Mean of ratios or ratio of means or both? ☆

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Abstract

Consider a finite population of units $(U_1, U_2, \ldots, U_N)$. On each unit U_i, variates of interest y and x are defined taking values Y_i and X_i, respectively, $i = 1, 2, \ldots, N$. In certain surveys, it is of interest to estimate the population ratio $R = Y/X$ (or equivalently, $\bar{Y}/\bar{X}$), where $Y = \sum_1^N Y_i$ and $X = \sum_1^N X_i$, based on a sample of size n selected according to a sampling design $p(s)$. Under simple random sampling scheme, the usual choices for the estimation of R are (i) a (single) ratio of sample means given by $\hat{R}_1 = \bar{y}/\bar{x}$ or (ii) the mean of (n) ratios, viz. $\hat{R}_n = \sum_1^n (y_i/x_i)/n$. It is well known that both $\hat{R}_1$ and $\hat{R}_n$ are biased for R. Using the extent of biases, we shall first discuss the role of $\hat{R}_1$ and $\hat{R}_n$ in the construction of unbiased ratio estimators. When y is considered as the study variate and x is an auxiliary variate related to y, the problem of estimation of the population mean $\bar{Y}$ or the population total Y is dealt by constructing $\hat{\bar{Y}} = \hat{R}\bar{X}$ or $\hat{Y} = \hat{R}X$.

For the estimation of the population total Y, we shall consider a class of Symmetrized Des Raj (SDR) strategies and look for a choice of a model-optimum estimator when design-unbiasedness is not demanded, among those utilising 'mean of ratios' and 'ratio of means'.

1. Introduction

Consider a finite population $\mathscr{U} = (U_1, U_2, \ldots, U_N)$ of size N. Two variables of interest y and x are defined taking values Y_i and X_i on U_i, respectively, $i = 1, 2, \ldots, N$. We are interested in estimating the population ratio $R = \sum_{i=1}^N Y_i / \sum_{i=1}^N X_i = Y/X$ (or equivalently $\bar{Y}/\bar{X}$) based on a sampling design $p(s)$.

Under a simple random sampling (srs) design, it is customary to use the estimators $\hat{R}_1 = \bar{y}/\bar{x}$, a single ratio of means or $\hat{R}_n = \sum_{i=1}^n (y_i/x_i)/n$, mean of n ratios as an estimate of R. It is well known that for srs design, both $\hat{R}_1$ and $\hat{R}_n$ are (design) biased for R. These estimators could also be interpreted as 'weighted averages of the ratios'

y_i/x_i, viz., $\sum_{i=1}^{n} w_i(y_i/x_i)$ and when $w_i = x_i/\sum_{i=1}^{n} x_i$ we have $\hat{R}_1$ and when $w_i = 1/n$, we get $\hat{R}_n$.

When y is the study variable and x is taken as an auxiliary variable related to y, one can use the ratio method of estimation to obtain the more efficient estimators of the population mean as $\hat{\bar{Y}}_1 = (\bar{y}/\bar{x})\bar{X}$ and $\hat{\bar{Y}}_n = (\Sigma y_i/x_i)\bar{X}/n$.

Both $\hat{\bar{Y}}_1$ and $\hat{\bar{Y}}_n$ are design (srs)-biased. However, when the relationship between y and x is expressed as a model for which $\exp(Y_i \mid X_i) = \beta X_i$, say, then both $\hat{\bar{Y}}_1$ and $\hat{\bar{Y}}_n$ are (model)-unbiased.

2. Role of $\hat{R}_1$ and $\hat{R}_n$ in the construction of unbiased ratio estimators

When $\hat{\bar{Y}}_1$ and $\hat{\bar{Y}}_n$ are design biased, several unbiased (or almost-unbiased) ratio estimators are suggested in the literature. We shall first briefly review some of these unbiased estimators and highlight the role of $\hat{R}_1$ and $\hat{R}_n$ in their construction and then suggest new classes of unbiased ratio estimators involving either $\hat{R}_n$ or $\hat{R}_1$.

