Indag. Mathem., N.S., 10 (1), 15–23 March 29, 1999

On a new asymptotic norming property

by Pradipta Bandyopadhyay and Sudeshna Basu

Stat-Math Division, Indian Statistical Institute, 203, B.T. Road, Calcutta 700 035, India,
e-mail: pradipta@isical.ac/ac/.in & res9415@isical.ac/ac/.in
Current address: Sudeshna Basu, Department of Mathematics, The George Washington University, Washington DC 20052, USA,
e-mail: sbasu@gwis2.circ.gwu.edu

Communicated by Prof. J. Korevaar at the meeting of January 26, 1998

ABSTRACT

In this work, we introduce a new Asymptotic Norming Property (ANP) which lies between the strongest and weakest of the existing ones, and obtain isometric characterisation of it. The corresponding w*-ANP turns out to be equivalent on the one hand, to Property (V) introduced by Sullivan, and to a ball separation property on the other. We also study stability properties of this new ANP and its w*-version.

1. INTRODUCTION

The Asymptotic Norming Property (ANP) was introduced by James and Ho [JH] to show that the class of separable Banach spaces with Radon–Nikodým Property (RNP) is larger than those isomorphic to subspace of separable duals. Three different Asymptotic Norming Properties were introduced and were shown to be equivalent in separable Banach spaces. Recently, Hu and Lin [HL1] have obtained isometric characterisations of the ANPs and shown that they are equivalent in Banach spaces admitting a locally uniformly convex renorming, a class larger than separable Banach spaces. In dual Banach spaces, they introduced a stronger notion called the w*-ANP, which turned out to be nice geometric properties.

Here, we introduce a new Asymptotic Norming Property which lies between the strongest and weakest of the existing ones, and obtain isometric characterisations of it. The corresponding w*-ANP, the main object of our study, turns out to be equivalent on the one hand, to Property (V) introduced by Sullivan

[S], and to a ball separation property *à la* Chen and Lin [CL] on the other. We also study stability properties of this new ANP and its w*-version.

2. THE ANPS: OLD AND NEW

Definition 2.1. For a Banach space X, let $S_X = \{x : \|x\| = 1\}$ and $B_X = \{x : \|x\| \leq 1\}$.

A subset Φ of B_{X^*} is called a norming set for X if $\|x\| = \sup_{x^* \in \Phi} x^*(x)$, for all $x \in X$. A sequence $\{x_n\}$ in S_X is said to be asymptotically normed by Φ if for any $\varepsilon > 0$, there exists a $x^* \in \Phi$ and $N \in \mathbb{N}$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geq N$.

For $\kappa =$ I, II or III, a sequence $\{x_n\}$ in X is said to have the property κ if

I. $\{x_n\}$ is convergent.

II. $\{x_n\}$ has a convergent subsequence.

III. $\bigcap_{n=1}^{\infty} \overline{co}\{x_k : k \geq n\} \neq \emptyset$, where $\overline{co}(A)$ is the closed convex hull of $A \subseteq X$.

For $\kappa =$ I, II or III, X is said to have the asymptotic norming property κ with respect to Φ (Φ-ANP-κ), if every sequence in S_X that is asymptotically normed by Φ has property κ.

Remark 2.1. In [HL1, Theorem 2.3], it is shown that Φ-ANP-III is equivalent to the apparently stronger property that every sequence in S_X asymptotically normed by Φ has a weakly convergent subsequence.

This motivates the following definition.

Definition 2.2. Let X be a Banach space and let $\Phi \subseteq B_{X^*}$ be a norming set for X. X is said to have Φ-ANP-II$'$ if any sequence $\{x_n\}$ in S_X which is asymptotically normed by Φ is weakly convergent.

Definition 2.3. X is said to have the asymptotic norming property κ (ANP-κ), $\kappa =$ I, II, II$'$ or III, if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ for $(X, \|\cdot\|)$ such that X has Φ-ANP-κ.

