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# VALID ASYMPTOTIC EXPANSIONS FOR THE LIKELIHOOD RATIO AND OTHER STATISTICS UNDER CONTIGUOUS ALTERNATIVES

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*SUMMARY.* Let $\{Z_n\}_{n\geqslant 1}$ be a sequence of i.i.d. random vectors. Let $W_n$ be a statistic based on the mean vector $\bar{Z}_n$ whose asymptotic null distribution is a central chi-square with $p$ degrees of freedom. It is shown that the distribution function under contiguous alternatives of $W_n$ possesses a valid asymptotic expansion in powers of $n^{-1/2}$, the leading term being a noncentral chi-square with $p$ degrees of freedom and the coefficients of $n^{-j/2}(j \geqslant 0)$ being finite linear combinations of noncentral chi-squares with same noncentrality parameter and with degrees of freedom $p, p+2, p+4, \ldots,$ provided the conditions of Chandra and Ghosh (1979) together with a uniform Cramér's condition and smoothness conditions on moments hold. The result is applied to get expansions for the likelihood ratio statistic, Wald's and Rao's statistics under contiguous alternatives. The similar expansion for the likelihood ratio statistic obtained formally by Hayakawa (1977) has been justified.

## 1. Introduction and main result

The limiting distributions of a large class of important statistics used in asymptotic theory are either the normal distribution or the $\chi^2$-distribution. It is often desirable (from the point of view of theoretical interest as well as of numerical accuracy) to improve upon the limiting distribution by obtaining asymptotic expansions. Bhattacharya and Ghosh (1978) settled this problem when the limiting distribution is normal, and Chandra and Ghosh (1979) when it is a central $\chi^2$. What are the limiting distributions of these statistics under contiguous alternatives ? The answer for the first case follows almost immediately from that of Bhattacharya and Ghosh (vide Remark 3 below, page 5). It is the second case that presents some novel features. The following discussion is confined to it.

*Let $\theta_0$ be a fixed element of $R^{k'}$ and $\{\theta_n\}_{n\geqslant 1}$ a fixed sequence in $R^{k'}$.* Let $\{\mathbf{Z}_n\}_{n\geqslant 1}$ be a sequence of $k$-dimensional random vectors which are independently and identically distributed (i.i.d.). Expectations under $\theta_n$ will be denoted by $E_n$ $(n \geqslant 0)$. Put for each $n \geqslant 0$

$$\mu(\theta_n) = E_n(\mathbf{Z}_1),\ V_n = E_n(\mathbf{Z}_1-\mu(\theta_n))^T(\mathbf{Z}_1-\mu(\theta_n))$$

where $T$ denotes transpose. *We shall assume that* $V_0$ **is nonsingular and that** $\mu(\theta_0)$ ***is the null vector.***

The following facts will be referred to frequently and are collected as a remark.

*Remark* 1 : We shall assume later (vide condition (c) of the Theorem) that $\{\theta_n\}_{n \geqslant 1}$ is "contiguous" to $\theta_0$ in the sense that the first $(s-1)$ moments under $\theta_n$ have expansions (in powers of $n^{-1/2}$) up to $o(n^{-(s-3)/2})$, whose leading terms are the corresponding moments under $\theta_0$. Suppose that $S_{\theta_0}$ is a sphere around $\theta_0$ such that the distribution of $\boldsymbol{Z}_1$ under $\theta$ is defined for each $\theta$ in $S_{\theta_0}$. Then under enough regularity conditions the $j$-th moment under $\theta$ will be $(s-3)$ times continuously differentiable; in this case if $\theta_n$ is contiguous in the commonly used sense, i.e., of the form $\theta_0 + n^{-1/2}\tilde{\Delta}$ $(n \geqslant 1)$, then the Taylor expansion for the $j$-th moment under $\theta_n$ (around $\theta_0$) will lead to the kind of expansion required by condition (c) of the Theorem. We shall also need a uniform Cramér's condition (vide 1.6)).

Let $H$ be a real-valued measurable function defined on $R^k$. Consider the statistic $H(\bar{\boldsymbol{Z}}_n)$ where $\bar{\boldsymbol{Z}}_n = n^{-1} \sum_{i=1}^{n} \boldsymbol{Z}_i$ $(n \geqslant 1)$. Assume that the limiting distribution under $\theta_0$ of the statistic

$$W_n = 2n[H(\bar{\boldsymbol{Z}}_n) - H(\mu(\theta_0))] \quad \ldots \quad (1.1)$$

is a central $\chi^2$. Chandra and Ghosh proved that under some conditions on $H$ and the distribution of $\boldsymbol{Z}_1$ under $\theta_0$, the distribution function of $W_n$ when $\theta_0$ obtains admits an asymptotic expansion in powers of $n^{-1}$, the coefficients of $n^{-j}$ $(j \geqslant 1)$ being finite linear combinations of the distribution functions of central $\chi^2$'s. To state these conditions, we need the following notations.

