

# Comprehensive analysis of conditionally exactly solvable models

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## Abstract

We study a quantum mechanical potential introduced previously as a conditionally exactly solvable (CES) model. Besides an analysis following its original introduction in terms of the point canonical transformation, we also present an alternative supersymmetric construction of it. We demonstrate that from the three roots of the implicit cubic equation defining the bound-state energy eigenvalues, there is always only one that leads to a meaningful physical state. Finally we demonstrate that the present CES interaction is, in fact, an exactly solvable Natanzon-class potential.

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## I. INTRODUCTION

Exactly solvable models have attracted much attention since the early years of quantum mechanics. Some solvable potentials have become standard examples of text books, but a lot more have been discovered by various approaches. Systematic work has been done to generate and classify these potentials using the factorization method [1], algebraic methods [2] and more recently, in terms of supersymmetric quantum mechanics (SUSYQM) [3]. These approaches were found to be interrelated with each other [4–6].

The most general family of solvable potentials is the six-parameter Natanzon class [7], which contains potentials with solutions expressible in terms of a single (confluent) hypergeometric function. A rather important subclass of this is that of the shape-invariant potentials [8], to which the most well-known potentials (such as the harmonic oscillator, Coulomb, Pöschl–Teller, etc.) potentials belong. Altogether 12 such potentials have been identified [5,9], but some of these, actually represent different forms of the same potentials, and their separate discussion is justified only for historical reasons. An important recent development was the introduction of supersymmetric quantum mechanics (SUSYQM), which can be considered a re-interpretation of the factorization method [1], and which links basically isospectral potentials in a pairwise manner. Shape-invariant potentials are defined in terms of SUSYQM: the functional form of the SUSYQM partner potentials has to be the same, and only the parameters appearing in them can be different.

SUSYQM has been found rather useful in generating new solvable potentials as SUSYQM partners from known solvable ones. A rather wide potential class is obtained as the SUSYQM partner of Natanzon potentials, but these are not Natanzon potentials themselves (except in the case of shape-invariance), since their solution is written as the linear combination of several (confluent) hypergeometric functions [10]. There are also further solvable potentials which are solved by functions other than the (confluent) hypergeometric type. Examples for this are the square well [11] and the exponential potential, which are solved by Bessel functions.

A different concept of solvability characterizes quasi-exactly solvable (QES) potentials [12]. In this case only part of the eigenstates can be obtained, by requiring termination of

a recursion relation defining the eigenfunctions in a polynomial form.

The most recent concept of solvability is related to conditionally exactly solvable (CES) potentials. The first models coined CES potentials [13,14] were characterized by the fact that the coupling constant of some potential term had to be fixed to a numerical constant value in order to obtain their solutions. These potentials were introduced by the point canonical transformation method [15]. Here we present the analysis of one of these CES potentials [14]. Our motivation is to clarify some inconsistencies in their treatment, and to determine their place in other classification schemes of solvable models. (We note that another class of CES potentials was also introduced using the techniques of SUSYQM [16,17], but we do not extend our analysis on this class.)

In Sec. II we give a re-interpretation of the potential of Ref. [14] in a supersymmetric context, and derive the bound-state energies determined implicitly by a cubic equation. In Sec. III the procedure is placed in a more general context of methods based on variable transformations, and the potential is identified as an exactly solvable member of the Natanzon potential class.

## II. THE MODEL OF DUTT, KHARE AND VARSHNI

We start with presenting the potentials introduced by Dutt et al. [14] as CES models. The two potentials defined on the full axis  $x \in (-\infty, \infty)$  can be written in a common form as

$$V^{(g_0, g_1, g_2, g_3)}(x) = \frac{g_0}{e^x z(x)} + \frac{g_1}{z(x)} + \frac{g_2}{z^2(x)} + \frac{g_3}{z^4(x)}, \quad (2.1)$$

with  $z(x) = (1 + e^{-2x})^{1/2} \in (1, \infty)$ . The explicit form of these potentials [14] is

$$V_1^{(DKV)}(x) = V^{(0, -B, A, -3/4)}(x), \quad V_2^{(DKV)}(x) = V^{(-B, 0, A, -3/4)}(x). \quad (2.2)$$

These potentials depend on two parameters ( $A$  and  $B$ ) which define the potential shape. The coupling constant of the third potential term has to be fixed to a constant value ( $-3/4$ ) in order to obtain exact solution of these models. This is why the authors of Ref. [14] identified these potentials as CES ones.

