A$ in the L_1-mean.

<u>Remark:</u> If we assume B_n is the same for all n, we obtain the dominated convergence theorem of Stinespring [8, Theorem 4.6] as a special case.

<u>Proof:</u>

We apply the following version of Fatou's Lemma due to Stinespring [8, Theorem 4.10]:-

'If T_n is a sequence of non-negative measurable operators converging grossly to a measurable operator T, then

$$m(T) \leq \liminf_{n \to \infty} m(T_n)\text{'}$$

As $A_n \to A$ grossly implies $|A_n - A| \to 0$ grossly, $S_n = B_n - |A_n - A| \to B$ grossly and $S_n \geq 0$.

Now

$$m(B) \leq \liminf_{n \to \infty} m(B_n - |A_n - A|)$$

$$\leq \limsup_{n \to \infty} m(B_n - |A_n - A|)$$

$$\leq \limsup_{n \to \infty} m(B_n)$$

$$= m(B).$$

Hence $m(B_n - |A_n - A|) \to m(B)$; or $m(|A_n - A|) \to 0$ or $A_n \to A$ in the L_1-mean.

The assumption of Theorem 1 is satisfied for the sequences of operators constructed in the previous example. That is, let A_n, P_n,... etc. be as given in that example. Set

$$B_n = A_n + \frac{1 - \frac{1}{n}}{1 - \frac{1}{2^n}} \cdot (I - P_n)$$

Clearly $0 \leq A_n \leq B_n$ for each n, (B_n) converges in measure, and hence grossly, to I, and further

$$m(B_n) = \frac{1}{n} + (1 - \frac{1}{n}) = 1.$$

<u>Corollary 1.1:</u> If $\{A_n\}$ and $\{B_n\}$ given by (*) and (**) respectively satisfy the condition $|A_n - A|^2 \leq B_n$, $n = 1,2,\ldots$, then $A_n \Rightarrow A$ in the L_2-mean.

To prove this, one has only to consider the sequence S_n where $S_n = B_n - |A_n - A|^2$, $(|A_n - A|^2 = (A_n - A)^*(A_n - A))$, and proceed as in the case of Theorem 1.

<u>Theorem 2:</u> Let each A_n given by (*) be self-adjoint. Suppose that there exist operators B and B_n satisfying (**) such that $|A_n| \leq B_n$, $n = 1,2,\ldots$ Then the limit-operator A of the sequence (A_n), is self-adjoint. Also A is integrable, and $m(A_n) \Rightarrow m(A)$.

<u>Proof:</u> It has been proved by Stinespring [8, Page 28], that, if a sequence $\{C_n\}$ of measurable operators converges grossly to a measurable operator C, then $\{C^*_n\}$ converges grossly to C^*. Hence the limit A is self-adjoint. Further, since for each n, $(B_n \pm A_n)$ is a positive operator, it follows, by a result of Stinespring [8 , Page 29], that $-B \leq A \leq B$. As B is integrable, so is A, so that $m(A)$ is finite. If $m(A_n) = m(A)$ for all but finitely many n, then, there is nothing to prove. Hence, in the most general case, $\{A_n\}$ consists of two subsequences $\{A_{i_n}\}$ and $\{A_{j_n}\}$

such that for all n, $m(A_{i_n}) \geq m(A) \geq m(A_{j_n})$. Set $S_n = B_{i_n} + |A| - (A_{i_n} - A)$. Each S_n is non-negative. And, on applying Fatou's Lemma,

$$\begin{aligned} m(B+|A|) &\leq \liminf_{n \to \infty} m(B_{i_n} + |A| - (A_{i_n} - A)) \\ &\leq \limsup_{n \to \infty} m(B_{i_n} + |A| - (A_{i_n} - A)) \\ &\leq \limsup_{n \to \infty} m(B_{i_n} + |A|) \\ &= m(B + |A|). \end{aligned}$$

Thus as $n \to \infty$, $m(S_n) \to m(B + |A|)$ and $m(A_{i_n} - A) \to 0$.

