Strategy-proof Social Choice Correspondences[1]

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Received February 17, 1999; final version received December 5, 2000;
published online August 17, 2001

We study the possibilities of constructing strategy-proof rules that choose sets of alternatives as a function of agents' preferences over such sets. We consider *two* restrictions on the domain of individual preferences over sets. Assuming that all singletons are in the range of the rule, we show that only dictatorial rules can be strategy-proof on the larger domain. The smaller domain also allows for rules which select the set of best elements of two *fixed* agents. *Journal of Economic Literature* Classification Number: D71.

1. INTRODUCTION

Social choice processes which result in the selection of sets of alternatives can be modelled as social choice correspondences. Sets of alternatives admits a very wide range of interpretations. For instance, they can stand for collections of mutually compatible decisions, as in Barberà, Sonnenschein and Zhou [5], or in Miyagawa [17]. Alternatively, these sets can represent collections of incompatible decisions, all of which have passed a first screening but are pending final resolution. Whatever the interpretation given to these sets of alternatives, social choice correspondences give rise to incentive problems, analogous to those which arise when the choice process

[1] Barberà acknowledges financial support from DGES under Project PB98-0870 and Generalitat de Catalunya under Project SGR 98-62. Dutta gratefully acknowledges financial support from Program Catedra Fundacion BBV. The authors are very grateful to the referee for remarkably detailed comments on the first version of the paper. They also thank the Associate Editor for helpful suggestions.

results in the selection of a single object. Our specific concern is the possibility of constructing choice procedures which choose sets of alternatives and are strategy-proof.

Following the Gibbard–Satterthwaite theorem (Gibbard [13], Satterthwaite [19]), the possibility of constructing strategy-proof social choice correspondences was explored by several authors.[2] The common framework adopted in these papers has been to consider aggregation rules whose domain consists of n-tuples of individual preference orderings *over the basic set of alternatives*, with nonempty subsets of alternatives as typical elements in the range. Unfortunately, while the definition of strategy-proofness for choice procedures which always select a single object is unambiguous, its extension to rules which may choose sets of alternatives becomes controversial. This is because the definition of strategy-proofness must now depend upon how individuals rank sets of alternatives given the ranking of the alternatives themselves. To appreciate this point, suppose that an individual can obtain the set $\{a_2\}$ by telling the truth and the set $\{a_1, a_3\}$ by lying. Assume that the individual prefers a_1 to a_2 and a_2 to a_3. Clearly, this information is not enough to pin down the individual's ranking over the sets $\{a_2\}$ and $\{a_1, a_3\}$. It is largely this ambiguity which has given rise to different papers, with different authors adopting different extensions of the individual preferences on alternatives to their power set.

The main purpose of this paper is to propose an alternative formulation of the question, one that eliminates any ambiguity of this sort. Suppose, for instance, that the set of feasible alternatives is the set $A = \{a_1, a_2, a_3\}$. We consider the aggregation rules whose *domain* is the set of all possible profiles of individual rankings over the set of seven possible non-empty subsets of A. The *range* of the aggregation rule is the set of seven possible non-empty subsets of A. Thus, the crucial difference between our framework and that of the papers referred to earlier is that the latter set considered aggregation rules whose domain consisted of n-tuples of preference rankings, each ranking being an ordering over the set A.

In other words, the domain of our aggregation rules includes that postulated in the earlier set of papers, while the *range* is identical. This implies that our domain permits the construction of *new* aggregation rules—the social choice rules can now utilise information about individuals' rankings over sets in a more meaningful manner. In contrast, the previous papers impose a strong invariance requirement on the social choice rule: the social outcome is not allowed to change even if the rankings of individuals over sets of alternatives change, so long as their rankings over

[2] See, for instance, Barberà [1], Gärdenfors [12], Feldman [9–11], Kelly [16], Pattanaik [18]. There has been recent revival of interest in this issue with the work of Ching and Zhou [7] and Duggan and Schwartz [8]. See Section 4 for a brief discussion of this literature.

singleton sets remain the same. Of course, we do not impose any such invariance requirement.

Notice also that our framework is formally identical to that of Gibbard or Satterthwaite—one can simply identify feasible "outcomes" to be non-empty *subsets* of A rather than *elements* of A. The only difference is that we do not insist on rules that operate for a *universal* domain. Obviously, the Gibbard–Satterthwaite impossibility result would have applied directly in the absence of some domain restriction.

In this paper we consider two natural domain restrictions, which are associated with specific interpretations of what it means for a society to choose a set of alternatives. In our first interpretation, we view social outcomes as sets of alternatives which have passed an initial test, the final decision (of a unique alternative among those which have passed) to be made in a second and later stage. Individuals may not be aware of how the second-stage decision is to be made. Alternatively, they may believe that the final outcome will be chosen on the basis of a lottery. In either case, we assume that each agent subjectively assesses *one* probability distribution over the set of feasible alternatives and then associates conditional probabilities to each subset. We also assume that individuals have von Neumann–Morgenstern preferences over lotteries and that each agent ranks sets according to the expected utilities associated with each set.

In the second scenario, the "final" outcome itself may be a set of outcomes. For instance, different candidates competing in a popularity contest may tie for first place and thus have to share the first prize. An interesting example of this phenomenon occurred when Barbara Streisand and Katherine Hepburn shared the 1968 Oscar for Best Actress.[3] It then makes sense to assume that individuals rank sets on the basis of the "equal-weighted average" utility associated to each set. Of course, this is formally equivalent to saying that agents associate each set with an even chance lottery over all its alternatives and then rank sets according to expected utilities. In this sense, our second domain restriction is also consistent with the first interpretation described earlier.

We concentrate on social choice functions whose domains are restricted to orderings of sets compatible with the above interpretations. We characterize the sets of strategy proof social choice functions respecting unanimity on each one of the two domains. For the larger one, associated with our first scenario of conditional expected utility maximizers, the only unanimous and strategy-proof social choice functions are dictatorial. We also show that drastically reducing the domain, as we do by assuming even-chance lotteries, does not expand the menu by very much: only dictatorial

[3] This example (and others of a similar type) is due to Benoit [6].

or bi-dictatorial[4] rules are strategy-proof. We interpret our results to demonstrate the remarkable robustness of the Gibbard–Satterthwaite result.

This conclusion is similar to those drawn in the papers that we have cited earlier.[5] However, as we have pointed out earlier, our results are derived in a considerably more general framework. It is also clear that *any* strategy-proof rule which only utilises information about individual rankings over singleton sets is strategy-proof in our more general framework. Hence, any impossibility result in our framework translates into a corresponding impossibility result in the framework used earlier. Since the converse is not true, the results in this paper are more general than many of those proved earlier about the existence of strategy-proof social choice correspondences.[6]

Before concluding, we should mention another related line of research. Some authors have considered social choice rules whose images are lotteries over alternatives. Early work on these probabilistic "decision schemes" characterized strategy-proof rules defined on preferences over alternatives (Gibbard [14], Barberà [2]). Extending these rules by allowing agents to declare preferences over lotteries opens the door to a plethora of new rules (Barberà, Bogomolnaia, van der Stel (BBS) [3]).

This has to be contrasted to our results which show that no gain is obtained by allowing for rules that take preferences over sets into account. In order to gain some insight into the differences between the two frameworks, consider the analogue of the "random dictatorship" rule in our framework. Suppose there are three individuals $\{1, 2, 3\}$ and $A = \{a_1, a_2, a_3\}$. Suppose the aggregation rule is to choose the *union* of the top-ranked alternatives of the individuals.[7] Let the most preferred sets of the three individuals be $\{a_1\}$, $\{a_1\}$ and $\{a_3\}$ respectively. If individuals declare their preferences correctly, then $\{a_1, a_3\}$ will be the outcome since this is the union of the most-preferred sets. Now, suppose individual 2 declares $\{a_2\}$ to be his or her most-preferred set. Then, the outcome will be $\{a_1, a_2, a_3\}$. However, $\{a_1\}\ P_2\{a_2, a_1\}\ P_2\{a_2\}\ P_2\{a_1, a_2, a_3\}\ P_2\{a_1, a_3\}\ P_2\{a_3\}$ is a feasible preference ordering for individual 2. Hence, this example shows that the rule of selecting the union of top-ranked sets can be manipulated.

[4] A rule is bi-dictatorial if it chooses the union of the best elements of two fixed agents.

[5] While several authors have shown that only dictatorial rules are strategy-proof, Feldman [11] shows that only bi-dictatorial rules are strategy-proof when individuals rank sets on the basis of an even-chance lottery.