One can easily demonstrate that the two potentials, in fact, are equivalent in the sense that

$$V^{(0,-B,A,-3/4)}(x) = V^{(-D,0,C,-3/4)}(-x) + \varepsilon, \quad (2.3)$$

where

$$\varepsilon = -A + 3/4, \quad C = -A + 3/2, \quad D = B. \quad (2.4)$$

Thus, in what follows it is sufficient to deal with only one of the potentials, so we pick  $V_1^{(DKV)}(x)$  for our analysis.

### A. Conventional approach via the point canonical transformation

In Ref. [14] potentials (2.2) were introduced using the point canonical transformation method [15], by which a Schrödinger-type differential equation can be transformed into another equation of this type, applying an invertible parametrization  $r = r(x)$ . With this change of variables, dating back to Liouville [18] a given asymptotically free equation

$$\left[ -\frac{d^2}{dr^2} + U(r) \right] \chi(r) = -\kappa^2 \chi(r) \quad (2.5)$$

can be transformed into an apparently different bound state problem

$$\left[ -\frac{d^2}{dx^2} + V(x) \right] \psi(x) = -k^2 \psi(x). \quad (2.6)$$

After we denote the derivative by a prime ( $x'(r)$  etc.), an extremely elementary correspondence between the potentials and/or energies is obtained,

$$U(r) + \kappa^2 = [x'(r)]^2 \{ V[x(r)] + k^2 \} + \left( \frac{3x''(r)}{4x'(r)} \right)^2 - \frac{1}{2} \frac{x'''(r)}{x'(r)}. \quad (2.7)$$

Obviously, the “old” energy eigenvalues are related to the parameters of the “new” potential, and vice versa. The formal definition of the new wave functions is also virtually trivial,

$$\psi(x) = (x'[r(x)])^{1/2} \chi[r(x)]. \quad (2.8)$$

In any situation of practical interest one may just pick up a suitable exactly solvable (ES) problem (2.5) and derive quickly its partner (2.6). Setting out from two shape-invariant [8]

ES potentials defined on the positive half axis, Dutt et al. [14] used the variable transformation  $x = \ln(\sinh r)$  to obtain potentials (2.2). The particular initial potentials and their energies were

$$U_1(r) = -2b \frac{\cosh r}{\sinh r} + a(a-1) \frac{1}{\sinh^2 r}, \quad \kappa^2 = \kappa_m^2 = (a+n)^2 + b^2/(a+n)^2 \quad (2.9)$$

[with  $b > (a + n_{\max})^2$ ] and

$$U_2(r) = -(2a+1)b \frac{\cosh r}{\sinh^2 r} + [a(a+1) + b^2] \frac{1}{\sinh^2 r}, \quad \kappa^2 = \kappa_n^2 = (a-n)^2 \quad (2.10)$$

(with  $b > a > n_{\max}$ ).

Recalling the bound-state wave functions of potentials  $U_j(r)$ , the solutions to potentials  $V_j^{(DKV)}(x)$  in (2.2) readily follow from Eq. (2.8). Without the loss of generality we can consider the  $j = 1$  case and recall the solutions of  $U_1(r)$  (see e.g. [9,5]) in terms of Jacobi polynomials,

$$\chi(z) = (z-1)^{-\frac{1}{2}(a+n-s)} (z+1)^{-\frac{1}{2}(a+n+s)} P_n^{(-a-n+s, -a-n-s)}(z), \quad s = b/(a+n) \quad (2.11)$$

with  $z = z(r) = \coth r$ . Using this function in (2.8), substituting it into the Schrödinger equation and matching parameters  $a$  and  $b$  with  $A$  and  $B$  appearing in  $V_1^{(DKV)}(x)$  in (2.2), we find  $B = 2b$  and

$$A = n^2 + 1/2 + (2n+1)a + b^2/(a+n)^2. \quad (2.12)$$

This equation will ultimately determine the energy eigenvalues of quantum number  $n$ , through a cubic equation as described also in Ref. [14]. We postpone the analysis of this formula to Sec. [III C], where our new results concerning the energy spectrum of the  $V_j^{(DKV)}(x)$  potentials are presented. Before that, we present an alternative interpretation of the same problem in terms of a supersymmetric framework.

## B. Supersymmetric construction

An interesting SUSY re-interpretation of the solvability of Schrödinger equations has been described by Nag et al. [19]. They have employed the two Dutra's models [13] in order to

illustrate their main idea. Unfortunately, the spectrum of states in the latter potentials can only be determined purely numerically [20]. Strictly speaking, the potentials do not belong to the CES class [21,22]. At best, only their incomplete (= quasi-exact) non-numerical solution can be obtained at certain *exceptional* energies and couplings [21,23]. Within the SUSY methodical framework, they seem less suitable for illustrative purposes.