For the other part, one need only consider $S_n = B_{j_n} + |A| - (A - A_{j_n})$ and proceed as before. Hence the Theorem.

For any measurable operator T, set $\mathrm{Re}\, T = \frac{T+T^*}{2}$ and $\mathrm{Im}\, T = \frac{T-T^*}{2i}$. Applying the proof of Theorem 2 separately to $\{\mathrm{Re}\, A_n\}$ and $\{\mathrm{Im}\, A_n\}$, in the place of $\{A_n\}$ given by (*) we immediately have the following:-

<u>Corollary 2.1</u> Each A_n is dominated by B_n, satisfying (**) such that $|\mathrm{Re}\, A_n| \leq B_n$, and $|\mathrm{Im}\, A_n| \leq B_n$, $n=1,2,\ldots$ Then $m(A_n) \to m(A)$.

Now let A_n and A satisfy the assumptions of Corollary 2.1. It is worthwhile examining whether $m(|A_n - A|) \to 0$ as $n \to \infty$. The method of proof of Theorem 2 fails to apply even when each A_n is self-adjoint. We encounter the following difficulty:- $|A_n| \leq |B_n|$ for each n. Still it does not follow that $|A_n - A| \leq B_n + |A|$, since the inequality $|A_n - A| \leq |A_n| + |A|$ is not in general valid for operators. Hence we cannot conclude that $S_n = B_n + |A| - (|A_n - A|)$ is positive, so that Fatou's lemma is not applicable to the sequence S_n.

The results of Section 1 will appear in [5].

Section 2: Some Dominated Convergence Theorems in a Finite Rings O Operators

In this part, we prove two dominated convergence theorems for operators measurable with respect to a finite gage space (H, α, m) where m(I) = 1. And in the case of a finite gage space, our theorems generalise some results of Stinespring.

Definitions: A sequence $\{A_n\}$ of operators is said to be U-continuous, if, given any $\varepsilon > 0$, there exists a 'δ' > 0, such that, for any projection P with $m(P) < \delta$, one has

$|m(A_n P)| < \epsilon$ for all n, and V-continuous, if $||A_n P||_2 < \epsilon$ for all n. Out definitions of convergence in measure and gross convergence are the same as those of Stinespring [8, Pages 23 and 26]. It is known [8, Page 32] that for a finite gage space, these two concepts are equivalent. The orthogonal complement of any projection P will be denoted either by $P^{\perp}$ or by $I - P$.

For a sequence $\{A_n\}$ of operators converging grossly to an operator A, the following two theorems have been proved by Stinespring, (without assuming the finiteness of the gage).

Theorem 1: [8, Page 29]. If there exists a non-negative integrable operator T such that $-T \leq \text{Re } A_n \leq T$, and $-T \leq \text{Im } A_n \leq T$ for all n, then $A_n \to A$ in the L_1 - mean.

Theorem 2: [8, Page 30, and the remark in Page 31]. If there exists a non-negative integrable operator T with $(\text{Re } A_n)^2 \leq T$ and $(\text{Im } A_n)^2 \leq T$ for all n, then $A_n \to A$ in the L_2- mean.

However, even in finite gage spaces, there are cases, which are not covered by these theorems. In section 1 of this chapter we have given an example of a sequence $\{A_n\}$ of operators converging grossly to zero (the zero operator) and $m(A_n) \to m(0) = 0$, but there does not exist any integrable operator T with $A_n \leq T$ for all n.

In order to cover such exceptional cases, also, we shall state and prove the following theorems and show how they apply to the above example.

In what follows, $\{T_n\}$ will denote an arbitrary seque sequence of operators, converging in measure to an operator T, and $\{S_n\}$ will denote an arbitrary sequence of non-negative, integrable operators converging in measure to an integrable operator S, and such that $m(S_n) \to m(S)$.

