BATALIN-TYUTIN QUANTISATION OF THE SPINNING PARTICLE MODEL

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Abstract:
The Spinning Particle Model for anyon is analysed in the Batalin-Tyutin scheme of quantisation in extended phase space. Here additional degrees of freedom are introduced in the phase space such that all the constraints in the theory are rendered First Class that is commuting in the sense of Poisson Brackets. Thus the theory can be studied without introducing the Dirac Brackets which appear in the presence of non-commuting or Second Class constraints. In the present case the Dirac Brackets make the configuration space of the anyon non-canonical and also being dynamical variable dependent, poses problems for the quantisation programme. We show that previously obtained results, (e.g. gyromagnetic ratio of anyon being 2), are recovered in the Batalin-Tyutin variable independent sector in the extended space. The Batalin-Tyutin variable contributions are significant and are computable in a straightforward manner. The latter can be understood as manifestations of the non-commutative space-time in the enlarged phase space.

I: Introduction

Theoretical models for anyons [1], excitations of arbitrary spin and statistics in 2+1-dimensions, have been proposed from different perspectives, the most popular ones being the point charge- Chern-Simons gauge field interacting theory [2] and the later Symplectic Models (SM) [3, 4] and the Spinning Particle Models (SPM) [5, 6]. The SM and SPM can be related to the anyon field equation proposed in [7]. It has been questioned [8] whether the former interacting theory is a minimal description of the anyon. The latter models are free from complications of this type and both of them agree regarding results, (such as rigidity of the angular momentum and the gyromagnetic ratio (g) being equal to 2), for free anyons as well as anyons interacting with abelian external gauge field, respectively.

However, the SM or SPM's are also interesting from another point of view. They have a close connection with the non-commutative space-time theories, which have created a lot of interest lately [9].

In SM [4], the symplectic structure has been posited, which gives rise to the non- commutativity in the particle configuration space. On the other hand, in SPM [5, 6], the above mentioned feature is a result of the constraint structure of the model. The set of Second Class Constraints (SCC), with non-commuting Poisson Brackets (PB), (in the sense of Dirac classification scheme [10]), induces non-trivial Dirac Brackets (DB) among the coordinate variables. Since we will concentrate on SPM [6] in the present work, these comments will be elaborated.

Let us now impose a bit of caution on the methodology of evaluation of the g value of anyon in SM and SPM. One compares the Hamiltonian (in the former [4]) and equations of motion (in the latter [6]) with the corresponding expressions obtained for a charged particle with spin in backgroung electromagnetic field *in conventional (commuting) configuration space*. The validity of this matching procedure can be questioned on the grounds that the nature of the configuration spaces in the two systems that are being compared are qualitatively different. But more importantly, the quantisation procedure runs into trouble due to operator ordering ambiguities in the DB's in SP and SPM. This complication arises since in theories with non-linear constraints, the DBs can involve dynamical variables which is true for the present case. When these DBs are elevated to commutation relations in the process of quantisation, the above mentioned feature creates problems.

Finally, (as has been pointed out before [5, 6]), the induced non-vanishing DB algebra $[A_a(\mathbf{x},t),\ A_b(\mathbf{y},t)]$ between supposedly "external" electromagnetic field variables makes us wonder as to how far the treatment of the gauge fields being non-dynamical is justified. The problems related to the non-canonical nature of the coordinate system have also been discussed in [11].

Now we come to the motivation of the present work. All of the above problems can be naturally addressed in the Batalin-Tyutin (BT) quantisation scheme [12], (which is a particular type of the more general quantisation method [13]), where additional phase space degrees of freedom, (BT variables), are introduced in such a way that existing SCC's are converted into (abelian) First Class Constraints (FCC, see [10]), thereby enhancing the gauge symmetry of the extended system. Thus the problematic DB's can be avoided. One can resort to Dirac quantisation of FCC system by requiring that the physical states obey the FCCs, (i.e. $FCC \mid physical\ state\rangle = 0$). Or else one can work in a convenient gauge where the problematic DBs do not appear. Later on we will use the first alternative.

Apart from the above reasons, we will find that by its own right, the problem of BT extension of SPM has several interesting features, some of them not addressed in the literature. Firstly, the SCC system in SPM can be expressed [5, 6] in a covariant (and neat) form which however is reducible that is the SCCs are not independent. The BT formulation has been developed for irreducible or independent set of SCCs only. This forces us to choose an independent set of SCC thereby losing manifest covariance. As un unsolved problem, it will be interesting to study the BT extension of a generic reducible SCC system.

Let us put the present work in its proper perspective. We have developed a framework for treating the Spinning Particle Model as an FCC system in a BT-extended phase space where the background space-time is normal, (that is commuting), so that comparison between results obtained for anyons and analogous models in conventional space-time do not pose any ambiguity. (In particular, we will recover the result that for anyons $g = 2$ but will indicate the presence of correction terms as well.) Also the canonical quantisation programme is rigorous since DBs do not appear and so operator ordering problems are absent in the canonical commutation relations.