Let $s$ be an integer $\geqslant 4$. Denote the *partial derivatives of* $H$ *at* $\mu(\theta_0)$ by

$$l_{i_1 i_2 \ldots i_j} = D^{i_1} D^{i_2} \ldots D^{i_j} H(\mu(\theta_0)) \quad \ldots \quad (1.2)$$

$(1 \leqslant i_1, \ldots, i_j \leqslant k;\ j \geqslant 1)$ where $D^i$ stands for differentiation with respect to the $i$-th coordinate variable. Write

$$\boldsymbol{l} = (l_1, \ldots, l_k), \quad \underset{k \times k}{\boldsymbol{L}} = ((l_{ij})).$$

*Let* $p$ *be the rank of* $L$. Denote by $H_{s-1}(\boldsymbol{z})$ *the Taylor expansion around* $\mu(\theta_0)$ *of* $H(\boldsymbol{z})$ *up to and including the terms involving the* $(s-1)$*-th order derivatives of* $H$. The symbols $\| \ \|$ and $<,>$ will denote *Euclidean norm* and *inner product* respectively. Let $\hat{Q}_{1,n}$ be *the characteristic function of* $\boldsymbol{Z}_1$ under $\theta_n$ i.e., let

$$\hat{Q}_{1,n}(t) = E_n(\exp\{i < t, \boldsymbol{Z}_1 >\}) \qquad n \geqslant 0,\ t \in R^k. \quad \ldots \quad (1.3)$$

$Z_1$ is said to satisfy *Cramér's condition under* $\theta_n$ if

$$\sup\{|\hat{Q}_{1,n}(t)| : \|t\| > b\} < 1 \qquad \ldots \text{ (1.4)}$$

for each $b > 0$ (see Bhattacharya and Ranga Rao, 1976, page 207).

Chandra and Ghosh assumed that under $\theta_0$, $\boldsymbol{Z}_1$ satisfies Cramér's condition, $E_n(\|\boldsymbol{Z}_1\|^{s-1})$ is finite and that the following assumptions $\mathbf{A}_s$(i)–(iv) hold.

*Assumption* $\mathbf{A}_s$ : (i) all derivatives of $H$ of order $s$ and less are continuous in a neighbourhood of $\mu(\theta_0)$;

(ii) the vector $\boldsymbol{l}$ is null;

(iii) the matrix $\boldsymbol{L}$ is non-null and satisfies the equation $\boldsymbol{L}\boldsymbol{V}_0\boldsymbol{L} = \boldsymbol{L}$;

(iv) if under some nonsingular linear transformation $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{z}$, $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(k)})$, $\boldsymbol{z}^T \boldsymbol{L}\boldsymbol{z}$ becomes a positive-definite quadratic form in $\boldsymbol{x}^1 = (x^{(1)}, \ldots, x^{(p)})$ then

$$H_{s-1}(\boldsymbol{A}^{-1}\boldsymbol{x}) = \sum_{i,j=1}^{p} x^{(i)}x^{(j)}P_{ij}(x) \qquad \ldots \text{ (1.5)}$$

for some polynomials $\{P_{ij}\}$.

$\boldsymbol{Z}_1$ is said to satisfy *Cramér's condition uniformly* if

$$\sup\{|\hat{Q}_{1,n}(t)| : \|t\| > b,\ n \geqslant 0\} < 1 \qquad \ldots \text{ (1.6)}$$

for each $b > 0$.

If $\boldsymbol{Z}_1$ satisfies Cramér's condition under $\theta_n$ and the distribution of $Z_1$ under $\theta_n$ converges in variation norm to that under $\theta_0$, then the uniform Cramér's condition holds. Under the set-up described in Remark 1, suppose that for each $\theta$ in $S_{\theta_0}$ the distribution of $\boldsymbol{Z}_1$ under $\theta$ admits a density $f_\theta$ such that the map $\theta \to f_\theta$ is continuous in $S_{\theta_0}$; then using Scheffé's theorem (see Lemma 2.1 of Bhattacharya and Ranga Rao, 1976), it is easy to see that the above sufficient condition holds.

In this paper $\boldsymbol{\nu}$ will always denote a vector of nonnegative integers and $\boldsymbol{z}$ an element of $R^k$. For each $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ and for each $\boldsymbol{z} = (z^{(1)}, \ldots, z^{(k)})$, put

$$\begin{aligned} \boldsymbol{z}^{\boldsymbol{\nu}} &= (z^{(1)})^{\nu(1)} \ldots (z^{(k)})^{\nu(k)} \\ |\boldsymbol{\nu}| &= \nu^{(1)} + \ldots + \nu^{(k)} \\ \boldsymbol{z}^1 &= (z^{(1)}, \ldots, z^{(p)}),\ \boldsymbol{z}^2 = (z^{(p+1)}, \ldots, z^{(k)}) \end{aligned} \qquad \ldots \text{ (1.7)}$$

$(1 \leqslant p \leqslant k)$. Let $\mu_{\nu,n} = E_n(\mathbf{Z}_1^\nu)$ be *the $\nu$-th moment* of $\mathbf{Z}_1$ under $\theta_n$, in case the latter is finite. Let $f_{j,\delta}$ and $\hat{f}_{j,\delta}$ be respectively *the density and characteristic function of a noncentral $\chi^2$ with degrees of freedom $j$ and noncentrality parameter* $\delta (j \geqslant 1, \ \delta > 0)$.

