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# NOTES

## ON ASYMPTOTICALLY MINIMAX TEST OF MANOVA

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*SUMMARY.* A test based on Pillai's (1955) trace criterion for the general linear hypothesis in MANOVA model has been shown to be asymptotically minimax in a restricted class of alternatives. Local minimaxity of the same test has been established by Schwartz (1967a) under the general alternative.

### 1. Introduction

For the general linear hypothesis in multivariate analysis of variance (MANOVA) model, Schwartz (1967a, b) has established that the test based on Pillai's (1955) trace criterion is admissible and locally minimax.

In this article an attempt is made to study the asymptotic minimax property, in the sense of Giri and Kiefer (1964), hereafter called G–K (1964), of the test based on Pillai's trace criterion. It has been shown that this test is asymptotically minimax in a restricted class of alternatives. Asymptotic minimaxity for the general class of alternatives is still an open problem.

### 2. Manova problem and invariance

In the canonical form of the MANOVA problem $W = (Y, U, Z)$ is $p\times(r+n+m)$ matrix and it will always be assumed that $r+m > p$ (so that $YY'+ZZ'$ will be non-singular with probability one). The columns of $W$ are independently normally distributed $p$-vectors with common unknown non-singular covariance matrix $\Sigma$. Let $EY = \xi(p\times r)$, $EU = \gamma(p\times n)$ and $EZ = 0(p\times m)$. The problem is to test $H_0[\xi = 0]$ against the alternative $H[\xi \neq 0]$.

Then this problem remains invariant under $GL(p)\times 0(r)$ acting on $W$ by $(A, H)W = (AYH, AU, AZ)$ and also under $R^{pn}$ acting by translation on $U$, where $GL(p)$ is full linear group, $0(r)$ is orthogonal group. A maximal invariant under $R^{pn}$ is $(Y, Z)$, the matrix $U$ having been eliminated by invariance under $R^{pn}$. A convenient choice of maximal invariant in the sample space of sufficient statistic and in the parameter space are, respectively, the set of ordered latent roots $l_1 \geqslant \ldots \geqslant l \geqslant 0$ of $Y'(YY'+ZZ')^{-1}Y$ and the set of ordered latent roots $\delta_1 \geqslant \ldots \geqslant \delta \geqslant 0$ of $\xi'\Sigma^{-1}\xi$.

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The first step in verifying the asymptotic minimaxity in the sense of G-K (1964), is to reduce the original problem using the Hunt-Stein theorem. The lower triangular group $G_T(p)$ satisfies the Hunt-Stein theorem.

Thus we shall require the ratio of the probability of the maximal invariant both under $GL(p)\times 0(r)$ and $G_T(p)$. Let $\mu_G$ denote the (left) Haar measure on $G$. $G$ acts transitively on $H_0$ so that under $H_0$ the maximal invariant has a single probability distribution. There exists a unique $A \epsilon G_T^+(p)$ such that $A\Sigma A' = I_p$ and if $\xi^* = A\xi$, the probability density of the maximal invariant under $(\xi, \Sigma)$ is the same as under $(\xi^*, I_p)$, where $G_T^+(p)$ is a group of lower triangular matrices with all the diagonal elements positive.

Following the development of Schwartz (1967a, equation (3) and (4)), the ratio of the probability of maximal invariant under $G$ (which may be $GL(p)\times 0(r)$ or $G_T(p)$) is given by

$$I = D^{-1}\exp[-\tfrac{1}{2}\operatorname{tr}\xi^{*\prime}\xi^*]\int_G |gg'|^{\frac{m+r}{2}}\exp[(\tfrac{1}{2}\operatorname{tr} gg' + \operatorname{tr}\xi^{*\prime}gg_0Y]d\mu_G(g) \quad \ldots \quad (2.1)$$

where

$$D^{-1} = \int_G |gg'|^{\frac{m+r}{2}}\exp[(\tfrac{1}{2}\operatorname{tr} gg']d\mu_G(g),\ g\,\epsilon\, G,\ g_0\,\epsilon\, G_T^+(p),$$

such that

$$g_0(YY'+ZZ')g_0' = I_p,$$

which implies that

$$\operatorname{tr} g_0YY'g_0' = \operatorname{tr} Y'(YY'+ZZ')^{-1}Y,$$

the trace criterion, which is due to Pillai (1955).

## 3. Main results

To establish the asymptotic minimax property of a test based on $\operatorname{tr} Y'(YY'+ZZ')^{-1}Y$ we are only to verify the conditions (4.1) and (4.2) of G-K (1964, p. 33).