<u>Theorem 3</u>: If, corresponding to each n, $-S_n \leq \mathrm{Re}\, T_n \leq S_n$ and $-S_n \leq \mathrm{Im}\, T_n \leq S_n$, then $T_n \to T$ in the L_1-mean.

<u>Remark</u>: In the case of a finite gage space, this generalises a result of Stinespring [Theorem 1, mentioned at the beginning of this section].

<u>Theorem 4</u>: If, corresponding to each n, $(\mathrm{Re}\, T_n)^2 \leq S_n$, and $(\mathrm{Im}\, T_n)^2 \leq S_n$, then $T_n \to T$ in the L_2-mean.

Remark: In the case of a finite gage space, this generalises a result of Stinespring. [Theorem 2 mentioned at the beginning of this section].

These two theorems will be proved with the help of the following lemmas

Lemma 1: [8, Page 32]. A sequence $\{A_n\}$ of operators converges in measure to an operator A, if and only if for any positive number ϵ, $m(R_n^{\epsilon}) \to 0$ as $n \to \infty$, where R_n^{ϵ} is the spectral projection of $|A_n - A|$ corresponding to the interval (ϵ, ∞).

Lemma 2: Let $\{A_n\}$ be a sequence of non-negative integrable operators converging in measure to an operator A, and let $m(A_n) \to m(A)$. Then $A_n \to A$ in the L_1-mean.

Proof: This lemma is well-known in the commutative case. However, the method by which it is proved in standard textbooks on Probability Theory, [2, Page 140, Problem 17] does not directly extend to this general case, for which the proof is as follows. Let an arbitrary positive number ϵ be given. For any n, let $C_n = A - A_n$, and G_n^{ϵ}, F_n^{ϵ}, and H_n^{ϵ} denote the spectral projections of C_n corresponding to the intervals $[\epsilon, \infty)$, $(-\infty, -\epsilon]$ and $(-\epsilon, +\epsilon)$

respectively. It is easily seen that H_n^ϵ is the spectral projection of $|C_n|$ corresponding to the interval $[0, \epsilon)$.

Now

$$C_n = C_n G_n^\epsilon + C_n F_n^\epsilon + C_n H_n^\epsilon \qquad \ldots \qquad (1)$$

and

$$|C_n| = C_n G_n^\epsilon - C_n F_n^\epsilon + |C_n| H_n^\epsilon \qquad \ldots \qquad (2)$$

Hence $0 \leq m(C_n G_n^\epsilon) \leq m(G_n^\epsilon AG_n^\epsilon) = m(AG_n^\epsilon)$. Since (C_n) converges in measure to zero, $m(G_n^\epsilon) \to 0$ as $n \to \infty$, and A being integrable, this implies that $m(AG_n^\epsilon) \to 0$. Hence $m(C_n G_n^\epsilon) \to 0$. Again $m(|C_n| H_n^\epsilon) \leq \| \, |C_n| . H_n^\epsilon \| = \epsilon$, and similarly $|m(C_n H_n^\epsilon)| < \epsilon$. As $m(C_n) \to 0$, it follows from (1) that

$$\limsup_{n \to \infty} |m(C_n F_n^\epsilon)| \leq \epsilon. \text{ Hence}$$

$$\limsup_{n \to \infty} m(|C_n|) \leq \epsilon + \epsilon = 2\epsilon. \ \epsilon$$

being arbitrary, the desired result follows.

<u>Lemma 3:</u> Let $\{A_n\}$ be any sequence of non-negative, integrable operators. Then $A_n \to A$ in the L_1-mean, if and only if, $\{A_n\}$ is U-continuous and converges in measure to A.

