Structure of $\mathbb{A}^*$-Fibrations over One-Dimensional Seminormal Semilocal Domains

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In this paper we give a structure theorem for an $\mathbb{A}^*$-fibration over a one-dimensional noetherian seminormal semilocal domain and show that, in this situation, any $\mathbb{A}^*$-fibration whose spectrum occurs as an affine open subscheme of the spectrum of an $\mathbb{A}^1$-fibration (equivalently, an affine line $\mathbb{A}^1$) is actually $\mathbb{A}^*$. The structure theorem provides examples of $\mathbb{A}^*$-fibrations over one-dimensional noetherian seminormal semilocal domains whose spectra are not affine open subschemes of any affine line $\mathbb{A}^1$ over the base ring. We also construct examples of nontrivial $\mathbb{A}^*$-fibrations over one-dimensional noetherian non-seminormal local domains whose spectra are open subschemes of $\mathbb{A}^1$-fibrations over the base ring.

Key Words: $\mathbb{A}^*$-fibrations; $\mathbb{A}^1$-fibrations; seminormality; open subscheme; module of differentials.

1. INTRODUCTION

Let R be a noetherian domain. A finitely generated flat R-algebra A is said to be an $\mathbb{A}^1$-*fibration* (respectively, an $\mathbb{A}^*$-*fibration*) over R if, at each point P of Spec R, the fibre ring $k(P) \otimes_R A$ is a polynomial ring $k(P)[T]$ (respectively, a Laurent polynomial ring $k(P)[T, T^{-1}]$).

From results of Asanuma and Hamann it can be deduced easily (see [3, 3.3]) that if R is a noetherian seminormal semilocal domain, then any $\mathbb{A}^1$-fibration over R is $\mathbb{A}^1$ over R (i.e., a polynomial ring in one variable over R). From [4, 3.11], it follows that if R is a noetherian normal semilocal domain, then any $\mathbb{A}^*$-fibration over R is $\mathbb{A}^*$ over R (i.e., a Laurent polynomial ring in one variable over R). Thus, if A is an $\mathbb{A}^*$-fibration over a noetherian normal semilocal domain R, then, as $A = R[T, T^{-1}]$, Spec A is isomorphic to an open affine subscheme of Spec $R[T]$.

Now, in view of the similarity observed between $\mathbb{A}^1$-fibration and $\mathbb{A}^*$-fibration over a noetherian normal semilocal domain, one would ask:

QUESTION. *Are $\mathbb{A}^*$-fibrations over noetherian seminormal semilocal domains necessarily $\mathbb{A}^*$?*

In this paper we investigate this question. We first give an explicit description of any $\mathbb{A}^*$-fibration A over a noetherian seminormal one-dimensional semilocal domain R as a two-generated R-algebra satisfying a certain relation (see 3.4). The structure theorem provides examples of *nontrivial* $\mathbb{A}^*$-fibrations over a noetherian one-dimensional seminormal local domain thereby answering the above question in the negative (see 3.9). Moreover, we show (3.8) that an $\mathbb{A}^*$-fibration over a one-dimensional noetherian seminormal semilocal domain R is trivial if and only if its spectrum occurs as an affine open subscheme of an affine line $\mathbb{A}^1$ over Spec R. Finally we construct examples (3.10, 3.11) of *nontrivial* $\mathbb{A}^*$-fibrations over arbitrary one-dimensional noetherian local domains such that the spectra of the $\mathbb{A}^*$-fibrations occur as affine open subschemes of the spectra of $\mathbb{A}^1$-fibrations over the base rings.

In Section 2, we recall relevant definitions and results. In Section 3, we prove our main results and construct the examples.

2. PRELIMINARIES

In this section we set up the notations, define the terms used in the paper, recall a few well-known results, and prove a few lemmas and a result on retracts of $\mathbb{A}^*$-fibrations. Throughout the paper we will assume the rings to be commutative.

Notations

For a ring R, R^* denotes the multiplicative group of units of R. For a prime ideal P of R, $k(P)$ denotes the residue field R_P/PR_P. The notation

$A = R^{[n]}$ would mean that A is a polynomial ring in n variables over R. For an R-algebra A, $\Omega_{A/R}$ denotes the universal module of R-differentials of A.

Definitions

An R-algebra A is said to be $\mathbb{A}^*$ if $A = R[T, T^{-1}]$ for some invertible element T in A which is algebraically independent over R.

A finitely generated flat R-algebra A is said to be an $\mathbb{A}^*$-*fibration* if, at each point P of Spec R, the fibre ring $k(P) \otimes_R A$ is $\mathbb{A}^*$ over $k(P)$. We shall call an $\mathbb{A}^*$-fibration *nontrivial* if it is not $\mathbb{A}^*$.

A finitely generated flat R-algebra B is said to be an $\mathbb{A}^1$-*fibration* over R if $k(P) \otimes_R B = k(P)^{[1]}\ \forall P \in \operatorname{Spec} R$.

An integral domain R with quotient field K is said to be *seminormal* if it satisfies the condition: an element $t \in K$ will belong to R if $t^2, t^3 \in R$. Equivalently, R is seminormal if it satisfies the condition: for $b, c \in R$ with $b^3 = c^2$, there is an $a \in R$ such that $a^2 = b$, $a^3 = c$.

Let A be an R-algebra. R is said to be a *retract* of A if there exists an R-algebra homomorphism from A to R.

We now quote a few results which would be needed in the paper. The following result is well-known—it follows easily from [7, 1.3].