[6] However, see Section 4, where we point out that Duggan and Schwartz's dictatorship result cannot be derived as a corollary of ours in view of their definition of strategy-proofness.

[7] Note that if individual i ranks $\{a_i\}$ over $\{a_j\}$ for each $j \neq i$, then the *set* $\{a_i\}$ must be his or her most-preferred amongst all subsets of A.

However, the random dictatorship rule cannot be manipulated in the Gibbard or BBS framework. This is because in their framework, the chosen lottery itself depends on the preferences announced by individuals. So, in the example given above (and assuming that the weights attached to all individuals are equal), if individuals announce their preferences correctly, then the probability associated with a_1 is $\frac{2}{3}$ and that to a_3 is $\frac{1}{3}$. If individual 2 announces that a_2 is his or her most-preferred alternative, then the outcome would be the equal-probability lottery on A. The reader can check that given that individual 2 prefers a_1 to a_2 to a_3, he or she will not have any incentive to manipulate the random dictatorship rule.

The plan of this paper is the following. In Section 2, we describe the basic frame-work. Section 3 contains our results. In the concluding section, we discuss how our results are related to the existing literature.

2. THE FRAMEWORK

Consider a society of N individuals, with $N \geqslant 2$. Let A be a finite set of outcomes, with $|A| \geqslant 3$. Elements of A will be denoted as a_1, a_2, a_k, etc. Let $\mathscr{A}$ denote the set of all non-empty subsets of A. We will use $\mathscr{A}_r$, $r = 1, ..., |A|$ to denote the set of all subsets of A which have exactly r elements. For example, $A_1 = \{\{a_1\}, ..., \{a_k\}, ...\{a_{|A|}\}\}$, while A_2 is the set of all sets of the type $\{a_j, a_k\}$ where a_j, $a_k \in A$.

Each individual i has a preference ordering over the set $\mathscr{A}$. Let $\mathscr{R}$ be the set of all orderings over $\mathscr{A}$. Individual orderings will be denoted by R_i, R_i', etc. As we have mentioned earlier, we are particularly interested in those individual orderings over $\mathscr{A}$ that can be obtained from the hypothesis that individual preferences over A satisfy the postulate of expected utility maximization. We specify this more formally.

DEFINITION 2.1. A *utility function* for individual i is a mapping v_i: $A \rightarrow \Re$.

Assumption 2.2. For all $i \in N$, for all distinct elements a_j, $a_k \in A$, $v_i(a_j) \neq v_i(a_k)$.

Assumption 2.2, which rules out indifference between any pair of alternatives, will be maintained throughout the paper.

DEFINITION 2.3. An *assessment* λ is a function λ: $A \rightarrow [0, 1]$ such that $\lambda(a_j) > 0$ for all $a_j \in A$ and $\sum_{a_j \in A} \lambda(a_j) = 1$.

DEFINITION 2.4. An ordering R_i over $\mathscr{A}$ is *conditionally expected utility consistent* (CEUC) if there exists a utility function v_i and an assessment λ_i such that:

$\forall X, Y \in \mathscr{A}$,

$$XR_iY \Leftrightarrow \sum_{a_j \in X} v_i(a_j)\left(\frac{\lambda_i(a_j)}{\sum_{a_k \in X} \lambda_i(a_k)}\right) \geqslant \sum_{a_j \in Y} v_i(a_j)\left(\frac{\lambda_i(a_j)}{\sum_{a_k \in Y} \lambda_i(a_k)}\right).$$

Let $\mathscr{D}_U$ be the set of all CEUC orderings.

Definition 2.5. The ordering R_i is *conditionally expected utility consistent with equal probabilities* (CEUCEP) if there exists a utility function v_i such that:

$$\text{for all } X, Y \in \mathscr{A}, \qquad XR_iY \Leftrightarrow \sum_{a_j \in X} v_i(a_j)\left(\frac{1}{|X|}\right) \geqslant \sum_{a_j \in Y} v_i(a_j)\left(\frac{1}{|Y|}\right).$$

Let $\mathscr{D}_E$ be the set of all CEUCEP orderings.

Definition 2.4 says the following. Suppose v_i is individual i's utility function over A. The assessment λ_i represents individual i's (subjective) beliefs about the probabilities with which any element a_k can be selected out of A. So, if individual i has to rank two sets X and Y, he or she assesses the conditional probabilities associated to the different elements when the final choice is to be out of the sets X and Y. Individual i prefers the set X over the set Y if the conditional expected utility associated with X is higher than the conditional expected utility associated with Y.

Definition 2.5 is in the same spirit but much more restrictive, since it assumes that the individual assessments λ_i assign an equal probability to every element in A.

Remark 2.6. Clearly $\mathscr{D}_E \subset \mathscr{D}_U \subset \mathscr{R}$.

We will henceforth represent the asymmetric components of R_i, R'_i by P_i, P'_i, and so on. A profile of individual preferences, $(R_1, ..., R_N)$ will be represented by R. Similarly, $R' = (R'_1, ..., R'_N)$, and so on.

The object of interest in this paper is an aggregation procedure or rule, which for each admissible profile of individual preferences over $\mathscr{A}$ selects a *single* element of $\mathscr{A}$. Notice that though the elements of $\mathscr{A}$ are sets of outcomes, the aggregation rule (or social choice function in more familiar terminology) is formally identical to the aggregation procedure used in the traditional Gibbard–Satterthwaite framework, except for the domain restriction. Since $\mathscr{D}_U$ and hence $\mathscr{D}_E$ are strict subsets of $\mathscr{R}$, the *domains* of our aggregation procedure are smaller.

From now on, $\mathscr{D}$ will stand for either $\mathscr{D}_U$ or $\mathscr{D}_E$.

DEFINITION 2.7. An N-person *Social Choice Function* (SCF) on $\mathscr{D} \subseteq \mathscr{R}$ is a mapping $f\colon \mathscr{D}^N \to \mathscr{A}$.[8]

DEFINITION 2.8. The SCF f is *dictatorial* over the domain $\mathscr{D}$ if there exists $i \in N$ such that:

$$\text{for all } R \in \mathscr{D}^N, \qquad f(R) \in \max(R_i, \mathscr{A}).$$

DEFINITION 2.9. The SCF $f\colon \mathscr{D}^N \to \mathscr{A}$ is *bi-dictatorial* if there exist individuals $i, j \in N$ such that for all $R \in \mathscr{D}^N$, $f(R) = \max(R_i, \mathscr{A}) \cup \max(R_j, \mathscr{A})$.

DEFINITION 2.10. The SCF $f\colon \mathscr{D}^N \to \mathscr{A}$ is *manipulable* if there exist $R \in \mathscr{D}^N$, $i \in N$ and $R'_i \in \mathscr{D}$ such that $f(R'_i, R_{-i})\, P_i\, f(R_i, R_{-i})$.

A SCF is *strategy-proof* if it is not manipulable.

DEFINITION 2.11. The SCF $f\colon \mathscr{D}^N \to \mathscr{A}$ satisfies *unanimity* if for all $R \in \mathscr{D}^N$ such that $\max(R_i, \mathscr{A}) = B$ for all $i \in N$, $f(R) = B$.

3. CHARACTERIZATION RESULTS

In this section, we explore the possibilities of constructing strategy-proof SCFs on $\mathscr{D}_U$ and $\mathscr{D}_E$ which satisfy unanimity. This is equivalent to restricting attention to SCFs whose range includes all *singleton* sets due to the following remark.

Remark 3.1. Although Assumption 2.2 rules out indifference between elements of $\mathscr{A}_1$, it is obvious that there can be $X, Y \in \mathscr{A}$ such that XR_iY and YR_iX. Yet, the best and the worst elements of any preferences in $\mathscr{D}_U$ and $\mathscr{D}_E$ will be unique. Therefore, in all unanimous profiles of $\mathscr{D}_U$ and $\mathscr{D}_E$, agents will agree that some singleton set is the best, and hence $\mathscr{A}_1 \subseteq \text{range } f$[9] for any f satisfying unanimity on these domains.

We show that on the domain $\mathscr{D}_U$, the only strategy-proof social choice function satisfying unanimity is the dictatorial one. Since $\mathscr{D}_U$ is a strict subset of the set of all possible orderings over $\mathscr{A}$, this result again demonstrates the remarkable robustness of the Gibbard–Satterthwaite results. Our second result shows that the range of possibilities is not widened in any essential way if the domain is restricted further to $\mathscr{D}_E$. On

[8] Note that subsequently we are going to consider social choice functions where the size of the society is not fixed.