Lastly we note that the present BT extension is classical in the sense that operator ordering problems in the extended system have not been addressed [14].

The paper is organised as follows: in section II we briefly reproduce the skeleton of BT formalism. This will also help us to fix the notations. We quickly introduce the SPM in section III for completeness. Section IV is devoted to developing the BT extension of SPM. This constitutes the main body of our work. Section V deals with the quantisation procedure and recovering of previous results. In section VI we conclude with a discussion and some future lines of work.

II: Batalin-Tyutin Formulation

In this section we state the main results of BT prescription [12] to be used. The basic idea behind the scheme is to introduce additional phase space variables (BT variables) ϕ^α_a, besides the existing physical degrees of freedom (q, p), such that *all* the constraints in the extended system are reduced to FCCs. This means that one has to modify the original constraints and Hamiltonian accordingly by putting BT-extension terms in them. The way to achieve this at the classical level has been provied in [12]. Let us consider a set of constraints $(\Theta^a_\alpha, \Psi_l)$ and a Hamiltonian operator H with the following Poisson Bracket (PB) relations,

$$\{\Theta^a_\alpha(q,p), \Theta^b_\beta(q,p)\} \approx \Delta^{ab}_{\alpha\beta}(q,p) \neq 0 \;\; ; \;\; \{\Theta^a_\alpha(q,p), \Psi^l_\beta(q,p)\} \approx 0$$

$$\{\Psi_l(q,p), \Psi_n(q,p)\} \approx 0 \;\; ; \;\; \{\Psi_l(q,p), H(q,p)\} \approx 0. \tag{1}$$

In the above (q, p) are referred to as physical variables and "$\approx$" means that the equality holds on the constraint surface and $\{p_a, x_b\} = g_{ab} \;\; ; \;\; g_{ab} = diag.(1, -1, -1)$. Clearly Θ^a_α and Ψ_l are SCC and FCC [10] respectively. The latter are responsible for gauge invariance and the former can be used as operator identities provided one uses the DBs [10] as defined below,

$$\{A(q,p), B(q,p)\}_{DB} = \{A, B\} - \{A, \Theta^a_\alpha\}\Delta^{\alpha\beta}_{ab}\{\Theta^b_\beta, B\} \;,\;\; \Delta^{ab}_{\alpha\beta}\Delta^{\beta\gamma}_{bc} = \delta^a_c\delta^\alpha_\gamma. \tag{2}$$

However, in systems with non-linear SCCs, in general the DBs can become dynamical variable dependent [5, 6, 15] due to the $\{A, \Theta^a_\alpha\}$ and $\Delta^{\alpha\beta}_{ab}$ terms, leading to problems for the quantisation programme. To cure this type of pathology, BT formalism is a systematic framework where one introduces the BT variables ϕ^{α}_{a}, obeying

$$\{\phi^{\alpha}_{a}, \phi^{\beta}_{b}\} = \omega^{\alpha\beta}_{ab} = -\omega^{\beta\alpha}_{ba}, \tag{3}$$

where $\omega^{\alpha\beta}_{ab}$ is a constant (or at most a c-number funtion) matrix, with the aim of modifying the SCC $\Theta^{a}_{\alpha}(q,p)$ to $\tilde{\Theta}^{a}_{\alpha}(q,p,\phi^{\alpha}_{a})$ such that,

$$\{\tilde{\Theta}^{a}_{\alpha}(q,p,\phi), \tilde{\Theta}^{b}_{\beta}(q,p,\phi)\} = 0 \ \ ; \ \ \tilde{\Theta}^{a}_{\alpha}(q,p,\phi) = \Theta^{a}_{\alpha}(q,p) + \Sigma^{\infty}_{n=1}\tilde{\Theta}^{a(n)}_{\alpha}(q,p,\phi) \ \ ; \ \ \tilde{\Theta}^{a(n)} \approx O(\phi^{n}) \tag{4}$$

The explicit terms in the above expansion are [12],

$$\tilde{\Theta}^{a(1)}_{\alpha} = X^{ab}_{\alpha\beta}\phi^{\beta}_{b} \ \ ; \ \ \Delta^{ab}_{\alpha\beta} + X^{ac}_{\alpha\gamma}\omega^{\gamma\delta}_{cd}X^{bd}_{\beta\delta} = 0 \tag{5}$$

$$\tilde{\Theta}^{a(n+1)}_{\alpha} = -\frac{1}{n+2}\phi^{\delta}_{d}\omega^{dc}_{\delta\gamma}X^{\gamma\beta}_{cb}B^{ba(n)}_{\beta\alpha} \ \ ; \ \ n \geq 1 \tag{6}$$

$$B^{ba(1)}_{\beta\alpha} = \{\tilde{\theta}^{b(0)}_{\beta}, \tilde{\theta}^{a(1)}_{\alpha}\}_{(q,p)} - \{\tilde{\theta}^{a(0)}_{\alpha}, \tilde{\theta}^{b(1)}_{\beta}\}_{(q,p)} \tag{7}$$

$$B^{ba(n)}_{\beta\alpha} = \Sigma^{n}_{m=0}\{\tilde{\theta}^{b(n-m)}_{\beta}, \tilde{\theta}^{a(m)}_{\alpha}\}_{(q,p)} + \Sigma^{n}_{m=0}\{\tilde{\theta}^{b(n-m)}_{\beta}, \tilde{\theta}^{a(m+2)}_{\alpha}\}_{(\phi)} \ \ ; \ \ n \geq 2 \tag{8}$$