Consider

$$\hat{\bar{Y}} = \Theta \frac{\bar{y}}{\bar{x}} \bar{X} + (1 - \Theta) \frac{\bar{X}}{n} \sum_{1}^{n} \frac{y_i}{x_i} \tag{2.1}$$

for the estimation of the population mean $\bar{Y}$. The estimator $\hat{\bar{Y}}$ in (2.1) is unbiased for $\bar{Y}$ if

$$\Theta = B_n/(B_n - B_1), \tag{2.2}$$

where $B_1 = \text{Bias}(\hat{\bar{Y}}_1)$ and $B_n = \text{Bias}(\hat{\bar{Y}}_n)$.

Murthy and Nanjamma (1959) demonstrated that $B_n \simeq nB_1$ based on n interpenetrating subsamples. Thus Θ of (2.2) reduces to $\Theta_{\text{MN}} = n/(n-1)$ and the resulting (almost) unbiased ratio estimator is given by

$$\hat{\bar{Y}}_{\text{MN}} = \Theta_{\text{MN}} \hat{R}_1 \bar{X} + (1 - \Theta_{\text{MN}}) \hat{R}_n \bar{X}, \quad \text{with } \Theta = n/(n-1). \tag{2.3}$$

It is also known (see, for example, Rao (1981)) that Hartley and Ross (1954) estimator could be written as

$$\hat{\bar{Y}}_{\text{HR}} = \Theta_{\text{HR}} \hat{R}_1 \bar{X} + (1 - \Theta_{\text{HR}}) \hat{R}_n \bar{X} \tag{2.4}$$

where $\Theta_{\text{HR}} = (n/n-1)\frac{N-1}{N}\frac{\bar{x}}{\bar{X}}$ and the estimator due to Nieto de Pascual (1961) could be expressed as

$$\hat{\bar{Y}}_p = \Theta_p \hat{R}_1 \bar{X} + (1 - \Theta_p) \hat{R}_n \bar{X} \quad \text{with } \Theta_p = -(N-1)/(n-1)N \frac{\bar{x}}{\bar{X}}. \tag{2.5}$$

Note that Θ_{MN} the factor combining $\hat{R}_1 \bar{X}$ and $\hat{R}_n \bar{X}$ is a constant but Θ_{HR} and Θ_p are random variables with the property that $\Theta_{\text{MN}} > 1$ while $E(\Theta_{\text{HR}}) > 1$ and $E(\Theta_p) < 0$. Thus, Murthy–Nanjamma type unbiased ratio estimators are non-convex combinations of $\hat{R}_1 \bar{X}$ and $\hat{R}_n \bar{X}$ while Hartley–Ross, Pascual type unbiased ratio estimators are stochastic non-convex combinations of $\hat{R}_1 \bar{X}$ and $\hat{R}_n \bar{X}$ (i.e., $\lambda \hat{R}_1 \bar{X} + (1-\lambda) \hat{R}_n \bar{X}$ with λ such that $E(\lambda) \notin (0, 1)$, see Rao (1981) for details.).

Next consider the estimator $\hat{R}_n\bar{X}$ and let

$$T(s) = [(N-1)n\bar{x} - N(n-1)\bar{X}]R_n\bar{X}, \quad \text{for } T(s) > 0. \tag{2.6}$$

Consider the strategies

H_1: (srswor, $e_1 = T(s)/(N-n)\bar{X}$)
H_2: (Midzuno–Sen (MS) scheme, $e_2 = T(s)/(N-n)\bar{x}$)
and H_3: (Deshpande, 1982 scheme, $e_3 = T(s)/(N\bar{X} - n\bar{x})$).

Deshpande and Prabhu Ajgaonkar, 1993 (DP, 1993) have shown that all the above are (design) unbiased for $\bar{Y}$.