Remark 2.2. Clearly, Φ-ANP-I $\Rightarrow$ Φ-ANP-II$'$ $\Rightarrow$ Φ-ANP-III. Thus all the ANPs are equivalent in Banach spaces admitting a locally uniformly convex renorming, in particular, in separable Banach spaces.

Definition 2.4. A Banach space X is said to have the Kadec property (K) if the weak and the norm topologies coincide on the unit sphere, i.e., $(S_X, w) = (S_X, \|\cdot\|)$.

X is said to have Kadec-Klee property (KK) if for any sequence $\{x_n\}$ and x in B_X with $\lim_n \|x_n\| = \|x\| = 1$ and w-$\lim_n x_n = x$, $\lim_n \|x_n - x\| = 0$.

The proofs of the following two theorems are evidently similar to those of

Theorems 2.4 and 2.5 of [HL1]. We include the details only when we feel some elaboration is needed.

Theorem 2.1. *Let Φ be a norming set for a Banach space X. The following are equivalent:*

(a) *X has* Φ-ANP-I
(b) *X has* Φ-ANP-II$'$ *and X has (K)*
(c) *X has* Φ-ANP-II$'$ *and X has (KK)*

Proof. Since Φ-ANP-II implies (K) [HL1, Theorem 2.4, (1) $\Rightarrow$ (2)], so does Φ-ANP-I. Thus $(a) \Rightarrow (b)$ follows and $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ Since X has Φ-ANP-II$'$, any sequence $\{x_n\}$ in S_X asymptotically normed by Φ is weakly convergent to some $x \in X$. Then by [HL1, Lemma 2.2], $\|x\| = 1$ and hence by (KK) we have $x_n \to x$ in norm. $\square$

Theorem 2.2. *Let Φ be a norming set for a Banach space X. The following are equivalent:*

(a) *X has* Φ-ANP-II$'$.
(b) *X has* Φ-ANP-III *and X is strictly convex.*

Proof. $(a) \Rightarrow (b)$ Strict convexity of X follows similarly as in the proof of [HL1, Theorem 2.5, (1) $\Rightarrow$ (2)].

$(b) \Rightarrow (a)$ Let $\{x_n\}$ be a sequence in S_X asymptotically normed by Φ. Since X has Φ-ANP-III, $D = \cap\overline{co}\{x_k : k \geq n\} \neq \emptyset$. Now X has Φ-ANP-III implies $\{x_n\}$ has weak cluster points and all of them must be in D. Since $D \subseteq S_X$ is convex and X is strictly convex, D is a singleton. Moreover, since every subsequence of $\{x_n\}$ is also asymptotically normed by Φ, that singleton is the weak limit of $\{x_n\}$. Hence X has Φ-ANP-II$'$. $\square$

Some renorming results similar to Theorem 2.7 of [HL1] can easily be obtained from our results. But in this work, we concentrate on the ANPs as isometric properties.

3. W*-ANPS

Definition 3.1. Let X^* be a dual Banach space. X^* is said to have w*-ANP-κ (κ = I, II, II$'$ or III) if there exists an equivalent norm $\|\cdot\|$ on X and a norming set Φ for X^* in B_X such that X^* has Φ-ANP-κ.

Remark 3.1. If $\Phi \subseteq B_X$ is a norming set for X^*, then $\overline{co}(\Phi \cup -\Phi) = B_X$. Hence, by [HL3, Lemma 3] and similar arguments, Φ-ANP-κ is equivalent to B_X-ANP-κ (κ = I, II, II$'$ or III). Thus, we can and do work with $\Phi = B_X$.

Definition 3.2. For a Banach space X, let $X^\perp = \{x^\perp \in X^{***} : x^\perp(x) = 0$ for all $x \in X\}$. A Banach space is said to be Hahn–Bahach smooth if for all

$x^* \in X^*, \|x^* + x^\perp\| = \|x^*\| = 1$ implies $x^\perp = 0$, i.e., $x^* \in X^{***}$ is the unique norm preserving extension of $x^*|_X$.

Remark 3.2. It is shown in [HL2, Theorem 1] that X is Hahn–Banach smooth if and only if X^* has B_X-ANP-III if and only if $(S_{X^*}, w^*) = (S_{X^*}, w)$.