In the above, we have defined,

$$X^{ab}_{\alpha\beta}X^{\beta\gamma}_{bc} = \omega^{ab}_{\alpha\beta}\omega^{\beta\gamma}_{bc} = \delta^{\gamma}_{\alpha}\delta^{a}_{c}. \tag{9}$$

A very useful idea is to introduce the Improved Variable $\tilde{f}(q,p)$ [12] corresponding to each $f(q,p)$,

$$\tilde{f}(q,p,\phi) \equiv f(\tilde{q},\tilde{p}) = f(q,p) + \Sigma^{\infty}_{n=1}\tilde{f}(q,p,\phi)^{(n)} \ \ ; \ \ \tilde{f}^{(1)} = -\phi^{\beta}_{c}\omega^{cb}_{\beta\gamma}X^{\gamma\delta}_{bd}\{\theta^{a}_{\delta}, f\}_{(q,p)} \tag{10}$$

$$\tilde{f}^{(n+1)} = -\frac{1}{n+1}\phi^{\beta}_{c}\omega^{cb}_{\beta\gamma}X^{\gamma\delta}_{bd}G(f)^{d(n)}_{\delta} \ \ ; \ \ n \geq 1 \tag{11}$$

$$G(f)^{b(n)}_{\beta} = \Sigma^{n}_{m=0}\{\tilde{\theta}^{b(n-m)}_{\beta}, \tilde{f}^{(m)}\}_{(q,p)} + \Sigma^{(n-2)}_{m=0}\{\tilde{\theta}^{b(n-m)}_{\beta}, \tilde{f}^{(m+2)}\}_{(\phi)} + \{\tilde{\theta}^{b(n+1)}_{\beta}, \tilde{f}^{(1)}\}_{(\phi)} \tag{12}$$

which have the property $\{\tilde{\Theta}^{a}_{\alpha}(q,p,\phi), \tilde{f}(q,p,\phi)\} = 0$. It can be proved that extensions of the original FCC Ψ_{l} and Hamiltonian H are simply,

$$\tilde{\Psi}_{l} = \Psi(\tilde{q},\tilde{p}) \ \ ; \ \ \tilde{H} = H(\tilde{q},\tilde{p}). \tag{13}$$

One can also reexpress the converted SCCs as $\tilde{\Theta}^{a}_{\alpha} \equiv \Theta^{a}_{\alpha}(\tilde{q},\tilde{p})$. The following identification theorem,

$$\{\tilde{A}, \tilde{B}\} = \{A, \tilde{B}\}_{DB} \ \ ; \ \ \{\tilde{A}, \tilde{B}\}\mid_{\phi=0} = \{A, B\}_{DB} \ \ ; \ \ \tilde{0} = 0, \tag{14}$$

will play a crucial role in our later application. Hence the outcome of the BT extension is the closed system of FCCs with the FC Hamiltonian given below,

$$\{\tilde{\Theta}^{a}_{\alpha}, \tilde{\Theta}^{b}_{\beta}\} = \{\tilde{\Theta}^{a}_{\alpha}, \tilde{\Psi}_{l}\} = \{\tilde{\Theta}^{a}_{\alpha}, \tilde{H}\} = 0 \ \ ; \ \ \{\tilde{\Psi}_{l}, \tilde{\Psi}_{n}\} \approx 0 \ ; \ \{\tilde{\Psi}_{l}, \tilde{H}\} \approx 0. \tag{15}$$

We will see that due to the non-linearity in the SCCs, the extensions in the improved variables, (and subsequently in the FCCs and FC Hamiltonian), turn out to be infinite series. This type of situation has been encountered before [15].

III: Spinning Particle Model revisited

The SPM proposed by us [6] where an anyon interacts with a $U(1)$ gauge field in 2+1-dimensions is given by the Lagrangian,

$$L = (m^2 U^a U_a + \frac{j^2}{2}\sigma^{ab}\sigma_{ab} + mj\epsilon^{abc}U_a\sigma_{bc})^{\frac{1}{2}} + eU_a A^a, \tag{16}$$

where

$$U^a = \frac{dx^a}{d\tau} \;\; ; \;\; \sigma^{ab} = \frac{1}{2}\epsilon^{abc}\sigma_c = \Lambda_c{}^a \frac{d\Lambda^{cb}}{d\tau} \;\; : \;\; \Lambda_c{}^a \Lambda^{cb} = \Lambda^a{}_c \Lambda^{bc} = g^{ab}.$$