[9] Henceforth, the restriction of $\mathscr{A}_1 \subseteq \text{range } f$ will be referred to as the *range condition*.

this domain, only dictatorial and bi-dictatorial social choice functions can satisfy unanimity and strategy-proofness.

The two theorems are stated formally.

THEOREM 3.2. *Let* $f\colon [\mathscr{D}_U]^N \to \mathscr{A}$. *Then* f *is strategy-proof and satisfies unanimity iff* f *is dictatorial.*

THEOREM 3.3. *Let* $f\colon [\mathscr{D}_E]^N \to \mathscr{A}$. *Then* f *is strategy-proof and satisfies unanimity iff* f *is dictatorial or bi-dictatorial.*

Before we prove these results, we want to make an important remark.

Remark 3.4. The two theorems are not valid in the case where $|A|=2$. Notice that even when $|A|=2$, $|\mathscr{A}|=3$. Hence, if the domain of preferences is *unrestricted*, then the Gibbard–Satterthwaite result will apply. However, if the domain is $[\mathscr{D}_U]^N$, then the dictatorship result no longer holds. For, suppose $A=\{a_1, a_2\}$, and consider the SCF which selects the majority winner if one exists, and A itself if a_1 and a_2 tie. The reader can check that this rule is strategy-proof.

Obviously, a dictatorial SCF is strategy-proof on $\mathscr{D}_U$ and hence on $\mathscr{D}_E$. It is also relatively straightforward to check that a bi-dictatorial SCF is strategy-proof on $\mathscr{D}_E$. So, we only prove the "only if" part of both theorems.

We begin with a lemma on the existence of various admissible preference orderings.

LEMMA 3.5. (i) *For all* $a_j, a_k \in A$ *and for all* $R_i \in \mathscr{D}_U$, $\{a_j\}\ P_i\{a_k\} \Rightarrow \{a_j\}\ P_i\{a_j, a_k\}\ P_i\{a_k\}$.

(ii) *For all distinct elements* $a_j, a_k, b_1, ..., b_L \in A$ *and for all* $R_i \in \mathscr{D}_E$, $\{a_j, b_j, b_2, ..., b_L\}\ P_i\{a_k, b_1, b_2, ..., b_L\} \Leftrightarrow \{a_j\}\ P_i\{a_k\}$.

(iii) *For all* $a_j, a_k, a_l \in A$, *there exists* $R_i \in \mathscr{D}_U$ *such that* $\{a_j\}\ P_i\{a_k\}\ P_i\{a_l\}$ *and* $\{a_k, a_l\}\ P_i\{a_j, a_l\}$.

(iv) *For all* $a_j, a_k \in A$, *there exists* $R_i \in \mathscr{D}_E$ *such that* $\max(R_i, \mathscr{A}) = \{a_j\}$, *and* $\{a_j\}\ P_i\{a_j, a_k\}\ P_i\{a_k\}\ P_i X$ *for all* $X \in \mathscr{A} - \{\{a_j\}, \{a_k\}, \{a_j, a_k\}\}$.

(v) *For any set* $X=\{b_1, b_2, ..., b_L\} \in \mathscr{A}$ *with* $L \geqslant 3$ *and* $Y \in \mathscr{A}$ *which is distinct from* X, *there exists* $R_i \in \mathscr{D}_E$ *with* $\max(R_i, \mathscr{A}) = \{b_1\}$, $\min(R_i, X) = \{b_L\}$ *such that* $XP_i\{b_1, b_L\}\ P_i Y$ *if either* $|Y| \geqslant L$ *or* $Y=\{a_k, b_L\}$ *where* $a_k \in A \setminus \{b_1\}$.

Proof. Parts (i) and (ii) follow immediately from the definitions.

In order to prove (iii), let $v_i(a_j)=1$, $v_i(a_k)=1-\varepsilon$, $v_i(a_l)=0$, $\lambda_i(a_j) = \lambda_i(a_l)=\alpha$ and $\lambda_i(a_k)=1-2\alpha-\delta$. Then, the expected utility of $\{a_k, a_l\}$ is

$\frac{1-2\alpha-\delta}{1-\alpha-\delta}(1-\varepsilon)$, and that of $\{a_j, a_l\}$ is $\frac{1}{2}$. By choosing α and ε sufficiently small, it is clear that one can get the desired result.

In order to prove (iv), take sufficiently small $\varepsilon > 0$, and let $v_i(a_j) = 1$, $v_i(a_k) = 1 - \varepsilon$, and $v_i(a_l) < \varepsilon$ for all $a_l \in A \setminus \{a_j, a_k\}$. Routine calculation yields the desired result.

We now prove (v). Construct the following utility function.

$$v_i(b_1) = 1 \tag{1}$$

$$\sum_{k=2}^{L-1} v_i(b_k) = (L-2)(1-\varepsilon) \tag{2}$$

$$v_i(b_L) = \delta \in \left(\frac{L-2}{L}, 1-2\varepsilon\right) \tag{3}$$

$$v_i(a) < \gamma < \delta \qquad \text{for all} \quad a \notin X. \tag{4}$$

Note that there is $\varepsilon > 0$ such that $1 - 2\varepsilon > \frac{L-2}{L}$. Assume that v_i satisfies (1) to (4) with such an ε. Now, the conditional expected utility[10] associated with X is $\frac{1+(L-2)(1-\varepsilon)+\delta}{L}$, while that of $\{b_1, b_L\}$ is $\frac{1+\delta}{2}$. Since $\delta < 1 - 2\varepsilon$, we get $XP_i\{b_1, b_L\}$.

Suppose $|Y| = 2$ and $b_L \in Y$. Then, $\{b_1, b_L\}\ P_i Y$ from Lemma 3.5(ii). Suppose now that $Y \neq X$, and $|Y| = L$. The highest conditional expected utility from such Y is obtained when Y contains $\{b_1, ..., b_{L-1}\}$ and b_{L+1} where $v_i(b_{L+1}) = \max_{b_k \notin X} v_i(b_k)$. Then, $\{b_1, b_L\}\ P_i Y$ if

$$\frac{1+\delta}{2} > \frac{1+(L-2)(1-\varepsilon)}{L} + \frac{1}{L}\gamma. \tag{5}$$

Since $\delta > \frac{L-2}{L}$, (5) will hold for sufficiently small values of ε and γ.

The proof that $\{b_1, b_2\}\ P_i Y$ when $|Y| > L$ is similar and omitted. ∎

We adopt the following strategy in proving Theorems 3.2 and 3.3. We first present a series of lemmas which prove Theorem 3.3 for the case $N = 2$. An extension of these lemmas prove Theorem 3.2 for $N = 2$. Induction arguments are then used to prove Theorems 3.2 and 3.3 in the general case.

Let $f: \mathscr{D}^2 \to \mathscr{A}$ be a 2-person SCF. The *option set*[11] of individual 2, given $R_1 \in \mathscr{D}$, is the set $O_2(R_1) = \{f(R_1, R_2) \mid R_2 \in \mathscr{D}_E\}$. For all $R_2 \in \mathscr{D}$, $O_1(R_2)$ is defined analogously. We note the following without proof.

[10] Since the required R_i must be in $\mathscr{D}_E$, we are assuming that $\lambda_i(a_k) = \frac{1}{L}$ for all $a_k \in A$.

[11] The use of option sets has proved to be a useful technique in characterizing strategy-proof social choice functions. See, for example, Barberà and Peleg [4], who proved the Gibbard–Satterthwaite theorem using option sets. The reader should keep in mind that option sets are relative to a given function on a given domain, even if this is not explicit in the notation.

Fact 3.7. If f is strategy-proof, then for all $R_1, R_2 \in \mathcal{D}$, $f(R_1, R_2) \in \max(R_2, O_2(R_1)) = \max(R_1, O_1(R_2))$.

In what follows, it will be interesting to study the structure of option sets. In particular, we will need to distinguish what singletons belong to an option set, what doubletons, etc. By slightly abusing language, we will denote, for $R_1 \in \mathcal{D}$ and $Z \subset \mathcal{A}$, $O_2(R_1, Z) = O_2(R_1) \cap Z$. Hence, for example, $O_2(R_1, \mathcal{A}_1)$ will stand for the elements of the option set $O_2(R_1)$ that are singletons. To keep notation consistent, we'll write $O_2(R_1) = O_2(R_1, \mathcal{A})$. Lemmas 3.8 to 3.12 refer to 2-person SCF's f: $[\mathcal{D}_E]^2 \to \mathcal{A}$ satisfying unanimity and strategy-proofness.