PROPOSITION 2.1. *Let R be a noetherian seminormal one-dimensional semilocal domain and R_1 a finite birational extension of R. Then the Jacobson radical of R_1 is also the Jacobson radical of R and hence contained in the conductor of R_1 in R. Therefore, if R_1 (and hence R) is local and the residue fields of R_1 and R are same, then $R_1 = R$.*

The following result is due to Chevalley (see [5, p. 222]).

THEOREM 2.2. *Let $f\colon X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X an affine scheme. Then Y is an affine scheme.*

As a consequence, one can deduce the following result.

COROLLARY 2.3. *Let $B \subseteq D$ be affine domains over a field k such that D is a finite extension of B. Suppose that P is a prime ideal in B of height one such that $\sqrt{(PD)}$ is principal. Then $U = \operatorname{Spec} B \setminus V(P)$ is an affine scheme. In fact, if $P = (f_1, \ldots, f_n)B$ and A is the ring of regular functions on U, then $A = B_{f_1} \cap \cdots \cap B_{f_n}$.*

Proof. Since D is a finite extension of B, the map $\pi\colon \operatorname{Spec} D \to \operatorname{Spec} B$ is a finite surjective morphism of noetherian separated schemes. Moreover, $\pi^{-1}(V(P)) = V(PD) = V(\sqrt{(PD)})$. Since $\sqrt{(PD)}$ is a principal ideal, say, generated by g, $\operatorname{Spec} D \setminus V(\sqrt{(PD)})$ is an affine scheme, namely, $\operatorname{Spec}(D[1/g])$. Thus, $\pi^{-1}(U) = \operatorname{Spec} D \setminus V(\sqrt{(PD)}) = \operatorname{Spec}(D[1/g])$.

Hence, the result follows from (2.2) by taking $X = \operatorname{Spec}(D[1/g])$ and $Y = U$. ∎

We now recall a few facts about $\mathbb{A}^1$-fibrations. The following result would follow easily from results of Asanuma and Hamann (see [3, 3.3]).

THEOREM 2.4. *Let R be a noetherian seminormal semilocal domain and B an $\mathbb{A}^1$-fibration over R. Then $B = R^{[1]}$.*

The next result (essentially due to Yanik ([8, 3.4])) gives a recipe for constructing $\mathbb{A}^1$-fibrations.

PROPOSITION 2.5. *Let R be a noetherian domain and let R_1 be a finite birational extension of R. Let C be an ideal in R_1 contained in R. Let $z \in R_1[X]$ be such that $(R_1/C)[\bar{z}] = (R_1/C)[X]$, where $\bar{z}$ denotes the image of z in $(R_1/C)[X]$. Let $B = R[z] + CR_1[X]$. Then B is an $\mathbb{A}^1$-fibration over R.*

Proof. From [8, 3.4], it follows that B is a retract of $R^{[n]}$ for some n. Hence B is a finitely generated flat R-algebra. Therefore, $R_1 \otimes_R B = R_1[X]$ and $B/CB = (R/C)[\bar{z}] = (R/C)^{[1]}$. Let $P \in \operatorname{Spec} R$. If P does not contain C, then $R_P = (R_1)_P$ and hence $R_P \otimes_R B = (R_1)_P[X] = R_P[X]$. Now the result follows, since $B/CB = (R/C)^{[1]}$. ∎

Below we give an example of Yanik ([8, 4.1]) of a nontrivial $\mathbb{A}^1$-fibration B over a noetherian local domain R containing the field of rationals.

EXAMPLE 2.6. Let k be a field of characteristic zero and $\widetilde{R}$ be a discrete valuation ring with k as the coefficient field. Let t be a uniformising parameter of $\widetilde{R}$ and let $R = k + t^n\widetilde{R}$ for some integer $n \geq 2$. Let $B = R[X + tX^2] + t^n\widetilde{R}[X]$. Then B is an $\mathbb{A}^1$-fibration over R and $B \neq R^{[1]}$.

We now state a few results on $\mathbb{A}^*$-fibrations. The following two results are consequences of [4, 3.11] and [4, 3.13], respectively.

THEOREM 2.7. *Let S be a noetherian normal semilocal domain and A an $\mathbb{A}^*$-fibration over S. Then A is $\mathbb{A}^*$ over R.*

THEOREM 2.8. *Let R be a noetherian semilocal domain and A an $\mathbb{A}^*$-fibration over R. Then there exists a finite birational extension R_1 of R such that $R_1 \otimes_R A$ is $\mathbb{A}^*$ over R_1.*

Arguing as in the proof of ([1, 5.1]), one can prove the following lemma.

LEMMA 2.9. *Let A be an $\mathbb{A}^*$-fibration over a noetherian domain R. Then $\Omega_{A/R}$ is a projective A-module of rank one.*

We shall conclude this section with a result on the existence of retracts of $\mathbb{A}^*$-fibrations over one-dimensional noetherian semilocal domains. We first prove a lemma.

LEMMA 2.10. *Let R be a noetherian zero-dimensional ring and let A be an $\mathbb{A}^*$-fibration over R. Then A is $\mathbb{A}^*$ over R.*

Proof. Let N denote the nilradical of R. Then R/N is a direct product of fields and hence $A/NA = (R/N)[Y, Y^{-1}]$. Let z, w be lifts of Y and Y^{-1}, respectively. Then $zw = 1 + f$, where $f \in NA$. Since NA is nilpotent, $1 + f$ is a unit in A. Let $v = (1+f)^{-1}z$. Since v, w are lifts of Y, Y^{-1}, respectively, and N is a nilpotent ideal of R, therefore, $A = R[v, w] + NA = R[v, w]$. Now as $vw = 1$ and A is R-flat, it follows that A is $\mathbb{A}^*$ over R. ∎

Note that if B is an $\mathbb{A}^1$-fibration over a noetherian domain R, then, from Asanuma's theorem ([2, 3.4]), it follows that R is a retract of B. If R is normal and semilocal and A is an $\mathbb{A}^*$-fibration over R, then, as A is $\mathbb{A}^*$ over R by (2.7), R is a retract of A. On the other hand, the result [4, 3.5] shows that $\mathbb{A}^*$-fibrations over a non-semilocal noetherian domain R need not have any retraction map to R, even when R is normal. In this connection we prove the following result.