In this paper

$$\delta = (\mathbf{\Delta}^1)^T \bar{\mathbf{V}}_0^{11} \mathbf{\Delta}^1$$

where $\bar{\mathbf{V}}_0^{11}$ is the submatrix consisting of the first $p$ rows and $p$ columns of the inverse of $\mathbf{V}_0$ and

$$\Delta = \lim_{n \to \infty} n^{1/2}(E_n(\mathbf{Z}_1) - E_0(\mathbf{Z}_1)).$$

We can now state the main result.

Theorem : *Assume that for some integer $s \geqslant 4$,*

(a) *$H$ and $V_0$ satisfy conditions* $A_s$(i)–(iv);

(b) *$\mathbf{Z}_1$ satisfies the uniform Cramér's condition* (1.6); *and*

(c) $\sup_{n \geqslant 0} E_n(\|\mathbf{Z}_1\|^s)$ *is finite and $\mu_{\nu,n}$ admits an expansion (in powers of $n^{-1/2}$) up to $o(n^{-(s-3)/2})$, $1 \leqslant |\nu| \leqslant (s-1)$.*

*Then there exist nonnegative integers $k_1, \ldots, k_{s-3}$ and constants $\{P_{i,j}\}$, not depending on $n$, $(0 \leqslant i \leqslant k_j, \ 1 \leqslant j \leqslant s-3)$ such that the following holds uniformly in $u \in [u_0, \infty)$, $u_0 > 0$.* One can replace $u_0$ by zero if $p > 1$.†

$$P_{\theta_n}(W_n \leqslant u) = \sum_{j=0}^{s-3} n^{-j/2} \int_{-\infty}^{u} \psi_j(v) dv + o(n^{-(s-3)/2}). \quad \ldots \ (1.8)$$

*where*

$$\psi_j = \sum_{i=0}^{k_j} P_{i,j} \ f_{p+2i,\delta} \quad j \geqslant 1,$$

$$\psi_0 = f_{p,\delta}. \quad \ldots \ (1.9)$$

*Remark* 2 : The theorem in its present form is often unsuitable for statistical applications because of the assumption that $W_n$ is a function of the mean vector $\bar{Z}_n$ based on some sequence of i.i.d. random variables. One can however easily verify that the Theorem remains valid if the normalised

† A similar modification is needed to correct a mistake in Theorem 1 in Chandra and Ghosh (1979) ; specifically the uniformity in $u$ asserted there fails for $p = 1$ unless $u \in [u_0 \ \infty) u_0 > 0$.

deviation $n^{1/2}(\bar{Z}_n-\mu(\theta_0))$ there is replaced by $n^{1/2}(U_n-E_0(U_n))$ where $\{U_n\}_{n\geq 1}$ is an arbitrary sequence of random variables possessing a uniform Edgeworth expansion.

*Remark* 3 : Suppose assumptions (b) and (c) (with $s$ in place of $(s-1)$) of the Theorem hold. Assume $\mathbf{A}_s$(i) and instead of the rest of $\mathbf{A}_s$, assume as in Bhattacharya and Ghosh that the vector $\boldsymbol{l}$ is non-null. Let

$$W_n^* = \sqrt{n}(H(\bar{Z}_n)-H(\mu(\theta_0)).$$

Then $P_{\theta_n}(W_n^* \leqslant u)$ has an asymptotic expansion (in powers of $n^{-1/2}$) valid up to $o(n^{-(s-2)/2})$ with the leading term of normal distribution with *nonzero* mean.

One may prove this by applying Theorem 2(b) of Bhattacharya and Ghosh (1978) with $P_{\theta_n}$ in place of $P$ and then expanding the coefficients in the expansion. Alternatively one may proceed as in the proof of Proposition 1 of Section 2 up to a relation analogous to (2.7) and then proceed as in the proof of Theorem 2(b) of Bhattacharya and Ghosh (1978).

It is easy to check that this expansion agrees with the formal Edgeworth expansion obtained by evaluating the first $s$ moments of $W_n^*$ formally up to $o(n^{-(s-2)/2})$ by the delta-method.

*Remark* 4 : It can be shown that each of the coefficients $\{P_{i,j}\}$ is a polynomial in the constants appearing in the expansions of $\{\mu_{\nu,n} : 1 \leqslant |\nu| \leqslant s-1\}$ and that for each $j \geqslant 1$, the sum $\sum_{i=0}^{k_j} P_{i,j}$ always vanishes; for a proof of the last fact, one needs only to verify that the Fourier-Stieltjes transform $\hat{\psi}_j(t)$ of $\psi_j$ vanishes at $t=0$. Also one may note that if $\theta_n \equiv \theta_0$ then the assumptions of the Theorem reduce essentially to those of Theorem 1(b) of Chandra and Ghosh and that in this case each of $\{P_{i,j} : 0 \leqslant i \leqslant k_j\}$ vanishes whenever $j$ is odd; this follows from Proposition 2(a), relation (2.19) and the fact that the coefficient of $n^{-j/2}$ in $h_{s-1}(\boldsymbol{z})$ or $\xi_{2,s,n}(z)$ is a polynomial $P_j(\boldsymbol{z})$ in $\boldsymbol{z}$ with the property that the degree of each term of $P(\boldsymbol{z})$ is odd or even according as $j$ is odd or even. Thus Theorem 1(b) of Chandra and Ghosh can be regarded as a special case of the above theorem. The present derivation can be used as an alternative to the involved arguments given in Remark 2.5 of our earlier paper.