Lemma 3.8. *Let $R_1, R'_1 \in \mathcal{D}_E$ and $a_j \in A$ be such that* $\max(R_1, \mathcal{A}) = \max(R'_1, \mathcal{A}) = \{a_j\}$. *Then $O_2(R_1, \mathcal{A}_1) = O_2(R'_1, \mathcal{A}_1)$.*

Proof. Suppose not. Assume w.l.o.g. that $\{a_k\} \in O_2(R_1, \mathcal{A}_1) - O_2(R'_1, \mathcal{A}_1)$. It follows from unanimity that $\{a_k\} \neq \{a_j\}$. Using Lemma 3.5(iv), we can pick $R_2 \in \mathcal{D}_E$ such that $\max(R_2, \mathcal{A}) = \{a_k\}$ and $\{a_k\}\ P_2\{a_j, a_k\}\ P_2\{a_j\}\ P_2 X$ for all $X \in \mathcal{A} - \{\{a_j\}, \{a_k\}, \{a_j, a_k\}\}$. Since $\{a_k\} \in O_2(R_1, \mathcal{A}_1)$, an application of Fact 3.7 yields $f(R_1, R_2) = \{a_k\}$. Since $\{a_j\} \in O_2(R'_1, \mathcal{A}_1)$ and $\{a_k\} \neq O_2(R'_1, \mathcal{A}_1)$, strategy-proofness implies that $f(R'_1, R_2)$ is either $\{a_j, a_k\}$ or $\{a_j\}$. According to Lemma 3.5(i), $\{a_j\}\ P_1\{a_j, a_k\}\ P_1\{a_k\}$. Therefore, in either case, player 1 will manipulate at (R_1, R_2). ∎

Next, we show that the option set of an agent, given the preference of the other, must either contain one singleton or all of them.

Lemma 3.9. *For all $R_1 \in \mathcal{D}_E$, either $O_2(R_1, \mathcal{A}_1) = \mathcal{A}_1$ or $O_2(R_1, \mathcal{A}_1) = \max(R_1, \mathcal{A})$.*

Proof. Suppose not. Let $R_1 \in \mathcal{D}_E$ be such that $\{a_j\} = \max(R_1, \mathcal{A})$, and let $a_k, a_l \in A \backslash \{a_j\}$ be such that $\{a_k\} \in O_2(R_1, \mathcal{A}_1)$ and $\{a_l\} \notin O_2(R_1, \mathcal{A}_1)$. Since Lemma 3.8 implies that $O_2(R_1, \mathcal{A}_1)$ depends only on the maximal element of R_1, we may assume w.l.o.g. that $\{a_l\}\ P_1\{a_k\}$.[12] Using Lemma 3.5(iv), pick $R_2 \in \mathcal{D}_E$ such that $\max(R_2, \mathcal{A}) = \{a_l\}$ and $\{a_l\}\ P_2\{a_k, a_l\}\ P_2\{a_k\}\ P_2 X$ for all $X \in \mathcal{A} - \{\{a_k\}, \{a_l\}, \{a_k, a_l\}\}$. Since 2 does not manipulate at (R_1, R_2), either $f(R_1, R_2) = \{a_k, a_l\}$ or $\{a_k\}$. But, from Lemma 3.5(i), we have $\{a_l\}\ P_1\{a_k, a_l\}\ P_1\{a_k\}$. Since $\{a_l\}$ is R_2-maximal for player 2, player 1 can force the outcome to be $\{a_l\}$ by announcing R'_1 where $\{a_l\}$ is R'_1-maximal. This follows from unanimity. Therefore player 1 manipulates at (R_1, R_2). ∎

[12] If $\{a_k\}\ P_1\{a_l\}$, let $\bar{R}_1$ be another ordering where $\{a_l\}\ \bar{P}_1\{a_k\}$, and $\{a_j\} = \max(\bar{R}_1, \mathcal{A})$.

Now, we show that if the option set of an agent contains all singletons for some preference of the other agent, then all singleton are *always* in his or her option set.

LEMMA 3.10. *If* $O_2(R_1, \mathscr{A}_1) = \mathscr{A}_1$ *for some* $R_1 \in \mathscr{D}_E$, *then* $O_2(R'_1, \mathscr{A}_1) = \mathscr{A}_1$ *for all* $R'_1 \in \mathscr{D}_E$.

Proof. Suppose not. In view of Lemma 3.9, we must have $O_2(R_1, \mathscr{A}_1) = \mathscr{A}_1$ and $O_2(R'_1, \mathscr{A}_1) = \{a_j\}$ for some $R_1, R'_1 \in \mathscr{D}_E$ where $\max(R'_1, \mathscr{A}) = \{a_j\}$. Suppose $\max(R_1, \mathscr{A}) = \{a_k\}$. We must have $\{a_j\} \neq \{a_k\}$ in order not to contradict Lemma 3.8. Pick a_l distinct from a_j and a_k and assume w.l.o.g. (using Lemma 3.8) that $\{a_k\}\ P_1\{a_j\}\ P_1\{a_l\}$. Let $R_2 \in \mathscr{D}_E$ be such that $\max(R_2, \mathscr{A}) = \{a_l\}$ and $a_l P_2\{a_l, a_j\}\ P_2\{a_j\}\ P_2 X$ for all $X \in \mathscr{A} - \{\{a_j\}, \{a_l\}, \{a_j, a_l\}\}$. Then, $f(R_1, R_2) = \{a_l\}$ and $f(R'_1, R_2)$ is either $\{a_j, a_l\}$ or $\{a_j\}$. Since $\{a_j\}\ P_1\{a_j, a_l\}\ P_1\{a_l\}$ from Lemma 3.5(i), player 1 manipulates at (R_1, R_2). ▮

Observe that if $O_2(R_1, \mathscr{A}_1) = \mathscr{A}_1$ for all $R_1 \in \mathscr{D}_E$ is true, then player 2 is a dictator since his or her maximal element in $\mathscr{A}_1$ must also be his or her maximal element in $\mathscr{A}$. Symmetrically, $O_1(R_2, \mathscr{A}_1) = \mathscr{A}_1$ for all $R_2 \in \mathscr{D}_E$ would also imply that 1 is a dictator. So, we are left with only one possibility, namely that the only singleton in the option set of one agent is the maximal element of the other agent. That is,

(*) For all $R_1, R_2 \in \mathscr{D}_E$ we have $O_2(R_1, \mathscr{A}_1) = \max(R_1, \mathscr{A})$ and $O_1(R_2, \mathscr{A}_1) = \max(R_2, \mathscr{A})$.

In this case, we show by the following lemma that the two-element sets containing the best element for an agent must be options for the other agent, and that they are the only two-element options.

LEMMA 3.11. *Suppose* (*) *holds. Then, for all* $R_1 \in \mathscr{D}_E$ *and* $a_j \in A$ *such that* $\max(R_1, \mathscr{A}) = \{a_j\}$, *we must have* $O_2(R_1, \mathscr{A}_2) = \{\{a_j, a_k\} \mid a_k \in A\}$.

Proof. We first show that $\{\{a_j, a_k\} \mid a_k \in A\} \subseteq O_2(R_1, \mathscr{A}_2)$. Suppose that there exists $a_k \in A$ such that $\{a_j, a_k\} \notin O_2(R_1, \mathscr{A}_2)$. Let $R_2 \in \mathscr{D}_E$ be such that $\max(R_2, \mathscr{A}) = \{a_k\}$ and $\{a_k\}\ P_2\{a_j, a_k\}\ P_2\{a_j\}\ P_2 X$ for all $X \in \mathscr{A} - \{\{a_j\}, \{a_k\}, \{a_j, a_k\}\}$. Since (*) holds, observe that $f(R_1, R_2)$ cannot be either $\{a_j\}$ or $\{a_k\}$. Furthermore, $\{a_j, a_k\} \notin O_2(R_1, \mathscr{A}_2)$. So, $f(R_1, R_2) \neq \{a_j, a_k\}$. Hence $f(R_1, R_2) = X$ for some X where $\{a_j\}\ P_2 X$. But then player 2 can manipulate at (R_1, R_2) by announcing R'_2 where $\{a_j\}$ is R'_2-maximal and thereby obtain $\{a_j\}$. Note that the last conclusion follows from unanimity.

