Some Geometrical Aspects of the Cone Linear Complementarity Problem

Madhur Malik

Thesis submitted to Indian Statistical Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Indian Statistical Institute
7, S. J. S. Sansanwal Marg
New Delhi-110016, India
India

Some Geometrical Aspects of the Cone Linear Complementarity Problem

Madhur Malik

Indian Statistical Institute

New Delhi, India

E. mail: madhur@isid.ac.in

Thesis supervised by:
Late Dr. S. R. Mohan

Indian Statistical Institute

New Delhi, India

Thesis submitted to Indian Statistical Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Submitted: December 2004

Revised: January 2006

Acknowledgements

I am indebted to Late Dr. S. R. Mohan, my thesis supervisor, for introducing me to the field of linear complementarity problems and also for his excellent guidance and encouragement to carry out this work.

I thank Dr. S. K. Neogy, who was appointed as my new thesis supervisor after the sudden demise of Dr. S. R. Mohan, for his comments while writing this revised version.

My sincere thanks are due to Professor M. S. Gowda, University of Maryland, Baltimore County, and Dr. Richard E. Stone, Northwest Airlines, for their helpful comments about my research papers included in this thesis.

I also like to thank all the referees of my papers and thesis examiners for their comments, leading towards considerable improvement in the presentation of this work.

Finally, I would like to thank Indian Statistical Institute for giving me financial assistance and providing me all the facilities.

Madhur Malik

New Delhi, January 2006.

Contents

Chapter 1

Introduction

1.1 Cone Linear Complementarity Problem

Let V be a finite dimensional real inner product space and K be a closed convex cone in V. Given a linear transformation $L : V \rightarrow V$ and a vector $q \in V$ the *cone linear complementarity problem* or *linear complementarity problem over* K, denoted as LCP(K, L, q), is to find a vector $x \in K$ such that

$$L(x) + q \in K^* \text{ and } \langle x, L(x) + q\rangle = 0,$$

where $\langle ., .\rangle$ denotes an inner product on V and K^* is the dual cone of K defined as:

$$K^* := \{y \in V : \langle x, y\rangle \geq 0 \,\forall\, x \in K\}.$$

Note that a subset C of V is a cone if $x \in C \Rightarrow \lambda x \in C$ for every $\lambda \geq 0$.

The cone LCP is the special case of a variational inequality problem which is formally stated as follows:

Let K be a closed convex set in V. Given a continuous function $f : V \rightarrow V$, the *variational inequality problem,* denoted VI(K, f), is to find a $x \in K$ such that

$$\langle y - x, f(x) \rangle \geq 0, \ \forall \, y \in K.$$

When K is a closed convex cone the VI (K, f) reduces to a (*cone*) *complementarity problem* (CP) and with the additional condition of f being affine we get a cone LCP. Volumes I and II of the recent book by Facchinei and Pang [9] provides an up to date account of finite dimensional variational inequalities and complementarity problems along with various applications and algorithmic details. Interested readers can look at the bibliography of [9] for more details.

Though cone LCP is a special case of a variational inequality problem, its usefulness as a modelling framework for various practical problems and the availability of an additional structure puts it in a distinguished position. See, for example, Çamlibel et al. [5] and Heemels et al. [29], where switched piecewise linear networks are modelled as cone LCP and [49, 50] for the reformulation of a Bilinear Matrix Inequality as a cone LCP on the cone of semidefinite matrices. Furthermore, [12] and [65] provide an excellent survey of various applications of complementarity problems in engineering and economics, and complementarity systems in optimization. The lecture notes [33] study complementarity problems in abstract spaces. Some early references related to a cone LCP (CP) include [25, 36, 37, 38]. For a recent work on cone LCP (CP) one can see [14, 15, 24, 40] and the references therein.

1.1.1 Examples of a Cone LCP

Various special cases of a cone LCP (CP) are found to be of fundamental importance in the literature. We discuss briefly some of these cases in the examples below.

Example 1.1.1 Given a real square matrix $M \in R^{n\times n}$ and a vector $q \in R^n$, the *linear complementarity problem*, denoted LCP(R^n_+, M, q), is to find a $x \in R^n_+$ such that $Mx+q \in R^n_+$ and $x^T(Mx+q) = 0$. The study of linear complementarity problem began in 1960's for solving convex quadratic programming problems [6]. Thereafter, Nash equilibrium problem for the bimatrix game was formulated as an LCP over R^n_+, and some efficient algorithms like Lemke's method has been proposed, see [41, 42]. The monograph [8] by Cottle, Pang and Stone provides a comprehensive treatment of various aspects of a LCP over R^n_+. This monograph also contains an extensive bibliography of the LCP over R^n_+ up to year 1990 with detailed notes and comments.

The study of LCP over R^n_+ has lead to interesting matrix classes and their characterizations. These matrix classes play an important role in the study of an existence and uniqueness of a solution, and the algorithmic aspects of a LCP over R^n_+. We shall define and discuss some of these matrix classes which will be used in the sequel. Their details can be found in [8].

- $M \in R^{n\times n}$ is a P-matrix if every principal minor of M is positive. The notion of a P-matrix was introduced by Fiedler and Pták [11], in 1962. It is central to the study of linear complementarity problems and characterizes the uniqueness of a solution to LCP(R^n_+, M, q) for all $q \in R^n$, and has found many applications in various fields, particularly in optimization [9].

- $M \in R^{n\times n}$ is strictly semimonotone if

$$x \in R^n_+,\ x * (Mx) \leq 0 \Rightarrow x = 0,$$

where $x * Mx$ is the componentwise product of x and Mx and the inequality is defined componentwise. Strictly semimonotone matrices are found useful in studying the solution properties as well as the algorithmic aspects of a LCP over R^n_+, [32].

- A matrix $M \in R^{n \times n}$ is a Q-matrix if LCP(R^n_+, M, q) over R^n_+ has a solution for all $q \in R^n$. It is known that P and strictly semimonotone matrices are Q-matrices.

- M is a R_0-matrix if LCP$(R^n_+, M, 0)$ has a unique (zero) solution. It can be shown that a matrix M is R_0 if and only if LCP(R^n_+, M, q) has a compact solution set (may be empty) for all $q \in R^n$.