PROPOSITION 2.11. *Let R be a noetherian one-dimensional semilocal domain and let A be an $\mathbb{A}^*$-fibration over R. Then R is a retract of A.*

Proof. By (2.8), there exists a finite birational extension R_1 of R such that $R_1 \otimes_R A = R_1[T, T^{-1}]$. Let C denote the conductor of R_1 in R and let J denote the Jacobson radical of R_1. Let $I = J \cap C$. Then I is an ideal of R. Therefore, we have the cartesian square of rings

$$\begin{array}{ccc} R & \hookrightarrow & R_1 \\ \downarrow & & \downarrow \\ R/I & \hookrightarrow & R_1/I \end{array}$$

where the vertical maps are surjective. Since A is flat over R, the above square induces the cartesian square of rings

$$\begin{array}{ccc} A & \hookrightarrow & R_1 \otimes_R A = R_1[T, T^{-1}] \\ \downarrow & & \downarrow \\ A/IA & \hookrightarrow & (R_1/I)[T, T^{-1}] \end{array}$$

By (2.10), $A/IA = (R/I)[Y, Y^{-1}]$. We identify A/IA as an R/I-subalgebra of $(R_1/I)[T, T^{-1}]$. Let z_1, z_2 be lifts of Y and Y^{-1}, respectively, in $R_1[T, T^{-1}]$. Since the second diagram is cartesian, it follows that $A = R[z_1, z_2] + IR_1[T, T^{-1}]$. Since $(R_1/I)[T, T^{-1}] = (R_1/I)[Y, Y^{-1}]$, there exists a (surjective) R_1/I-algebra homomorphism ψ: $(R_1/I)[T, T^{-1}] \twoheadrightarrow R_1/I$ such that $\psi(Y) = \psi(Y^{-1}) = 1$. Obviously, $\psi(T)$ is a unit in R_1/I. Since I is contained in the Jacobson radical of R_1, $\psi(T)$ can be lifted to a unit λ in R_1. Therefore, ψ can be lifted to a surjective R_1-algebra homomorphism Ψ: $R_1[T, T^{-1}] \to R_1$ by defining $\Psi(T) = \lambda$. From the construction of z_1, z_2, and Ψ, it follows that $1 - \Psi(z_1), 1 - \Psi(z_2) \in I$. Therefore,

$\Psi(z_1), \Psi(z_2) \in R$. Since $A = R[z_1, z_2] + IR_1[T, T^{-1}]$ and $IR_1 \subseteq R$, it follows that $\Psi(A) = R$. Thus R is a retract of A. ∎

3. MAIN RESULTS AND EXAMPLES

We shall first prove the structure theorem for an $\mathbb{A}^*$-fibration over a noetherian seminormal one-dimensional semilocal domain (3.4). We begin with a few lemmas.

LEMMA 3.1. *Let $\overline{S}$ be a reduced zero-dimensional noetherian ring. Let $\overline{D} = \overline{S}[W, W^{-1}] = \overline{S}[Z, Z^{-1}]$ for some indeterminates W and Z. Then there exists $\lambda \in (\overline{S})^*$ such that, putting $T = \lambda Z$, we have $W + W^{-1} = T + T^{-1}$ and $W = eT + (1-e)T^{-1}$ for some idempotent $e \in \overline{S}$.*

Proof. Let $M_1, \ldots, M_n$ denote the maximal ideals of $\overline{S}$ and let $L_i = \overline{S}/M_i$, $1 \le i \le n$. Then $L_i[W, W^{-1}] = L_i[Z, Z^{-1}]$ $\forall i, 1 \le i \le n$. Therefore, given any i, $1 \le i \le n$, either $W \equiv \lambda_i Z$ (mod $M_i\overline{D}$), or $W \equiv \lambda_i^{-1} Z^{-1}$ (mod $M_i\overline{D}$) for some unit $\lambda_i \in L_i$.

Suppose that, for each i, $W \equiv \lambda_i Z$ (mod $M_i\overline{D}$). Since $\overline{S} \cong L_1 \times \cdots \times L_n$, by the Chinese Remainder theorem, there exists a unit λ in $\overline{S}$ such that the image of λ in $L_i(= \overline{S}/M_i)$ is λ_i, $1 \le i \le n$. Let $T = \lambda Z$. Then $W \equiv T$ (mod $M_i\overline{D}$), for every i, $1 \le i \le n$, and hence $W = T$. Similarly, if $W \equiv \lambda_i^{-1} Z^{-1}$ (mod $M_i\overline{D}$) for every i, $1 \le i \le n$, then by choosing λ and T as before, $W = T^{-1}$. In either case, the relations between W and T hold.