We shall now obtain the spectrum of the potential  $V_1^{(DKV)}(x)$  in (2.2) in a manifestly supersymmetric fashion. Before doing this we recall that in supersymmetric quantum mechanics [9] a pair of Hamiltonians  $H_{\pm}$  defined by

$$H_{\pm} = -\frac{d^2}{dx^2} + V_{\pm}(x) = -\frac{d^2}{dx^2} + W^2(x) \pm W'(x) \quad (2.13)$$

are isospectral except for the zero energy ground state, which, for unbroken supersymmetry exists only for one of the partner potentials,  $V_-(x)$ . The ground-state solution of  $H_-$  is related to the  $W(x)$  superpotential through

$$W(x) = -\frac{d}{dx} \ln \psi_0^{(-)}(x) . \quad (2.14)$$

One can also extend the concept of superpotential to the excited states of  $V_-(x)$ , simply using  $\psi_n^{(-)}(x)$  in (2.14). In this case  $W(x)$  has singularities at the nodes of  $\psi_n^{(-)}(x)$ , and one can talk about singular superpotentials [9]. (Note that such singularities cannot occur using the nodeless ground-state wave function  $\psi_0^{(-)}(x)$ .) Despite these singularities of  $W(x)$ , it can be shown [9] that  $V_-(x)$  will be singularity-free in this case too, and these will appear only for the partner potential  $V_+(x)$ . Our purpose is, however, to discuss only  $V_-(x)$ , which we identify with  $V_1^{(DKV)}(x)$  in Eq. (2.2), in a supersymmetric form, therefore we shall avoid the problems arising due to the singularities of  $W(x)$ .

For this purpose, let us consider the superpotential

$$W(z) = \frac{B_1}{z} - \frac{1}{2z^2} - C_0 + \sum_{i=1}^n \frac{g_i'(z)}{g_i(z)} , \quad (2.15)$$

where  $z = (1 + e^{-2x})^{1/2}$  as in (2.2) and (2.1), and  $g_i(z)$  is given by

$$g_i(z) = \frac{1}{1 + g_i z} \quad , \quad C_0 = \epsilon_0^{1/2} , \quad (2.16)$$

where  $\epsilon_n$  is related to the (negative) bound-state energies of potential  $V_1^{(DKV)}(x)$  via  $\epsilon_n = -E_n$ . Note that the zero-energy wave function  $\psi_0^{(-)}(x) = N_0 \exp[-\int W(x)dx]$  is always

normalizable for our choice of  $W(x)$ , irrespective of the values of  $g_i$ . It may be noted that if we had omitted the last term in (2.15) i.e.,

$$W_0(z) = \frac{B_1}{z} - \frac{1}{2z^2} - C_0 \quad (2.17)$$

we would have obtained only the ground state. Insertion of the last term containing the sum ensures that we would get the excited states also.

It is straightforward to show that  $W(z)$  can be written in the form

$$W(z) = \frac{B_1 - \sum_{i=1}^n g_i}{z} - \frac{1}{2z^2} + C'_0 + \frac{\sum_{i=1}^n (g_i^2 - 1)}{(1 + g_i z)}, \quad (2.18)$$

where we have defined  $C'_0 = n - C_0$ .

Using (2.18) we obtain

$$\begin{aligned} W^2(x) - W'(x) &= [(B_1 - \sum_{i=1}^n g_i)^2 - C'_0 - \sum_{i=1}^n (g_i^2 - 1) + 1]/z^2 + [2(B_1 - \sum_{i=1}^n g_i)C'_0 \\ &\quad + 2(B_1 - \sum_{i=1}^n g_i)(\sum_{i=1}^n (g_i^2 - 1)) - (B_1 - \sum_{i=1}^n g_i) + 2g_i(g_i^2 - 1)]/z \\ &\quad - \frac{3}{4z^4} + \sum_{i=1}^n \frac{1}{1 + g_i z} [-2(B_1 - \sum_{i=1}^n g_i)(g_i^2 - 1)g_i - g_i^2(g_i^2 - 1) \\ &\quad + 2C'_0(g_i^2 - 1) - (g_i^4 - 1) + \sum_{j \neq i} \frac{(g_j^2 - 1)(g_i^2 - 1)g_i}{g_i - g_j}] + (C'_0)^2. \end{aligned} \quad (2.19)$$

We now make the following identification

$$W^2(x) - W'(x) = V_1^{(DKV)}(x) - E, \quad (2.20)$$

where  $E$  is the energy of the states in potential  $V_1^{(DKV)}(x)$ .