We now observe that e_1 can be written as

$$e_1 = \left(\frac{(N-1)n\,\bar{x}}{N(n-1)\,\bar{X}} - 1\right)\left(\frac{N(n-1)\,\bar{X}}{(N-n)\,\bar{X}}\right)\hat{R}_n\bar{X} = \left(\frac{\Theta_{\mathrm{HR}} - 1}{E(\Theta_{\mathrm{HR}}) - 1}\right)\hat{R}_n\bar{X}.$$

Motivated by this, we consider in general

$$\hat{\hat{Y}} = \left(\frac{1 - \Theta(s)}{1 - E(\Theta(s))}\right)\hat{R}_n\bar{X} \tag{2.7}$$

whenever the multiplier is positive.

For $\hat{\hat{Y}}$ to be unbiased for $\bar{Y}$, we should have

$$E(\hat{\hat{Y}}) = E\left(\frac{1 - \Theta(s)}{1 - E(\Theta(s))}\frac{\bar{X}}{n}\sum_1^n \frac{y_i}{x_i}\right) = \bar{Y}$$

or

$$E\left((1 - \Theta(s))\sum_1^n \frac{y_i}{x_i}\right) = \frac{n\bar{Y}}{\bar{X}}(1 - E(\Theta(s)))$$

or

$$\sum_1^N Y_i\left\{\frac{\pi_i}{X_i} - \sum_{s\ni i}\frac{\Theta(s)\,p(s)}{X_i}\right\} = \frac{n}{X}\sum_1^N Y_i(1 - E(\Theta(s))) \quad \forall \underset{\sim}{Y}$$

which implies the condition

$$\frac{1}{X_i}\sum_{s\ni i} p(s)(1 - \Theta(s)) = \frac{n}{X}(1 - E(\Theta(s)))$$

or

$$\sum_{s\ni i} p(s)\lambda(s) = \pi_\lambda = nX_i/X, \tag{2.8}$$

where $\lambda(s) = (1 - \Theta(s))/(1 - E(\Theta(s)))$ and π_λ is the sample weight function as defined by Srivastava (1985). Note that $E\lambda(s) = 1$ which is outside the open interval $(0, 1)$.

Table 1 gives the estimators e_i of DP (1993) as special cases whenever they are defined.

Table 1
Special cases of unbiased ratio estimators

$\Theta(s)$	$p(s)$	$E(\Theta(s))$	$\hat{\bar{Y}}$
c	πPS	c	$\hat{R}_n\bar{X}$
Θ_{HR}	$1/\binom{N}{n}$	$\frac{(N-1)n}{N(n-1)}$	e_1
$\frac{N(n-1)}{n(N-1)}\frac{\bar{X}}{\bar{x}}$	MS	$\frac{N(n-1)}{n(N-1)}$	e_2
$\frac{Nn(\bar{X}-\bar{x})}{N\bar{X}-n\bar{x}}$	$\frac{N\bar{X}-n\bar{x}}{N\bar{X}\binom{N-1}{n}}$	0	e_3

Furthermore, letting $T'(s)=(N-n)\bar{y}\bar{X}$, Deshpande and Prabhu–Ajgaonkar (DP, 1993) observed that
H_1': (srswor, $e_1'=T'(s)/(N-n)\bar{X}=\bar{y}$)
H_2': (MS scheme, $e_2'=T'(s)/(N-n)\bar{x}=(\bar{y}/\bar{x})\bar{X}$)
and H_3': (Deshpande scheme, $e_3'=T'(s)/(N\bar{X}-n\bar{x})=(\bar{y}/\bar{x}^c)\bar{X}$),

where $\bar{x}^c$ is the average of x_i's *not* in the sample, are all design-unbiased for $\bar{Y}$.