To verify (4.1) of G-K (1964), for a test $R : \operatorname{tr} Y'(YY'+ZZ')^{-1}Y \geqslant C_\alpha$, at level $\alpha$, we are to evaluate $I$ in (2.1) over $GL(p)\times 0(r)$ for large $\lambda = \sum_{i=1}^{r}\delta_i$ and hence we have the following

Theorem 3.1 : *For testing* $H_0[\xi = 0]$ *against* $H\left[\xi, \Sigma \mid \delta_i = \delta + 0\left(\frac{1}{\lambda}\right), i = 1 \ldots r\right]$, *the power function of the test* $R : T = \text{tr } Y'(YY'+ZE')^{-1}Y \geqslant C_\alpha$, *at level p, for large,* $\lambda$, *is given by*

$$P[T > C_\alpha \mid H] = 1 - \exp \frac{\lambda}{2}\left[-1 + \frac{C_\alpha}{r}\right][1+0(1)] \qquad \ldots \quad (3.1)$$

*where*

$$\lambda = r\delta + 0\left(\frac{1}{\lambda}\right), \; C_\alpha \leqslant r.$$

*Remark* 1 : Here we have considered a restricted class of alternatives, where we assume that when $\lambda$ is large all the roots are large so as to satisfy the relation $\delta_i = \delta + 0\left(\frac{1}{\lambda}\right)$ for $i = 1, \ldots, r$. It may be remarked that the asymptotic minimax property of the test $R$ is obtainable only in this class of alternatives.

*Remark* 2 : It may be noted that when $r = p$, for testing $H_0[\xi = 0]$ against $H_1[\xi, \Sigma \mid (\delta_1, \ldots \delta_p) = \lambda(\bar{\delta}_1, \ldots \bar{\delta}_p), \; \lambda > 0]$, where $\bar{\delta}_i$'s are known, the test, $\Sigma \bar{\delta}_i l_i > C_\alpha$, which is admissible (vide Schwartz (1967b, p. 707)) can be shown to satisfy (4.1) of G-K (1964). This follows directly from equ. (8.4) of Muirhead (1978) for large $\lambda$. It is clear that $H_o$ is also restrictive but more general than $H$ considered in theorem 3.1.

To prove the theorem we consider the following

Lemma 3.1 : *Let* ${}_1F_1(a, b, \Omega, L)$ *be the generalised confluent hypergeometric function of matrix arguments, where* $\Omega$ *is the matrix of parameters having roots* $\delta_1 \geqslant \ldots \geqslant \delta_r \geqslant 0$ *and* $L$ *is the random matrix having root* $l_1 \geqslant \ldots \geqslant l_r \geqslant 0$. *Then under the assumption that* $\delta_1 = \ldots = \delta_r = \delta$ *when* $\delta$ *is sufficiently large, the asymptotic form of* ${}_1F_1$ *is given by*

$$e^{\delta \Sigma l_i} \delta^{\frac{r(a-b-r+1)}{2}} \prod_i l_i^{\frac{a-b}{2}} \prod_{i<j} C_{ij}^{-\frac{1}{2}}[1+0(\Omega^{-1})], \qquad \ldots \quad (3.2)$$

*where* $C_{ij} = (l_i - l_j)$.

*Outline of the Proof* : When all the roots are large the asymptotic form of ${}_1F_1(a, b, \Omega, L)$ can be easily obtained from Muirhead (1978, equ. (8.4)) which was derived by Constantne and Muirhead (1976, Sec. 3). Refering to the derivation of (2.5) of Anderson (1965, p. 1160) and introducing the assumption that $\delta_1 = \ldots = \delta_r = \delta$, where $\partial$ is large, the result (3.2) follows immediately from the line of derivation of Constantine and Muirhead follows (1976, Sec. 3).

*Proof of the theorem* : Exact expression for $I$ in (2.1) over $GL(p)\times 0(r)$ is the well-known form

$$I_1 = \exp[-\tfrac{1}{2}\operatorname{tr}\xi^{*\prime}\xi^{*}]\,{}_1F_1\left(\frac{m+r}{2},\ \frac{r}{2},\ \frac{1}{2}\xi^{*\prime}\xi^{*}, L\right) \qquad \ldots \quad (3.3)$$

where

$$L = Y'(YY'+ZE')^{-1}Y.$$

Let

$$\lambda = \operatorname{tr}\xi^{*\prime}\xi^{*} = \sum_{i=1}^{r}\delta_i.$$

Then assuming that when $\lambda$ is large all the roots are large so as to satisfy the relation $\delta_i = \delta + 0\left(\frac{1}{\lambda}\right)$, the asymptotic form of (3.3), by using (3.2) of lemma 3.1, is

$$I_1 = \text{const. } e^{-\frac{\lambda}{2}+\frac{\lambda}{2}\sum_i \frac{l_i}{r}[1+O(1)]}\left[1+0\left(\frac{1}{\lambda}\right)\right]$$

$$\simeq e^{\frac{\lambda}{2}\left[-1+\sum_i \frac{l_i}{r}\right][1+O(1)]} \qquad \ldots \quad (3.4)$$

where constant term is free from $\lambda$.