Extending a characterisation of rotundity of X^* due to Vlasov [V], Sullivan [S] introduced the following stronger property:

Definition 3.3. A Banach space X is said to have the Property (V), if there do not exist an increasing sequence $\{B_n\}$ of open balls with radii increasing and unbounded, and norm one functionals x^* and y_k^* such that for some constant c,

$x^*(b) > c$ for all $b \in \cup B_n$,

$y_k^*(b) > c$ for all $b \in B_n, n \leq k$ and

$\operatorname{dist}(co(y_1^*, y_2^*, \ldots), x^*) > 0$.

Definition 3.4. Let $W \subseteq X^*$ be a closed bounded convex set.

(a) A point $x^* \in W$ is said to be a weak*-weak point of continuity (w*-w pc) of W if x^* is a point of continuity of the identity map from (W, w^*) to (W, w)

(b) A point $x^* \in W$ is said to be a w*-strongly extreme point of W if the family of w*-slices containing x^* forms a local base for the weak topology of X^* at x^* (relative to W).

Now we have our main characterisation theorem.

Theorem 3.1. *For a Banach space X, the following are equivalent:*

(a) *X^* has B_X-ANP-II$'$.*

(b) *X^* is strictly convex and X is Hahn–Banach smooth.*

(c) *X has Property (V).*

(d) *All points of S_{X^*} are w^*-strongly extreme points of B_{X^*}.*

Proof. $(a) \Leftrightarrow (b)$ is immediate from Theorem 2.2 and Remark 3.2, while $(b) \Leftrightarrow (c)$ is just [S, Theorem 4].

$(b) \Leftrightarrow (d)$ Since $(S_{X^*}, w^*) = (S_{X^*}, w)$, and the norm is lower semi-continuous with respect to both weak and weak* topology of X^*, any $x^* \in S_{X^*}$ is a w*-w pc of B_{X^*}. Now, since X^* is strictly convex, every $x^* \in S_{X^*}$ is an extreme point of B_{X^*}. By a classical result of Choquet [C, Proposition 25.13], for any $x^* \in S_{X^*}$, the family of w*-slices containing x^* forms a local base for the weak*, and therefore the weak, topology of X^* relative to B_{X^*}.

$(d) \Rightarrow (b)$ From (d), it is immediate that X^* is strictly convex and any $x^* \in S_{X^*}$ is a w*-w pc of B_{X^*} $\square$

Definition 3.5. (a) The duality mapping D for a Banach space X is the set-valued map from S_X to S_{X^*} defined by

$$D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}, \quad x \in S_X.$$

(b) A Banach space X is said to be very smooth if every $x \in S_X$ has a unique norming element in X^{***}.

(c) [HL2] A Banach space X is said to be Quasi–Fréchet differentiable if for any convergent sequence $\{x_n\}$ in S_X and any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$, the sequence $\{x_n^*\}$ has a norm convergent subsequence.

(d) X is said to be weakly Hahn–Banach smooth, if for all $x \in S_{X^*}$, and $x_n^* \in S_{X^*}$, $\lim_{n\to\infty} x_n^*(x) = 1$ implies that $\{x_n^*\}$ has a weakly convergent subsequence.

It is known that X is Fréchet differentiable (very smooth) if and only if the duality mapping D is single-valued and is norm-norm (norm-weak) continuous. And from [HL2, Theorem 4], it is known that if X^* has B_X-ANP-I (B_X-ANP-II) then X is Hahn–Banach smooth and Fréchet differentiable (Quasi–Fréchet differentiable). The following question posed in [HL2] still seems to be open.

Question 3.1. Let X be a Banach space which is Hahn–Banach smooth and Fréchet differentiable (Quasi–Fréchet differentiable). Does it follow that X^* has B_X-ANP-I (B_X-ANP-II)?

Theorem 3.2. *If X^* has B_X-ANP-II$'$, then X is very smooth.*

Proof. That Property (V) implies very smooth was already observed in [S]. We, however, prefer the following direct and ANP-like argument similar to [HL2, Theorem 4(1)].