Now, by reindexing if necessary, we may assume that $W \equiv \lambda_i Z$ (mod $M_i\overline{D}$) for $1 \le i \le s$ and $W \equiv \lambda_i^{-1} Z^{-1}$ (mod $M_i\overline{D}$) for $s+1 \le i \le n$ and that the ideals $J_1 = M_1 \cap \cdots \cap M_s$ and $J_2 = M_{s+1} \cap \cdots \cap M_n$ are both proper comaximal ideals with $J_1 \cap J_2 = 0$. By the previous arguments, $W \equiv \mu_1 Z$ (mod $J_1\overline{D}$) and $W \equiv \mu_2^{-1} Z^{-1}$ (mod $J_2\overline{D}$), where μ_1, μ_2 are units in $\overline{S}/J_1$ and $\overline{S}/J_2$, respectively. Now, using the Chinese Remainder theorem, choose a unit λ in $\overline{S}$ such that its images in $\overline{S}/J_1$ and $\overline{S}/J_2$ are μ_1 and μ_2, respectively. Put $T = \lambda Z$. Then $W \equiv T$ (mod $J_1\overline{D}$) and $W \equiv T^{-1}$ (mod $J_2\overline{D}$). Thus, $W + W^{-1} = T + T^{-1}$ as $J_1 \cap J_2 = 0$. Since $\overline{S} \cong \overline{S}/J_1 \times \overline{S}/J_2$, there exists an idempotent $e \in \overline{S}$ whose image is 1 in $\overline{S}/J_1$ and 0 in $\overline{S}/J_2$. Then clearly $W = eT + (1-e)T^{-1}$. Hence the result. ∎

LEMMA 3.2. *Let S be a noetherian semilocal domain and let J be the Jacobson radical of S. Let $D = S[Z, Z^{-1}]$. Suppose that $D/JD(= (S/J)[Z, Z^{-1}]) = (S/J)[W, W^{-1}]$. Then there exist elements $z_1, z_2 \in D$ such that*

(i) $\overline{z}_1 = W + W^{-1}$ *and* $\overline{z}_2 = W$ *where bar denotes reduction modulo JD.*

(ii) $D = S[z_1, z_2]$.

(iii) $z_2^2 - z_2z_1 - b(b-1)z_1^2 + (2b-1)^2 = 0$ *for some* $b \in S$ *such that* $b(b-1) \in J$.

Proof. Let $\overline{S} = S/J$. Then, by (3.1), there exists $\overline{\lambda} \in (\overline{S})^*$ and an idempotent $e \in \overline{S}$ such that

$$W + W^{-1} = (\overline{\lambda}Z) + (\overline{\lambda}Z)^{-1} \quad \text{and}$$
$$W = e(\overline{\lambda}Z) + (1-e)(\overline{\lambda}Z)^{-1} \quad \text{in} \quad D/JD.$$

Let λ and b be lifts of $\overline{\lambda}$ and e, respectively. As J is the Jacobson radical of S, we have $\lambda \in S^*$. Since e is idempotent, $b(b-1) \in J$ and hence $(2b-1)^2 = 1 + 4b(b-1) \in S^*$, i.e., $2b-1 \in S^*$. Put

$$T = \lambda Z, z_1 = T + T^{-1}, z_2 = bT + (1-b)T^{-1}.$$

Then, as $\lambda, 2b-1 \in S^*$, clearly $D(= S[Z, Z^{-1}] = S[T, T^{-1}]) = S[z_1, z_2]$ and conditions (i) and (ii) hold. A routine verification shows that (iii) holds as well. ∎

LEMMA 3.3. *Let R be a noetherian seminormal one-dimensional semilocal domain and S the normalisation of R. Then S is a finite module over R.*

Proof. Clearly, it is enough to prove the result when R is local. So we assume that R is non-normal and local with maximal ideal M. We show that $MS \subset R$. Let $\alpha \in S \setminus R$. Since $R[\alpha]$ is a finite birational extension of the one-dimensional seminormal local domain R, from (2.1), it follows that M is the conductor of $R[\alpha]$ in R. Thus $\alpha M \subset R$ $\forall \alpha \in S$, i.e., $MS \subset R$. Since $M \neq 0$ and R is noetherian, it follows that S is a finite R-module. ∎

We now prove the structure theorem.

THEOREM 3.4. *Let R be a noetherian seminormal one-dimensional semilocal domain with Jacobson radical J and quotient field K. Then the* A*-*fibrations over R are precisely the algebras of the type $R[X, Y]/(Y^2 - YX - aX^2 - \lambda)$, where $R[X, Y] = R^{[2]}$, λ is a unit in R, and a is an element of J for which there exists $b \in K$ such that $b(b-1) = a$. Moreover, such an algebra would be* A* *if and only if $b \in R$.*

Proof. We first show that if $A' = R[X, Y]/(Y^2 - YX - aX^2 - \lambda)$ for some $\lambda \in R^*$ and $a \in J$ for which there exists $b \in K$ such that $b(b-1) = a$, then A' is an A*-fibration over R. Let $\overline{X}$ and $\overline{Y}$ denote the images of X and Y, respectively, in A'. A' is a finitely generated R-algebra and, being a free module over $R[\overline{X}]$ $(= R^{[1]})$, is R-flat. We now show that all the fibres of A' are A*. Each closed fibre $k(P) \otimes_R A'$ is of the form $k(P)[X, Y]/(Y(Y - X) - \lambda_P)$, where λ_P is the image of λ in $k(P)$. Since $\lambda_P \neq 0$, it follows that $k(P) \otimes_R A' \cong k(P)[Y, Y^{-1}]$.

Let S be the normalisation of R. We now show that $S \otimes_R A'$ is $\mathbb{A}^*$ over S. Note that $b \in S$ and $(2b-1)^2 = 4a+1 \in R^*$ as $a \in J$, so that $2b-1 \in S^*$. Now let

$$U = Y - bX, V = Y + (b-1)X.$$

Then $S[U, V] = S[X, Y]$ since $2b - 1 \in S^*$ and $UV = Y^2 - YX - aX^2$ so that

$$\begin{aligned} S \otimes_R A' &= S[X, Y]/(Y^2 - YX - aX^2 - \lambda) \\ &= S[U, V]/(UV - \lambda) \cong S[U, U^{-1}]. \end{aligned}$$

In particular, the generic fibre $K \otimes_R A'$ is $\mathbb{A}^*$ over K. Thus, A' is an $\mathbb{A}^*$-fibration over R.