The proof of the Theorem is given in Section 2. In Section 3, we consider applications to the likelihood ratio and other related statistics. In Section 4, expansions under a fixed alternative of these statistics have been obtained (a) when the null hypothesis is simple and (b) when the null hypothesis is composite and the observations are coming from an exponential family of distributions.

## 2. Proof of the theorem

*We assume throughout this section that the assumptions of the Theorem hold.* We may then assume without loss of generality that

$$V_0 = I,\ z^T L z = \|z^1\|^2 \qquad \dots \quad (2.1)$$

and that $A_s$(iv) holds with $A = I$, where $I$ is *the* $k \times k$ *identity matrix* (see Remark 2.4 of Chandra and Ghosh). Let

$$g_n(z) = 2n[H(\mu(\theta_0) + n^{-1/2}z) - H(\mu(\theta_0))]$$

and $h_{s-1}$ be a Taylor expansion around $\mu(\theta_0)$ of $g_n$ i.e.,

$$h_{s-1}(z) = 2\sum_{j=2}^{s-1} \frac{n^{-(j-2)/2}}{j\,!} \sum l_{i_1 \dots i_j} z^{(i_1)} \dots z^{(i_j)} \qquad \dots \quad (2.2)$$

(see (1.2)). Let $\phi_V$ denote the density of the normal distribution on $R^k$ with the mean vector $\mathbf{0}$ and the dispersion matrix $V$. Let $C_n(t)$ be *the characteristic function of* $W_n$ *under* $\theta_n$,

$$C_n(t) = E_n(\exp(itW_n)) \qquad t \in R^1 \qquad \dots \quad (2.3)$$

Write

$$M_n = \{z \in R^k : \|z\|^2 < (s-1)\log n\} \quad n \geqslant 1.$$

It is well-known that for any $x \in R^k$ and any integer $q \geqslant 0$

$$\int_{M_n^c} \|z\|^q \exp\left\{-\frac{1}{2}\|z-x\|^2\right\} dz = o(n^{-(s-2)/2}) \qquad \dots \quad (2.4)$$

where $M_n^c$ is the complement of $M_n$. *Throughout this section,* $\sum_i$ *will stand for summation over finitely many* $i$, *not necessarily over the same set of values of* $i$ *in each appearance.*

Before we start proving our Theorem, we shall state and prove three auxiliary propositions.

*Proposition* 1 : Let assumptions (a), (b) and (c) of the Theorem hold. Then there exist constants $\{\psi_{j,i}\}$, not depending on $n$, ($i \geqslant 0$ and $j = 1, 2, \dots, s-3$) such that the following holds uniformly in $u \in [u_0, \infty]$, $u_0 \geqslant 0$ if $p > 1$ and $u_0 > 0$ if $p = 1$,

$$P_{\theta_n}(W_n \leqslant u) = \sum_{j=0}^{s-3} n^{-j/2} \int_{-\infty}^{u} \psi_j(v)dv + o(n^{-(s-3)/2}) \qquad \dots \quad (2.5)$$

where

$$\psi_j = \sum_{i=0}^{\infty} \psi_{j,i} f_{p+i,0}, \qquad j \geq 1$$
$$\psi_0 = f_{p,s} \qquad \ldots \quad (2.6)$$

Thus once this proposition is established, the Theorem would follow if one could show that $\psi_j$ given by (2.6) can as well be expressed in the form (1.9). Proposition 1 will be used here to show essentially the existence of a valid expansion for $W_n$ which is needed for the proof of Proposition 3 below.

*Proof of Proposition* 1 : Let $\Delta_n$ be defined by

$$\mu(\theta_n) = \mu(\theta_0) + n^{-1/2}(\Delta + \Delta_n)$$

The Edgeworth expansion of

$$n^{1/2}(\bar{Z}_n - \mu(\theta_n)) = n^{1/2}(\bar{Z}_n - \mu(\theta_0)) - \Delta - \Delta_n$$

under $\theta_m$ can be written as

$$\xi_{1,s,n,m}(z) = \phi_{V_m}(z)\left[1 + \sum_{j=1}^{s-3} n^{-j/2} \sum_i R_{ij}(z)\right]$$

where $\{R_{ij}\}$ are polynomials in $z$ with coefficients *rational functions* of $\{\mu_{v,m} : 1 \leq |\nu| \leq s-1\}$ *with nonvanishing denominators* (at least for all sufficiently large $n$). Because of the uniform Cramér's condition and the uniform boundedness of $\{\mu_{v,m}\}$ (guaranteed by condition (c)), we get, setting $\xi_{1,s,n}(z) = \xi_{1,s,n,n}(z)$,

$$P_{\theta_n}(W_n \leq u) = \int_{\{g_n \leq u\}} \xi_{1,s,n}(z - \Delta - \Delta_n)dz + o(n^{-(s-3)/2})$$

uniformly in $u \in R^1$. We use here Theorem 1.5 of Bhattacharya (1977) and the first observation following its proof, page 11; one needs the following estimate

$$\sup_u \int_{|g_n - u| < \epsilon_n} \phi_{V_n}(z)dz = o(n^{-(s-3)/2}), \quad \epsilon_n = \exp(-dn)$$