- M is a nondegenerate matrix if every principal minor of M is nonzero. Nondegenerate matrices characterize the finiteness of the solution set of a LCP(R^n_+, M, q) for all $q \in R^n$.

The normal map and fixed point map, see Page 83 in [9] and Page 24 in [8], corresponding to LCP over R^n_+ are piecewise affine. By observing the above fact, Robinson [60] generalized the P-matrix property from LCP over R^n_+ to affine variational inequalities over polyhedral sets. Motivated by the above, various researchers have studied the local and global invertibility of piecewise affine maps, see [53, 17, 9].

Example 1.1.2 Let S^n be the space of $n \times n$ real symmetric matrices and S^n_+ be the cone of $n \times n$ real symmetric positive semidefinite matrices. Given a linear transformation $L : S^n \to S^n$ and a $Q \in S^n$, the linear complemenatrity problem over S^n_+, with the inner product defined as $\langle X, Y\rangle := \text{trace}\,(XY)$, is called a *semidefinite linear complementarity problem*, denoted SDLCP(L, Q).

The above form of the SDLCP is due to Gowda and Song [18]. The SDLCP was introduced in a slightly different form by Kojima, Shindoh and Hara [39] as a unified mathematical model for various problems arising from dynamical systems and control theory and combinatorial optimization. This problem includes, as a special case, the primal-dual pair of semidefinite programs (SDP), see [57], as described below. Also see [39].

Consider the Primal SDP problem:

$$\begin{aligned} &\text{Minimize } \langle C, X\rangle := \text{trace}\,(CX) \\ &\text{subject to} \quad \langle A_i, X\rangle \geq b_i,\ 1 \leq i \leq m, \\ &\qquad\qquad\quad X \in S^n_+ \end{aligned}$$

where $C \in S^n$, $b_i \in R$ and $A_i \in S^n$ with $1 \leq i \leq m$ are given. Let b be the vector in R^m whose i^{th} coordinate is b_i. The dual of the above problem is given by:

$$\begin{aligned} &\text{Maximize } b^T y \\ &\text{subject to} \quad C - \textstyle\sum_{i=1}^m y_i A_i \in S^n_+, \\ &\qquad\qquad\quad y \geq 0. \end{aligned}$$

Let B denote the $m \times m$ diagonal matrix whose i^{th} diagonal entry is b_i and Q be the $(m+n) \times (m+n)$ matrix defined as

$$Q := \begin{pmatrix} -B & 0 \\ 0 & C \end{pmatrix}.$$

Assuming that there is a complementary pair of solutions to the above optimization problem, the SDLCP formulation of the pair of primal and dual SDP's is given by:

Find $X^{'} \in S_{+}^{m+n}$ such that $L^{'}(X^{'}) + Q \in S_{+}^{m+n}$ and $\langle X^{'}, L^{'}(X^{'}) + Q \rangle = 0$, where the transformation $L^{'}$ is:

$$L^{'}\left(\begin{pmatrix} Y & * \\ *^T & X \end{pmatrix}\right) = \begin{pmatrix} \langle A_1, X\rangle & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \langle A_m, X\rangle & 0 \\ 0 & \cdots & 0 & -\sum_{i=1}^{m} y_i A_i \end{pmatrix}$$

where Y is an $m \times m$ symmetric matrix with y_i as the i^{th} diagonal entry, X is an $n \times n$ symmetric matrix and $*$ is a matrix of order $m \times n$.

The SDLCP can also be regarded as a generalization of the LCP over R_{+}^{n}, see [59]. But strikingly, the properties of LCP over R_{+}^{n} do not carry over to SDLCP trivially, because the semidefinite cone is nonpolyhedral and the matrix product, in general, is not commutative.

Various aspects of the above problem have been studied in recent years. Gowda et al. study the solution properties (existence and uniqueness) of a general SDLCP as well as with special transformations like Lyapunov and Stein transformations (for $A \in R^{n\times n}$ and $X \in S^n$, the *Lyapunov and Stein transformations* are defined by $L_A(X) := AX + XA^T$ and $S_A(X) := X - AXA^T$, respectively), making connection between control theory/dynamical systems, complementarity problems and matrix theory, see [18, 19, 21, 22]. They introduce and study various properties of linear transformations in the semidefinite setting, similar to the matrix classes studied in LCP over R^n_+. The connection between bilinear matrix inequality problem and SDLCP has been made in [49].

Algorithmic aspects and interior point algorithms with a more general form of a SDLCP are discussed in [39, 51].

EXAMPLE 1.1.3 In the space R^n, the second-order or Lorentz cone is defined as

$$\Lambda^n_+ := \{(x_0, x_1, \ldots, x_{n-1})^T : x_0 \geq 0,\, x_0^2 \geq x_1^2 + \ldots + x_{n-1}^2\}.$$

Given a matrix $M \in R^{n\times n}$ and $q \in R^n$, the linear complementarity problem over Λ^n_+, with the usual inner product, is called a *second-order cone linear complementarity problem*, denoted SOCLCP(M, q).

Second-order cone programming and complementarity problems have been subjects of some recent studies. Pang et al. have studied the stability of solutions to semidefinite and second-order cone complementarity problems in [54]. Study of smoothing functions for second-order cone complementarity problems to develop noninterior continuation methods has been made by Fukushima et al. in [13]. Smoothing and regularization methods for solving monotone second-order cone complementarity problems are discussed in [27]. In [26], Hayashi et al. proposed a matrix-splitting method to solve a linear complementarity problem over the direct product of second-order cones. An application of a LCP over the direct product of second-order cones in studying three-dimensional quasi-static frictional contact problems has been given by Kanno et al. in [34]. One can see the same paper and the references therein for various applications of second-order cone (linear) complementarity problems. In another application, Hayashi et al. [28] have studied the Robust Nash equilibria in the framework of a second-order cone complementarity problems. For a comprehensive exposition to various applications and algorithmic aspects of second-order cone programming problems, the reader is advised to refer to Alizadeh and Golfarb, [1].