Since X^* is strictly convex, X is smooth. Now let $\{x_n\} \subseteq S_X$ be such that $x_n \to x$. Let $\{x_n^*\} = D(x_n)$, we have $|x_n^*(x) - 1| \leq |x_n^*(x) - x_n^*(x_n)| \leq \|x_n^*\| \|x - x_n\| \leq \|x - x_n\| \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} x_n^*(x) = 1$. So $\{x_n^*\}$ is asymptotically normed by B_X, and hence, is weakly convergent to x^* (say). Clearly, $x^* \in D(x)$ and since X is smooth, $\{x^*\} = D(x)$. Hence X is very smooth. □

Analogous to [HL2], we now have the following question:

Question 3.2. Let X be Hahn–Banach smooth and very smooth. Does X^* have B_X-ANP-II$'$?

Remark 3.3. Let us say that a Banach space X has property P_1^*-κ ($\kappa =$ I, II, II$'$ or III), if for any convergent sequence $\{x_n\}$ in S_X, and any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$, the sequence $\{x_n^*\}$ has property κ (recall that property III means having a weakly convergent subsequence). Then clearly, w*-ANP-$\kappa \Rightarrow P_1^*$-κ and

X has property P_1^*-III $\iff$ X is weakly Hahn–Banach smooth.
X has property P_1^*-II $\iff$ X is Quasi–Fréchet differentiable.
X has property P_1^*-II$'$ $\iff$ X is very smooth.
X has property P_1^*-I $\iff$ X is Fréchet differentiable.

Thus Questions 3.1 and 3.2 are essentially whether the implication w*-ANP-$\kappa \Rightarrow P_1^*$-κ ($\kappa =$I, II and II$'$) can be reversed under Hahn–Banach smoothness.

Observe that if we can reverse the implications for $\kappa =$ II and II$'$, the result for $\kappa =$ I would follow. Observe also that since X is smooth and Hahn–Banach smooth implies that X is very smooh [S, Corollary to Lemma 5.2], the Question 3.2 (i.e., $\kappa =$ II$'$) actually boils down to

Question 3.3. If X is smooth and Hahn–Banach smooth, is X^* strictly convex?

As for $\kappa =$ II, observe that since $D(S_X)$ is dense in S_{X^*}, w*-ANP-II is equivalent to the apparently weaker property that any sequence $\{x_n^*\}$ in $D(S_X)$ that is asymptotically normed by B_X has a convergent subsequence. Now, if $\{x_n\} \subseteq S_X$ and $x_n^* \in D(x_n)$ is such that the sequence $\{x_n^*\}$ is weakly convergent, must $\{x_n\}$ have a convergent subsequence?

Example 3.1. In general, for a Banach space X, the properties ANP-I, II, II$'$ and III are all distinct, i.e., except for the obvious implications Φ-ANP-I $\Rightarrow$ Φ-ANP-II $\Rightarrow$ Φ-ANP-III and Φ-ANP-I $\Rightarrow$ Φ-ANP-II$'$ $\Rightarrow$ Φ-ANP-III, no other implication is generally true.

Proof. Clearly, it suffices to show that none of Φ-ANP-II of Φ-ANP-II$'$ implies the other.

(1) Let $X = c_0$, $X^* = \ell_1$. Since $(S_{X^*}, w) = (S_{X^*} \| \cdot \|)$ on ℓ_1, by [HL1, Theorem 3.1], X^* has been B_X-ANP-II. But X^* is not strictly convex.