Here $(x^a \; , \; \Lambda^{ab})$ is a Poincare group element and also a set of dynamical variables of the theory. The canonical momenta are defined in the following way [5, 6],

$$p^a = -\frac{\partial L}{\partial U_a} \equiv \pi^a - eA^a \; , \; S^{ab} = -\frac{\partial L}{\partial \sigma_{ab}} \equiv \frac{1}{2}\epsilon^{abc}S_c. \tag{17}$$

The phase space algebra is,

$$\{x_a, x_b\} = 0 \;\; ; \;\; \{p_a, x_b\} = g_{ab} \;\; ; \;\; \{\pi_a, \pi_b\} = eF_{ab} = e(\partial_a A_b - \partial_b A_a), \tag{18}$$

$$\{S^a, S^b\} = \epsilon^{abc}S_c \;\; ; \;\; \{S^a, \Lambda^{0b}\} = \epsilon^{abc}\Lambda^0{}_c \;\; ; \;\; \{\Lambda^{0a}, \Lambda^{0b}\} = 0. \tag{19}$$

In the free theory, one encounters the following set of FCC and SCC respectively,

$$\Psi_1 \equiv \pi_a \pi^a - m^2 \approx 0 \;\; ; \;\; \Psi_2 \equiv \pi.S - \frac{mj}{2} \approx 0,$$

$$\Theta_1^a \equiv \Lambda^{0a} - \frac{p^a}{m} \approx 0 \;\; ; \;\; \theta_2^a \equiv S^{ab}p_b = \epsilon^{abc}p_b S_c \approx 0.$$

In the free theory, Ψ_1 and Ψ_2 are respectivly the mass-shell condition and the Pauli-Lubanski relation.

In the interacting theory, One obtains to $O(e)$ the following set of FCCs and SCCs ,

$$\Psi_1 \equiv \pi_a \pi^a - m^2 + \frac{2eF_{ab}\pi^a}{\pi^2}\Theta_2^b \approx 0 \;\; ; \;\; \Psi_2 \equiv \pi.S - \frac{mj}{2} + \frac{ejF_{ab}\pi^a}{2m\pi^2}\Theta_2^b \approx 0, \tag{20}$$

$$\Theta_1^a \equiv \Lambda^{0a} - \frac{\pi^a}{m} \approx 0 \;\; ; \;\; \theta_2^a \equiv S^{ab}\pi_b = \epsilon^{abc}\pi_b S_c \approx 0. \tag{21}$$

Actually one can check that $S^a S_a = \frac{j^2}{4}$ but this is not independent of Ψ_l given above (20). To verify the constraint algebra after computing the PBs, one has to invoke the relation $S^a = \frac{j}{2m}\pi^a$, which is consistent with the SCC Θ_2^a and the normalisation agrees with the free theory. The above relation is valid also at the level of DBs [6].

Note that in the above set of SCC, Θ_1^a has been imposed from outside such that the angular coordinates are properly restricted. It is easy to ascertain that the SCCs Θ_α^a form a reducible set, since

$$\pi^a \Theta_{1a} = -\frac{1}{2}m\Theta^{1a}\Theta_{1a} \;\; ; \;\; \pi^a \Theta_{2a} = 0.$$

(With $e = 0$, these features are true for the free theory stated above.) However, one works with this reducible set because the manifestly covariant structure simplifies calculations and one can invert [5, 6] the SCC algebra matrix perturbativly to get the DBs. To $O(e)$ we obtain the DB between two generic field as [6],

$$\{A, B\}_{DB} = \{A, B\} + \frac{eF_{ab}}{m^2}\{A, \Theta_2^a\}\{\Theta_2^b, B\}$$

$$-\frac{1}{2}\epsilon^{abc}S_c\{A, \Theta_{1a}\}\{\Theta_{1b}, B\} + \frac{1}{2m}\{A, \Theta_2^a\}\{\Theta_{1a}, B\}$$

$$-\frac{1}{2m}\{A, \Theta_1^a\}\{\Theta_{2a}, B\}. \quad (22)$$

In particular the following DB

$$\{x^a, x^b\}_{DB} = -\frac{1}{2m^2}\epsilon^{abc}S_c + O(e), \quad (23)$$

gives rise to the non-commutative space-time, which in turn generates the arbitrary spin contribution in the angular momentum. Note that the above algebra (23) is non-trivial in the free theory as well. Fixing the gauge condition $x_0 = \tau$ as the proper time and using Ψ_1 one ends up with the Hamiltonian,

$$H = (m^2 - \pi^i\pi_i)^{\frac{1}{2}} - eA_0, \quad (24)$$

where the Θ_2^a dependent term has been dropped since we used DBs [6, 5]. To $O(e)$ the 2+1-dimensional analogue of the 3+1-dimensional Bargmann-Michel- Telegdi equation [16] is,

$$\dot{S}^a = \{H, S^a\}_{DB} = \frac{e}{m}F^{ab}S_b. \quad (25)$$

Comparison with the original equation [16] reveals that $g = 2$ for the particle.