Then it follows that

$$-2(B_1 - \sum_{i=1}^n g_i)g_i - g_i^2 + 2C'_0 - (g_i^2 + 1) + 2 \sum_{i \neq j} \frac{(g_j^2 - 1)g_i}{g_i - g_j} = 0, \quad (2.21)$$

$$2(B_1 - \sum_{i=1}^n g_i)C'_0 + 2(B_1 - \sum_{i=1}^n g_i)(\sum_{i=1}^n (g_i^2 - 1)) - (B_1 - \sum_{i=1}^n g_i) + 2 \sum_{i=1}^n g_i(g_i^2 - 1) = -B, \quad (2.22)$$

$$(B_1 - \sum_{i=1}^n g_i)^2 - C'_0 - \sum_{i=1}^n (g_i^2 - 1) + 1 = A, \quad (2.23)$$

$$(C'_0)^2 = -E . \quad (2.24)$$

Multiplying (2.21) by  $g_i$  and summing over  $i$  we obtain

$$-2(B_1 - \sum_{i=1}^n g_i) \sum_{i=1}^n g_i - 2 \sum_{i=1}^n g_i^3 + 2 \sum_{i=1}^n C'_0 g_i - (2n-1) \sum_{i=1}^n g_i = 0 . \quad (2.25)$$

From (2.22) and (2.25) we get

$$B_1 = \frac{B}{1+2C_0} = \frac{B}{1+2\epsilon_n^{1/2}} . \quad (2.26)$$

It can be verified by insertion that the wave functions  $\psi_n^{(-)}(x) = N \exp[-\int W(x)dx]$  are normalizable. Equations (2.21) and (2.23) also imply that

$$A = \frac{(B/2)^2}{(n + \frac{1}{2} + \epsilon_n^{1/2})^2} + n^2 + n + 1 + (2n+1)\epsilon_n^{1/2} , \quad (2.27)$$

where in obtaining the above relation we have taken  $C'_0 = -\epsilon_n^{1/2}$ . We can summarize that our supersymmetric construction reproduces exactly the results obtained in Ref. [14]. Once we take  $a = \epsilon_n^{1/2}$  it proves equivalent to Eq. (2.12) of our preceding subsection [IA].

### C. The allowed bound-state energies

Let us now continue with the analysis of the energy eigenvalues based on the formula (2.12) and its equivalent form (2.27) obtained in two different ways. The key element of our approach is the strict observation of the constraints imposed on the parameters by the boundary conditions of the wave functions. By this we mean both the solutions of the “old” potential  $U_1(r)$  (2.9) and those of the “new” one  $V_1^{(DKV)}(x)$  (2.2).

The appropriate physical boundary condition for (2.11) near the threshold  $r \rightarrow 0$  is standard, though a bit counterintuitive [21,24]. Its implementation implies that we have to choose  $a > 1/2$ . Then, after the transition from  $r$  to  $x$  we get the wave functions still safely normalizable near the left infinity  $x \rightarrow -\infty$ . Similarly, our explicit wave functions remain asymptotically normalizable near the right infinities  $r \rightarrow \infty$  and  $x \rightarrow +\infty$  if and only if we have  $a+n < b/(a+n)$ . This means that the eligible quantum numbers  $n = 0, 1, \dots, M$  have to be such that  $0 \leq M < b^{1/2} - a$ , i.e.,



$$(n + 1/2)^2 < (a + n)^2 < b. \quad (2.28)$$

As mentioned in Sec. [II A](#), for transition to the “new” potential  $V_1^{(DKV)}(x)$  we have to re-parametrize  $g_1 = -B \equiv -2b$  and define the “new” CES energy in terms of the “old” ES coupling,  $k = a - 1/2 > 0$ . The second CES coupling  $A > 2(n + 1/2)^2 + 3/4$  is then defined by [\(2.12\)](#), which is equivalent to [\(2.27\)](#) in Sec. [II B](#). The  $n$ -dependence of the “new” energy  $a = a(n) = k_n + 1/2 > 1/2$  is fully consistent with the  $n$ -independence of the coupling  $A$ . For each level the CES potential  $V(x)$  is a map of a *different* ES potential  $U(r)$ . The energies are determined by the cubic algebraic equation. In order to make this definition unique we have to tell which one of the three roots of Eq. [\(2.12\)](#) is “physical”. In Ref. [\[14\]](#) we find an advice that “from the three roots we can discard two by demanding that the spectrum must reduce to the standard one for  $B = 0$ ”. Such a vague recipe is misleading since it is in manifest contradiction with the above normalizability condition [\(2.28\)](#) which implies that  $b = B/2 > 1/4$  cannot lie too close to zero.