Thus asymptotically, for large $\lambda$ and under the alternative hypothesis stated in the theorem, we have from (3.4), the power function of the test, $T \geqslant C_\alpha$. viz.,

$$P[T > C_\alpha \mid H] = 1 - e^{\frac{\lambda}{2}\left[-1+\frac{C_\alpha}{r}\right][1+O(1)]}.$$

Hence the theorem.

Since $\frac{C_\alpha}{r} \leqslant 1$, it follows that from (3.1), the condition (4.1) of G-K (1964) is satisfied.

To verify the condition (4.2) of G-K (1964), we consider $I$ in (2.1) for $G = G_T(p)$. Then following the derivation of Schwartz (1967a, equ. (19), p. 346), we have

$$I_2 = e^{-\frac{\lambda}{2}+\frac{\lambda}{2}\sum_{i>j}(\eta_i v'_j)^2} \prod_{j=1}^{p} {}_1F_1\left(\frac{m+r-j+1}{2},\ \frac{1}{2},\ \frac{\lambda}{2}(\eta_j v'_j)^2\right)$$

where

$$\eta_i = (\eta_{i1}, \ldots, \eta_{ir}),\ \eta_{ik} = \xi^*_{ik}/\sqrt{\lambda},$$

$$v_j = (v_{j1}, \ldots, v_{jr}).$$

Then for large $\lambda$ (remembering that for large $x$, ${}_1F_1(a, b, x) = e^{x[1+0(1)]}$), we have

$$I_2 = e^{-\frac{\lambda}{2}+\frac{\lambda}{2}\sum_{i>j}(\eta_i v_j')^2+\frac{\lambda}{2}\sum_{j}(\eta_j v_j')^2[1+O(1)]}$$

$$= e^{-\frac{\lambda}{2}+\frac{\lambda}{2}\sum_{i\geqslant j=1}^{p}(\eta_i v_j')^2[1+O(1)]}$$

$$= e^{-\frac{\lambda}{2}+\frac{\lambda}{2}\sum_{i=1}^{p}\eta_i'\eta_i\sum_{j\leqslant i}v_j'v_j[2+O(1)]}. \qquad \ldots \quad (3.5)$$

To consider the asymptotic minimax property of a test based on $\operatorname{tr} v'v$, where $v = g_0 Y$ (in 2.1), we observe that $R^{pn}$ and $G_T(p)$ invariant test is a function of $v$, where $g_0(YY'+ZE')g_0' = I$ and that its distribution under $(\xi, \Sigma)$ depends only on $\xi^* = A\xi$, where $A\Sigma A = I_p$ and $g_0$, $A \epsilon G_T^k(p)$. Even though $\xi^*$ and $v$ are not $G_T(p)$ invariant but only $G_T^+(p)$ invariant it will make the development simpler to construct a priori measures on $\{\eta\}$ for fixed $\lambda$. Any such measure induces a corresponding measure on the space of a $G_T(p)$ maximal invariant.

Now $I_2$ in (3.5) represents the ratio of the probability density of the maximal invariant w.r.t. $G_T(p)$, viz., $f_{\xi^*,I}/f_{0,I}$, which can be written as $f_{\lambda,\eta,I}/f_{0,\eta,I}$ in terms of the parameters in (3.5).

To verify (4.2) of G-K (1964), we observe that letting $\xi_{0,\lambda}$ give measure 1 to the point $\eta = 0$ and $\xi_{1,\lambda}$ give measure 1 to the countour

$$(\eta_1'\eta_1 = \ldots = \eta_{p-1}'\eta_{p-1} = 0,\ \eta_p'\eta_p = I),$$

we have from (3.5)

$$\frac{\int f_{\lambda,\eta,I}\xi_{1,\lambda}d\eta}{\int f_{0,\eta,I}\xi_{0,\lambda}d\eta} = e^{\frac{\lambda}{2}[-1+T][1+O(1)]} \quad \ldots \quad (3.6)$$

where

$$T = \text{tr } Y(YY+ZE)^{-1}Y, \ \lambda = \sum_{i=1}^{r} \delta_i.$$

Thus (3.6) satisfies (4.2) of G-K (1964) and hence we have the following.

Theorem 3.2 : *For testing* $H_0[\xi = 0]$ *against* $H[\xi, \Sigma \,|\, \delta_i = \delta + 0\left(\frac{1}{\lambda}\right)$, $i = 1, \ldots, r$ *for large* $\lambda$, *the test with critical region* $tr\ Y'(YY'+ZZ')^{-1}Y \geqslant C_\alpha$, *is asymptotically minimax in the sense of Giri and Kiefer* (1964).

*Remark* 3 : It may be noted that although the test : $\Sigma\bar{\delta}\, l_i > C_\alpha$, proposed in remark 2 for the $H_0$ against $H_1$, satisfies (4.1) of G-K (1964), it can be shown that this test does not satisfy (4.2) of G-K (1964) under suitable choice of prior on $\{\eta\}$. Hence this test is not asymptotically minimax in the sense of G—K (1964).

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