As we have mentioned before, the comparison has been carried out between the present system in non-commutative configuration space and the original Bargmann-Michel-Telegdi equation in normal coordinate system.

The connection between our result and that of SM [4] is very direct but subtle. Remember that in SM [4], a modified expression, (with an $O(eF)$ term), for the "mass-shell" condition is used. (In our analysis, an analogous Θ_2^a-dependent term in (24) was "strongly" put to zero since it was proportional to the SCCs and we use DBs.) The analysis in SM is for FCC system, according to Dirac, where one demands that the FCCs annihilate the physical states. (In our BT extended theory, we will also follow this route.) Now in [4], one solves perturbativly, for small e the non-canonical symplectic algebra, (which is equivalent to our DBs), in terms of a set of canonical phase space variables and rewrites the Hamiltonian in terms of these *new* variables and finally compares this expression with the Hamiltonian of a charged particle in ordinary space time. Again the $g = 2$ result [4] is reproduced. Obviously this derivation [4] also suffers from the same conceptual drawback as the previous one [6]. With this background, we now move on to the BT extension of SPM.

IV:Batalin-Tyutin extension of Spinning Particle Model

This section comprises of the main body of our work where we introduce the BT machinary [12] in order to take into account the SCCs of the theory but at the same time avoid using a non-canonical coordinates. As mentioned before, we now use a smaller set of SCCs which are irreducible. Also note that in the extended space the constraints will be modified and we can no longer use the original constraints to simplfy the SCC algebra. Choosing Θ^i_α as the irreducible set of SCCs, the PB algebra is,

$$\{\Theta^i_\alpha, \Theta^j_\beta\} = \Delta^{ij}_{\alpha\beta} \tag{26}$$

$$\Delta^{ij}_{11} = \frac{eF^{ij}}{m^2} \tag{27}$$

$$\Delta^{ij}_{12} = -[(\pi.\Lambda)g^{ij} - \pi^i\Lambda^{0j} + \frac{e}{m}\epsilon^{jbc}F^i{}_bS_c] = -\Delta^{ji}_{21} \;\; ; \;\; \pi.\Lambda = \pi_a\Lambda^{0a} \tag{28}$$

$$\Delta^{ij}_{22} = [\pi_0(\pi.S)\epsilon^{ij} + e(F^{ij}S^2 + S_bF^{bi}S^j + S^iF^{jb}S_b] \;\; ; \;\; S^2 = S_aS^a. \tag{29}$$

Next, following (5), we need to compute $X^{ij}_{\alpha\beta}$ as the whole calculational scheme rests on this quantity and its inverse. This we do perturbativly by first considering the free theory, (i.e. $e = 0$), where,

$$\Theta^i_1 \equiv \Lambda^{0i} - \frac{p^i}{m} \;\; ; \;\; \Theta^i_2 \equiv \epsilon^{ibc}p_bS_c \tag{30}$$

$$\{\Theta^i_\alpha, \Theta^j_\beta\} = \Delta^{ij}_{\alpha\beta} = \begin{pmatrix} 0 & -((p.\Lambda)g^{ij} - p^i\Lambda^{0j}) \\ ((p.\Lambda)g^{ij} - p^j\Lambda^{0i}) & p_0(p.S)\epsilon^{ij} \end{pmatrix}. \tag{31}$$

For the free theory we propose,

$$X^{ij}_{\alpha\beta}\mid_{e=0} \equiv x^{ij}_{\alpha\beta} = \begin{pmatrix} 0 & -((p.\Lambda)g^{ij} - p^i\Lambda^{0j}) \\ g^{ij} & \frac{1}{2}p_0(p.S)\epsilon^{ij} \end{pmatrix}. \tag{32}$$

One can check that $x^{ij}_{\alpha\beta}$ satisfies (5) for the free theory, provided we choose $\omega^{\alpha\beta}_{ab} = \epsilon^{\alpha\beta}g_{ab}$, $\epsilon^{12} = 1$. Since there are some artitrariness involved in $x^{ij}_{\alpha\beta}$ and $\omega^{\alpha\beta}_{ab}$, their choices are dictated by convenience. The inverse is defined below,

$$x^{\alpha\beta}_{ij}x^{jk}_{\beta\gamma} = \delta^\alpha_\gamma g^i_k,$$

$$x^{\gamma\delta}_{jk} = \begin{pmatrix} \frac{p_0(p.S)}{2(p.\Lambda)}(\epsilon_{jk} - \frac{\epsilon_{jl}p_l\Lambda_{0k}}{p_0\Lambda_{00}}) & g_{jk} \\ -\frac{1}{(p.\Lambda)}(g_{jk} + \frac{p_j\Lambda_{0k}}{p_0\Lambda_{00}}) & 0 \end{pmatrix}. \tag{33}$$