(for any $d > 0$) which can be proved by first approximating $g_n$ by $h_{s-1}$, and then using arguments similar to those used in (2.16) through (2.19) of Bhattacharya and Ghosh (1978), their (2.8) is to be replaced by the third equality in the proof of Theorem 1(b) of Chandra and Ghosh (1979).

We now make use of assumption (c) and expand $R_{ij}$'s and $\phi_{V_n}(z)$, getting uniformly in $u \in R^1_+$,

$$P_{\theta_n}(W_n \leqslant u) = \int_{\{g_n \leqslant u\}} \xi_{2,s,n}(z)dz + o(n^{-(s-3)/2}) \qquad \ldots \quad (2.7)$$

where

$$\xi_{2,s,n}(z) = \phi(z-\Delta) \sum_{j=0}^{s-3} n^{-j/2} P'_{i,j}(z) \qquad \ldots \quad (2.8)$$

$P'_{1,i}(z)$ being suitable polynomials (free from $n$) in $z$ ($j \geqslant 1$), $P'_{1,0}(z) \equiv 1$ and $\phi = \phi_I$ (recall that $V_0 = I$; cf. (2.1)). The coefficients of the polynomials $P'_{1,j}$ are themselves polynomials in the constants appearing in the expansions of $\{\mu_{v,n} : 1 \leqslant |\nu| \leqslant s-1\}$.

We now proceed as in the proof of Theorem 1 of Chandra and Ghosh. Since

$$\sup_{z \in M_n} |g_n(z) - h_{s-1}(z)| = 0(\varepsilon_n)$$

with $\varepsilon_n = n^{-(s-2)/2}(\log n)^{s/2}$ and since

$$\sup_{u \in R^1} \left| \int_{A_n} z^v \phi(z-\Delta) dz \right| = 0(\varepsilon_n),$$

where $A_n = \{z : |h_{s-1}(z) - u| \leqslant \varepsilon_n\} \cap M_n$, one gets from (2.4) and (2.7),

$$P_{\theta_n}(W_n \leqslant u) = \int_{\{h_{s-1} \leqslant u\} \cap M_n} \xi_{2,s,n}(z) + o(n^{-(s-3)/2}) \qquad \ldots \quad (2.9)$$

uniformly in $u \in R^1_+$.

One now applies successively the following three one-to-one transformations $T_1$, $T_2$ and $T_3$ on $R^k$. The transformation $T_1$ is an orthogonal transformation on $z^1 = (z^{(1)}, \ldots, z^{(p)})$ with the first transformed variable as $< \Delta^1, z^1 > /\|\Delta^1\|$, keeping the remaining $z$'s unchanged (if $\Delta^1$ is null, then the transformation $T_1$ is not needed). The transformation $T_2$ sends $z^1$ to $(r, \theta^{(1)}, \ldots, \theta^{(p-1)})$ by means of the standard polar transformation with $z^{(1)} = r \sin \theta^{(1)}$, keeping the remaining $z$'s unchanged. Finally, the transformation $T_3$ is defined by

$$T_3(r, \theta, z^2) = (r', \theta, z^2)$$

where

$$r' = (h_{s-1}(T_1^{-1} T_2^{-1}(r, \theta, z^2)))^{\frac{1}{2}},$$

$\boldsymbol{\theta} = (\theta^{(1)}, \ldots, \theta^{(p-1)})$ and $z^{2} = (z^{(p+1)} \ldots, z^{(k)})$. After some more computations along the lines of Chandra and Ghosh (1979) (see the proof of their Theorem 1(a)), (2.9) reduces to

$$P_{\theta_n}(W_n \leqslant u) = K \exp\left(-\frac{\delta}{2}\right) \int_{B_n} \exp\{-\tfrac{1}{2}(r')^2\} (r')^{p-1} \psi_n'(r', \theta) dr'\, d\theta$$
$$+o(n^{-(s-3)/2}),$$

where $K$ is a constant free from $n$,

$$B_n = \{(r', \theta) : (r')^2 \leqslant u,\ (r')^2 < (s-3/2) \log n\}$$

and

$$\psi_n'(r', \theta) = \left[\sum_{j=0}^{s-3} n^{-j/2} P'_{2,j}(r', \theta)\right] \exp\{\Delta r' \sin \theta^{(1)}\},$$

$P'_{2,j}(r', \theta)$ being a finite sum of products of powers of $r'$ and the trignometric functions $\sin \theta^{(i)}$, $\cos \theta^{(i)} (1 \leqslant i \leqslant p-1)$ of $\theta$. One now expands $\exp\{\Delta r' \sin \theta^{(1)}\}$ in an infinite series and integrate term by term the resulting infinite series which is permissible since the last series is uniformly convergent over the set $\{(r', \theta) : (r')^2 < (s-3/2) \log n\}$. The rest of the proof is easy.