<u>Proof:</u> If $A_n \to A$ in the L_1-mean, then it is known that it converges to A in measure, and it is easily verified that $\{A_n\}$ is U-continuous. For the converse, let an arbitrary positive number ϵ be given. By assumption, there exists a '∂' > 0, such that, for any projection P with $m(P) < \partial$, one has $m(A_nP) < \epsilon$ for all n. The sequence $\{PA_n P\}$ converges in measure to PAP. The involved operators being non-negative, it follows by the non-commutative version of Fatou's Lemma, [8 , Page 31] that $m(PAP) \leq \liminf_{n\to\infty} m(PA_nP) \leq \epsilon$ i.e. $m(AP) \leq \epsilon$.

Let P_n be the spectral projection of $|A_n - A|$ corresponding to the interval $[0, \epsilon)$ and $Q_n = I - P_n$. Then $m(P_n) \to 1$. So, for all $n \geq$ some positive integer N, $m(P_n) \geq 1 - \partial$. Thus for any $n \geq N$,

$$\begin{aligned} ||A_n - A||_1 &= m(|A_n - A|) \\ &= m(|A_n - A|P_n) + m(|A_n - A|Q_n) \\ &\leq \epsilon + m(|A_n - A|Q_n). \end{aligned}$$

Let U_n and W_n denote the spectral projection of $A_n - A$ corresponding to $[\epsilon, \infty)$ and $(-\infty, -\epsilon]$ respectively. Then $Q_n = U_n + W_n$ so that $m(U_n) \leq \partial$ and $m(W_n) \leq \partial$.

And

$$m(|A_n - A|Q_n) = m((A_n-A)U_n) - m((A_n-A)V_n)$$
$$\leq |m(A_nU_n)| + |m(AU_n)|$$
$$+|m(A_nV_n)| + |m(AV_n)|$$
$$\leq \epsilon + \epsilon + \epsilon + \epsilon = 4\epsilon$$

Hence $||A_n - A||_1 \leq \epsilon + 4\epsilon = 5\epsilon$. ϵ being arbitrary, the desired result follows.

Lemma 4: For any square-integrable operator A, and $\epsilon > 0$, one can find a 'δ' > 0, such that for any projection P with $m(P) < \delta$, one has $||AP||_2 < \epsilon$.

Proof: 'A' being square-integrable, there exists a bounded operator B with $||A - B||_2 < \frac{\epsilon}{2}$. Let $||B|| = k$, let $\delta = \frac{\epsilon}{2k}$, and P any projection with $m(P) < \delta$. Then $||BP||_2 \leq ||B||.\ m(P) \leq k.\frac{\epsilon}{2k} = \epsilon/2$. As $||AP - BP||_2 < \frac{\epsilon}{2}$, it follows that $||AP||_2 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence the lemma.

Lemma 5: For any square-integrable operator A and projection P, $m(A^2P) = (||AP||_2)^2$.

Proof:
$$(||AP||_2)^2 = m((AP)^*(AP))$$
$$= m((PA^*)(AP))$$
$$= m(P(A^*A)P)$$
$$= m(PA^2P)$$
$$= m(A^2P).$$

Lemma 6: Let $\{A_n\}$ be a sequence of square-integrable operators. Then $A_n \to A$ in the L_2-mean, if and only if $\{A_n\}$ converges in measure to A, and is also V-continuous.

