Distinguishability of Bell States

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More than two multipartite orthogonal states cannot always be discriminated if only local operations and classical communication (LOCC) are allowed. We show that four Bell states cannot be discriminated by LOCC, even probabilistically, using the separability properties of a four-party unlockable bound entangled state. Using an existing inequality among the measures of entanglement, we show that any three Bell states cannot be discriminated with certainty by LOCC. Exploiting the inequality, we calculate the distillable entanglement of a certain class of $4 \otimes 4$ mixed states.

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Nonorthogonal states cannot be discriminated with certainty. This is essentially the no-cloning theorem [1]. However, discrimination with certainty is not guaranteed even for multipartite orthogonal states, if only local operations and classical communication (LOCC) are allowed. Given a set of multipartite orthogonal states, at present it is not always possible (without further study) to say whether they can be discriminated or not, where only LOCC are allowed.

A landmark result was obtained in [2], where it was shown that any two multipartite orthogonal states can be distinguished with certainty by LOCC. Recently Virmani *et al.* [3] have shown that for inconclusive discrimination of any two multipartite pure nonorthogonal states the optimal probability of error can be attained by using LOCC only. Chen and Yang [4] have shown that the same is true even in the case of conclusive discrimination. In this Letter we solve the question of distinguishability of any set consisting of the orthogonal Bell states by LOCC (where a single copy of the Bell states are provided). It was noted in [2] that if *two copies* of a state are provided, which is known to be one of the four (orthogonal) Bell states,

$$|B_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|B_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|B_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

one can discriminate between them using LOCC only. In this Letter, analyzing the separability properties of the unlockable bound entangled state $\frac{1}{4}\sum_{i=1}^{4} P[|B_i\rangle|B_i\rangle]$ [5], we show that it is not possible to discriminate (deterministically or probabilistically) among the four Bell states if only a *single copy* is provided. We also show that it is not possible to discriminate deterministically among any three Bell states if only a single copy is provided and if only LOCC are allowed. That two Bell states can be distinguished follows from the result in [2].

Suppose now that it is possible to discriminate (with certainty) between the four Bell states using only LOCC. Assume that there is a four-party state

$$\rho^S = \frac{1}{4}\sum_{i=1}^{4} P[|B_i\rangle_{AB}|B_i\rangle_{CD}]$$

shared between Alice (A), Bob (B), Claire (C), and Danny (D) with all four being at distant locations [5]. If the four Bell states are locally distinguishable, Alice and Bob would be able to discriminate between them without meeting (i.e., by LOCC). And when Alice and Bob have found one state, say $|B_1\rangle_{AB}$, a classical communication to Claire and Danny would result in their (Claire and Danny's) sharing $|B_1\rangle_{CD}$. As Alice and Claire did not meet Bob and Danny, this would contradict the fact that ρ^S is separable in the AC:BD cut. (Note that ρ^S can be written as $\frac{1}{4}\sum_{i=1}^{4} P[|B_i\rangle_{AC}|B_i\rangle_{BD}]$.) This proves that it is not possible to locally discriminate (with certainty) among the four Bell states, when only a single copy of the Bell states is provided.

One may now ask whether these four Bell states can be discriminated probabilistically. If probabilistic discrimination of any one (of the Bell states) from the rest is possible, it would imply creating entanglement between Claire and Danny (in the same way described above) with nonzero probability. But this again contradicts the separability of the state ρ^S across the AC:BD cut. Hence even probabilistic discrimination among the four Bell states is not possible, if only LOCC is allowed and if only a single copy is provided.

We now proceed to study the local distinguishability of any *three* orthogonal Bell states. It is to be noted here that any *three* Bell states (shared by Alice and Bob) can be discriminated by probabilistic LOCC. For example, $|B_1\rangle$ can be discriminated from $|B_3\rangle$, $|B_4\rangle$ if Alice and Bob both perform projective measurements in the $\{|0\rangle, |1\rangle\}$ basis. Their results would either be correlated or anticorrelated. A correlated result would imply $|B_1\rangle$.

We now show that any three Bell states cannot be locally discriminated with certainty, if only a single copy is provided.

The relative entropy of entanglement for a bipartite quantum state σ is defined as [6]

$$E_r(\sigma) = \min_{\rho \in D} S(\sigma \parallel \rho),$$

where D is the set of all separable states on the Hilbert space on which σ is defined, and $S(\sigma \parallel \rho) \equiv \mathrm{tr}\{\sigma(\log_2 \sigma - \log_2 \rho)\}$ is the relative entropy of σ with respect to ρ.