Now we show that A' is $\mathbb{A}^*$ over R if and only if $b \in R$. The "if" part would follow from the preceding argument, since, if $b \in R$, then $R[U, V] = R[X, Y]$ so that $A' = R[U, V]/(UV - \lambda) \cong R[U, U^{-1}]$.

So now assume that $A'(= R[X, Y]/(Y^2 - YX - aX^2 - \lambda)) = R[T, T^{-1}]$, where a and λ are as above. We show that $b \in R$. We have already seen that $S \otimes_R A' = S[U, U^{-1}]$, where $U = \overline{Y} - b\overline{X}$. Therefore, as $S[T, T^{-1}] = S \otimes_R A' = S[U, U^{-1}]$, we have $T = \alpha U$ or $T = \alpha U^{-1}$ for some $\alpha \in S^*$. Without loss of generality, we may assume that $T = \alpha U$. Since A' is a free module over $R[\overline{X}]$ $(= R^{[1]})$ with basis 1 and $\overline{Y}$, we have

$$T = f(\overline{X}) + g(\overline{X})\overline{Y} \quad \text{where} \quad f, g \in R^{[1]}.$$

On the other hand, we have seen that

$$T = \alpha U = -\alpha b\overline{X} + \alpha\overline{Y}.$$

Comparing, we get $\alpha b \in R$ and $\alpha \in R$. Since α is a unit in S and S is integral over R, it follows that $\alpha \in R^*$. Hence $b \in R$.

We now prove the converse part of the theorem. Let A be an $\mathbb{A}^*$-fibration over R. We show that A has the structure described above.

Recall that, by (3.3), the normalisation S of R is a finite R-module. Moreover, by (2.1), J (the Jacobson radical of R) is also the Jacobson radical of S. Hence we have a cartesian square

$$\begin{array}{ccc} R & \hookrightarrow & S \\ \downarrow & & \downarrow \\ R/J & \hookrightarrow & S/J \end{array}$$

where the vertical maps are surjective.

Let $D = S \otimes_R A$. Since A is R-flat, we may regard A to be a subring of D by identifying A with its image under the map $x \to 1 \otimes x$. Since D is an $\mathbb{A}^*$-fibration over the semilocal PID S, by (2.7), we have $D = S[Z, Z^{-1}]$ for some indeterminate Z. Since A/JA is an $\mathbb{A}^*$-fibration over R/J and

R/J is a direct product of fields, we have $A/JA = (R/J)[W, W^{-1}]$ for some indeterminate W. Again, since $R/J \hookrightarrow S/J$ is injective and A is R-flat, we regard A/JA as a subring of D/JD. Since $D/JD = S \otimes_R (A/JA)$, we have $D/JD = (S/J)[W, W^{-1}] = (S/J)[Z, Z^{-1}]$. Therefore, by (3.2), there exist $z_1, z_2 \in D$, which are lifts of $W + W^{-1}$ and W, respectively, such that $D = S[z_1, z_2]$ and $z_2^2 - z_2 z_1 - b(b-1)z_1^2 + (2b-1)^2 = 0$ for some $b \in S$ satisfying $b(b-1) \in J$. Now since A is R-flat, the cartesian square above induces the cartesian square

$$\begin{array}{ccccl} & A & \hookrightarrow & D & (= S[Z, Z^{-1}]) \\ & \downarrow & & \downarrow & \\ ((R/J)[W, W^{-1}] =) & A/JA & \hookrightarrow & D/JD & (= (S/J)[Z, Z^{-1}] = (S/J)[W, W^{-1}]), \end{array}$$

where the vertical maps are surjective. Therefore, as z_1, z_2 are lifts of a system of generators of A/JA, and since J is contained in the conductor of S in R, it follows that

$$A = R[z_1, z_2] + JD = R[z_1, z_2] + JS[z_1, z_2] = R[z_1, z_2].$$

Let $a = b(b-1)$ and $\lambda = -(2b-1)^2$. Then $a \in J$ and $\lambda = -1 - 4a \in R^*$. Let $R[X, Y] = R^{[2]}$ and $A' = R[X, Y]/(Y^2 - YX - aX^2 - \lambda)$. Clearly the R-algebra homomorphism $R[X, Y] \to A$, defined by $X \to z_1$, $Y \to z_2$, induces a surjective R-algebra homomorphism ϕ: $A' \twoheadrightarrow A$. We have shown earlier that A' is an A*-fibration over R, in particular, A' is an integral domain and dim $A' = 2 =$ dim A. Therefore, ϕ is an isomorphism, i.e., $A \cong A'$. Hence the result. ∎

COROLLARY 3.5. *Let R be a noetherian seminormal one-dimensional semilocal domain with quotient field K and residue field k such that $1/2 \in R$. Then the A*-fibrations over R are precisely the algebras of the type $R[X, Y]/(Y^2 - \mu X^2 - \lambda)$, where $R[X, Y] = R^{[2]}$, $\lambda \in R^*$, and μ is an element of $R^* \cap (K^*)^2$ such that the image of μ in k is in $(k^*)^2$. Moreover, such an algebra would be A* over R if and only if $\mu \in (R^*)^2$.*