1.2 Preliminaries

1.2.1 Notations

A real n-dimensional space is denoted by R^n and R^n_+ is the nonnegative orthant in R^n. For any $x \in R^n$, x^T denotes the transpose of a vector x. Also, $x \in R^n_+$ ($-x \in R^n_+$) is represented by $x \geq 0$ ($x \leq 0$). The space of all $n \times n$ real matrices is denoted by $R^{n\times n}$. For any $A \in R^{n\times n}$ its transpose is denoted by A^T. We use the symbol $X \succeq 0$ ($\succ 0$) to say that X is symmetric and positive semidefinite (positive definite); the symbol $X \preceq 0$ means that $-X \succeq 0$.

A finite dimensional real inner product space is denoted by V. Orthogonal projection onto the subspace S of V is denoted by $\mathrm{Proj}_S(.)$ and $\mathrm{span}\, E$ represents the *linear span* of a subset E of a linear space V. The *inner product* and *norm* on V are denoted by $\langle ., . \rangle$ and $||.||$, respectively. For a convex cone $K \subseteq V$ its *dual* in the space V is denoted by K^*. For any set $S \subseteq V$ its *interior* and *boundary* are denoted by $\mathrm{int}\, S$ and $\mathrm{bd}\, S$, respectively. Its orthogonal complement is denoted by $S^\perp$. The *relative interior* $\mathrm{ri}\, C$ of a conex set $C \subset V$ is the interior of C for the topology relative to the affine hull of C. The *dimension* of a convex set C is the dimension of its affine hull. The *relative boundary* $\mathrm{rbd}\, C := \mathrm{cl}\, C \backslash \mathrm{ri}\, C$, where $\mathrm{cl}\, C$ denotes closure of C.

For a linear transformation L on V its *adjoint*, denoted L^T, is defined by the equality

$$\langle x, L(y)\rangle = \langle L^T(x), y\rangle \qquad \forall\, x,\, y \in V.$$

L is *self-adjoint* if $L = L^T$. The *determinant* of a linear transformation L is defined as the product of its all eigenvalues.

The set $\mathrm{SOL}(K, L, q)$ denotes the solution set of the $\mathrm{LCP}(K, L, q)$.

1.2.2 Classes of linear transformations

In the context of a LCP over a closed convex cone we have the following definitions.

Definition 1.2.1 Let $L : V \to V$ be a linear transformation and K be a closed convex cone in V. Then

(a) L is said to be *monotone* (*strictly monotone*) if $\langle x, L(x)\rangle \geq 0$ (> 0) $\forall\, 0 \neq x \in V$.

(b) L is *copositive* (*strictly copositive*) on K if

$$\langle x, L(x)\rangle \geq 0 \ (> 0) \ \forall\, 0 \neq x \in K.$$

(c) L has the $\mathbf{Q}$-*property* if $\mathrm{LCP}(K, L, q)$ has a solution for all $q \in V$.

(d) L has the $\mathbf{R_0}$-*property* if $\mathrm{LCP}(K, L, 0)$ has a unique (zero) solution.

Observation 1.2.1 *If L has the $\mathbf{R_0}$-property, then the set $SOL(K, L, q)$ is compact (may be empty) for all $q \in V$.*

Proof. Note that SOL(K, L, q) is always closed. Let $\{x_n\} \subset$ SOL(K, L, q) be an unbounded sequence of nonzero terms. Consider the subsequence $\{x_m\}$ of $\{x_n\}$ such that $\frac{x_m}{||x_m||}$ converges to some $x \in K$. Then the sequence $L(\frac{x_m}{||x_m||}) + \frac{q}{||x_m||}$ converges to $L(x) \in K^*$ with $\langle x, L(x)\rangle = 0$, contradicting the $\mathbf{R_0}$-property. ■

Below, we state some properties of a linear transformation in the semidefinite setting from Gowda et al. [18, 19, 21].

Definition 1.2.2 Let $L : S^n \to S^n$ be a linear transformation. Then

(a) L has the **P**-*property* if

$$X \in S^n,\ XL(X) = L(X)X \preceq 0 \Rightarrow X = 0.$$

(b) L has the *strictly semimonotone property* (**SSM**-*property*) if

$$X \in S^n_+,\ XL(X) = L(X)X \preceq 0 \Rightarrow X = 0.$$

(c) L is *nondegenerate* if

$$X \in S^n,\ XL(X) = 0 \Rightarrow X = 0.$$

1.2.3 Closed convex cones and Principal Subtransformations

Definition 1.2.3 ([4])

(a) A nonempty subset F of a (closed) convex cone K in V is called a *face* of K, denoted by $F \lhd K$, if F is a convex cone and

$$x \in K,\ y - x \in K \text{ and } y \in F \Rightarrow x \in F.$$

(b) The smallest subspace of V containing a closed convex cone K (affine hull of K in V) is the set

$$K - K = \{x - y : x \in K,\, y \in K\}.$$

(c) The *smallest face* of K containing a point $x \in K$ is defined as

$$\Phi(x) := \cap\{F : F \lhd K,\, x \in F\}.$$

(d) The *complementary face* of $F \lhd K$ is defined as

$$F^{\triangle} := \{x \in K^* : \langle x, y\rangle = 0\,\forall\, y \in F\}.$$

EXAMPLE 1.2.1 The faces of R^n_+ are $\{0\}$, R^n_+ and any set of the form

$$F := P\{(x_1, x_2, \ldots, x_k, 0, \ldots, 0)^T : x_i \geq 0,\ 1 \leq i \leq k\},$$

where P is a permutation matrix and $k \in \{1, \ldots, n\}$. The complementary face of F is given by

$$F^{\triangle} = P\{(0, \ldots, 0, x_{k+1}, \ldots, x_n)^T : x_i \geq 0,\ k+1 \leq i \leq n\}.$$

The complementary face of $\{0\}$ (R^n_+) is R^n_+ ($\{0\}$).

EXAMPLE 1.2.2 ([30]) Let $X \in S^n_+$ be a matrix of rank r. Then

(i) There exists an orthogonal U such that the smallest face of S^n_+ containing X is

$$F = \left\{ U \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} U^T : Y \in S^r_+ \right\}.$$

The complementary face of F is

$$F^{\triangle} = \left\{ U \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix} U^T : Z \in S^{n-r}_+ \right\}.$$

(ii) The dimension of the face F is $\frac{r(r+1)}{2}$ and the dimension of the complementary face $F^{\triangle}$ is $\frac{(n-r)(n-r+1)}{2}$.