Proof: Let $A_n \to A$ in the L_2-mean. Then, by a known result, [8 , Page 24], $\{A_n\}$ converges in measure to A. Also $A_n - A$ and A_n being square-integrable, it follows that A is square-integrable. By Lemma 4, there exists a '∂_1', such that $||AP||_2 < \frac{\epsilon}{2}$, for all projections P with $m(P) < \partial_1$. As $A_n \to A$ in the L_2-mean, there exists a positive integer N, with $||A_n - A||_2 < \frac{\epsilon}{2}$ for all $n \geq N$. Hence $||A_nP||_2 < \epsilon$ for all $n \geq N$. Also $A_1, \ldots, A_N$ being each square-integrable, and N being finite, there exists a ∂_2 such that for any projection P with $m(P) < \partial_2$, we have $||A_iP||_2 < \frac{\epsilon}{2}$, $i = 1,2,\ldots N$. Let $\partial = \min(\partial_1, \partial_2)$. Then for any projection P with $m(P) < \partial$, we have $||A_iP||_2 < \epsilon$, $i = 1,2,\ldots,$ which shows that $\{A_n\}$ is V-continuous. For the converse, note that since $\{A_n\}$ converges in measure to A, $\{A_n^2\}$ converges in measure to A^2. Let an arbitrary positive number ϵ be given. Since $\{A_n\}$ is V-continuous, it follows that there exists a $\delta > 0$, such that for any projection P with $m(P) < \delta$, $||A_nP||_2 < \epsilon$ for all n. Hence $(||A_nP||_2)^2 = m(A_n^2P) < \epsilon^2$. By Fatou's Lemma, it follows that $m(A^2P) < \epsilon^2$ (i.e.) $(||AP||_2)^2 < \epsilon^2$. As $\{A_n\}$ converges in measure to A, there exists a sequence $\{Q_n\}$ of projections such that $||(A_n-A)Q_n|| < \epsilon$ for all n, and $m(Q_n) \rightarrow 1$. Hence for all $n \geq$ some positive integer N, $m(Q_n^{\perp}) < \delta$, so that for any $n \geq N$,

$$
\begin{aligned}
||A_n-A||_2 &\leq ||(A_n-A)Q_n||_2 + ||(A_n-A)Q_n^{\perp}||_2 \\
&\leq \epsilon + ||A_nQ_n^{\perp}||_2 + ||A\,Q_n^{\perp}||_2 \\
&\leq \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned}
$$

Hence the lemma.

<u>Proof of Theorem 3:</u>

It suffices to prove the Theorem assuming each T_n to be self-adjoint, as the general case is reducible to this. Since $-S_n \leq T_n \leq S_n$, it follows that $-S \leq T \leq S$, which shows that T is integrable. Again, it follows by lemma 2, that $S_n \to S$ in the L_1-mean. Therefore, it also follows that $\{S_n\}$ and hence $\{T_n\}$, are U-continuous. Thus, given an arbitrary positive number ϵ, there exists a $\delta > 0$, such that for any projection P with $m(P) < \delta$, one has $|m(TP)| < \epsilon$ and $|m(T_nP)| < \epsilon$ for all n. Let G_n^{ϵ} and F_n^{ϵ} denote the spectral projections of $T_n - T$ corresponding to the intervals $[\epsilon/2, \infty)$ and $(-\infty, -\epsilon/2]$. Let $k_n = I - (G_n + F_n)$. Then k_n is the spectral projection of $|T_n - T|$ corresponding to $[0, \epsilon/2)$. In view of convergence in measure, $m(k_n) \to 1$ as $n \to \infty$; so that there exists a positive integer N, such that for $n \geq N$, $m(k_n) > 1 - \delta$, so that $m(G_n)$ and $m(F_n)$ are both $< \delta$. Now, for any $n \geq N$,

$$
\begin{aligned}
\| |T_n - T| \|_1 &= m(|T_n - T|) \\
&= m(|T_n - T| K_n) + m(|T_n - T| G_n) + m(|T_n - T| F_n) \\
&\leq \epsilon + m(|T_n - T| G_n) + m(|T_n - T| F_n) \\
&= \epsilon + m((T_n - T)G_n) - m((T_n - T)F_n) \\
&\leq \epsilon + |m(T_n G_n)| + |m(TG_n)| \\
&\qquad + |m(T_n F_n)| + |m(TF_n)| \\
&\leq \epsilon + \epsilon + \epsilon + \epsilon + \epsilon = 5\epsilon.
\end{aligned}
$$

Hence $T_n \to T$ in the L_1-mean. The general case follows by applying the previous result separately to the sequences $(Re\ T_n)$ and $(I_m\ T_n)$.