Proof. By (3.4), if A is an A*-fibration, then $A \cong R[U, V]/(V^2 - VU - aU^2 - \lambda)$, where $R[U, V] = R^{[2]}$, $\lambda \in R^*$, and $a \in J$ for which there exists $b \in K$ such that $b(b-1) = a$. Putting $Y = V - U/2$, $X = U$, and $\mu = (2b-1)^2/4 = a + 1/4$, we see that A has the desired form. Similarly the converse. ∎

COROLLARY 3.6. *Let A be an A*-fibration over a noetherian seminormal one-dimensional semilocal domain R. Then $\Omega_{A/R}$ is a free A-module of rank one.*

Proof. By (2.9), $\Omega_{A/R}$ is a finitely generated projective module of rank one. Since, by (3.4), $A = R[X, Y]/(F)$ (where $R[X, Y] = R^{[2]}$), $\Omega_{A/R}$ is stably free of rank one and hence free. ∎

We shall next show (3.8) that the only $\mathbb{A}^*$-fibration over a one-dimensional seminormal semilocal domain R, whose spectrum occurs as an affine open subscheme of Spec($R^{[1]}$), is the trivial $\mathbb{A}^*$-fibration. We first prove a technical result below.

PROPOSITION 3.7. *Let R be a one-dimensional noetherian local domain with residue field k and A an $\mathbb{A}^*$-fibration over R. Suppose that there exists an R-subalgebra B of A such that B is an $\mathbb{A}^1$-fibration over R and* Spec $A =$ Spec $B \setminus V(P)$, *where P is a radical ideal of B. Then*

(i) *P is a prime ideal of B of height one.*

(ii) *$P \cap R = 0$ and B/P is a finite birational extension of R.*

(iii) *B/P is local with k as the residue field.*

Moreover, A is $\mathbb{A}^*$ over R if and only if $B = R^{[1]}$ and $B/P = R$.

Proof. We first show that P is a prime ideal of B of height one. Since R is a one-dimensional domain, and B is an $\mathbb{A}^1$-fibration over R, it is easy to see that B is a Cohen–Macaulay domain. Hence, as Spec $B \setminus V(P)$ is an affine scheme, it follows that all the minimal prime ideals of P are of height one.

Let $\widetilde{R}$ denote the normalisation of R. Let $\widetilde{B} = \widetilde{R} \otimes_R B$ and $\widetilde{A} = \widetilde{R} \otimes_R A$. From the Krull–Akizuki theorem [6, 11.7, p. 84] and the fact that R is local, it follows that $\widetilde{R}$ is a semilocal PID. Therefore, by (2.4), $\widetilde{B}(= \widetilde{R} \otimes_R B) = \widetilde{R}[Z]$ and, by (2.7), $\widetilde{A}(= \widetilde{R} \otimes_R A) = \widetilde{R}[T, T^{-1}]$ for some indeterminates Z and T.

Since Spec $\widetilde{A} =$ Spec $\widetilde{B} \setminus V(P\widetilde{B})$, $\widetilde{A} = \widetilde{R}[T, T^{-1}]$ and $\widetilde{B}$ is a UFD, we see that the radical of $P\widetilde{B}$ is a principal prime ideal of $\widetilde{B}$, say, generated by f. It follows that $P = f\widetilde{B} \cap B$ and hence P is a prime ideal of height one.

Since f becomes a unit in $\widetilde{A}$, $f = \lambda T^m$ for some $\lambda \in (\widetilde{R})^*$ and for some non-zero integer m. Without loss of generality we may assume that $m > 0$. Then T is integral over $\widetilde{B}$ and hence $T \in \widetilde{B}(= \widetilde{R}[Z])$. Now it is easy to see that $\widetilde{R}[Z] = \widetilde{R}[T]$ and $T\widetilde{B} \cap B = P$. Therefore, $P \cap R = 0$ and we have $R \hookrightarrow B/P \hookrightarrow \widetilde{R}$. Thus, B/P is a finite birational extension of R.

Let M be the maximal ideal of R. Note that Spec$(A/MA) =$ Spec$(B/MB) \setminus V((P + MB)/MB)$. Since $B/MB = k[W]$ and $A/MA = k[Y, Y^{-1}]$, arguing as before, we may assume that $W = Y$ and the radical of $(P + MB)/MB$ in B/MB is generated by W. This shows that, since B/P is a finite extension of R, B/P is a local ring with residue field k.

Now assume that $B = R[X]$ and $B/P = R$. Then, without loss of generality, we may assume that $X \in P$. Since P is a prime ideal of height one, $P = XR[X]$ and hence $A = R[X, X^{-1}]$.

Now assume that $A = R[X, X^{-1}]$. Since B is Cohen–Macaulay and P is a prime ideal of B of height one, it follows that $B = B_P \cap A$. Let K

denote the quotient field of R. Since $\widetilde{B}(= \widetilde{R} \otimes_R B) = \widetilde{R}[T]$, $T\widetilde{B} \cap B = P$, and $P \cap R = 0$, it follows that $B_P = K[T]_{(T)}$, a discrete valuation ring. Therefore, either $X \in B_P$ or $X^{-1} \in B_P$. Without loss of generality, we may assume that $X \in B_P$ and hence in B. Thus $R[X] \subseteq B \subseteq R[X, X^{-1}]$. In particular, $X \in \widetilde{B}$. Since $\widetilde{R}[X, X^{-1}] = \widetilde{A} = \widetilde{R} \otimes_R A = \widetilde{R}[T, T^{-1}]$, it follows that $X \in T\widetilde{B} \cap B = P$. As $P \cap R = 0$, we have $P \cap R[X] = XR[X]$. Thus, the local ring B_P dominates the (birational) local ring $R[X]_{(X)}$ which is a discrete valuation ring. Therefore, $B_P = R[X]_{(X)}$ and hence $B = B_P \cap A = R[X]_{(X)} \cap R[X, X^{-1}] = R[X]$. As $X \in P$ and P is a prime ideal of height one, $P = XR[X]$. Thus $B/P = R$. ∎

Theorem 3.8. *Let R be a noetherian seminormal one-dimensional semilocal domain and A an $\mathbb{A}^*$-fibration over R. Then A is $\mathbb{A}^*$ over R if and only if* Spec A *is an open subscheme of an $\mathbb{A}^1$-fibration over R (or equivalently, Spec A is an open subscheme of* Spec$(R^{[1]})$.)