THEOREM 1.2.1 ([4]) *Let F be a face of a closed convex cone K and let $x \in K$. Then F is the smallest face of K containing x if and only if x lies in the relative interior of F.*

It is easy to see that for any $x \in \operatorname{ri} F$, $F^{\triangle}$ can equivalently be represented as $F^{\triangle} := \{y \in K^* : \langle x, y\rangle = 0\}$. For any face F of K, $F \subseteq (F^{\triangle})^{\triangle}$. Also $G \lhd F \lhd K$ implies that $G \lhd K$.

A closed convex cone K is *solid* if $\operatorname{int} K \neq \phi$. K is *pointed* if $K \cap (-K) = \{0\}$. A solid and pointed closed convex cone is called *proper*.

PROPOSITION 1.2.1 ([3, 9]) *For a convex cone K in V the following statements hold.*

(i) *K^* is a closed convex cone.*

(ii) *If K is closed, then $(K^*)^* = K$.*

(iii) *If K is closed, then K is solid (pointed) if and only if K^* is pointed (solid).*

A closed convex cone K is *self-dual* if $K = K^*$. By Proposition 1.2.1, every self-dual closed convex cone is solid and pointed.

Motivated by the concept of a principal submatrix of a matrix in $R^{n\times n}$ we introduce the notion of a principal subtransformation of a linear transformation on V corresponding to a face F of a closed convex cone K.

DEFINITION 1.2.4 Let $L : V \to V$ be a linear transformation and $F \lhd K$. Then a *principal subtransformation* of L with respect to F is a linear transformation $L_{FF} : \operatorname{span} F \to \operatorname{span} F$ such that $L_{FF}(x) = \mathrm{Proj}_{\operatorname{span} F} L(x) \, \forall \, x \in \operatorname{span} F$.

EXAMPLE 1.2.3 Consider a linear transformation $L : S^2 \to S^2$ defined as

$$L(X) := \begin{pmatrix} 2(x_{11} + x_{12}) & 2x_{12} + x_{22} \\ 2x_{12} + x_{22} & 2x_{22} \end{pmatrix},$$

and a face F of S^2_+ given as

$$F = \left\{ U \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} U^T : \alpha \in R_+ \right\},$$

where $U := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then the principal subtransformation $L_{FF} : \operatorname{span} F \to \operatorname{span} F$ is given by

$$L_{FF}(X) = U \begin{pmatrix} (U^T L(X) U)_{11} & 0 \\ 0 & 0 \end{pmatrix} U^T = U \begin{pmatrix} 3\alpha & 0 \\ 0 & 0 \end{pmatrix} U^T \quad \forall \, X \in \operatorname{span} F.$$

REMARK 1.2.1 The notion of a principal subtransformation in a semidefinite setting was also introduced independently by Gowda et al. [22]. Though the connection of their notion with our notion has been described in detail in [44], our notion of a principal subtransformation with respect to a face of K seems to be more general and geometric in nature.

With the above notion of a principal subtransformation, given a linear transformation L and a $q \in \operatorname{span} F$ the $\mathrm{LCP}(F, L_{FF}, q)$ is to find an $x \in F$ such that $L_{FF}(x) + q \in F^d$ and $\langle x, L_{FF}(x) + q \rangle = 0$, where F^d is the dual cone of F in $\operatorname{span} F$ defined as $F^d := \operatorname{span} F \cap F^*$.

Observation 1.2.2 *The following statements hold for any linear transformation $L : V \to V$ and a closed convex cone K in V.*

(i) $(L^T)_{FF} = (L_{FF})^T \ \forall \ F \lhd K$.

(ii) *If $G \lhd F \lhd K$, then $(L_{FF})_{GG} = L_{GG}$.*

(iii) *L is strictly monotone (strictly copositive on K) $\Rightarrow$ L_{FF} is strictly monotone (strictly copositive on F) $\forall \ F \lhd K$.*

(iv) *L is self-adjoint $\Rightarrow$ L_{FF} is self-adjoint $\forall \ F \lhd K$.*

Proof. The proof of the statements (i)-(iv) follows easily from the definition of a principal subtransformation and the fact that for any $x, y \in \operatorname{span} F$ we have

$$\begin{aligned}\langle x, L(y)\rangle &= \langle x, \mathrm{Proj}_{\operatorname{span} F} L(y) + \mathrm{Proj}_{F^\perp} L(y)\rangle \\ &= \langle x, \mathrm{Proj}_{\operatorname{span} F} L(y)\rangle \\ &= \langle x, L_{FF}(y)\rangle.\end{aligned}$$

1.3 Euclidean Jordan algebras

This section begins with a preliminary introduction to a Euclidean Jordan algebra. For a complete treatment one can read the book [10] by Faraut and Koranyi. Brief summaries can also be found in the articles [63, 64].

A *Euclidean Jordan algebra* $\mathcal{A}$ is a finite dimensional real vector space equipped with an inner product $\langle x, y\rangle$ and a bilinear map $(x, y) \to x \circ y$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$,

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ where $x^2 = x \circ x$, and

(iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$,

for all $x, y, z \in \mathcal{A}$.

The product $x \circ y$ is called a Jordan product. A Euclidean Jordan algebra $\mathcal{A}$ has an identity element, if there exists a (unique) element $e \in \mathcal{A}$ such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{A}$. Henceforth, we shall assume the existence of an identity element in a Euclidean Jordan algebra.

In a finite dimensional inner product space V, a self-dual closed convex cone K is *symmetric* if for any two elements $x, y \in \operatorname{int} K$ there exists an invertible linear transformation $\Theta : \mathcal{A} \to \mathcal{A}$ such that $\Theta(K) = K$ and $\Theta(x) = y$. The *cone of squares* $\mathcal{K}$ associated with $\mathcal{A}$ is a self-dual closed convex cone defined as

$$\mathcal{K} := \{x^2 : x \in \mathcal{A}\}.$$

Theorems III.2.1 and III.3.1 in [10] give the following characterization of symmetric cones in a Euclidean vector space.