This is an exact result. Now, for the interacting theory, we find to $O(e)$,

$$X^{ij}_{\alpha\beta} = x^{ij}_{\alpha\beta}(p_a \to \pi_a) + ey^{ij}_{\alpha\beta} \tag{34}$$

$$y^{ij}_{11} = \frac{F^{il}\Lambda_{0l}p^j}{m^2p_0\Lambda_{00}(p^k\Lambda_{0k})} \;\; ; \;\; y^{ij}_{22} = \frac{1}{2}(S_0)^2F^{ij} \tag{35}$$

$$y_{21}^{ij} = \frac{S_0 \epsilon^{rl} F^{0r} S^l}{p_0(p.S)} g^{ij} \tag{36}$$

$$y_{12}^{ij} = -\frac{1}{m} F^i{}_b \epsilon^{jbc} S_c + \frac{F^{il} \Lambda_{0l} p_k \epsilon^{jk} p_0(p.S)}{2m^2 p_0 \Lambda_{00}(p^k \Lambda_{0k}} + \frac{S_0 \epsilon^{rl} F^{0r} S^l}{p_0(p.S)}((p.\Lambda) g^{ij} - p^i \Lambda^{0j}) \tag{37}$$

The inverse, to $O(e)$, is of the form,

$$X_{ij}^{\alpha\beta} = x_{ij}^{\alpha\beta}(p_a \to \pi_a) - e x_{ik}^{\alpha\mu}(p) y_{\mu\nu}^{kl}(p) x_{lj}^{\nu\beta}(p). \tag{38}$$

In the present work the explicit form of $X_{ij}^{\alpha\beta}$ will not be utilised. From the relation,

$$\tilde{\Theta}_\alpha^{i(1)} = X_{\alpha\beta}^{ij} \phi_j^\beta, \tag{39}$$

the explicit expressions for $\tilde{\Theta}_\alpha^{i(1)}$ (i.e. one BT variable extension term) are,

$$\tilde{\Theta}_1^{i(1)} = -(g^{ik}(\pi.\Lambda) - \pi^i \Lambda^{0k})\phi_k^2 + e(y_{12}^{ik}\phi_k^2 + y_{11}^{ik}\phi_k^1), \tag{40}$$

$$\tilde{\Theta}_2^{i(1)} = \frac{1}{2}\pi_0(\pi.S)\epsilon^{ik}\phi_k^2 + \phi^{1i} + e(y_{21}^{ik}\phi_k^1 + y_{22}^{ik}\phi_k^2), \tag{41}$$

We emphasise that the series for $\tilde{\Theta}_1^i$ and $\tilde{\theta}_2^i$ do not terminate and the higher order terms in ϕ_i^α can be derived by a straightforward but extremely tedious calculation. However, to check whether our target of converting SCCs to FCCs has really been achieved (to $O(e)$), one can convince oneself that,

$$\{(\Theta_\alpha^i + \tilde{\Theta}_\alpha^{i(1)}) \ , \ (\Theta_\beta^j + \tilde{\Theta}_\beta^{j(1)})\} = 0 + O(\phi).$$

To check the cancellation of $O(\phi)$ terms, $\Theta_\alpha^{i(2)}$ terms are required.

Now we move on to the *one BT extension* of the physical degrees of freedom, i.e the Improved Variables. The BT extensions of the BT variables themselves will vanish. Let us start by constructing the extensions for π^a. From the generic expression given in (10), we have,

$$\tilde{\pi}^{a(1)} = -\phi_c^\beta \omega_{\beta\gamma}^{cb} X_{bd}^{\gamma\delta} \{\Theta_\delta^d, \pi^a\}_{(q,p)}$$

$$= \frac{e}{m}(\phi^{2j} x_{ji}^{11} - \phi^{1j} x_{ji}^{21}) F^{ia} - e\phi_i^2 \epsilon^{ide} S_e F_d{}^a$$

$$= \frac{e}{m} F^{ia}[(\phi^{2j} \frac{p_0(p.S)}{2(p.\Lambda)}(\epsilon_{ji} - \frac{\epsilon_{jl} p_l \Lambda^0{}_i}{p_0 \Lambda_{00}})$$

$$+ \phi^{ij} \frac{1}{(p.\Lambda)}(g_{ij} + \frac{p_j \Lambda_{0i}}{p_0 \Lambda_{00}})] - e\phi_i^2 \epsilon^{ide} S_e F_d{}^a. \tag{42}$$