Extending these, one can consider as in (2.7)

$$\hat{\bar{Y}}' = \frac{\Theta'(s)}{E(\Theta'(s))}\hat{R}_1\bar{X} \tag{2.9}$$

whenever $\Theta'(s)/E(\Theta'(s))$, the multiplier is positive and the condition for unbiasedness is given by

$$E(\Theta'(s)\hat{R}_1\bar{X}) = \bar{Y}E(\Theta'(s))$$

or

$$\sum_{s\in\mathscr{S}}\Theta'(s)\frac{\bar{y}}{\bar{x}}\bar{X}\,p(s) = \left(\sum_1^N Y_i/N\right)E(\Theta'(s))$$

or

$$\sum_1^N Y_i\left\{\sum_{s\ni i}\left(\Theta'(s)p(s)\Big/\sum_{i\in s}x_i\right)\right\} = \left(\sum_1^N Y_i/X\right)E(\Theta'(s)) \quad \forall \underset{\sim}{Y}$$

which implies that

$$\sum_{s\ni i}\left(\Theta'(s)p(s)\Big/\sum_{i\in s}x_i\right) = E(\Theta'(s))/X$$

or

$$\sum_{s\ni i}\left(\lambda'(s)p(s)\Big/\sum_{i\in s}x_i\right) = 1/X \tag{2.10}$$

similar to (2.8), where $\lambda'(s)=\Theta'(s)/E(\Theta'(s))$.

Note that $E(\lambda'(s))=1$ in all these cases and $\notin$ open $(0,1)$.

Table 2
Special cases of unbiased ratio estimators

$\Theta'(s)$	$p(s)$	$E(\Theta'(s))$	$\hat{\hat{Y}}'$
$\frac{\bar{x}}{\bar{X}}$	$1/\binom{N}{n}$	1	e_1'
1 (or c)	$\frac{\sum x_i}{\binom{N-1}{n-1}X}$	1 (or c)	e_2'
$\frac{(N-n)\bar{x}}{N\bar{X}-n\bar{x}}$	$\frac{N\bar{X}-n\bar{x}}{N\bar{X}\binom{N-1}{n}}$	1	e_3'

Table 3
r.m.s.e.'s of different estimators

Estimator	r.m.s.e	
	Population I	Population II
$\hat{\hat{Y}}_{\mathrm{MN}}$	0.0105	0.0101
$\hat{\hat{Y}}_{\mathrm{HR}}$	0.0127	0.0287
$\hat{\hat{Y}}_{\mathrm{P}}$	0.0114	0.0175
$\hat{\hat{Y}}_1$	0.0255	0.0066
$\hat{\hat{Y}}_n$	0.0641	0.0140
$e_1'(=\bar{y})$	0.0741	0.3003
e_2'	0.0230	0.0051
e_3'	0.3211	0.8306

Remark 2.1. One can further obtain Hartley–Ross type unbiased ratio estimators by considering

$$\alpha\{\lambda'(s)\hat{R}_1\bar{X}\} + (1-\alpha)\{\lambda(s)\hat{R}_n\bar{X}\}$$

for a chosen $p(s)$. For example, when $p(s) = \binom{N}{n}^{-1}$ (i.e. srswor), $\lambda'(s) = \bar{x}/\bar{X}$, $\lambda(s) = (1-\Theta_{\mathrm{HR}})/(1-E(\Theta_{\mathrm{HR}}))$, we have

$$\hat{\hat{Y}}^*_{\mathrm{HR}} = \alpha\left\{\frac{\bar{x}}{\bar{X}}\right\}\hat{\hat{Y}}_1 + (1-\alpha)\left\{\frac{1-\Theta_{\mathrm{HR}}}{1-E(\Theta_{\mathrm{HR}})}\right\}\hat{\hat{Y}}_n$$

which gives a class of Hartley–Ross type unbiased ratio estimators for $\bar{Y}$. In particular, the value $\alpha = n(N-1)/N(n-1) = E(\Theta_{\mathrm{HR}})$ gives

$$\hat{\hat{Y}}^*_{\mathrm{HR}} = \Theta_{\mathrm{HR}}\hat{\hat{Y}}_1 + (1-\Theta_{\mathrm{HR}})\hat{\hat{Y}}_n$$

which is the same as $\hat{\hat{Y}}_{\mathrm{HR}}$ of (2.4). Similar classes of estimators could be obtained for other $p(s)$ values (Table 2). We shall omit the details here.