THEOREM 1.3.1 ([10]) *A cone is symmetric iff it is the cone of squares of some Euclidean Jordan algebra.*

For each $x \in \mathcal{A}$ let d be the smallest positive integer such that the set $\{e, x, x^2, \ldots, x^d\}$ is linearly dependent. Then d is called the *degree* of x. The *rank* of $\mathcal{A}$ is defined as the largest degree of any $x \in \mathcal{A}$. An element $c \in \mathcal{A}$ is an *idempotent* if $c^2 = c$. It is *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set $\{f_1, f_2, \ldots, f_n\}$ of primitive idempotents in $\mathcal{A}$ is a *Jordan frame* if

$$f_i \circ f_j = 0 \text{ if } i \neq j \text{ and } \textstyle\sum_{i=1}^n f_i = e.$$

THEOREM 1.3.2 **(Spectral theorem)** *Let $\mathcal{A}$ be a Euclidean Jordan algebra of rank n. Then for every x in $\mathcal{A}$, there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that*

$$x = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n.$$

The numbers λ_i (with their multiplicities) are uniquely determined by x and are called the eigenvalues of x.

With the above decomposition we shall define *determinant* of $x \in \mathcal{A}$ as follows.

$$\det\,(x) := \lambda_1 \lambda_2 \ldots \lambda_n.$$

x is said to be invertible if $\det(x) \neq 0$. In which case inverse of x is defined as $x^{-1} := \lambda_1^{-1} f_1 + \ldots + \lambda_n^{-1} f_n$. Also from the spectral theorem it is clear that $x \in \mathcal{K}\,(\text{int}\,\mathcal{K})$ if and only if all its eigenvalues are nonnegative (positive).

EXAMPLE 1.3.1 In the space S^n of $n \times n$ real symmetric matrices the inner product and the Jordan product are defined by

$$\langle X, Y \rangle := \text{trace}(XY) \text{ and } X \circ Y := \tfrac{1}{2}(XY + YX)\ .$$

The cone of squares in S^n is the cone of all positive semidefinite matrices denoted by S^n_+. The identity matrix is the identity element and the set of matrices $\{E_{11}, E_{22}, \ldots, E_{nn}\}$ where E_{ii} is the diagonal matrix with $(i,i)^{th}$ entry 1 and others zero, constitutes a Jordan frame. Thus S^n is a rank n algebra. Any other Jordan frame in S^n is of the form $\{UE_{11}U^T, UE_{22}U^T, \ldots, UE_{nn}U^T\}$ where U is any $n \times n$ orthogonal matrix.

EXAMPLE 1.3.2 Consider the space R^n $(n > 1)$ whose elements $x = (x_0, \bar{x}^T)^T$, where $\bar{x} = (x_1, \dots, x_{n-1})^T$, are indexed from zero, equipped with the usual inner product and the Jordan product defined as

$$x \circ y = (\langle x, y \rangle, x_0 \bar{y}^T + y_0 \bar{x}^T)^T.$$

Then R^n is a Euclidean Jordan algebra, denoted by Λ^n with the cone of squares, a *Lorentz cone* (second-order cone) which is seen to be $\Lambda^n_+ := \{x \in R^n : ||\bar{x}|| \leq x_0\}$. The identity element in this algebra is given by $e = (1, 0, \dots, 0)^T$. Also the spectral decomposition of any x with $\bar{x} \neq 0$ is given by $x = \lambda_1 e_1 + \lambda_2 e_2$, where

$$\lambda_1 = x_0 + ||\bar{x}||,\ \lambda_2 = x_0 - ||\bar{x}||$$

and

$$e_1 := \tfrac{1}{2}(1, \bar{x}^T/||\bar{x}||)^T, \text{ and } e_2 := \tfrac{1}{2}(1, -\bar{x}^T/||\bar{x}||)^T.$$

In a Euclidean Jordan algebra a linear transformation $L_x : \mathcal{A} \rightarrow \mathcal{A}$ corresponding to any x is defined as $L_x(y) := x \circ y$ for all $y \in \mathcal{A}$. See [10, 64] for the discussion on the spectral decomposition of L_x. We say two elements x, y of a Euclidean Jordan algebra $\mathcal{A}$ *operator commute* if $L_x L_y = L_y L_x$, which means that $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in \mathcal{A}$. Lemma X.2.2, [10] or Theorem 27, [64], give the following characterization of operator commutativity.

LEMMA 1.3.1 *Two elements x, y of a Euclidean Jordan algebra $\mathcal{A}$ operator commute iff there is a Jordan frame $\{f_1, f_2, \dots, f_n\}$ such that $x = \sum_{i=1}^n \lambda_i f_i$ and $y = \sum_{i=1}^n \mu_i f_i$.*

EXAMPLE 1.3.3 In the space S^n it is easy to observe that matrices X and Y operator commute iff $XY = YX$. Also in Λ^n, vectors x and y operator commute iff either $\bar{y}$ is a multiple of $\bar{x}$ or $\bar{x}$ is a multiple of $\bar{y}$.

DEFINITION 1.3.1 (a) A linear transformation $\Psi : \mathcal{A} \to \mathcal{A}$ is said to be an *automorphism* of $\mathcal{A}$ if Ψ is invertible and $\Psi(x \circ y) = \Psi(x) \circ \Psi(y)$ for all $x, y \in \mathcal{A}$. The set of all automorphisms of $\mathcal{A}$ is denoted by $Aut(\mathcal{A})$.

(b) A linear transformation $\Theta : \mathcal{A} \to \mathcal{A}$ is said to be an *automorphism* of $\mathcal{K}$ if $\Theta(\mathcal{K}) = \mathcal{K}$. The set of all automorphisms of $\mathcal{K}$ is denoted by $Aut(\mathcal{K})$.

We shall now illustrate the above concepts through some examples.

EXAMPLE 1.3.4 ([62]) In the Euclidean Jordan algebra S^n, corresponding to any automorphism $\Psi \in Aut(S^n)$, there exists a real orthogonal matrix U such that

$$\Psi(X) = UXU^T \quad (\forall X \in S^n).$$

For an automorphism $\Theta \in Aut(S^n_+)$ there exists an $n \times n$ invertible matrix Q such that

$$\Theta(X) = QXQ^T \quad (\forall X \in S^n).$$

EXAMPLE 1.3.5 ([43]) In a Euclidean Jordan algebra Λ^n an $n \times n$ matrix A (or $-A$) belongs to $Aut(\Lambda^n_+)$ iff there exist a $\mu > 0$ such that

$$A^T J_n A = \mu J_n$$

where $J_n = \text{diag}(1, -1, -1, \ldots, -1)$. Also any $A \in Aut(\Lambda^n)$ can be written as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

where $D : R^{n-1} \to R^{n-1}$ is an orthogonal matrix.