Proof of Theorem 4:

To begin with, let us assume that each T_n is self-adjoint. As $S_n \to S$ in the L_1-mean, it follows that $\{S_n\}$ is U-continuous. Since $T_n^2 \leq S_n$, $\{T_n^2\}$ is also U-continuous. Further, for any projection P, $m(T_n^2\ P) = (||T_n P||_2)^2$ so that the sequence $\{T_n\}$ is V-continuous. Also $\{T_n\}$ converges in measure to T. Hence by Lemma 6, $T_n \to T$ in the L_2-mean.

The general case can be proved by applying the previous result, separately to $(Re\ T_n)^2$ and $(Im\ T_n)^2$.

We shall now show how our theorems apply to the example immediately preceding Theorem 3. Let A_n, P_n,... etc., be as in that example. Define $B_n = A_n + P_n^{\perp}$. Then $\{B_n\}$ converges in measure to the identity operator I and $m(B_n) \to m(I) = 1$. As $A_n \leq B_n$, for all n, and $\{A_n\}$ converges in measure to zero, it follows by Theorem 3 that $A_n \to 0$ in the L_1-mean.

The question arises as to whether these results are valid even when the gage is not finite. All the above theorems depend on lemma 1, which has been proved to be true only for finite gage spaces. However, it will be interesting to examine, whether the above results hold for an arbitrary gage space.

Applications to Operator-Entropy

I. Let $\{T_n\}$ be a sequence of non-negative, square-integrable operators, converging in the L_2-mean to an operator T. Then the operator-entropy of T_n $(= - T_n \log T_n)$, tends in the L_1-mean, to the operator-entropy of T ($= - T \log T$), and in particular, the numerical entropy of T_n ($= - m(- T_n \log T_n)$) tends to the numerical entropy of T.

<u>Proof</u>:

Let $\sigma(\lambda) = \lambda \log \lambda$, where $\sigma(0)$ is defined to be zero. Hence $\sigma(\lambda)$ is continuous in the closed unit interval so that there exists a positive constant c, such that $-c < \sigma(\lambda) < c$, whenever $0 \leq \lambda \leq 1$. Hence it follows that for any non-negative λ, $- \lambda^2 - c \leq \lambda \log \lambda \leq \lambda^2 + c$.

Hence it also follows that for any non-negative operator A

$$-A^2 - cI \leq A \log A \leq A^2 + cI.$$

Since $\{T_n\}$ converges in the L_2-mean to T, it converges in measure to T. Hence $\{T_n^2\}$ converges in measure to T^2. Also as $||T_n - T||_2 \to 0$, it follows that $m(T_n^2) \to m(T^2)$. Thus if we set $S_n = T_n^2 + cI$, and $S = T^2 + cI$, then (S_n) is a sequence of non-negative integrable operators converging in measure to a non-negative integrable operator S and also $m(S_n) \to m(S)$. Moreover $-S_n \leq T_n \log T_n \leq S_n$ for each n. As $\{T_n\}$ converges in measure to T, it follows by Theorem 2 of Chapter I, that $\{T_n \log T_n\}$ converges in measure to $T \log T$. Now as a consequence of our dominated convergence theorem, it follows that $\{T_n \log T_n\}$ converges in the L_1-mean to $T \log T$.

Before proceeding further, we shall prove a lemma.

<u>Lemma:</u> Let β be any ring, contained in α. Let A be any square-integrable operator and $K = E[A|\beta]$ denote the conditional expectation of A, given β. Then, one has $||K||_p \leq ||A||_p$, $p = 1,2$,

Proof:

We shall firstly show this for $p = 1$. Note that for any U in β, $m(AU) = m(KU)$. Let U_1 denote the unit sphere of α and U_2 that of β. Then,

$$\begin{aligned} ||K||_1 &= \sup_{U \text{ in } U_2} (|m(KU)|) \\ &= \sup_{U \text{ in } U_2} (|m(AU)|) \\ &\leq \sup_{U \text{ in } U_1} (|m(AU)|) \\ &= ||A||_1. \end{aligned}$$

For $p = 2$: $G = L_2(H, \alpha, m)$ is a Hilbert Space, of which $N = L_2(H, \beta, m)$ is a closed subspace 'A' is an element of G, and K is its projection on the subspace N. Hence the norm of K regarded as an element of $N(=||K||_2)$ is less than or equal to the norm of A, regarded as an element of G $(=||A||_2)$. Thus $||K||_2 \leq ||A||_2$.