The problem is not too difficult to disentangle. Equation [\(2.12\)](#) has very transparent graphical interpretation in terms of the intersection of the left-hand side horizontal line with the right-hand side curve with three branches. The latter shape is a sum of a growing linear term with a spike oriented upwards. Figure [1](#) indicates how one gets a triplet of roots in the  $n = 0$  ground state at  $b = 6.25$  and  $A = 10.25$ . Always, only one of them is compatible with the normalizability condition [\(2.28\)](#) and lies in the “admissible” interval  $(0.5, 2.5)$ .

The general rule is that we always have to pick up the middle root as physical. Let us give a proof of this assertion. Firstly we re-name  $b = \beta^2$  and re-scale our three roots,  $Z = [a_1(n) + n]/\beta < X = [a_2(n) + n]/\beta < Y = [a_3(n) + n]/\beta$ . As long as  $a(n) \in (1/2, \beta - n)$  we may re-write Eq. [\(2.12\)](#) in the significantly simplified form

$$\tau = \mu X + \frac{1}{X^2}$$

(and, similarly for  $Y$  and  $Z$ ) with abbreviations

$$\tau = \tau(A, b, n) \equiv \frac{A + n^2 + n - 1/2}{b}, \quad 0 < \mu = \frac{2n + 1}{\beta} < 2.$$

The leftmost root  $Z$  will be always negative and can be discarded immediately. Knowing that the acceptable root  $X$  is constrained,  $X \in (T, 1)$ ,  $T = (n + 1/2)/\beta \in (0, 1)$ , it is now sufficient to prove that the third root  $Y$  *always* violates our condition [\(2.28\)](#),

$$[X \in (T, 1) \ \& \ T \in (0, 1)] \implies Y > 1. \quad (2.29)$$

For this purpose we eliminate  $\tau$  and get the quadratic equation

$$\mu = \frac{X + Y}{X^2 Y^2}.$$

We can skip the negative alternative and have the unique definition of the root  $Y$ ,

$$Y = \frac{1 + (1 + 4\mu X^3)^{1/2}}{2\mu X^2}.$$

As a smooth function of  $\mu \in (0, 2)$  and  $X \in (T, 1)$  it satisfies our rule (2.29) *everywhere* within a two-dimensional domain containing all points with  $\mu < 1$  and *not containing* any point of the sign-changing boundary. This is demonstrated quite easily. The boundary curve can be implicitly defined as a set  $X = \xi(\mu)$ ,

$$1 + (1 + 4\mu \xi^3)^{1/2} = 2\mu \xi^2.$$

Only on it the sign of  $Y - 1$  can change. This set is a part of the curve defined by the square of the latter equation,

$$\mu \xi^2 = \xi + 1.$$

In the graphical language it is trivial to find that for the positive  $\xi > 0$  the right-hand side straight line intersects the left-hand side parabola in a point which is a decreasing function of  $\mu$ . Hence, the curve touches the boundary of our open simplex of normalizability (with  $\mu \in (0, 2)$  and  $X < 1$ ) in a single point ( $\mu = 2, \xi = 1$ ). QED.

### III. INTERPRETATION OF THE POTENTIAL

The potentials of Ref. [14] derived in two different ways in Section II can be placed into a more general context by realizing that both the point canonical transformation method [15] presented in Sec. II A and the supersymmetric construction of Sec. II B can be formulated in terms of a rather general approach based on the change of variables [25, 7, 5]. In this section we specify these connections with the formulation of Ref. [5], which can be considered a simplified treatment of the general Natanzon-class potentials [7].

Following the discussion of Ref. [5] one considers the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi(x) = 0 \quad (3.1)$$

and assumes that its solutions can be written in the form

$$\psi(x) = f(x)F(z(x)) , \quad (3.2)$$

where  $F(z)$  satisfies a second-order differential equation

$$\frac{d^2F}{dz^2} + Q(z)\frac{dF}{dz} + R(z)F(z) = 0 . \quad (3.3)$$

The function  $F(z)$  can be any special function of mathematical physics, e.g. the (confluent) hypergeometric function [26], or any other function satisfying a second-order differential equation of the type (3.3). Simple calculation shows [5] that the function  $E - V(x)$  can be written as

$$E - V(x) = \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left( \frac{z''(x)}{z'(x)} \right)^2 + (z'(x))^2 \left[ R(z(x)) - \frac{1}{2} \frac{dQ(z)}{dz} - \frac{1}{4} Q^2(z(x)) \right] , \quad (3.4)$$

where the only unknown element is the function  $z(x)$ , which basically governs the change of variables connecting the two differential equations (3.1) and (3.3). Expressing  $f(x)$  in (3.2) in terms of  $z(x)$  and  $Q(z)$ , the solutions of the Schrödinger equation can be written [5] as

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}} \exp\left(\frac{1}{2} \int^{z(x)} Q(z) dz\right) F(z(x)) . \quad (3.5)$$

We are left with the task of finding such a functional form of  $z(x)$  which takes our Schrödinger equation (3.4) into an exactly and *completely* solvable problem.