PROPOSITION 1.3.1 (Proposition 6, [24]) *For $x, y \in \mathcal{A}$, the following conditions are equivalent.*

(i) $x \in \mathcal{K}, y \in \mathcal{K},$ *and* $\langle x, y \rangle = 0.$

(ii) $x \in \mathcal{K}, y \in \mathcal{K},$ *and* $x \circ y = 0.$

In each case, elements x and y operator commute.

1.4 Summary of the Thesis

Various matrix classes (P, Q, strictly semimonotone, R_0, copositive, etc.) have been found useful in the study of existence and uniqueness of solutions, and the working of various algorithms for LCP over R^n_+. A similar approach has been made by Gowda and Song [18] who initiated the study of solution properties of a SDLCP particularly for Lyapunov and Stein transformations. Following the same approach, in this thesis, we shall define various transformation classes for a general cone LCP as well as for LCP over specialized cones. We shall concentrate on the solution properties (existence and uniqueness) of a cone LCP when the transformations belong to such classes, along with the geometrical aspects.

In **Chapter 2** we introduce the notion of a complementary cone for a cone LCP, motivated by a similar notion in LCP over R^n_+, see [8, 52]. We shall also introduce the notion of a nondegenerate linear transformation and study the finiteness of the solution set of a cone linear complementarity problem.

(Results of this chapter have appeared in [48].)

In **Chapter 3** we consider the linear complementarity problem over the cone of squares (symmetric cone) in a Euclidean Jordan algebra. The Euclidean Jordan algebraic framework unifies various linear complementarity problems such as semidefinite and second-order cone linear complementarity problems. We study the facial structure of a symmetric cone and use it in generalizing the notion of a strictly semimonotone matrix and **SSM** linear transformation from the theory of LCP [8] and SDLCP [18, 47] respectively, to a Euclidean Jordan algebraic setting. We shall relate the **SSM**-property to the uniqueness of a solution to LCP($\mathcal{K}, L, q$) for all $q \in \mathcal{K}$ and the strict copositivity of self-adjoint linear transformations on $\mathcal{A}$. Finally, we study the **Q**-property of all principal subtransformations in the context of a LCP over a proper cone in a finite dimensional inner product space and connect it to the strict semimonotonicity when specialized in the setting of a Euclidean Jordan algebra.

(A part of this chapter has appeared in [45].)

In **Chapter 4** we shall discuss relationship between **Q** and $\mathbf{R_0}$ properties of a second-order cone invariant transformations. Our study is motivated by a result of Murty [52] in the context of a LCP over R^n_+ with a nonnegative square matrix. It states that LCP(R^n_+, M, q) is solvable for all $q \in R^n$ if and only if the diagonal entries of M are positive (equivalent to M being R_0), where M is a $n \times n$ nonnegative matrix. Though we do not have a complete generalization of Murty's result to a second-order cone, in this chapter we characterize the $\mathbf{R_0}$-property of a quadratic representation $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$ of Λ^n for $a, x \in \Lambda^n$ where '$\circ$' is a Jordan product and show that the $\mathbf{R_0}$-property of P_a is equivalent to stating that SOCLCP(P_a, q) has a solution for all $q \in \Lambda^n$.

(Results of this chapter have appeared in [46].)

In **Chapter 5** we shall study whether the matrix representation of a linear transformation $L : S^n \rightarrow S^n$, denoted as $\mathcal{N}(L)$, with the **P**-property, with respect to the canonical basis in S^n, is a P-matrix [8]. The motivation for asking the above is partly the issue studied in Theorem 8 of [18] (also see [20]). Also, when L is self-adjoint we have the following equivalence:

L_{FF} has the **P**-property for all $F \Leftrightarrow L$ has **P**-property $\Leftrightarrow L$ is strictly monotone $\Leftrightarrow \mathcal{N}(L)$ is symmetric positive definite, (see Theorem 1 in [22]).

(A part of this chapter has appeared in [47].)

Finally, **Chapter 6** finishes with some open problems and concluding remarks.

Chapter 2

Complementary Cones and Nondegenerate Transformations

In this chapter we introduce and study the notion of a *complementary cone*, *nondegenerate* complementary cone and *nondegenerate linear transformation* in connection with the LCP over a closed convex cone K, generalizing the notion of a complementary cone and nondegenerate matrix studied in linear complementarity theory, see [8, 52]. We study the closedness and the boundary structure of a complementary cone in the cone LCP. We show that unlike complementary cones in LCP over R^n_+, complementary cones in a cone LCP need not be closed. However, closedness of complementary cones is shown to be a necessary condition for many of the important solution properties of a cone LCP to hold. Finally, we unify and prove the results of Gowda and Song [21], Malik [45] and Tao [67] characterizing the finiteness of the solution set of a SDLCP, LCP over a symmetric cone and SOCLCP, respectively, for any cone LCP.

2.1 Complementary cones

The notion of a complementary cone has been introduced by Murty [52] in relation to a LCP over R^n_+. This notion is well studied in the literature on the LCP theory, see [8]. It has been found useful in studying the existence and multiplicity of solutions to LCP over R^n_+ and in studying a geometric interpretation of pivoting algorithms to solve the LCP. The book by Cottle, Pang and Stone [8] provides a detailed study of all the above concepts. The notion of a complementary cone has been extended to the semidefinite linear complementarity problems in [47]. It is further studied in the context of a second-order cone linear complementarity problem [67] and LCP over a symmetric cone in a Euclidean Jordan algebra [45].

Motivated by the above we present the following generalization of the concept of a complementary cone. Subsequently, we show how complementary cones explain the geometry and the solution properties of a cone LCP.