To state the next result, let $j_1(\nu)$ denote the number of odd components of $\nu$ (see also (1.7)).

*Proposition* 2 :

(a) $$\int_{\|z^1\|^2 < u} z^{\nu} \exp\{-\tfrac{1}{2}\|z-\Delta\|^2\} dz = \sum_{j=m_1}^{m_2} \alpha_j^1(\Delta) \int_{-\infty}^{u} f_{p+2j,\delta}(v) dv, \quad \ldots \quad (2.10)$$

where $\{\alpha_j^1(\Delta)\}$ are suitable polynomials in $\Delta$ and

$$m^1 = \tfrac{1}{2}(|\nu^1| + j_1(\nu_1)), \quad m_2 = |\nu^1| \quad \ldots \quad (2.11)$$

(b) Let $r^*$ be a nonnegative integer. Then

$$(it)^{r^*} \int_{R^1} \exp\{it\|z^1\|^2\} z^{\nu} \exp\{-\tfrac{1}{2}\|z-\Delta\|^2\} dz = \sum_{j=m_1-r^*}^{m_2} \alpha_j^2(\Delta) \hat{f}_{p+2j,\delta}(t)$$

provided that $|\nu^1| \geqslant 2r^*$, where $\{\alpha_j^2(\Delta)\}$ are suitable polynomials in $\Delta$ and $m_1, m_2$ are defined by (2.11).

It follows from (2.7) and (2.8) that Proposition 2(a) establishes the special case of the Theorem when

$$W_n = 2n(\bar{Z}_n - \mu(\theta_0))^T L(\bar{Z}_n - \mu(\theta_0)).$$

*Proof of Proposition* 2 : Without any loss of generality, we may assume that $p = k$. We need the following fact :

*Fact* (**A**) : Let

$$g(x; b, r) = x^r \exp\{-\tfrac{1}{2}(x-b)^2\} \qquad x \in R^1,\ b \in R^1$$

where $r$ is a non-negative integer. Then there exist (numerical) constants $a_0, a_1, \ldots, a_m$ such that

$$(g(\sqrt{x}; b, r) + g(-\sqrt{x}; b, r))2\sqrt{x} = \sum_{j=0}^{m} a_j b^{2j+j_1} f_{q+2j,\, b^2}(x) \quad \ldots \quad (2.12)$$

wjere $j_1 = 0$ or 1 according as $r$ is even or odd and

$$q = r+1+j_1, \quad m = \tfrac{1}{2}(r-j_1). \quad \ldots \quad (2.13)$$

The fact (**A**) can be established as follows : the left side of (2.12) is

$$\exp\{-\tfrac{1}{2}(x+b^2)\} \sum_{i=0}^{\infty} \frac{b^{2i+j_1}}{(2i+j_1)\,!}\, x^{i+\frac{q}{2}-1}$$

while the coefficient of $x^{i+\frac{q}{2}}$ in the right side of (2.12) is,

$$\frac{a_0 + a_1(2i) + a_2(2i)(2(i-1)) + \ldots}{2^{2i+q/2}\, \Gamma(i+1)\, \Gamma\left(i+\frac{q}{2}\right)} \cdot b^{2i+j_1}.$$

One therefore verifies, using the duplication formula for the gamma function, that

$$\frac{2^{(2i+q)/2}\Gamma(i+1)\Gamma\left(i+\frac{q}{2}\right)}{(2i+j_1)\,!}$$

is a polynomial (in $i$) of degree $(r-j_1)/2$. This completes the proof of the fact (**A**).

To prove Proposition 2, observe that the integral on the left side of (2.10) can be written as

$$\int_A \prod_{i=1}^{k} \{(g_i(\sqrt{z^{(i)}}) + g_i(-\sqrt{z^{(i)}}))/2\sqrt{z^{(i)}}\}\, dz$$

where

$$A = \{z : z^{(1)} + \ldots + z^{(k)} \leqslant u,\ z^{(1)} > 0, \ldots, z^{(k)} > 0\}$$

and

$$g_i(z^{(i)}) = g(z^{(i)};\ \Delta^{(i)},\ \nu^{(i)}) \quad i \geqslant 1.$$

The fact (A), the fact that the family of noncentral $\chi^2$'s is closed under convolution and relation (2.13) complete the proof of Proposition 2(a).