We now show that $O_2(R_1, \mathscr{A}_2) \subseteq \{\{a_j, a_k\} \mid a_k \in A\}$. Suppose not. Let $\{a_k, a_l\} \in O_2(R_1, \mathscr{A}_2)$ where a_j, a_k and a_l are all distinct. Let $R_2 \in \mathscr{D}_E$ be such that $\max(R_2, \mathscr{A}) = \{a_k\}$ and $\{a_k\}\ P_2\{a_k, a_l\}\ P_2\{a_l\}\ P_2 X$ for all $X \in \mathscr{A} - \{\{a_k\}, \{a_k, a_l\}, \{a_l\}\}$. Since $\{a_k\} \notin O_2(R_1, \mathscr{A}_1)$ by hypothesis (since case (*) holds), it must be the case that $f(R_1, R_2) = \{a_k, a_l\}$. By applying the arguments in the previous paragraph to $O_1(R_2, \mathscr{A}_2)$, we know that $\{a_j, a_k\} \in O_1(R_2, \mathscr{A}_2)$. But $\{a_j, a_k\}\ P_1\{a_k, a_l\}$ from Lemma 3.5(iii). So, 1 can manipulate at (R_1, R_2). ∎

LEMMA 3.12. *Suppose* (*) *holds. Then, for all* $R_1 \in \mathscr{D}_E$, *we have* $O_2(R_1, \mathscr{A}) = O_2(R_1, \mathscr{A}_1) \cup O_2(R_1, \mathscr{A}_2)$.

Proof. Suppose not. Assume without loss of generality that $X = \{b_1, ..., b_L\}$ is the set of smallest cardinality greater than 2 such that $X \in O_2(R_1, \mathscr{A})$ for some $R_1 \in \mathscr{D}_E$. If more than one such set exists, select one arbitrarily.

Let $\max(R_1, \mathscr{A}) = \{a_j\}$. We first show that $a_j \in X$. Suppose not. Let $\mathscr{X}$ represent the set of all non-empty subsets of X, and let $\{b_1\} = \max(R_1, \mathscr{X})$. Pick $R_2 \in \mathscr{D}_E$ such that $\{b_1\} = \max(R_2, \mathscr{A})$, $\{a_j\} = \min(R_2, \mathscr{A})$, and $\{b_1\}\ P_2\{b_2\} \cdots P_2\{b_L\}\ \ P_2\{a_k\}$ for all $a_k \notin X$. Moreover, for all Y, $Z \in \mathscr{A}$, if $a_j \notin Y$ and $a_j \in Z$, then YP_2Z. Note that such an ordering can be constructed by choosing a utility function v_2 such that $v_2(a_j) = 0$ and $v_2(a_k) \geqslant 1 - \varepsilon$ for all $a_k \neq a_j$, where $\varepsilon > 0$ is sufficiently small. From Lemma 3.11, $O_2(R_1, \mathscr{A}_2) = \{\{a_j, a_k\} \mid a_k \in A\}$. Since $XP_2\{a_j, b_1\}$, we must have $XP_2 Y$ for all $Y \in O_2(R_1, \mathscr{A}_2)$. In addition, $XP_2 Y$ for all Y such that $|Y| \geqslant L$. Since X is the set of smallest cardinality greater than 2 in $O_2(R_1, \mathscr{A})$, this ensures that $f(R_1, R_2) = X$. But, since $\{b_1\}\ P_1 X$ and $f(R_1', R_2) = \{b_1\}$ if $\max(R_1', \mathscr{A}) = \{b_1\}$, 1 can manipulate by announcing R_1'. Therefore, $\max(R_1, \mathscr{A}) \in X$.

From now on, we assume that $X = \{b_1, ..., b_L\} \in O_2(R_1, \mathscr{A})$, where $\max(R_1, \mathscr{A}) = \{b_1\}$ and $\max(R_1, \mathscr{X} \backslash \{b_1\}) = \{b_2\}$. Note that this implies that $\{b_1, b_2\}\ P_1 X$.

We claim that for all $R_1' \in \mathscr{D}_E$ such that $\max(R_1', \mathscr{A}) = \{b_1\}$, it must be true that $X \in O_2(R_1', \mathscr{A})$. Suppose not. Pick $R_2 \in \mathscr{D}_E$ such that

(a) $\max(R_2, \mathscr{A}) = \{b_2\}$ and $\{b_k\}\ P_2\{b_{k+1}\}$ for all $k \in \{2, ..., L-1\}$.

(b) $\min(R_2, X) = \{b_1\}$ and $\{b_1\}\ P_2\{a\}$ for all $a \notin X$.

(c) $XP_2\{b_1, b_2\}\ P_2 Y$ if $|Y| \geqslant L$ or both $b_1 \in Y$ and $|Y| = 2$, but $Y \neq \{b_1, b_2\}$.

Note that Lemma 3.5(v) ensures the existence of such an ordering. The following must be true.

(i) $\{b_2\} \notin O_2(R_1, \mathscr{A})$ from (*).

(ii) $\{b_1, b_2\} \in O_2(R_1, \mathscr{A})$ from Lemma 3.11.

These, together with (c) above, ensure that $f(R_1, R_2) = X$. Otherwise, 2 will manipulate at (R_1, R_2).

Now, consider $f(R'_1, R_2)$. Lemma 3.11 ensures that $\{b_1, b_2\} \in O_2(R'_1, \mathscr{A})$. Again, (*) and (c), together with the supposition that $X \notin O_2(R'_1, \mathscr{A})$ ensure that $f(R'_1, R_2) = \{b_1, b_2\}$. But then 1 manipulates at (R_1, R_2) since $\{b_1, b_2\}\ P_1 X$.

Finally, we show that $X \notin O_2(R_1, \mathscr{A})$. Let $R''_1 \in \mathscr{D}_E$ be such that $\{b_1\} = \max(R''_1, \mathscr{A})$ and $\{b_1\}\ P''_1\{b_1, b_2\}\ P''_1\{b_2\}\ P''_1 Z$ for all $Z \in \mathscr{A} - \{\{b_1\}, \{b_1, b_2\}, \{b_2\}\}$. By our earlier argument, $f(R''_1, R_2) = X$. But, 1 can manipulate by announcing $\hat{R}_1$ such that $\max(\hat{R}_1, \mathscr{A}) = \{b_2\}$, because Fact 3.7 yields $f(\hat{R}_1, R_2) = \{b_2\}$ and $\{b_2\}\ P''_1 X$. ∎

Remark 3.13. Lemmas 3.11 and 3.12 ensure that if (*) holds, then f is bi-dictatorial. Since f is either dictatorial or (*) holds, we have proved Theorem 3.3 for the case $N = 2$. We now prove Theorem 3.2.

Proof of Theorem 3.2. We first prove the theorem for the case $N = 2$.

Let f: $[\mathscr{D}_U]^2 \to \mathscr{A}$ be a 2-person strategy-proof SCF satisfying unanimity. Since $\mathscr{D}_E \subset \mathscr{D}_U$, we claim that Lemmas 3.8, 3.9 and 3.10 remain valid for the corresponding option sets.[13] Therefore, we have once again that f must be dictatorial or (*) must hold.

Step 1. We claim that (*) cannot hold. Note first that the first step in the proof of Lemma 3.11 remains valid, i.e. for all $R_1 \in \mathscr{D}_U$ and $a_j \in A$ such that $\max(R_1, \mathscr{A}) = \{a_j\}$, we must have $\{\{a_j, a_k\} \mid a_k \in A\} \subseteq O_2(R_1, \mathscr{A})$. Now pick $a_j, a_k, a_l \in A$ and $R_1 \in \mathscr{D}_U$ such that $\{a_j\}\ P_1\{a_k\}\ P_1\{a_l\}$ and $\{a_k, a_l\}\ P_1\{a_j, a_l\}$ (using Lemma 3.5(iii)). Let $R_2 \in \mathscr{D}_E$ be such that $\{a_l\}\ P_2\{a_j, a_l\}\ P_2\{a_j\}\ P_2 X$ for all $X \in \mathscr{A} - \{\{a_j\}, \{a_l\}, \{a_j, a_l\}\}$. Since (*) holds, $f(R_1, R_2)$ cannot be a singleton. From our earlier remark $\{a_j, a_l\} \in O_2(R_1, \mathscr{A})$ so that $f(R_1, R_2) = \{a_j, a_l\}$. Let $R'_1 \in \mathscr{D}_E$ be such that $\max(R'_1, \mathscr{A}) = \{a_k\}$. Since $R'_1, R_2 \in \mathscr{D}_E$, we know from our $N = 2$ version of Theorem 3.3 that $f(R'_1, R_2) = \{a_l, a_k\}$. Since $\{a_l, a_k\}\ P_1\{a_l, a_j\}$, individual 1 manipulates at (R_1, R_2). Therefore (*) cannot hold and Theorem 3.2 holds when $N = 2$.