DEFINITION 2.1.1 Given a linear transformation $L : V \to V$ a *complementary cone* of L corresponding to the face F of K is defined as

$$\mathcal{C}_F := F^{\triangle} - L(F)$$

OBSERVATION 2.1.1 *The linear complementarity problem LCP(K, L, q) has a solution if and only if there exists a face F of K such that $q \in \mathcal{C}_F$.*

Proof. Suppose $x \in V$ solves the LCP(K, L, q). Then $x \in K$, $y := L(x) + q \in K$ and $\langle x, y \rangle = 0$. Taking F to be the smallest face of K containing x we get $y = L(x) + q \in F^{\triangle}$. Hence $q \in \mathcal{C}_F$. The converse is obvious. ∎

In the literature on LCP theory, see [8, 9], the set of all vectors q for which the LCP(K, L, q) has a solution is called the *range set* of a LCP defined by a cone K and a linear transformation L. By the above observation, the range set of a LCP defined by K and L is the union of all complementary cones of L. The notion of a range set is also studied in the context of a variational inequality problem VI(K, f) where K is a closed convex set and f is an affine map. It is closely related to the existence and stability theory of the VI problem, see [15, 16, 58]. The following example shows that complementary cones and their union are not closed in general. However, it is easy to see that complementary cones and the range are closed when K is a polyhedral cone.

EXAMPLE 2.1.1 Let Λ^3_+, the second-order cone in R^3, be defined as $\Lambda^3_+ := \{x = (x_0, x_1, x_2)^T \in R^3 : (x_1^2 + x_2^2)^{\frac{1}{2}} \leq x_0\}$. Let $M : R^3 \to R^3$ be a matrix defined as

$$M(x) = \begin{pmatrix} (x_0 + x_1)/2 \\ (x_0 + x_1)/2 \\ x_2 \end{pmatrix}.$$

Then $M \begin{pmatrix} \frac{1}{2}(\epsilon + \frac{1}{\epsilon}) \\ \frac{1}{2}(\epsilon - \frac{1}{\epsilon}) \\ -1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ as $\epsilon \downarrow 0$. However, there exist no $x \in \Lambda^3_+$ such that $M(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Thus the complementary cone of M corresponding to the face Λ^3_+ is not closed.

Also SOCLCP(M, q) with $q = (0, 0, 1)^T$ does not have a solution. However, for the sequence $q_t = (-t/2, -t/2, 1)^T$ converging to q as $t \downarrow 0$ note that SOCLCP(M, q_t) has a solution for all $t > 0$. Thus the union of complementary cones is also not closed in general.

In our next proposition we give a sufficient condition for the closedness of a complementary cone of a given linear transformation L and corresponding to a given face F. For this we appeal to a result which can be deduced from Theorem 9.1 in [61], but an independent proof has been supplied here.

Lemma 2.1.1 *Let K be a closed convex cone in R^n and $A : R^n \to R^m$ be a $m \times n$ real matrix. If $Az = 0,\ z \in K$ implies $z = 0$, then $A(K)$ is closed.*

Proof. Let $\{x_t\}$ be a sequence in K such that $A(x_t) \to y$. We shall show that $y \in A(K)$. When $y = 0$ the result is trivial, so consider the case $y \neq 0$. Then $x_t \neq 0$ for all large t and hence there exists a subsequence $\{x_s\}$ of $\{x_t\}$ such that $\frac{x_s}{||x_s||}$ converges to some $w \in K$. Thus $\frac{A(x_s)}{||x_s||} \to A(w)$. Note that the sequence $\{x_s\}$ is bounded, otherwise we have $A(w) = 0$, which by the hypothesis gives $w = 0$, a contradiction. Since $\{x_s\}$ is bounded, it has a subsequence converging to some $x \in K$. Thus we have $y = A(x) \in K$, which completes the proof. ■

Proposition 2.1.1 *Given a linear transformation $L : V \to V$ and a face F of a closed convex cone K, the complementary cone $\mathcal{C}_F$ is closed if*

$$x \in F,\ L(x) \in F^{\triangle} \text{ implies } x = 0.$$

Proof. By Lemma 2.1.1 and the condition described above, it is apparent that $L(F)$ is closed. Let $\widetilde{L} : V \times V \to V$ be defined as $\widetilde{L}(x, y) = x + y$. Let $\mathcal{C}_F = \{y - L(x) : x \in F \text{ and } y \in F^{\triangle}\}$ be a complementary cone corresponding to the face F. Let $K_1 := \{y : y \in F^{\triangle}\}$ and $K_2 := \{-L(x) : x \in F\}$. Then $\widetilde{L}(K_1 \times K_2) = K_1 + K_2 = \mathcal{C}_F$. Now, $\widetilde{L}(y, -L(x)) = 0$ for some $x \in F$ and $y \in F^{\triangle}$ implies that $y - L(x) = 0 \Rightarrow L(x) \in F^{\triangle}$, which by the given condition yields $x = 0$. Thus we have $y = L(x) = 0$. Appealing to Lemma 2.1.1 again, we conclude that $\mathcal{C}_F$ is closed. ∎

REMARK 2.1.1 L has the $\mathbf{R_0}$-property if and only if for every $F \triangleleft K$ the following relation holds:

$$x \in F,\ L(x) \in F^{\triangle} \Rightarrow x = 0.$$

PROPOSITION 2.1.2 *Let L has the $\mathbf{R_0}$-property. Then the union of all complementary cones is closed.*

Proof. The union of all complementary cones of L can be represented by the set $S = \{q : \mathrm{SOL}(K, L, q) \neq \phi\}$. Let $\{q_n\}$ be a sequence in S such that $q_n \to q$ and $\{x_n\}$ is a corresponding sequence of solutions to $\mathrm{LCP}(K, L, q_n)$. Note that $\{x_n\}$ is bounded, otherwise we can construst a sequence $\frac{x_m}{||x_m||}$ converging to some nonzero $x \in K$, contradicting the $\mathbf{R_0}$-property. Since $\{x_n\}$ is bounded, there exists a convergent subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y \in \mathrm{SOL}(K, L, q)$.

∎

2.2 Nondegenerate linear transformations

Murty [52] first used the term "nondegenerate matrix" in the context of a LCP over R^n_+ to study the finiteness of the set $\mathrm{SOL}(R^n_+, M, q)$ for all $q \in R^n$. It is important to note that the nondegeneracy of matrices is neither related to the concept of nondegeneracy of basic solutions of equations nor to the concept of nondegenerate solution in LCP over R^n_+. ($x \in R^n$ is a nondegenerate solution of LCP(R^n_+, M, q) if x is a solution with $x + Mx + q \in \text{int } R^n_+$).