Obviously, the transformation employed in Sec. IIA (i.e. the point canonical transformation [15] or the Liouvillean method [18]) is a special case of the above construction. Taking

$$Q(z) = 0 \quad R(z) = -\kappa^2 - U(z) , \quad (3.6)$$

Eq. (3.4) reduces to the inverted version of Eq. (2.12) (with  $r$  and  $-k^2$  there replaced with  $z$  and  $E$  here). Similarly, (3.5) also reduces to the equivalent of (2.8), where  $\chi(r)$  is playing the role of  $F(z)$ .

From here the approaches applied in Refs. [25,5] and in the point canonical transformation [15] emphasize somewhat different strategies of deriving solvable potentials within the Natanzon potential class [7]. In Refs. [25,5] the main point is to identify some term on the right-hand side of Eq. (3.4), to account for the constant (i.e. the energy) term on the left-hand side. With this, a differential equation of the type

$$\left(\frac{dz}{dx}\right)^2 \phi(z) = C \quad (3.7)$$

was obtained (see also [27]), and this determined the function  $z(x)$  describing the variable transformation. In some cases the  $z(x)$  function could not be determined explicitly from (3.7), only the inverse  $x(z)$  function, therefore a number of solvable models obtained this way turned out to be “implicit” potentials [28,29]. On the other hand, following the point canonical transformation method [15], the  $z(x)$  function is always available in an explicit form, however, it is not guaranteed that any  $z(x)$  function would lead to a Schrödinger-like equation in which all the  $n$ -dependence can be absorbed into the constant (energy) term. Equation (2.12) might turn out to have Sturm–Liouvillean form, where  $n$  typically appears in coordinate-dependent terms. Simply stated, the approach of Ref. [5] focuses on having the energy in a simple form, even on the expense of leaving the solutions in a complicated (implicit) form, while in the point canonical transformation the preference is having the solutions in an explicit form, rather than getting the energy expression in a simple way. We stress that despite this difference, the two approaches are interrelated, and are special cases of deriving Natanzon-class potentials. We shall come back to this point later on.

### A. Conventional construction

Let us now see how potential  $V_1^{(DKV)}(x)$  in Eq. (2.2) can be obtained from the method described in Ref. [5]. For this,  $F(z)$  should be identified with a Jacobi polynomial:  $F(z) = P_n^{(\alpha,\beta)}(z)$ . Equation (4.2) in Ref. [5] is an explicit form for  $E - V(x)$  in this case:

$$\begin{aligned} E - V(x) = & \frac{z'''(x)}{2z'(x)} - \frac{3}{4} \left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1-z^2(x)} n(n+\alpha+\beta+1) \\ & + \frac{(z'(x))^2}{(1-z^2(x))^2} \left[ \frac{1}{2}(\alpha+\beta+2) - \frac{1}{4}(\beta-\alpha)^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(z'(x))^2 z(x)}{(1-z^2(x))^2} \frac{1}{2} (\beta - \alpha)(\beta + \alpha) \\
& + \frac{(z'(x))^2 z^2(x)}{(1-z^2(x))^2} \left[ \frac{1}{4} - \left( \frac{\alpha + \beta + 1}{2} \right)^2 \right].
\end{aligned} \tag{3.8}$$

As discussed in Ref. [5], one selects differential equations of the type (3.7) for  $z(x)$  to get constant terms on the right-hand side of (3.8). In [5] the first two non-trivial terms were picked, leading to the PI and PII potential classes, typical representations of which are, for example,  $U_2(r)$  and  $U_1(r)$  in Eqs. (2.10) and (2.9), respectively. The defining differential equation of these is  $(z')^2(1-z^2)^{-1} = C$  and  $(z')^2(1-z^2)^{-2} = C$ . Later in Ref. [29] the third ‘‘PIII’’ possibility,  $z(z')^2(1-z^2)^{-2} = C$  was also discussed, resulting in an ‘‘implicit potential’’. All these potentials are exactly solvable Natanzon-class potentials, furthermore, those discussed in Ref. [5] also have the property of shape-invariance.