Using these two results, the two dominated convergence theorems, which we have proved, can be rewritten as follows:-

<u>Theorem</u> 5: Let all the assumptions of Theorem 3 be satisfied Then, in the notation of Theorem 3, $E(T_n|\beta) \to E(T|\beta)$ in the L_1-mean.

<u>Theorem</u> 6: Let all the assumptions of Theorem 4 be satisfied. Then, (in the notation of Theorem 4), $E(T_n|\beta) \to E(T|\beta)$ in the L_2-mean.

II. Let T be any non-negative square-integrable operator and R any bounded self-adjoint operator. Let β be the ring generated by R, and let $S = E(T|\beta)$. Then the number $m(T \log T) - m(S \log S)$ is defined by Nakamura and Umegaki [3] to be the information about T contained in R. This we shall denote by $I(T; R)$. $I(T; R)$ is finite, T being square-integrable.

As another application of our theorems, we state the following result:-

Let $\{T_n\}$ be a sequence of non-negative, and square-integrable operators, converging in the L_2-mean to T. Then, in the above notation, $I(T_n; R) \to I(T; R)$.

<u>Proof:</u>

Let $E(T_n \mid \beta) = S_n$ and $E(T|\beta) = S$. Then, by lemma 1, $S_n \to S$ in the L_2-mean. Hence, by an earlier result, $m(T_n \log T_n) \to m(T \log T)$ and

$m(S_n \log S_n) \to m(S \log S)$.

Thus $I(T_n; R) = m(T_n \log T_n) - m(S_n \log S_n)$

$$\to m(T \log T) - m(S \log S)$$

$$= I(T; R).$$

REFERENCES

1. Kolmogorov, A.N.: 'Foundations of the Theory of Probability. Chelsea Publishing House, New York, 1956.

2. Loeve, M. : 'Probability Theory'. D. Van Nostrand Company, Inc., New York, (Third Edition).

3. Nakamura, M. and Umegaki, H.: 'A Note on the Entropy for Operator Algebra'. Proc. Japan Acad. 37, (1961), 149.

4. Padmanabhan, A.R.: 'Stability and Mixing in von Neumann Algebras' (To appear in the Kodai. Math. Sem. Rep.).

5. ------- 'Some Dominated Convergence Theorems in a von Neumann Algebra'. (To appear in the Proc. Japan. Acad.

6. Renyi, A.: 'On Stable Sequences of Events'. Sankhya Series A, Vol. 25, Part 3, 293.

7. Segal, I.E.: 'A Non-Commutative Extension of Abstract Integration'. Ann. Math. Vol. 57, (1953), 401.

8. Stinespring, W.F.: 'Integration Theorems for Gages and Duality for Unemodular Groups.' Trans. Am. Math. Vol. 90 (1959), 15.

9. Sucheston, L.: 'On Mixing and the Zero-one Law,' Math. Anal. and Appl. 6, (1963), 447.

10. Umegaki, H. : 'Conditional Expectation in an Operator Algebra - I', Tohoku Math. J., Vol. 6, (1954), 177.

11. -------- : 'Conditional Expectation in an Operator Algebra - II', Tohoku Math. J., Vol. 8, (1956), 86.

12. ------------- : 'Conditional Expectation in an Operator Algebra - IV.' Kodai.Math. Sem. Rep. Vol. 14, 59.