(2) On $X = \ell_2$, define an equivalent norm as $\|x\|_0 = \max\{1/2(\|x\|_2), \|x\|_\infty\}$. And define $T : \ell_2 \to \ell_2$ by $T(\alpha_k) = \alpha_k/k$, for $(\alpha_k) \in \ell_2$. Then T is an 1-1 continuous linear map. Hence the equivalent dual norm $\|x\|_3 = \|x\|_0 + \|Tx\|_2$ is strictly convex [D1]. Also since ℓ_2 is reflexive, it has B_X-ANP-III with respect to $\| \cdot \|_3$. Thus by Theorem 2.2, $(\ell_2, \| \cdot \|_3)$ has B_X-ANP-II$'$. But, it was observed in [S] that $(\ell_2, \| \cdot \|_3)$ lacks (KK). $\square$

Example 3.2. The above two examples show that a space may have ANP-III, but may lack either ANP-II or II$'$. The following is an example of a Banach space which has ANP-III but lacks both ANP-II and II$'$.

Proof. Let $X = \ell_2 \oplus_1 \mathbb{R}$. It is clear that $X^* = \ell_2 \oplus_\infty \mathbb{R}$ is reflexive, and hence, has B_X-ANP-III. However X^* is not strictly convex, and hence cannot have B_X-ANP-II$'$. Also the weak and the norm topologies do not coincide on S_{X^*}. Indeed, since ℓ_2 is infinite dimensional, by Riesz' lemma, there exists a sequence $\{x_n\}$ in S_{ℓ_2} such that $\|x_n - x_m\|_2 \geq 1$, $n \neq m$. Let $z_n = (x_n, 1)$. So $\|z_n\|_\infty = 1$ and $\|z_n - z_m\|_\infty \geq 1$. Clearly, $\{z_n\}$ cannot have any norm convergent subsequence. But as ℓ_2 is reflexive, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging weakly to some $x \in B_{\ell_2}$. Then obviously $(x_{n_k}, 1) = z_{n_k}$ converges weakly to $(x, 1) = z$ (say) and $\|z\|_\infty = 1$. $\square$

In a recent work, Chen and Lin [CL] have obtained certain ball separation properties which, in equivalent formulations, characterise B_X-ANP-κ ($\kappa =$ I, II and III).

Here we obtain a similar characterisation of B_X-ANP-II$'$. In fact, as in [CL], we also take a local approach, i.e., we characterise w*-strongly extreme points of B_{X^*}. And for that we need the following characterisation for w*-w pc of B_{X^*} which is immediate from Theorem 3.1 of [CL].

Theorem 4.1. *For a Banach space X and $f_0 \in S_{X^*}$, the following are equivalent:*

(i) *f_0 is a w*-w pc of B_{X^*}.*

(ii) *for any $x_0^{**} \in X^{**}$ and $\alpha \in \mathbb{R}$, if $f_o(x_0^{**}) > \alpha$, then there exists a ball B^{**} in X^{**} with centre in X such that $x_0^{**} \in B^{**}$ and* $\inf f_0(B^{**}) > \alpha$.

From [CL, Theorem 1.3] and the arguments of [B, Corolary 2], we get

Theorem 4.2. *For a Banach space X and $f_0 \in S_{X^*}$, the following are equivalent:*

(i) *f_0 is an extreme point of B_{X^*}.*

(ii) *for any compact set $A \subseteq X$ if* $\inf f_0(A) > 0$, *then there exists a ball B in X such that $A \subseteq B$ and* $\inf f_0(B) > 0$.

(iii) *for any finite set $A \subseteq X$ if* $\inf f_0(A) > 0$, *then there exists a ball B in X such that $A \subseteq B$ and* $\inf f_0(B) > 0$.

Theorem 4.3. *For a Banach space X and $f_0 \in S_{X^*}$, the following are equivalent:*

(a) *f_0 is a w*-strongly extreme point of B_{X^*}.*

(b) *f_0 is a w*-w pc and an extreme point of B_{X^*}.*

(c) *for any compact set $A \subseteq X^{**}$ if* $\inf f_0(A) > 0$, *then there exists a ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and* $\inf f_0(B^{**}) > 0$.

(d) *for any finite set $A \subseteq X^{**}$ if* $\inf f_0(A) > 0$, *then there exists a ball $B^{**} \subseteq X^{**}$ with centre in X such that $A \subseteq B^{**}$ and* $\inf f_0(B^{**}) > 0$.