Consider now the state

$$\rho^{(3)} = \frac{1}{3} \sum_{i=1}^{3} P[|B_i\rangle_{\mathrm{AB}} |B_i\rangle_{\mathrm{CD}}],$$

where the Bell states involved are *any* three of the four. One can easily check that this state has nonzero distillable entanglement in the AC:BD cut, which easily follows from the fact that one can discriminate one particular Bell state from the other two by LOCC (see also [7]).

Let $E_r(\rho^{(3)}_{\mathrm{AC:BD}})$ denote the relative entropy of entanglement of the state $\rho^{(3)}$ in the AC:BD cut. Then

$$E_r(\rho^{(3)}_{\mathrm{AC:BD}}) \leq S\left(\rho^{(3)}_{\mathrm{AC:BD}} \parallel \frac{1}{4} \sum_{i=1}^{4} P[|B_i\rangle_{\mathrm{AC}} |B_i\rangle_{\mathrm{BD}}]\right) = 2 - \log_2 3 < 1.$$

But distillable entanglement is bounded above by E_r [9,10]. Consequently, the distillable entanglement of $\rho^{(3)}$, in the AC:BD cut, is strictly less than unity.

Suppose now that it is possible to discriminate (with certainty) between any three Bell states when only LOCC are allowed and when only a single copy is provided. So if Alice, Bob, Claire, and Danny share the state $\rho^{(3)}$, then Alice and Bob, without meeting, would again be able to make Claire and Danny share 1 ebit of distillable entanglement by using LOCC. Therefore distillable entanglement of $\rho^{(3)}$ in the AC:BD cut is at least unity. And again we have reached a contradiction. Therefore even three Bell states are not locally distinguishable with certainty if only a single copy is provided. Here we remark that the state $\rho^{(3)}$ can also be called "unlockable" [5] *in the sense* that it would not be possible to produce 1 ebit between C and D when A and B are separated although 1 ebit can be generated between C and D when A and B come together.

We now show that the above method can be employed to calculate the distillable entanglement, in the AC:BD (or AD:BC) cut, of any state of the form

$$\rho'^{(2)} = \tfrac{1}{2} P[(a|00\rangle + b|11\rangle)_{\mathrm{AB}} |B_1\rangle_{\mathrm{CD}}] + \tfrac{1}{2} P[(\overline{b}|00\rangle - \overline{a}|11\rangle)_{\mathrm{AB}} |B_2\rangle_{\mathrm{CD}}],$$

where $|a|^2 + |b|^2 = 1$. As two orthogonal states can always be locally discriminated [2], one can produce 1 ebit between C and D by operating locally on A and B. Thus the distillable entanglement of the above state would at least be 1 in the AC:BD as well as the AD:BC cut. But the relative entropy of entanglement (which is an upper bound of the distillable entanglement [9,10]) of $\rho'^{(2)}$ is 1 in these cuts [11]. Thus the distillable entanglement of the above state in the AC:BD (and AD:BC) cut is 1. In particular, the relative entropy of entanglement as well as the distillable entanglement of the state

$$\rho^{(2)} = \frac{1}{2} \sum_{i=1}^{2} P[|B_i\rangle_{\mathrm{AB}} |B_i\rangle_{\mathrm{CD}}]$$

is 1 in the AC:BD (and AD:BC) cut.

Earlier it was shown that any two Bell states can be discriminated by LOCC even if only a single copy is provided. Four Bell states can also be discriminated if *two* copies of the state are supplied [2]. Our result combined with these previous studies completes the issue of local discrimination among the Bell states. Interestingly we obtained the distillable entanglement of a certain class of $4 \otimes 4$ mixed states exploiting an existing inequality between entanglement measures. The issue of discrimination among a more general class of entangled states will be addressed in a forthcoming paper.

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$$\mathrm{rank}(\rho^{(3)}_{\mathrm{AC:BD}}) < \max\{\mathrm{rank}[\mathrm{tr}_{\mathrm{BD}}(\rho^{(3)}_{\mathrm{AC:BD}})], \mathrm{rank}[\mathrm{tr}_{\mathrm{AC}}(\rho^{(3)}_{\mathrm{AC:BD}})]\},$$

$\rho^{(3)}_{\mathrm{AC:BD}}$ will have nonzero distillable entanglement [8].
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