Proof. The "only if" part follows trivially. For the "if" part, by [4, 3.3], we may assume that R is local with residue field k, say. Let B be an $\mathbb{A}^1$-fibration over R such that Spec $A =$ Spec $B \setminus V(P)$ for some radical ideal P of B. In view of (3.7), it is enough to show that $B = R^{[1]}$ and $B/P = R$. Since R is seminormal local, by (2.4), $B = R^{[1]}$. By (3.7), P is a prime ideal of B of height one, B/P is a local domain with residue field k and B/P is a finite birational extension of R. Therefore, by (2.1), $B/P = R$. ∎

We now use the structure theorem (3.4) to construct an explicit example of a nontrivial $\mathbb{A}^*$-fibration over a noetherian seminormal one-dimensional local domain.

Example 3.9. Let k be a field and let $\widetilde{R}$ be a semilocal noetherian normal domain of dimension one with precisely two maximal ideals M_1 and M_2 such that $\widetilde{R}/M_1 = \widetilde{R}/M_2 = k$ and $k \hookrightarrow \widetilde{R}$. (For instance, we may take $\widetilde{R} = S^{-1}k[t]$, where k is a field and $S = k[t] \setminus (I_1 \cup I_2)$, where $I_1 = tk[t]$ and $I_2 = (t-1)k[t]$.) Let $J = M_1 \cap M_2$ and let $R = k + J$. Then J is the conductor ideal of $\widetilde{R}$ in R, $R/J = k$, and $\widetilde{R}/J(= k \oplus k)$ is a finite module over $R/J(= k)$. Therefore, $\widetilde{R}$ is a finite module over R. Hence, as $\widetilde{R}$ is noetherian, by the Eakin–Nagata theorem [6, 3.7, p. 18], R is noetherian. Now it is easy to see that R is a local domain with maximal ideal J and residue field $R/J = k$. Moreover, $\widetilde{R}$ is the normalisation of R and R is seminormal in $\widetilde{R}$. Since $M_1 + M_2 = \widetilde{R}$, there exists $b \in M_1$ such that $1 - b \in M_2$. Let $a = b(b-1)$. Then $b \notin R$ but $b(b-1) = a \in J \subset R$. Now let $A = R[X, Y]/(Y^2 - YX - aX^2 - 1)$. Then, by (3.4), A is a *nontrivial* $\mathbb{A}^*$-fibration over R and hence by (3.8), Spec A is not an open subscheme of any affine line over R. Note that, by (3.6), $\Omega_{A/R}$ is free.

In the following examples (3.10, 3.11), R will be a noetherian one-dimensional (non-seminormal) local domain, A a nontrivial $\mathbb{A}^*$-fibration over R, and B an $\mathbb{A}^1$-fibration over R such that $\operatorname{Spec} A = \operatorname{Spec} B \setminus V(P)$ for some $P \in \operatorname{Spec} B$. Note that, by (3.7), either $B/P \neq R$ or $B \neq R^{[1]}$. In (3.10), $B = R^{[1]}$ whereas in (3.11), $B/P = R$.

Example 3.10. Let k be a field, $\widetilde{S} = k[t]$, and $S = k[t^2, t^3]$. Consider the S-algebra surjection ϕ: $S[W] \twoheadrightarrow k[t]$ defined by $\phi(W) = t$. Let $Q = \ker(\phi)$. Then $Q = (W^2 - t^2, t^2W - t^3, t^3W - t^4)$ and $\sqrt{(Q\widetilde{S}[W])} = (W - t)$. Let $R = S_M$ where M is the maximal ideal $(t^2, t^3)S$. Let $B = R[W]$ and $P = QB$. Then $B/P = k[t]_{(t)}$. Now $\operatorname{Spec} S[W] \setminus V(Q)$ is an affine scheme by (2.3) so that $\operatorname{Spec} R[W] \setminus V(P)$ is also an affine scheme, say, $\operatorname{Spec} A$. Now we show that A is an $\mathbb{A}^*$-fibration over R. Obviously, A is a finitely generated flat algebra over R. Moreover, $\operatorname{Spec}(K \otimes_R A) = \operatorname{Spec}(K[W]) \setminus V(W - t)$, where $K = k(t)$ is the quotient field of R. Thus the generic fibre $K \otimes_R A = K[W, (W - t)^{-1}]$ is $\mathbb{A}^*$ over K. Since $MS[W] + P = MS[W] + (W^2)$, we have $k \otimes_R A = k[W, W^{-1}]$. Thus A is an $\mathbb{A}^*$-fibration over R. Since $B/P \neq R$, by (3.7), A is a *nontrivial* $\mathbb{A}^*$-fibration. Note that, since $\operatorname{Spec} A$ is an affine open subscheme of $\operatorname{Spec}(R^{[1]})$, $\Omega_{A/R}$ is free.