Proof. $(a) \Leftrightarrow (b)$ is just the local version of Theorem 3.1 $(b) \Leftrightarrow (d)$.

$(b) \Rightarrow (c)$ Since f_0 is a w*-strongly extreme point of B_{X^*}, it is easily seen that it remains extreme in $B_{X^{***}}$. Thus by Theorem 4.2, for any compact set A in X^{**} with $\inf f_0(A) > 0$, there exists a ball in $B^{**} = B^{**}(x_0^{**}, r) \subseteq X^{**}$ such that $A \subseteq B^{**}$ and $\inf f_0(B^{**}) > 0$. Now, $\inf f_0(B^{**}(x_0^{**}, r)) > 0$ implies $f_0(x_0^{**}) > r$. Since f_0 is a w*-w pc, by Theorem 4.1, there exists a ball $B^{**}(x, t) \subseteq X^{**}$ such that $x_0^{**} \in B^{**}(x, t)$ and $\inf f_0(B^{**}(x, t)) > r$. This implies $f_0(x) > r + t$. Thus, $A \subseteq B^{**}(x_0^{**}, r) \subseteq B^{**}(x, r + t)$ and $\inf f_0(B^{**}(x, r + t) > 0$.

$(c) \Rightarrow (d)$ is trivial.

$(d) \Rightarrow (b)$ Taking $A \subseteq X$, it follows from Theorem 4.2 that f_0 is extreme in B_{X^*}. And taking A to be a singleton, it follows from Theorem 4.1 that f_0 is an w*-w pc. □

Corollary 4.4. *For a Banach space X, the following are equivalent:*

(i) X^* *has* B_X-ANP-II$'$.

(ii) *for any* w^**-closed hyperplane* H *in* X^{**}, *and any compact convex set* A *in* X^{**} *with* $A \cap H = \emptyset$, *there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $A \subseteq B^{**}$ *and* $B^{**} \cap H = \emptyset$.

(iii) *for any* w^**-closed hyperplane* H *in* X^{**}, *and any finite dimensional convex set* A *in* X^{**} *with* $A \cap H = \emptyset$, *there exists a ball* B^{**} *in* X^{**} *with centre in* X *such that* $A \subseteq B^{**}$ *and* $B^{**} \cap H = \emptyset$.

5. STABILITY RESULTS

Theorem 5.1. *Let* X *be a Banach space with* Φ-ANP-κ, $\kappa =$ I, II, II$'$ *or* III. *Then any closed subspace* Y *of* X *has* $\Phi|_Y$-ANP-κ *where* $\Phi|_Y = \{y^* : y^* = x^*|_Y, x^* \in \Phi\}$.

Theorem 5.2. *Let* X *be a Banach space such that* X^* *has* B_X-ANP-κ, $\kappa =$ I, II, II$'$ *or* III. *Then for any closed subspace* Y *of* X, Y^* *has* B_Y-ANP-κ.

Proof. Let $\{y_n^*\} \subseteq S_{Y^*}$ be asymptotically normed by B_Y. For every $n \geq 1$, let x_n^* be a norm preserving extension of y_n^* to X. Then $\{x_n^*\}$ is asymptotically normed by B_X, and hence has property κ. Now the restriction map $x^* \to x^*|_Y$ brings property κ back to $\{y_n^*\}$. □

Corollary 5.3. *Hahn–Banach smoothness and Property* (V) *are hereditary.*

Remark 5.1. This observation appears to be new. Note that we do not need the stability of the ANPs under quotients to prove the above theorem. In fact, it is not clear whether the ANPs are indeed stable under quotients.

Let X be a Banach space, $1 < p, q < \infty$ with $1/p + 1/q = 1$ and (Ω, Σ, μ) be a positive measure space so that Σ contains an element with finite positive measure. Let Φ be a norming set for X. Then define $\Phi_1 = co(\Phi \cup \{0\}) \backslash S_X$ and

$$\Delta_n = \{\sum_{i=1}^{m} \lambda_i x_i^* \chi_{E_i} : x_i^* \in \Phi_1, \frac{(n-1)}{n} \leq \|x_i^*\| \leq \frac{n}{(n+1)}, E_i \in \Sigma,$$
$$E_i \cap E_j = \emptyset, \text{ for } i \neq j, \lambda_i > 0 \text{ with } \sum_{i=1}^{m} \lambda_i^q \mu(E_i) = 1\}$$

Then $\Delta(\Phi, \mu, q) = \bigcup_{n \geq 1} \Delta_n$ is a norming set for $L^p(\mu, X)$ [HL3].