In a straightforward manner, one can compute the rest of the Improved Variables, from the structures given below,

$$\tilde{x}^{a(1)} = -\phi_c^\beta \omega_{\beta\gamma}^{cb} X_{bd}^{\gamma\delta} \{\Theta_\delta^d, x^a\}_{(q,p)}$$

$$= \frac{g^{ai}}{m}(\phi^{2j} x_{ji}^{11} - \phi^{1j} x_{ji}^{21}) + \epsilon^{aic} S_c(\phi^{2j} X_{ji}^{12} - \phi^{1j} X_{ji}^{22}), \tag{43}$$

$$\tilde{S}^{a(1)} = -\phi_c^\beta \omega_{\beta\gamma}^{cb} X_{bd}^{\gamma\delta} \{\Theta_\delta^d, S^a\}_{(q,p)}$$

$$= \epsilon^{aic}\Lambda_{0c}(-\phi^{1j}X^{21}_{ji} + \phi^{2j}X^{11}_{ji}) - (g^{ai}(\pi.S) - S^i\pi^a)(\phi^{2j}X^{12}_{ji} - \phi^{1j}X^{22}_{ji}) \quad (44)$$

$$\tilde{\Lambda}^{0a(1)} = -\phi^\beta_c\omega^{cb}_{\beta\gamma}X^{\gamma\delta}_{bd}\{\Theta^d_\delta, \Lambda^{0a}\}_{(q,p)}$$
$$= (g^{ai}(\pi.\Lambda) - \Lambda^{0i}\pi^a)(\phi^{1j}X^{22}_{ji} - \phi^{2j}X^{12}_{ji}). \quad (45)$$

These Improved Variables also comprise of infinite sequences of higher order ϕ^α_i terms. As a non-trivial consistency check, we have tested the validity of the assertion that, to $O(\phi)$, $\tilde{\Theta}^a_\alpha \equiv \Theta^a_\alpha(\tilde{q},\tilde{p})$ holds. To examine the $two-\phi$-term, $O(\phi^2)$-terms in $\tilde{\Theta}^i_\alpha$ and $\tilde{f}(q,p)$ are required. In the next section we will make use of these results to redetermine the g-value for the anyon in the extended phase space. However, as we have stressed before, the main achievement is that a consistent framework has been provided wherein quantisation of the SPM is unambiguous and derivation of the previous results are more transparant.

V: Application - g in extended space

Let us start by recovering the "mass shell" condition $\tilde{\Psi}_1$ in BT extended space, which is simply given by

$$\tilde{\Psi}_1 \equiv \tilde{\pi}_a\tilde{\pi}^a - m^2 + \frac{2e\tilde{F}_{ab}\tilde{\pi}^a}{\tilde{\pi}^2}\tilde{\Theta}^b_2 \approx 0. \quad (46)$$

Remembering that *all* the Improved Variables are of the form

$$\tilde{\mathcal{A}}(q,p,\phi) = \mathcal{A}(q,p) + O(\phi) \quad + .. \ ; \quad \tilde{F} = F(\tilde{x}) = F(x) + (\partial F)\phi + \quad .. \ ,$$

and neglecting ∂F terms, it is clear that ϕ-independent *polynomial* expressions of relevant operators will remain intact and so up to non-ϕ terms, the previous relations survive. Hence proceeding in the same way as before, in the extended space, one can compare the Schrodinger equation to derive $g = 2 + O(\phi)$ for anyons. However, more work is to be done to ascertain whether the ϕ-terms can contribute to g even in the non-relativistic limit $p_0 \approx m >> \mid \mathbf{p} \mid$ used in [4]. Notice that the Pauli-Lubanski relation is modified to,

$$\tilde{\Psi}_2 \equiv \tilde{\pi}.\tilde{S} - \frac{mj}{2} + \frac{ej\tilde{F}_{ab}\tilde{\pi}^a}{2m\tilde{\pi}^2}\tilde{\Theta}^b_2 \approx 0.$$

This discussion corresponds to the SM model [4].

Returning to our SPM model [6], we follow the discussion in Section III and introduce the gauge fixing condition for $\tilde{\Psi}_1$ to be $\tilde{x}_0 - \tau = 0$ and obtain the Hamiltonian as,

$$\tilde{H} = (m^2 - \tilde{\pi}^i\tilde{\pi}_i + \tilde{\Theta}^a_2 - term)^{\frac{1}{2}} - e\tilde{A}_0. \quad (47)$$

We immedietly notice that the Hamiltonian in extended space is not a rational function of the phase space variables and also that the Θ^a_2 term can not be dropped. But the theorem in (14) comes to the rescue. The Θ^a_2 can contribute to the equations of motion of the generic form $\dot{\tilde{\mathcal{A}}} = \{\tilde{H}, \tilde{\mathcal{A}}\}$ by terms proportional to $\tilde{\Theta}^a_2$ only since since PBs in extended space with $\tilde{\Theta}^i_\alpha$ vanish. So long as we are concerned with computing PBs, the theorem (14) can be applied without going to the details of the explicit structure of the respective operators in the PB. Hence utilising (14), we can simply transform the BMT equation of motion for the spin variable [6], in terms of the Improved Variables and get,

$$\dot{\tilde{S}}^a = \frac{e}{m}\tilde{F}^{ab}\tilde{S}_b \approx \frac{e}{m}F^{ab}\tilde{S}_b. \quad (48)$$

This is our cherished Bargmann-Michel-Telegdi equation for anyon in BT- extended space. Obviously in the the BT extended space, the expression for g is modified to $g = 2 + O(\phi)$ for anyons, with the additional terms coming from the BT- variable dependent terms.