The fourth possibility,

$$z^2(z')^2(1-z^2)^{-2} = C \tag{3.9}$$

was not discussed in detail in Ref. [5], only the generic form of the solution was mentioned. However, it turns out, that the function  $z(x) = (1 + \exp(2C^{1/2}x + D))^{1/2}$  satisfies (3.9), and it leads to the same variable transformation as that discussed in Ref. [14], if the  $C^{1/2} = -1$  and  $D = 0$  choice is made. The actual form of (3.4) is now (in the ‘‘PIV’’ case)

$$\begin{aligned}
E_n - V(x) = & - \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 + \frac{1}{2} (\beta - \alpha)(\beta + \alpha) z^{-1}(x) + \frac{3}{4} z^{-4}(x) \\
& + \left[ \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 - \left( \frac{\alpha + \beta}{2} \right)^2 - \frac{3}{4} - \frac{1}{4} (\beta - \alpha)^2 \right] z^{-2}(x).
\end{aligned} \tag{3.10}$$

This leads to a solvable potential if the  $n$ -dependence can be canceled in the coordinate-dependent (i.e. potential) terms by a suitable change of the parameters. Comparing (3.10) with (2.2) we get

$$A = - \left[ \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 - \left( \frac{\alpha + \beta}{2} \right)^2 - \frac{3}{4} - \frac{1}{4} (\beta - \alpha)^2 \right], \tag{3.11}$$

$$B = \frac{1}{2} (\beta - \alpha)(\beta + \alpha), \tag{3.12}$$

and

$$E_n = - \left( n + \frac{\alpha + \beta + 1}{2} \right)^2 . \quad (3.13)$$

Obviously,  $\alpha$  and  $\beta$  depend on  $n$  and also on the potential parameters  $A$  and  $B$ . Substituting (3.13) in (3.11) and combining it with (3.12) we arrive at (2.12), the equation defining the energy eigenvalues in the two approaches of Sec. II.

The bound-state wave functions are found to be

$$\psi(x) \sim z^{1/2}(x)(z(x) + 1)^{\beta_n/2}(z(x) - 1)^{\alpha_n/2} P_n^{(\alpha_n, \beta_n)}(z(x)) , \quad (3.14)$$

which (apart from some misprints), corresponds to Eqs. (15), (16) and (18) in Ref. [14], if we substitute  $\alpha_n = B/(2c) - c$  and  $\beta_n = -B/(2c) - c$ .

## B. Supersymmetric connection

In the knowledge of the bound-state wave functions, constructing the superpotential  $W(x)$  is a simple matter using Eq. (2.14). From (3.14) with  $n = 0$  one obtains

$$W(x) = \frac{1}{2}(\alpha_0 + \beta_0 + 1) + \frac{\alpha_0 - \beta_0}{2z(x)} - \frac{1}{2z^2(x)} . \quad (3.15)$$

In order to get closer to the methods described in Sec. II B, we also introduce the singular superpotentials obtained in a similar way from the wave functions with  $n > 0$ . The Jacobi polynomial appearing in these functions is best expressed in a product form

$$P_n^{(\alpha_n, \beta_n)}(z) \sim \prod_{i=1}^n (z - c_i) , \quad (3.16)$$

where the  $c_i$  are at the roots (nodes) of the polynomial. Obviously, the logarithmic derivative of this product will reduce to a sum form

$$\frac{d}{dx} (\ln P_n^{(\alpha_n, \beta_n)}(z)) = \frac{dz}{dx} \frac{d}{dz} \sum_{i=1}^n \ln(z - c_i) = (z^{-1} - z) \sum_{i=1}^n \frac{1}{z - c_i} . \quad (3.17)$$

Here we used the differential equation (3.9) to express  $z'$  in terms of  $z$ . This explains the sum appearing in the superpotential (2.18) in Sec. II B. A similar construction can readily be presented for the superpotential used in Ref. [19] describing the potential of Ref. [13] in a supersymmetric framework. The polynomial there is of the Hermite type.

### C. Relation to the Natanzon potentials

Our discussion in the present Section was based on the approach of Ref. [5], which is general enough to incorporate both the conventional and the supersymmetric formulation of potential (2.2) in a relatively straightforward way. One can, however, put the whole subject into an even more general framework, that of the Natanzon potentials [7]. Although the discussion could have been presented using the formalism of this potential class, we decided to follow the easier route of Ref. [5] for several reasons. First, the general formalism was too heavy for demonstrative purposes, and second, its relation to the machinery of supersymmetric quantum mechanics [3,9] is less transparent. However, to conclude this Section we present the essential facts about Natanzon potentials, and their relevance to the potentials we investigated.