Theorem 5.4. *Let* X *be a Banach space,* $\Phi \subseteq B_{X^*}$ *be a norming set for* X. X *has* Φ-ANP-II$'$ *if and only if* $L^p(\mu, X)$ *has* $\Delta(\Phi, \mu, q)$-ANP-II$'$.

Proof. It is well-known that X is strictly convex if and only if $L^p(\mu, X)$ is strictly convex [D2]. And in [HL3, Theorem 6], it is shown that X has Φ-ANP-III if and only if $L^p(\mu, X)$ has $\Delta(\Phi, \mu, q)$-ANP-III. Now, the result follows from Theorem 2.2. □

Remark 5.2. Let X be a Banach space. If $(X, \|\cdot\|)$ has ANP-II, the space $(L_p(\mu, X), \|\cdot\|)$ may not have ANP-II. For an example, see [HL3]. Thus we have nicer stability results for ANP-II$'$ which was lacking in ANP-II.

Theorem 5.5. *Let X be a Banach space. X has Property (V) if and only if $L^p(\mu, X)$ has (V) $(1 < p < \infty)$.*

Proof. By Corollary 5.3, X inherits Property (V) from $L^p(\mu, X)$.

Conversely, if X has Property (V), by [S, Theorem 4], X is Hahn–Banach smooth. Hence X is an Asplund space. Thus, $L^p(\mu, X)^* = L^q(\mu, X^*)$, where $1/p + 1/q = 1$. From [HL3, Theorem 6], $L^p(\mu, X)$ is Hahn–Banach smooth. Also, X^* strictly convex implies $L^q(\mu, X^*)$ is strictly convex. The result now follows from Theorem 2.2. □

REFERENCES

[B] Bandyopadhyay, P. – The Mazur Intersection Property for Families of Closed Bounded Convex Sets in Banach spaces. Colloq. Math. **LXIII**, 45–56 (1992).

[CL] Chen, D. and B.-L. Lin – Ball Separation Properties in Banach spaces. Preprint (1995).

[C] Choquet, G. – Lectures on Analysis. Vol. II, W A Benjamin, Inc. New York (1969).

[D1] Day, M.M. – Normed Linear Spaces. Springer-Verlag, Berlin (1972).

[D2] Day, M.M. – Strict Convexity And Smoothness Of Normed Spaces. Trans. Amer. Math. Soc. **78**, 516–528 (1955).

[HL1] Hu, Z. and B.-L. Lin – On the Asymptotic Norming Properties of Banach spaces. Proc. Conference on Function Spaces, SIUE: Lecture notes in Pure and Applied Maths. **136**, Marcel and Dekker, 195–210 (1992).

[HL2] Hu, Z. and B.-L. Lin – Smoothness and Asymptotic Norming Properties in Banach spaces. Bull. Austral. Math. Soc. **45**, 285–296 (1992).

[HL3] Hu, Z. and B.-L. Lin – Asymptotic Norming and Mazur's Intersection Property in Lebesgue Bochner Function Spaces. Bull. Austral. Math. Soc. **48**, 177–186 (1993).

[JH] James, R.C. and A. Ho – The Asymptotic Norming and the Radon Nikodým Properties for Banach Spaces. Ark. Mat. **19**, 53–70 (1981).

[S] Sullivan, F. – Geometric Properties determined by the higher duals of Banach Space. Illinois Journal of Math. **21**, 315–331 (1977).

[V] Vlasov, L.P. – Approximate properties of sets in Normed Linear Spaces. Russian Math. Surveys **28**, 1–66 (1973).

Received July, 1997