To facilitate a comparison with the results obtained in the physical variable sector, (without introducing BT variables), we can rewrite the Improved Variables by their physical counterpart and the BT extension terms. Since the phase space PB algebra (18,19, 3) is trivially known, we can compute $\dot{S}^a$ from (51) in a straightforward way. Obviously it will be of the form,

$$\dot{\tilde{S}}^a \approx \frac{e}{m}F^{ab}\tilde{S}_b + O(\phi). \quad (49)$$

In a restricted way, we can eliminate the BT variables in terms of the physical variables if we adopt the Dirac quantisation prescription for systems with FCCs only, where the physical sector of Hilbert space is required to satisfy $FCC \mid Physical\ State\rangle = 0$ [17]. In the present case, this reduces to

$$\tilde{\Theta}^i_\alpha \mid Ph.\ St.\rangle = \tilde{\Psi}_l \mid Ph.\ St.\rangle = 0. \quad (50)$$

Since we are interested in substituting the BT variables in the expression of the Hamiltonian in (47), we can solve the constraints for $e = 0$ as the terms involving BT variables are already of $O(e)$. This leads to the set of equations,

$$\tilde{\Theta}^i_1 \mid_{e=0} = \Theta^i_1 \mid_{e=0} - (g^{ij}(p.\Lambda) - p^i\Lambda^{oj})\phi^2_j = 0,$$

$$\tilde{\Theta}^i_2 \mid_{e=0} = \Theta^i_2 \mid_{e=0} + \phi^{1i} + \frac{1}{2}m^2 S_0 \epsilon^{ik}\phi^2_k = 0. \quad (51)$$

From the solutions of the above equations, we find,

$$\phi^{1i} = -\Theta^i_2 - \frac{1}{2}m^2 S_0 \epsilon^{ij}\phi^2_j,$$

$$\phi^{2i} = \frac{\Theta_{1j}}{(p.\Lambda)}(g^{ij} + \frac{p^i\Lambda^{0j}}{p_0\Lambda_{00}}) \quad (52)$$

Notice that the BT-variables in this restricted derivation, being proportional to Θ^i_α, are non-vanishing even in the free theory since in the extended sector Θ^i_α are no longer the constraints. It should be emphasised that this is not the whole story since $\tilde{\Theta}^i_\alpha$ consists of an infinite number of terms. Also there are indications that higher order terms will survive even if, on top of our small e restriction, we incorporate the non-relativistic limit, (that is $p_0 \approx m >>\mid \mathbf{p} \mid$, as has been done in [4]). On the other hand, putting these expressions in the BMT equation shows the leading term in $g = 2$ survives but there can be non-trivial velocity dependent corrections. Putting the BT variables back in (49) we get the BMT equation completely in the physical sector.

VI: Conclusion

To conclude, in the present work we have formulated an extension of the Spinning Particle Model for anyon [6] along the lines of Batalin-Tyutin quantisation scheme [12]. The reason behind working in the extended phase space lies in the presence of non-linear Second Class Constraints in the model, which induce a non-canonical structure in the particle coordinate. Specifically, the position coordinates become non-commuting which creates problem for the quantisation programme. Also this makes the comparison between results obtained here and in conventional space-time models, difficult. The gyromagnetic ratio of anyon was obtained to be 2 [4, 6] by invoking precisly this type of matching.

To avoid this type of non-trivial structure in the space-time, the Batalin-Tyutin formalism [12] is adopted where extra BT-variables are introduced in the phase space in such a way that all the constraints become First Class in the extended space and the problematic Dirac Brackets can be avoided. Hence the commuting space-time structure is kept intact. However, the price to pay is that the extensions of the constraints and relevant variables turn out to be infinite serieses, (even to lowest order in e, the electromagnetic coupling), with higher powers of BT variables. One has to be very careful in taking the non-relativistic limit [4] and *a priory* it is difficult to determine whether the serises will terminate or not.

In the present work we have computed explicitly the one-ϕ extensions of *all* the constraints and degrees of freedom. From the nature of the extensions, it is clear the BT-variable independent relations remain the same as the original ones. Hence one might say that the BT terms are effects of the non-commuting space-time algebra. However, we note that these results are partial in the sense that higher BT-variable terms may also contribute to this order of accuracy. One of our immediate goals in this area is to compute explicitly effect of the BT variable terms in observable quantities, *e.g.* g.

From another point of view, this work is significant since it may provide a mapping between theories in *non-commuting* space-time on one hand, and theories in conventional configuration space with extra *spin* degrees of freedom. Subsequently, Batalin-Tyutin extension can be introduced and one can check if results of simple non-commutative models [9] are reproduced.

Another interesting problem of the formal kind is to develop the Batalin-Tyutin scheme for reducible Second Class Constraint systems, such as the Spinning Particle Model in manifestly covariant form. One has to be careful in introducing the BT variables since reducibility in the system will be reflected in the number of these degrees of freedom.

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