The following table presents the relative mean squared errors (r.m.s.e.) of some of these estimators for Population I, considered by Nieto de Pascual (1961) and by Sukhatme (1954, p. 165) when $n = 2$ (Table 3).

For population I, $\hat{\hat{Y}}_{\mathrm{MN}}$ behaves well while $\hat{\hat{Y}}_{\mathrm{HR}}$ and $\hat{\hat{Y}}_P$ are quite close. For population II for which $\rho(=0.995)$ is very close to unity, $\hat{\hat{Y}}_1$ as well as e_2' fare better while e_3' is very inefficient, as expected. (e_1, e_2, e_3 are not calculated since they take negative values for some samples for these populations.)

Remark 2.2. Such numerical comparisons as above may not give much insight into the reason why an estimator clicks in a given situation. However, from the wide applicability based on interpenetrating subsamples and the relatively very simple expression, the Murthy–Nanjamma almost unbiased ratio estimator $\hat{\bar{Y}}_{\text{MN}}$ is preferable to others. Even though exactly unbiased, $\hat{\bar{Y}}_{\text{HR}}$ could be less efficient compared to even the conventional biased ratio estimator $\hat{\bar{Y}}_1$, if β is nearer to R than to $(\sum Y_i/X_i)/N$.

3. Optimality of estimators based on $\hat{R}_1$ and $\hat{R}_n$

Suppose that we assume a model ξ in which y_i's are considered to be realised values of independent random variables Y_i's where Y_i has $\exp_\xi(Y_i|X_i) = \beta X_i$ and $\text{var}_\xi(Y_i|X_i) = \sigma^2 v(x_i)$; β and σ^2 being constants. Royall (1970) has proved that the estimator

$$\hat{T}_v = \sum_{i\in s} y_i + \frac{\sum_{i\in s} y_i x_i/v(x_i)}{\sum_{i\in s} x_1^2/v(x_i)} \left(X - \sum_{i\in s} x_i\right) \tag{3.1}$$

is model (ξ)-optimum.

When $v(x_i) = x_i$, we have the optimum estimator

$$\hat{T}_1 = \hat{R}_1 X$$

and when $v(x_i) = x_i^2$ we have the corresponding optimum estimator

$$\hat{T}_2 = \sum y_i + \hat{R}_n (X - \sum x_i).$$

Furthermore, if design-unbiasedness is not demanded, the strategy consisting of the πPS plan and the estimator $\hat{T}_2$ would be better than the πPS plan and Horvitz–Thompson (HT) estimator as noted by Royall (1970):

$$E_\xi M(p_{\pi\text{PS}}, \hat{T}_2) \leqslant E_\xi\, M(p_{\pi\text{PS}}, \hat{T}_{\text{HT}}).$$

It is well known that $\hat{R}_1 X$ is optimum for $v(x_i) = x_i$. It is interesting to note that $\hat{R}_n X$ is optimum when $v(x_i) = x_i^2/(X - nx_i)$ (see Ramachandran and Rao (1973)) since

$$\hat{T}_v = \sum y_i + \frac{\sum y_i x_i/\{x_i^2/(X - nx_i)\}}{\sum x_i^2/\{x_i^2/(X - nx_i)\}} (X - \sum x_i)$$
$$= \hat{R}_n X. \tag{3.2}$$

Note here that $(\hat{T}_1 - \sum y_i)/(X - \sum x_i)\, X = \hat{R}_1 X$ while $(\hat{T}_2 - \sum y_i)/(X - \sum x_i)\, X = \hat{R}_n X$ and the implied estimator $\hat{\beta}_1 = \sum y_i \mid \sum x_i$ happens to give the estimator $\hat{T}_1 = \hat{R}_1 X$ whereas the implied estimator $\hat{\beta}_2 = \sum (y_i|x_i)/n$ *does not* give $\hat{R}_n X$, but gives $\hat{T}_2$ as mentioned above.