The general families of the Natanzon [7] and Natanzon confluent [30] potentials are characterized by the feature that their solutions are expressed in terms of a single (confluent) hypergeometric function. The general Natanzon potential depends on six parameters, three of which ( $f$ ,  $h_0$  and  $h_1$ ) appear explicitly in the expression

$$V(x) = -\frac{z'''(x)}{2z'(x)} + \frac{3}{4} \left( \frac{z''(x)}{z'(x)} \right)^2 + \frac{fz(x)(z(x)-1) + h_0(1-z(x)) + h_1z(x)}{\mathcal{R}(z(x))}, \quad (3.18)$$

while three others ( $a$ ,  $c_0$  and  $c_1$ ) enter implicitly through the  $z(x)$  function determined by the differential equation

$$z'(x) \equiv \frac{dz}{dx} = \frac{2z(1-z)}{(\mathcal{R}(z))^{1/2}} \quad (3.19)$$

with

$$\mathcal{R}(z) = az(x)(z(x)-1) + c_0(1-z(x)) + c_1z(x). \quad (3.20)$$

The construction of [5], when specified for the Jacobi polynomials (a special case of the hypergeometric function [26]) can easily be recognized as a particular reformulation of this change of variable method. (See also Ref. [31] and the Appendix of Ref. [27].) The energy spectrum is determined [7] by the implicit equation

$$2n+1 = (f+1-aE_n)^{1/2} - (h_0+1-c_0E_n)^{1/2} - (h_1+1-c_1E_n)^{1/2} \equiv \alpha_n - \beta_n - \delta_n, \quad (3.21)$$

while the bound-state wave functions are written as

$$\psi(x) \sim \mathcal{R}^{1/4}(z(x))(1 - z(x))^{\frac{\beta_n}{2}}(z(x))^{\frac{\beta_n}{2}} F(-n, \alpha_n - n; \beta_n + 1; z(x)) . \quad (3.22)$$

The form of (3.22) is again reminiscent of the construction of Ref. [3], while (3.21) is close to the implicit energy formula obtained for the potential of Ref. [14] in the point canonical transformation formalism.

Equations similar to those above are valid for the Natanzon confluent potential class [30] too.

It is instructive to examine the role of the 3+3 parameters appearing in the Natanzon potentials, as it is related to the concept of conditionally exact solvability. For the most commonly occurring potentials (like the shape-invariant ones [8]), the three parameters determining the  $z(x)$  function via (3.19) and (3.20), usually only one appears, and even that one is a trivial scaling parameter of the coordinate and/or the energy scale. (Trivial coordinate shifts can also appear through them.) Usually they play a non-trivial role only in the case of some “implicit” potentials [28].

The other three parameters appearing in (3.18) set the potential shape, and determine the relative strength of the individual potential terms. In most potentials only one or two of these parameters appear. The two parameters appearing in potential (2.2),  $A$  and  $B$  are of this type. (There could be one more parameter setting the length scale, but it is set to 1 in this case.) Obviously, when there are three potential terms, as in (2.2), and only two parameters, then the relative strength of the three potential terms cannot be arbitrary, and has to be constrained. This is why the third term of (2.2) is a numerical constant, i.e.  $-3/4$ . It is the presence of this numerical constant which earned potentials in Refs. [13,14] the name “conditionally exactly solvable”. In fact, based on the structure of their eigenfunctions, the potentials appearing in Ref. [14] are of the Natanzon type [7], while those in Ref. [13] belong to the Natanzon confluent class [30]. There are, however, further considerations regarding normalizability and regularity, which might impose restrictions on the solvability of certain potentials. Not surprisingly, these may play a more important role in the case of the less “trivial” potentials [32].

Finally, we note that the other class of CES potentials [16,17] has completely different



nature, and does not belong to the Natanzon class, rather it has features typical for SUSY partners of general Natanzon-class potentials. This again confirms our finding that the concept of conditionally exact solvability is not an alternative of exact solvability, rather it classifies potentials according to different principles.

#### IV. CONCLUSIONS

We analyzed the potentials introduced originally in Ref. [14] as conditionally exactly solvable (CES) potentials via the method of point canonical transformation. Our results concerned the following three areas.

*i)* We gave a supersymmetric re-interpretation of this potential class.

*ii)* We examined the cubic formula which determines implicitly the energy eigenvalues of the problem. We rigorously took into account boundary conditions of the eigenfunctions, and corrected certain inaccuracies presented in Ref. [14]. We demonstrated that from the three roots of the cubic equation there is only one (the middle one) which can lead to physically acceptable eigenstates.

*iii)* We interpreted this potential in the general framework of the Natanzon potential class, and demonstrated that this CES potential, in fact, belongs to this class, and therefore it is a *bona fide* exactly solvable problem.

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Figure 1. Graphical solution of eq. (2.12)