We now prove Theorem 3.2 for general N.

[13] The reader can verify that the only properties of orderings invoked in these lemmas are (i) and (iv) of Lemma 3.5. Property (i) is satisfied for all orderings in $\mathscr{D}_U$. Since (iv) postulates the existence of an ordering in $\mathscr{D}_E$ satisfying a certain property, $\mathscr{D}_E \subset \mathscr{D}_U$ implies that this ordering is admissible in $\mathscr{D}_U$.

Step 2. Assume that the theorem is true for all $k \leq N-1$. Let $f[\mathscr{D}_U]^N \to \mathscr{A}$ be a strategy-proof N-person SCF satisfying the range condition. Define a SCF g: $[\mathscr{D}_U]^{N-1} \to \mathscr{A}$ as follows.

$$\text{For all} \quad R_1, R_2, ..., R_{N-1} \in \mathscr{D}_U,$$

$$g(R_1, ..., R_{N-1}) = f(R_1, ..., R_{N-1}, R_{N-1}).$$

Note that Fact 3.7 implies that g satisfies the range condition. Also observe that for all $R_{-(N-1)} \in [\mathscr{D}_U]^{N-2}$, R_{N-1}, $R'_{N-1} \in \mathscr{D}_U$, $g(R_{-(N-1)}, R_{N-1}) = f(R_{-(N-1)}, R_{N-1}, R_{N-1})\ R_{N-1} f(R_{-(N-1)}, R'_{N-1}, R_{N-1})\ R_{N-1} f(R_{-(N-1)}, R'_{N-1}, R'_{N-1}) = g(R_{-(N-1)}, R'_{N-1})$. So, individual $(N-1)$ cannot manipulate g. Clearly, $i \in \{1, ..., N-2\}$ cannot manipulate g because that would directly contradict the assumption that f is strategy-proof. Therefore, g is strategy-proof. From the induction hypothesis, either $i \in \{1, ..., N-2\}$ or $(N-1)$ is a dictator.

Step 3. We claim that if $i \in \{1, ..., N-2\}$ is a dictator in g, then i is also a dictator in f. Suppose, for instance, that1 is a dictator in g. Select any R_1, $R_2, ..., R_N \in \mathscr{D}_U$. Let $\max(R_1, \mathscr{A}) = \{a_j\}$. Choose $R'_{N-1} \in \mathscr{D}_U$ such that $\min(R'_{N-1}, \mathscr{A}) = \{a_j\}$. Then, $f(R_1, ..., R_{N-2}, R'_{N-1}, R'_{N-1}) = g(R_1, ..., R_{N-2}, R'_{N-1}) = \{a_j\}$ since 1 is a dictator in g. Since f is strategy-proof, $f(R_1, ..., R_{N-2}, R'_{N-1}, R'_{N-1})\ R'_{N-1} f(R_1, ..., R_{N-2}, R'_{N-1}, R_N)\ R'_{N-1} f(R_1, ..., R_{N-1}, R_N)$. Since $\min(R'_{N-1}, \mathscr{A}) = \{a_j\}$, this implies that $f(R_1, ..., R_N) = \{a_j\}$. Hence,1 is a dictator in f.

Step 4. Suppose individual $(N-1)$ is a dictator in g. Choose any arbitrary profile $R \in [\mathscr{D}_U]^{N-2}$ for an $(N-2)$ society. Consider the two-person society $\{N-1, N\}$, and define a two-person SCF h: $[\mathscr{D}_U]^2 \to \mathscr{A}$ as follows. For all $(R_{N-1}, R_N) \in \mathscr{D}_U^2$, $h(R_{N-1}, R_N) = f(R, R_{N-1}, R_N)$. Since $(N-1)$ dictates in g, it follows that h satisfies the range condition. Since f is strategy-proof, it follows immediately that h is strategy-proof. Therefore, h is dictatorial. Without loss of generality, let N be the dictator in h.

Now choose any other profile $\bar{R} \in [\mathscr{D}_U]^{N-2}$, and consider the two-person SCF $\bar{h}$: $[\mathscr{D}_U]^2 \to \mathscr{A}$ such that for all $(R_{N-1}, R_N) \in \mathscr{D}_U^2$, $\bar{h}(R_{N-1}, R_N) = f(\bar{R}, R_{N-1}, R_N)$. Again, $\bar{h}$ must have a dictator. We want to show that N continues to be the dictator in $\bar{h}$. Since $\bar{R}$ is chosen arbitrarily, this will establish that N is a dictator in f.

Suppose instead that $(N-1)$ is a dictator in $\bar{h}$. Consider a sequence of profiles $\{R^0, ..., R^{N-2}\}$, each in $[\mathscr{D}_U]^{N-2}$, such that

(i) $R^0 = R$, $R^{N-2} = \bar{R}$.

(ii) For each $k = 1, ..., N-2$, $R^k_j = R^{k-1}_j$ for all $j \neq k$ and $R^k_k = \bar{R}_k$.

Thus, the sequence describes a movement from R to $\bar{R}$ such that one individual at a time switches from R_i to $\bar{R}_i$.

Let h^k be the two-person SCF "corresponding" to R^k. That is, for each (R_{N-1}, R_N), let $h^k(R_{N-1}, R_N) = f(R^k, R_{N-1}, R_N)$. Let j be the smallest integer such that N is the dictator in h^{j-1} and $N-1$ is the dictator in h^j. Clearly, such j must exist since the dictators in h and $\bar{h}$ are different.

Without loss of generality, let $\{a_k\} P_j \{a_l\}$. Pick R_{N-1} and R_N such that $\max(R_{N-1}, \mathscr{A}) = \{a_l\}$ and $\max(R_N, \mathscr{A}) = \{a_k\}$. Then, we must have $h^{j-1}(R_{N-1}, R_N) = \{a_k\}$ and $h^j(R_{N-1}, R_N) = \{a_l\}$. The definitions of h^{j-1} and h^j now imply that j manipulates f at $(\bar{R}_j, R_{N-1}, R_N)$ via R_j.

This completes the proof of Step 4. ∎

We now complete the proof of Theorem 3.3.

Proof of Theorem 3.3. We prove the theorem by induction on N. We have proved the theorem for $N=2$. Assuming that it is true for all $K \leqslant N-1$, we show that it is true for all SCFs f: $[\mathscr{D}_E]^N \to \mathscr{A}$.

Let f: $[\mathscr{D}_E]^N \to \mathscr{A}$ be an N person SCF. Pick $i, j \in \{1, ..., N\}$ and consider the $N-1$ society $N - \{j\}$. A typical profile for this society will be denoted by $(R_{-ij}, R_i) \in [\mathscr{D}_E]^{N-1}$. For any such profile, let $(R_{-ij}, R_i, R_j) \in [\mathscr{D}_E]^N$ denote its extension to a profile for an N society. The $N-1$ person SCF g_{ij}: $[\mathscr{D}_E]^{N-1} \to \mathscr{A}$ is defined as follows: for all $(R_{-ij}, R_i) \in [\mathscr{D}_E]^{N-1}$, $g_{ij}(R_{-ij}, R_i) = f(R_{-ij}, R_i, R_j)$ where $R_i = R_j$.

It follows from Step 2 in the proof of Theorem 3.2 that if f is strategy-proof and satisfies the range condition, then g_{ij} satisfies the same properties. Applying the induction hypothesis, we conclude that g_{ij} is either dictatorial or bi-dictatorial. Therefore, one of the four cases below must apply.

Case 1. $\exists k \in N - \{i, j\}$ such that k is the dictator in g_{ij}.

Case 2. $\exists k, l \in N - \{i, j\}$ such that k, l are bi-dictators in g_{ij}.

Case 3. i is a dictator in g_{ij}.

Case 4. $\exists k \in N - \{i, j\}$ such that k, i are bi-dictators in g_{ij}.

We will show that Theorem 3.3 is valid in each case.

Case 1. Arguments in Step 3 in the proof of Theorem 3.2 establish that if $k \in N \setminus \{i, j\}$ is a dictator in g_{ij}, then k is a dictator in f.