Example 3.11. Let k be a field of characteristic zero, $\widetilde{S} = k[t]$, and $S = k + t^n k[t]$, where $n \geq 2$. The conductor of $\widetilde{S}$ in S is the maximal ideal M of S given by $M = t^n k[t]$. Let $z = X + tX^2$ and y the image of z in $(\widetilde{S}/M)[X]$. Then $(\widetilde{S}/M)[y] = (\widetilde{S}/M)[X]$. Let $D = S[z] + M\widetilde{S}[X]$. By (2.5), D is an $\mathbb{A}^1$-fibration over S. It is easy to see that there exists $\phi \in \widetilde{S}^{[1]}$ such that $\phi(0) = 0$ and $X \equiv \phi(z)$ (modulo $M\widetilde{S}[X]$). Let $w = X - \phi(z)$. Then $w \in M\widetilde{S}[X] \subseteq D$ and $\widetilde{S}[z, w] = \widetilde{S}[X]$. It follows that $D = S[z, w]$ and $w \in MD$. Let $P = (z, w)D$. Then $D/P = S$ and hence $P \in \operatorname{Spec} D$. Moreover, $P\widetilde{S}[X] = X\widetilde{S}[X]$ so that, by (2.3), the open subscheme $\operatorname{Spec} D \setminus V(P)$ is an affine scheme. Now let $R = S_M$ and $B = R \otimes_S D$. Then B is an $\mathbb{A}^1$-fibration over the noetherian one-dimensional local domain R and $\operatorname{Spec} B \setminus V(PB)$ is an affine open subscheme of $\operatorname{Spec} B$, say, $\operatorname{Spec} A$. We show that A is an $\mathbb{A}^*$-fibration over R. Clearly, A is finitely generated and flat over R. Let $\widetilde{R}$ denote the normalisation of R and C the conductor of $\widetilde{R}$ in R. Then C is the maximal ideal of R and $\widetilde{R}/C = k[t]/(t^n)$. Let ϵ denote the image of t in $\widetilde{R}/C$. Note that $\widetilde{R} \otimes_R B = \widetilde{R}[X]$ and $\widetilde{R} \otimes_R A = \widetilde{R}[X, X^{-1}]$. In particular, if K denotes the quotient field $k(t)$ of R, then $K \otimes_R A = K[X, X^{-1}]$. Since $B/CB = k[y]$ and $P + CB = zB + CB$, we have $A/CA = k[y, y^{-1}]$. Since $\widetilde{R} \otimes_R A$ and $R/C \otimes_R A$ are both $\mathbb{A}^*$ over $\widetilde{R}$ and R/C, respectively, it follows that A is an $\mathbb{A}^*$-fibration over R. As in (2.6), $B \neq R^{[1]}$. Hence, by (3.7), A is not $\mathbb{A}^*$ over R.

Note that, as $B \neq R^{[1]}$, by results of Asanuma and Hamann (see [3, 3.4]), $\Omega_{B/R}$ is not free. We now show that $\Omega_{A/R}$ is free if and only if

$n = 2$. The cartesian square of rings (with the vertical maps surjective)

$$\begin{array}{ccc} A & \hookrightarrow & \widetilde{R} \otimes_R A \quad (= \widetilde{R}[X, X^{-1}]) \\ \downarrow & & \downarrow \\ (k[y, y^{-1}] =)\ A/CA & \hookrightarrow & \widetilde{R}/C \otimes_R A\ (= (\widetilde{R}/C)[X, X^{-1}] = (\widetilde{R}/C)[y, y^{-1}]) \end{array}$$

induces the cartesian square of modules,

$$\begin{array}{ccc} \Omega_{A/R} & \hookrightarrow & \widetilde{R}[X, X^{-1}] \otimes_A \Omega_{A/R}\ (= \Omega_{(\widetilde{R}[X, X^{-1}])/\widetilde{R}}) \\ \downarrow & & \downarrow \\ (\Omega_{(k[y, y^{-1}])/k} =)\ \Omega_{A/R}/C\Omega_{A/R} & \hookrightarrow & \Omega_{((\widetilde{R}/C)[X, X^{-1}])/(\widetilde{R}/C)} \end{array}$$

Since $\Omega_{(\widetilde{R}[X, X^{-1}])/\widetilde{R}}$ and $\Omega_{(k[y, y^{-1}])/k}$ are free modules with bases dX and dy, respectively, and $dy = (1 + 2\epsilon X)dX$, $\Omega_{A/R}$ is free if and only if there exist $f \in (k[y, y^{-1}])^*$ and $g \in (\widetilde{R}[X, X^{-1}])^*$ such that $f\overline{g} = 1 + 2\epsilon X$, where $\overline{g}$ denotes the image of g in $(\widetilde{R}/C)[X, X^{-1}]$.

If $n = 2$, then taking $f = y^2$ and $g = X^{-2}$, we see that $\Omega_{A/R}$ is free.

If $n \geq 3$, we show that $1 + 2\epsilon X$ cannot be split as above. Suppose that $f\overline{g} = 1 + 2\epsilon X$, where f, g are as before. Then $f = \lambda y^l$ and $g = \mu X^m$ for some $\lambda \in k^*$, $\mu \in \widetilde{R}^*$, and l, m are integers. Since the images of y and X are same in $(\widetilde{R}/t\widetilde{R})[X, X^{-1}]$, it follows that $l + m = 0$ and the image of μ in $\widetilde{R}/t\widetilde{R} (= k)$ is λ^{-1}. Thus $1 + 2\epsilon X = \mu\lambda(y/X)^l = \mu\lambda(1 + \epsilon X)^l$ in $(\widetilde{R}/C)[X, X^{-1}]$ and hence in $(\widetilde{R}/C)[X]$. One can easily check that this is not possible as $\epsilon^2 \neq 0$ and characteristic of k is 0. Thus $\Omega_{A/R}$ is not free.

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