Let s^* be a sample of those n labels for which the sizes x_i are the largest. Further, let p^* be the sampling plan which selects s^* with certainty i.e. $p^*(s^*) = 1$. Then following Royall (1970), when $v(x_i) = x_i$, the strategy consisting of the purposive sampling plan p^* and the estimator $\hat{T}_1 = \hat{R}_1 X$ is optimum for Y.

We have already noted in (3.2) that under the variance function $v(x_i) = x_i^2/(X - nx_i)$, $\hat{T}_v = \hat{R}_n X$ is the optimum estimator. Furthermore, we have

$$E_\xi(\hat{T}_v - Y^2) = (X - \sum x_i)^2 E_\xi(\hat{\beta}_n - \beta)^2 + \sigma^2 \sum_{\bar{s}} v(x_i)$$

$$= \frac{1}{n}\left(X - \sum_s x_i\right) + \sigma^2 \left(\sum_1^N \frac{x_i^2}{X - nx_i} - \sum_s \frac{x_i^2}{X - nx_i}\right). \tag{3.3}$$

Also

$$x_i > x_j \Rightarrow \frac{x_i^2}{X - nx_i} > \frac{x_j^2}{X - nx_j}$$

which shows that (3.3) is minimum for the largest x_i values. Thus, when $v(x_i) = x_i^2/(X - nx_i)$, the strategy p^* together with $\hat{R}_n X$ is optimum (cf. Royall (1970)).

4. Symmetrized Des Raj Estimators

In this section we shall consider a class of symmetrized Des Raj (SDR) estimators and discuss the role of the 'mean of ratios' and 'ratio of means' in its construction.

Let $s = (i_1, i_2, \ldots, i_n)$ be an ordered sample with the corresponding values of the study variable $y = (y_{i_1}, y_{i_2}, \ldots, y_{i_n})$ and initial probabilities of selection $(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$. Des Raj's (Raj (1956)) ordered estimator for the population total Y is given by

$$\hat{Y}_{\mathrm{D}} = \sum_{k=1}^{n} c_k t_k, \tag{4.1}$$

where $t_k = y_{i_1} + y_{i_2} + \cdots + y_{i_{k-1}} + \frac{y_{i_k}}{p_{i_k}}(1 - p_{i_1} - p_{i_2} - p_{i_{k-1}})$ and c_k's are constants such that $\sum_{k=1}^{n} c_k = 1$.

Murthy (1957) constructed a symmetrized unordered estimator known as Symmetrized Des Raj (SDR) estimator given by

$$\hat{Y}_{\mathrm{SDR}} = \frac{\sum p_s(i_1, i_2, \ldots, i_n)\hat{Y}_{\mathrm{D}}}{\sum p_s(i_1, i_2, \ldots, i_n)}, \tag{4.2}$$

where $p_s(i_1, i_2, \ldots, i_n)$ is the probability of the ordered sample $s(i_1, i_2, \ldots, i_n)$ and the summation $\sum$ is over all $n!$ permutations of $(i_1, i_2, \ldots, i_n)$.