The part (b) with $r^* = 0$ is equivalent to the part (a). The case of the general $r^*$ follows from the part (a) and the following elementary fact :

*Fact (B)* : If $q \geqslant 2r^*+1$,

$$(it)^{r^*} \hat{f}_{q,\delta}(t) = \frac{1}{2^{r^*}} \sum_{j=0}^{r^*} \binom{r^*}{j} (-1)^j \hat{f}_{q-2j,\delta}(t) \quad t \in R^1.$$

*Proposition* 3 : Suppose that the assumptions of the Theorem hold. Suppose that for each real $t$,

$$C_n(t) = C_{n,1}(t) + o(n^{-(s-3)/2}) \qquad \ldots \quad (2.14)$$

where $C_{n,1}(t)$ is the Fourier-Stieltjes transform of

$$\sum_{j=0}^{s-3} n^{-j/2} g_j(v) \quad v \in R^1 \qquad \ldots \quad (2.15)$$

(with $g_j$ free from $n$). Then (2.15) is the valid expansion for $W_n$ under $\theta_n$ up to $o(n^{-(s-3)/2})$. (For the definition of $C_n(t)$, see (2.3)).

*Proof of Proposition* 3 : By Proposition 1, there exists a valid expansion for $W_n$ under $\theta_n$ which is of the form

$$\sum_{j=0}^{s-3} n^{-j/2} \psi_j(v) \qquad \ldots \quad (2.16)$$

(see (2.6)), $\psi_j$ being free from $n$. We shall first show that

$$C_n(t) = \int_{R^1} \exp(itv) \left\{ \sum_{j=0}^{s-3} n^{-j/2} \psi_j(v) \right\} dv + o(n^{-(s-3)/2}). \qquad \ldots \quad (2.17)$$

Now (vide Remark 2.7 of Chandra and Ghosh)

$$\begin{aligned} C_n(t) &= \int_{M_n} \exp(itg_n(z)) \xi_{2,s,n}(z) dz + o(n^{-(s-3)/2}) \\ &= \int_{M_n} \exp(ith_{s-1}(z)) \xi_{2,s,n}(z) dz + o(n^{-(s-3)/2}) \qquad \ldots \quad (2.18) \\ &= \int_{R^1} \exp(itv) \left\{ \sum_{j=0}^{s-3} n^{-j/2} \psi_j(v) \right\} dv + o(n^{-(s-3)/2}) \end{aligned}$$

($\xi_{2,s,n}$ is defined in (2.8)). The first equality follows by Theorem 20.1 of Bhattacharya and Ranga Rao (1976) and the fact that $P_{\theta_n}(M_n^c) = o(n^{-(s-3)/2})$. The proof of the last equality is similar to the argument following (2.9).

From (2.17), (2.14) and the definition of $C_{n,1}(t)$, it follows that $C_{n,1}(t)$ is the Fourier-Stieltjes transform of (2.16). The unicity property of the Fourier-Stieltjes transform then implies that the expressions (2.15) and (2.16) must be identical. This completes the proof of Proposition 3.

*Proof of the Theorem* : We make here the convention that $P(z)$ (with or without suffixes) will stand for a polynomial in $z$ with coefficients free from $n$. Now

$$C_n(t) = \int_{M_n} \exp(ith_{s-1}(z))\xi_{2,s,n}(z)dz + o(n^{-(s-3)/2})$$

$$= \int_{R^k} \exp(it\|z^1\|^2)\psi_n^1(z)dz + o(n^{-(s-3)/2}) \qquad \dots \quad (2.19)$$

where

$$\psi_n^1(z) = \left\{1 + \sum_{j_1=1}^{s-3} n^{-j_1/2} \sum_{j_2=1}^{j_1} (it)^{j_2} P^1_{j_1,j_2}(z)\right\} \xi_{2,s,n}(z).$$

The above equalities follow from (2.18) and from (2.1), (2.2) and (2.4) respectively.

From (2.8) we can rewrite $\psi_n^1(z)$ as

$$\psi_n^1(z) = \left\{\sum_{j_1=0}^{s-3} n^{-j_1/2} \sum_{j_2=0}^{j_1} (it)^{j_2} P^2_{j_1,j_2}(z)\right\} \phi(z-\Delta); \qquad \dots \quad (2.20)$$

($P^2_{0,0}(z) \equiv 1$). The coefficients of the polynomials $\{P^2_{j_1,j_2}(z)\}$ are themselves polynomials in the constants appearing in the expansions of $\{\mu_{\nu,n} : 1 \leqslant |\nu| \leqslant s-1\}$. In view of the assumption $A_s$(iv), note that if $z^\nu$ is any term of some $P^1_{j_1,j_2}(z)$, then $|\nu^1| \geqslant 2j_2$. Clearly this property is inherited by the polynomials $\{P^2_{j_1,j_2}(z)\}$. *Thus if we let $C_{n,1}(t)$ denote the integral on the right side of* (2.19), then Proposition 2(b) and (2.20) imply that

$$C_{n,1}(t) = \sum_{j_1=0}^{s-3} n^{-j_1/2} \sum_{j_2=0}^{kj_1} \beta_{j_1,j_2} \hat{f}_{p+2j_2,s}(t)$$

for suitable nonnegative integers $\{k_{j_1}\}$ and suitable constants $\{\beta_{j_1, j_2}\}$; $k_0 = 0$, $\beta_{0,0} = 1$. In other words,

$$C_{n,1}(t) = \int_{R^1} \exp(itv) \left\{ \sum_{j=0}^{s-3} n^{-j/2} \psi_j(v) \right\} dv \qquad \ldots \quad (2.21)$$

where $\{\psi_j\}$ are of the form (1.9). The definition of $C_{n,1}(t)$, relations (2.19) and (2.21) and Proposition 3 together complete the proof of the Theorem.