Case 2. Let $k, l \in N - \{i, j\}$ be bi-dictators in g_{ij}. We will show that k, l are bi-dictators in f, i.e. for an arbitrary profile (R_{-ij}, R_i, R_j), $f(R_{-ij}, R_i, R_j) = \{a_k, a_l\}$ where $\max(R_k, \mathscr{A}) = \{a_k\}$ and $\max(R_l, \mathscr{A}) = \{a_l\}$. We assume $a_l \neq a_k$; otherwise the argument for Case 1 suffices. Choose $\bar{R}_i \in \mathscr{D}_E$ such that $X \bar{P}_i \{a_k\} \; \bar{P}_i \{a_k, a_l\} \; \bar{P}_i \{a_l\}$ for all $X \in \mathscr{A} - \{\{a_k\}, \{a_l\}, \{a_k, a_l\}\}$. Let $\bar{R}_i = \bar{R}_j$. We have $f(R_{-ij}, \bar{R}_i, \bar{R}_j) = g_{ij}(R_{-ij}, \bar{R}_i) = \{a_k, a_l\}$. Observe that $f(R_{ij}, R_i, \bar{R}_j)$ is either $\{a_k, a_l\}$ or $\{a_l\}$; otherwise i will manipulate at $(R_{-ij}, \bar{R}_i, \bar{R}_j)$. By an identical argument $f(R_{-ij}, R_i, R_j)$ is either $\{a_k, a_l\}$ or $\{a_l\}$. Let $R'_i \in \mathscr{D}_E$ be such that $XP'_i\{a_l\}\ P'_i\{a_k, a_l\}\ P'_i\{a_k\}$ for all $X \in \mathscr{A} - \{\{a_k\}, \{a_l\}, \{a_k, a_l\}\}$. Let $R'_i = R'_j$ and observe that $f(R_{-ij}, R'_i, R'_j) = \{a_k, a_l\}$. By applying the previous argument, it follows that $f(R_{-ij}, R_i, R_j)$ is either $\{a_k, a_l\}$ or $\{a_k\}$. Clearly $f(R_{-ij}, R_i, R_j) = \{a_k, a_l\}$.

Case 3. Choose $R \in [\mathscr{D}_E]^{N-2}$ and construct h: $[\mathscr{D}_E]^2 \rightarrow \mathscr{A}$ such that $h(R_i, R_j) = f(R, R_i, R_j)$ for all $R_i, R_j \in \mathscr{D}_E$. As in Step 4 of Theorem 3.2, h is strategy-proof and satisfies the range condition since i is a dictator in g_{ij}. So, h is either dictatorial or bi-dictatorial. We need to show that (i) if $k \in \{i, j\}$ is a dictator in h, then k is a dictator in f; (ii) if i and j are bi-dictators in h, then they are bi-dictators in f.

The proof of (i) is almost identical to that of Step 4, but is given for completeness.

Without loss of generality, suppose j is a dictator in h. Consider the profile $\bar{R}$ in $[\mathscr{D}_E]^{N-2}$, where $R_l = \bar{R}_l$ for all $l \neq k$ and $\bar{R}_k \neq R_k$. Assume that $\{a_k\}\ P_k\{a_l\}$. Define the two-person SCF $\bar{h}$ such that $\bar{h}(R_i, R_j) = f(\bar{R}, R_i, R_j)$ for all $(R_i, R_j) \in [\mathscr{D}_E]^2$. Choose R_i, R_j such that $\max(R_i, \mathscr{A}) = \{a_l\}$ and $\max(R_j, \mathscr{A}) = \{a_k\}$. Now, $\bar{h}$ is dictatorial or bi-dictatorial. If j is not the dictator in $\bar{h}$, then $\bar{h}(R_i, R_j) = \{a_k\}$ or $\{a_l, a_k\}$. Hence, $f(R_{-k}, \bar{R}_k, R_i, R_j) = \{a_k\}$ or $\{a_l, a_k\}$, while $f(R_{-k}, R_k, R_i, R_j) = \{a_l\}$. But, then k manipulates at $(R_{-k}, \bar{R}_k, R_i, R_j)$. Repeated application of this argument yields that j must be a dictator in f.

The proof of (ii) is very similar to that of (i) and is omitted.

Case 4. Suppose $k \in N \backslash \{i, j\}$ and i are bi-dictators in g_{ij}. We will show that either the pair (k, i) or the pair (j, k) are bi-dictators in f.

Consider the $(N-1)$ person SCF g_{ki}. We know that one of Cases 1–4 must hold with respect to g_{ki}. However, given that k and i are bi-dictators in g_{ij}, there cannot be any dictator in g_{ki}. The only possible candidate for a dictator is j. But suppose j is the dictator in g_{ki}. Choose $R \in [\mathscr{D}_E]^N$ such that $R_k = R_i$ with $\max(R_i, \mathscr{A}) = \{a_i\}$ and $\max(R_j, \mathscr{A}) = \{a_j\}$. Then, $f(R_{-ki}, R_i, R_k) = \{a_j\}$. But, then since i and k are bi-dictators in g_{ij}, $f(R_{-ijk}, R_j, R_k, R_i^*) = \{a_i, a_j\}$ if $R_i^* = R_j$. Since $\{a_i, a_j\}\ P_i\{a_j\}$, i can manipulate at (R_{-ki}, R_i, R_k).

So, only one of the following cases can occur:

(i) k dictates in g_{ki}

(ii) $\exists l \in N \backslash \{k, i\}$ such that l and k are bi-dictators in g_{ki}.

Suppose (i) holds. We know from Case 3 for g_{ij} that either k or i dictates in f or k and i are bi-dictators in f. Since we have assumed that k and i are bi-dictators in g_{ij}, it follows that k and i must be bi-dictators in f.

Suppose (ii) holds. Since k and i are bi-dictators in g_{ij}, we must have $l = j$. That is, j and k are bi-dictators in f.

Now consider the $(N-1)$ SCF g_{jk}. By considering analogous arguments to that for g_{ki}, it follows that the only possible dictator for g_{jk} is j. But, then j and k would be bi-dictators in f.

So, the only other possibility is the following:

(**) k and i are bi-dictators in g_{ij}, k and j are bi-dictators in g_{ki}, and i and j are bi-dictators in g_{jk}.

The final step of the proof is to show that (**) cannot hold.

Suppose (**) holds. Pick $a_j, a_k, a_l \in \mathscr{A}$ and $R_j, \bar{R}_j \in \mathscr{D}_E$ satisfying the following for all $X \in \mathscr{A} \setminus \{\{a_j\}, \{a_k\}, \{a_l\}, \{a_k, a_j\}, \{a_k, a_l\}, \{a_j, a_l\}, \{a_j, a_k, a_l\}\}$.

(i) $\{a_k\}\ P_j\{a_k, a_l\}\ P_j\{a_l\}\ P_j\{a_j, a_k\}\ P_j\{a_j, a_l\}\ P_j\{a_j, a_k, a_l\}\ P_j\{a_j\}\ P_j X$.

(ii) $\{a_l\}\ \bar{P}_j\{a_k, a_l\}\ \bar{P}_j\{a_k\}\ \bar{P}_j\{a_j, a_l\}\ \bar{P}_j\{a_j, a_k\}\ \bar{P}_j\{a_j, a_k, a_l\}\ \bar{P}_j\{a_j\}\ \bar{P}_j X$.

The reader can check that these are permissible orderings in $\mathscr{D}_E$. Let $(R_k, R_i) \in [\mathscr{D}_E]^2$ be such that $\max(R_k, \mathscr{A}) = \{a_j\}$ and $\max(R_i, \mathscr{A}) = \{a_k\}$. Let $R_{-ijk} \in [\mathscr{D}_E]^{N-3}$ be an arbitrary profile for an $(N-3)$ society. Since k and i are bi-dictators in g_{ij}, we have $f(R_{-ijk}, R_i, R_j, R_k) = \{a_j, a_k\}$. We claim that $f(R_{-ijk}, R_i, \bar{R}_j, R_k)$ must either be $\{a_j, a_l\}$ or $\{a_j, a_k\}$. To see this, observe that if this outcome is in the set $\{\{a_k\}, \{a_k, a_l\}, \{a_l\}\}$, then j will manipulate at $(R_{-ijk}, R_i, R_j, R_k)$ via $\bar{R}_j$. If it is not in that set nor in $\{\{a_j, a_k\}, \{a_j, a_l\}\}$, then j will manipulate at $(R_{-ijk}, R_i, \bar{R}_j, R_k)$ via R_j.

Case A. $f(R_{-ijk}, R_i, \bar{R}_j, R_k) = \{a_j, a_l\}$.