As an estimator of the population total Y, Basu (1971) considered

$$t_n = \sum_{i_j \in s} y_{i_j} + \frac{y_{i_n}}{p_{i_n}}\left(1 - \sum_{i_j \in s} p_{i_j}\right) \tag{4.3}$$

and rewrites (4.3) as

$$t_n = S_1 + S_2,$$

where $S_1 = \sum_{i_j \in s} y_{i_j}$ and S_2 is an estimate of $Y - \sum_{i_j \in s} y_{i_j}$. It is clear that S_2 would be an 'exact' estimate of $Y - \sum_{i_j \in s} y_{i_j}$ iff $y_{i_n}/x_{i_n} = (Y - \sum_{i_j \in s} y_{i_j})/(X - \sum_{i_j \in s} x_{i_j}) =$ weighted average of the unobserved ratios, weights being the sizes of the corresponding units. Basu then suggests that it is 'natural' for the surveyor to estimate $(Y - \sum y_{i_j})/(X - \sum x_{i_j})$ by 'some sort of average' of observed ratios such as $\sum_{i_j \in s} y_{i_j} / \sum_{i_j \in s} x_{i_j}$ (ratio of means) or $1/n \sum_{i_j \in s} y_{i_j}/x_{i_j}$ (mean of ratios).

Thus we have the choices

$$\hat{Y}_{B_1} = \sum_{i_j} y_{i_j} + \frac{\sum y_{i_j}}{\sum x_{i_j}} \left(X - \sum_{i_j} x_{i_j} \right) = \sum_{s} y_i + \frac{\sum y_i}{\sum x_i} \left(X - \sum_{s} x_i \right) \tag{4.4}$$

or,

$$\hat{Y}_{B_2} = \sum_{s} y_i + \frac{1}{n} \left(\sum_{s} \frac{y_i}{x_i} \right) (X - \sum x_i) \tag{4.5}$$

both of which have 'greater face validity' than (4.1). With PPSWOR plan both $\hat{Y}_{B_1}$ and $\hat{Y}_{B_2}$ are design-biased.

Now, consider a more general estimator

$$\hat{Y}^{+} = \sum w_{i_1 i_2 \ldots i_n} t_n, \tag{4.6}$$

where t_n is the ordered Des Raj estimator based on $(i_1, i_2, \ldots, i_n)$ and w's are weights with $\sum w_{i_1 i_2 \ldots i_n} = 1$, summation being over all $n!$ permutations of $(i_1, i_2, \ldots, i_n)$. When $w_{i_1 i_2 \ldots i_n} = p_{i_n}/(n-1)! \sum_{i_j \in s} p_{i_j}$, (4.6) simplifies to $\hat{Y}^{+} = \hat{Y}_{B_1}, w_{i_1 i_2 \ldots i_n} = 1/n!, \hat{Y}^{+} = \hat{Y}_{B_2}$ and when $w_{i_1 i_2 \ldots i_n} = \frac{p_s(i_1 i_2 \ldots i_n)}{\sum p_s(i_1 i_2 \ldots i_n)}$, we get $\hat{Y}^{+} = \hat{Y}_{\mathrm{SDR}}$. Thus (4.6) generates a class of SDR estimators. We then have the problem of choosing the best in this class. If design-unbiasedness is not demanded, we note that

$$\hat{Y}^{+} = \sum_{i \in s} \beta_{si} y_i$$

is ξ-unbiased where

$$\beta_{si} = 1 + \sum{}' \frac{w_{i_1, i_2 \ldots i_{n-1} i}(1 - \sum_j p_j)}{p_i},$$

where $\sum'$ is summation over all $(n-1)!$ permutations of $(i_1, i_2, \ldots, i_{n-1})$ the first $(n-1)$ units in s with i as the last unit and where s is the ordered sample $i_1, i_2, \ldots, i_{n-1}, i$. Let $a_i = n \sum' w_{i_1 i_2, \ldots i_{n-1}}$. Then from Ramachandran and Rao (1973) it follows for any sampling plan p with fixed sample size n, under the model ξ (with variance function $v(x)$)

$$E_{\xi} M(p, \hat{Y}^{+}) \geqslant E_{\xi} M(p, \hat{Y}_{B_2})$$

if for $i, i' \in s$ and $s \in \mathscr{S}$ with $p(s) > 0, a_i \leqslant a_{i'}$ implies $V(x_i)/x_i^2 \leqslant V(x_{i'})/x_{i'}^2$, where for $i \in s, a_i = n \sum' w_{i_1 i_2 \ldots i_{n-1}} i$. This identifies $\hat{Y}_{B_2}$ which has $\hat{R}_n$ as the implied estimator as the optimum in this class under the given conditions.