## 3. Applications

The Theorem (suitably modified as indicated in Remark 2) of Section 1 can be used to obtain asymptotic expansions of the distribution functions of the likelihood ratio statistic, Wald's and Rao's statistics (see Rao (1965), pages 347–352) under contiguous alternatives, provided that the assumptions $(A_1)$–$(A_6)$ of Bhattacharya and Ghosh (1978) hold and that $E_{\theta_n}(Z^\nu)$, $1 \leqslant |\nu| \leqslant s-1$ are finite and admit asymptotic expansions in powers of $n^{-1/2}$ (for the definition of $\mathbf{Z}_1$, see (2.35) of Bhattacharya and Ghosh, 1978, page 448). It should be noted that the above assumptions are satisfied by the family of exponential distributions with $\theta$ as the natural parameter, provided the assumptions made in Section 3 of Chandra and Ghosh (1979) hold.

To prove this one constructs a set $A_n$ such that (i) $P_{\theta_n}(A_n^c) = o(n^{-(s-3)/2})$ and (ii) on $A_n$ the statistic under consideration can be sufficiently well approximated by a $W_n$ which is of the form (1.1) and which satisfies condition (a) of the Theorem. The $A_n$ used in Sections 4 and 5 of Chandra and Ghosh does this job; in fact the only new thing to be proved is $P_{\theta_n}(A_n^c) = o(n^{-(s-3)/2})$ which follows easily. Note that the possibility of the uniform Edgeworth expansion for maximum likelihood estimators (suitably normalised) is guaranteed by Theorem 3(b) of Bhattacharya and Ghosh (1978).

Hayakawa (1977) obtained an expansion, up to $o(n^{-1/2})$, for the likelihood ratio statistic under contiguous alternatives by a formal inversion of characteristic function. His formal expansion can be justified by suitably modifying Proposition 3 of Section 2; for details, see the last part of Section 4 of Chandra and Ghosh (1979).

## 4. Expansions under a fixed alternative

Suppose that $\{\mathbf{Z}_i\}_{i\geqslant 1}$ is a sequence of i.i.d. random vectors with common distribution either $P_{\theta_0}$ or $P_{\theta_1}$. Let $E_{P_{\theta_i}}(\mathbf{Z}^1) = \mu(\theta_i)$, $i = 0, 1$. Define $W_n$ by (1.1). We want to find expansions for $W_n$ under $P_{\theta_1}$. To this end, assume that $A$ (i) and (ii) hold with $l = \text{grad } H(\mu(\theta_0))$ and that $l_1 \equiv \text{grad } H(\mu(\theta_1)) \neq 0$ where $\text{grad } H = (D^1H, \ldots, D^kH)$. Then the distribution function under $P_{\theta_1}$ of

$$n^{-1/2}\{W_n - 2n[H(\mu(\theta_1)) - H(\mu(\theta_0))]\}$$

possesses an asymptotic expansion (in powers of $n^{-1/2}$) with *the leading term a normal distribution with zero mean and variance* $\mathbf{l}_1^T \mathbf{V} \mathbf{l}_1$ where

$$V = E_{P_{\theta_1}}(\mathbf{Z}^1 - \mu(\theta_1))^T(\mathbf{Z}^1 - \mu(\theta_1)).$$

($\mathbf{V}$ is assumed to be nonsingular). The result follows from Theorem 2(b) of Bhattacharya and Ghosh (1978).

Consider now the problem of testing a *simple null hypothesis*. One can apply the above result to get asymptotic expansions for the likelihood ratio and other related statistics under a fixed alternative. In the last section of his paper, Hayakawa has obtained (formally) such a result for the case of the likelihood ratio statistic. It can be shown that this formal expansion is in fact a valid one.

We assumed above that the null hypothesis is simple. Similar expansions are possible for the case of a composite null hypothesis provided the observations are coming from an exponential family of distributions with natural parameter space; one has to express the maximum likelihood estimators under the null hypothesis (we assume that these estimators exist) in terms of the sample mean. Since Wald's statistic depends only on the unrestricted maximum likelihood estimators, the asymptotic expansion for this statistic can be obtained even if the null hypothesis is composite and the parent population is not exponential.

For a fixed alternative, Siotani (1971) has obtained an expansion for Hotelling's generalized $T_0^2$ whose terms are *noncentral chi-squares*. The validity of his expansion can be justified by Theorem 1. For this, take $\mathbf{Y}_i = n^{-1/2}\mathbf{Z}_i$ where $\{\mathbf{Z}_i\}$ are as in Siotani (1971). Then our $\bar{Z}_n$, a $(mp+p(p-1)/2)\times 1$ vector, is $(Y_1, \ldots, Y_m, s_{ij}\ i \geqslant j)$. Then Siotani's statistic satisfies our conditions. One now applies Remark 2.

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