Let $\bar{R}_k = \bar{R}_j$ and $R'_k \in \mathscr{D}_E$ be such that $\{a_l\}\ P'_k\{a_j, a_l\}\ P'_k\{a_j\}\ P'_k X$ for all $X \in \mathscr{A} \setminus \{\{a_j\}, \{a_l\}, \{a_j, a_l\}\}$. Since i and j are bi-dictators in g_{jk}, we must have $f(R_{-ijk}, R_i, \bar{R}_j, \bar{R}_k) = \{a_k, a_l\}$. We claim that $f(R_{-ijk}, R_i, \bar{R}_j, R'_k) = \{a_k, a_l\}$. Suppose not. Since j can force the outcome to be $\{a_k, a_l\}$ by announcing $R'_j = R'_k$, we must have $f(R_{-ijk}, R_i, \bar{R}_j, R'_k) = \{a_l\}$. But then k will manipulate at $(R_{-ijk}, R_i, \bar{R}_j, \bar{R}_k)$ via R'_k. Therefore, $f(R_{-ijk}, R_i, \bar{R}_j, R'_k) = \{a_k, a_l\}$. Since $\{a_j, a_l\}\ P'_k\{a_k, a_l\}$, k will manipulate at $(R_{-ijk}, R_i, \bar{R}_j, R'_k)$ via R_k. Thus, f would not be strategy-proof if Case A were to hold.

Case B. $f(R_{-ijk}, R_i, \bar{R}_j, R_k) = \{a_j, a_k\}$.

Let $\tilde{R}_k = R_i$ and let $R^*_k \in \mathscr{D}_E$ be such that $\{a_k\}\ P^*_k\{a_j, a_k\}\ P^*_k\{a_j\}\ P^*_k X$ for all $X \in \mathscr{A} \setminus \{\{a_k\}, \{a_j\}, \{a_j, a_k\}\}$. Since k and j are bi-dictators in g_{ki}, we have $f(R_{-ijk}, R_i, \bar{R}_j, \tilde{R}_k) = \{a_k, a_l\}$. By replicating the appropriate arguments in Case A, it follows that $f(R_{-ijk}, R_i, \bar{R}_j, R^*_k) = \{a_k, a_l\}$. But

$\{a_j, a_k\}\ P_k^*\{a_k, a_l\}$. Therefore k will manipulate at $(R_{-ijk}, R_i, \bar{R}_j, R_k^*)$ via R_k. So, f would not be strategy-proof if Case B were to hold.

Thus (**) cannot hold. This completes the proof of Theorem 3.3. ∎

4. RELATED LITERATURE[14]

One of the early papers on strategy-proofness to which our work is related is Feldman (1980). He considers decision schemes that map profiles of preferences over basic alternatives (i.e. singleton sets) into even-chance lotteries over these alternatives. He then proves a bi-dictatorship result for decision schemes that are strategy-proof and satisfy a unanimity requirement. We claim that this result follows from our Theorem 3.3. Consider a Feldman decision scheme which satisfies unanimity and is strategy-proof. Construct a social choice function by associating, for every preference profile (defined over basic alternatives), the support of the value of the Feldman decision scheme (an even-chance lottery) for that profile. We now extend the domain of preferences to CEUCEP orderings by imposing the invariance requirement that the value of the SCF is unchanged if preferences over singleton sets is unchanged. It can be verified that if the Feldman decision scheme is strategy-proof, then so is the SCF that we have defined. Moreover, unanimity is satisfied as well, so that Theorem 3.3 applies. The SCF must therefore either be dictatorial or bi-dictatorial. This is exactly what Feldman's result states. We note that the step which allowed us to associate a lottery with its support was valid only because the lotteries were assumed to be even-chance. Once this assumption is relaxed as in the Gibbard (1977) model, this correspondence between a lottery and its support fails to preserve strategy-proofness either for CEUC or CEUCEP preferences. In the Introduction, we have given an example to illustrate this point.

Recently there has been a revival of interest in the issue of manipulable correspondences. Two papers of particular interest are Duggan and Schwartz [8] and Ching and Zhou [7]. We discuss each in turn. Duggan and Schwartz consider choice functions which map profiles of preferences on basic alternatives into sets of these alternatives. They define a choice function to be manipulable if there exists an individual, a "true" preference profile and a "false" ranking such that:

For every lottery over the set obtained by lying, and for every lottery over the set obtained from telling the truth, there is an expected utility function consistent with the voter's true ranking of basic alternatives for which

[14] We thank the referee for clarifying many of the issues discussed in this section.

the expected utility of the first lottery exceeds the expected utility of the second.

Duggan and Schwartz demonstrate the existence of such a manipulation provided that there are at least three basic alternatives and that the choice function satisfies Citizen Sovereignty, non-dictatorship and a condition called Residual Resoluteness. The last named condition requires the choice function to be singleton-valued in a special class of profiles. Is this result a special case of Theorem 3.2? We can, of course, extend a choice function from profiles of rankings over basic alternatives to profiles of CEUC rankings over sets by the invariance principle described previously. It is also easy to see that Citizen Sovereignty, Residual Resoluteness and non-manipulability imply unanimity. Therefore, according to Theorem 3.2, every such social choice function which is non-dictatorial must be manipulable in our sense. Restating our original definition in this context, it follows that there exists an individual, a "true" preference profile and a "false" ranking such that:

For some prior probability distribution over basic alternatives and for some expected utility consistent with the individual's true preferences over sets, the individual's expected utility conditional on the set obtained from lying is greater than the expected utility conditional on the set obtained from telling the truth.

It is clear from the two definitions that manipulability in our sense is not equivalent to manipulability in the Duggan–Schwartz sense. In the latter case, for *every* pair of lotteries obtained from lying and truthtelling, there is a utility representation of preferences, such that lying is more profitable than truthtelling in terms of expected utility. In our case, there need be only one lottery on the set obtained by lying and one on the set obtained by telling the truth (constrained to come from the same prior probability distribution) that allows an increase in expected utility. Therefore the Duggan–Schwartz result does not follow from ours. Since the Duggan–Schwartz result does not imply ours, the two sets of results are independent.

Ching and Zhou, like Duggan and Schwartz also consider social choice rules which associate sets of alternatives with profiles of preferences over basic alternatives. Using the same definition of manipulability as ours, they show that only *dictatorial* or *constant* social choice functions are strategy-proof. Hence, their result "almost"[15] follows from ours.

A paper closely related to ours is Benoit [7]. The paper considers social choice rules which, like our social choice functions, map profiles of preferences over sets of basic alternatives to these sets themselves. The

[15] The qualification is required because we rule out constant functions by assuming that all singleton sets are in the range of the social choice function.

notion of strategy-proofness in this paper is therefore identical to ours. However, Benoit does not assume that preferences over sets are consistent with utility maximization in any way. The main assumption made about the domain is that certain special orderings called "top" and "bottom" preferences are admissible. Roughly speaking, a "top" preference is one where sets are evaluated on the basis of their maximal elements while "bottom" preferences are those where minimal elements are critical. In addition, certain weak assumptions are made regarding all admissible orderings (for example, an assumption that the maximal element of an ordering is always a singleton, and a neutrality assumption). A less innocuous assumption made on social choice rules is that they satisfy the property of "near unanimity". This property states that if all but one individual have a common maximal singleton, then this singleton must be the value of the social choice rule at that profile. For instance, the rule which selects the union (over individuals) of maximal elements violates this property. Note also that in order for this property to be satisfied, there must be at least three individuals.

The main result in Benoit's paper is that if there are at least three individuals and three basic alternatives, any strategy-proof social choice function satisfying near unanimity, must be dictatorial. Although, or conclusions are similar, our results are logically unrelated. While Benoit makes no assumptions regarding expected utility maximization, we assume the weaker unanimity condition. These different assumptions also mean that our proof techniques are very different from each other. We build on the two-person case (ruled out by assumption in Benoit's model) while Benoit develops a line of reasoning first used by Geanokoplos (Geanokoplos [15]) to give a direct proof of Arrow's Impossibility Theorem. The connection between our domain assumptions is also intriguing. It is true that "top" and "bottom" preferences can be rationalized as CEUCEP (and therefore CEUC) preferences. However, a close reading of our proof will reveal that they are not required in any way for our results. The only assumptions on preferences that we require are specified completely in Lemma 3.5 and it can be verified that they have little to do with "top" or "bottom" preferences. It is likely therefore, that the results are even "less related" than they appear to be.

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