5. Conclusion

We have seen in the preceding sections the role played by the two estimators 'mean of ratios' and 'ratio of means' in the construction of unbiased ratio estimators or in determining an optimum in the class of SDR estimators. Historically, these two estimators have been of interest even in classical inference from infinite populations. For example, Rao (1952, pp. 154–155) discusses the role of the two statistics $\hat{R}_n = 1/n \sum y_i/x_i$ and $\hat{R}_1 = \bar{y}/\bar{x}$ based on n pairs of observations (y_i, x_i), $i = 1, 2, \ldots, n$ from Normal populations. $\hat{R}_n$ is considered 'inconsistent' as an estimator of the parametric function $E(y)/E(x)$ while $\hat{R}_1$ is 'consistent'. Rao cites examples of the uses of indices in biometric research where usually mean index is calculated by taking ratio of means. He also discusses why this is not strictly applicable with the mean of ratios.

We shall now discuss another interesting example from Rao (1989, pp. 116–118). The problem was, to estimate the unknown large number N of people of a minority community who took refuge in the Red Fort in Delhi during communal riots of 1947. A method which was 'unconventional and ingenious' suggested by Sengupta was used for the estimation of N.

If y_1, y_2, y_3 denote the quantities of rice, pulses and salt used daily to feed all the refugees and if from consumption surveys, the per person requirements of these commodities are known to be x_1, x_2, x_3, respectively, then each $\frac{y_i}{x_i}$ provides a parallel and valid estimate of N. Following the discussions of the above sections, it is but natural to consider the two alternatives

$$\hat{N}_{(1)} = \frac{y_1 + y_2 + y_3}{x_1 + x_2 + x_3}$$

the ratio of means

$$\text{or} \qquad \hat{N}_{(3)} = \frac{1}{3}\left(\frac{y_1}{x_1} + \frac{y_2}{x_2} + \frac{y_3}{x_3}\right)$$

the 'mean of three ratios'.

However, it was found by 'Cross-Examination of Data' that y_3/x_3 is the lowest compared to $\frac{y_1}{x_1}$ or $\frac{y_2}{x_2}$ since the figures for y_1, y_2 given by the supplier of these expensive items were exaggerated. Thus taking the figure y_3 to be 'more appropriate', the estimate suggested was

$$\hat{N} = \frac{y_3}{x_3}.$$

So in a situation like this, it is neither the 'mean of ratios' nor the 'ratio of means' that is suitable. As pointed out by the referee, it should be noted that this example pertains to an unconventional situation where the reported y-values are not the correct ones and hence does not relate to the earlier sections. However, given such a situation and extra information, the statistician should not routinely go in for either 'mean of ratios' or 'ratio of means' but 'cross-examine' the data to go in for an alternative estimate.

Remark. Also in those few situations of stratified ratio estimation where the stratum ratios are equal, it is well known (see Murthy, 1967) that the combined estimator $\hat{Y}_C$ is preferred to the separate estimator $\hat{Y}_S$ since the Bias, $B(\hat{Y}_C)$ is much smaller than $B(\hat{Y}_S)$ while the MSE's are equal. Furthermore, if the strata are such that the stratum totals X_i could be made equal (as in 'equi-stratification rule' suggested by Mahalanobis, 1952 and Kitagawa, 1956) we have $B_S = kB_C$. Then correcting for the Bias of $\hat{Y}_C$, we have an almost unbiased stratified ratio estimator given by $\hat{Y} = (k/(k-1))\hat{Y}_C - (1/(k-1))\hat{Y}_S$ on the lines of Murthy and Nanjamma (1959).

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