In the case of LCP over R^n_+, nondegeneracy of a matrix can be described by $x * Mx = 0 \Rightarrow x = 0$, where $x * Mx$ is the componentwise product of x and Mx. Gowda snd Song [21] extended this notion to SDLCP by the condition $X \in S^n,\ XL(X) = 0 \Rightarrow X = 0$. In this section we extend and study the notion of a nondegenerate linear transformation on V and study the facial structure of complementary cones in a cone LCP.

THEOREM 2.2.1 ([21]) *Let $A \in R^{n\times n}$. Then the Lyapunov transformation L_A is nondegenerate if and only if $0 \notin \sigma(A) + \sigma(A)$, where $\sigma(A)$ denotes the set of eigenvalues of A.*

DEFINITION 2.2.1 (a) A complementary cone $\mathcal{C}_F$ of L corresponding to the face F is called *nondegenerate* if

$$x \in \operatorname{span} F,\ L(x) \in \operatorname{span} F^{\triangle} \Rightarrow x = 0.$$

A complementary cone which is not nondegenerate is called *degenerate.*

(b) A linear transformation L is *nondegenerate* if $\mathcal{C}_F$ is nondegenerate for every $F \lhd K$.

REMARK 2.2.1 (i) By Proposition 2.1.1 every nondegenerate complementary cone is closed.

(ii) For $V = S^n$ and $K = S^n_+$ a linear transformation $L : V \to V$ is nondegenerate if and only if (see also [21])

$$X \in S^n,\ XL(X) = 0 \Rightarrow X = 0.$$

PROPOSITION 2.2.1 *For any face F of a closed convex cone K,* $\det L_{FF} \neq 0$ *implies that $\mathcal{C}_F$ is nondegenerate. Moreover, if* $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}$ *for a nonzero $F \lhd K$, then $\mathcal{C}_F$ is nondegenerate if and only if* $\det L_{FF} \neq 0$.

Proof. Let $\det L_{FF} \neq 0$ for $F \lhd K$. Let $x \in \operatorname{span} F$ such that $L(x) \in \operatorname{span} F^{\triangle}$. Then $L_{FF}(x) = 0$. Since L_{FF} is nonsingular, it implies that $x = 0$. Conversely, suppose that $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}$ and $\mathcal{C}_F$ is nondegenerate for some nonzero $F \lhd K$. Let $0 \neq x \in \operatorname{span} F$ such that $L_{FF}(x) = 0$. Then $L(x) \in \operatorname{span} F^{\triangle}$, which contradicts the fact that $\mathcal{C}_F$ is nondegenerate. This completes the proof. ■

COROLLARY 2.2.1 *Suppose that* $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}$ *for all nonzero $F \lhd K$. Then the following are equivalent.*

(i) *L is nondegenerate.*

(ii) $\det L_{FF} \neq 0$ *for every nonzero $F \lhd K$.*

Below we give an example of a nonpolyhedral convex cone to illustrate the hypothesis in the above corollary.

EXAMPLE 2.2.1 ([66]) Take C to be the compact convex set in R^2 with extreme points at $(0, 1)$, $(0, 0)$, and $(1/k, (1/k)^2)$, $k = 1, 2, ...$ Let K be the proper cone in R^3 given by

$$K := \{\lambda(x, 1) : \lambda \geq 0 \text{ and } x \in C\}.$$

Note that K is not polyhedral (K has countably infinite number of faces). K has the property that for each face F of K, $\dim F + \dim F^{\triangle} = 3$. Thus, any matrix $M \in R^{3\times 3}$ satisfies the hypothesis in Corollary 2.2.1.

REMARK 2.2.2 In particular, when $K = R^n_+$, we note that the condition of Corollary 2.2.1 is satisfied and hence an $n \times n$ real matrix M in LCP over R^n_+ is nondegenerate if and only if all the principal minors of M are nonzero. (Also see [8].)

Below we give an example to show that, in general, nondegeneracy of a linear transformation need not imply the nonsigularity of all principal subtransformations.

EXAMPLE 2.2.2 Consider a Lyapunov transformation $L_A : S^2 \to S^2$ for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

By Theorem 2.2.1, L_A is nondegenerate, however, $\det (L_A)_{FF} = 0$ corresponding to the face F of S^2_+ generated by $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Definition 2.2.1 (b) of a nondegenerate linear transformation is motivated by the uniqueness of a solution to a LCP on a given face and has been explained in the following proposition.

PROPOSITION 2.2.2 *Given a linear transformation* $L : V \to V$ *and a* $F \lhd K$, $\mathcal{C}_F$ *is nondegenerate if and only if for each* $q \in \mathcal{C}_F$ *there exist a unique* $x \in F$ *and* $y \in F^{\triangle}$ *such that* $q = y - L(x)$.

Proof. Suppose there exist $x_1,\ x_2 \in F$ and $y_1,\ y_2 \in F^{\triangle}$ such that $q = y_1 - L(x_1) = y_2 - L(x_2)$, which implies that $y_1 - y_2 = L(x_1 - x_2)$, where $x_1 - x_2 \in \operatorname{span} F$ and $y_1 - y_2 \in \operatorname{span} F^{\triangle}$. By the nondegeneracy of L, $x_1 = x_2$ and $y_1 = y_2$. Conversely, suppose there exists a $x \in \operatorname{span} F$ such that $L(x) \in \operatorname{span} F^{\triangle}$. Writing $x = x_1 - x_2$ with $x_1,\ x_2 \in F$ and $L(x) = y_1 - y_2$ with $y_1,\ y_2 \in F^{\triangle}$ we get

$$\bar{q} := y_1 - L(x_1) = y_2 - L(x_2).$$

Since each $q \in \mathcal{C}_F$ has a unique representation in $\mathcal{C}_F$ we get $x_1 = x_2$ and $y_1 = y_2$. ■

REMARK 2.2.3 If L has the property that LCP(K, L, q) has a unique solution for all $q \in V$, then L is nondegenerate and hence all the complementary cones of L are closed.

COROLLARY 2.2.2 *Given $L : V \to V$ and $q \in V$, LCP(K, L, q) has infinitely many solutions only if either q is contained in a degenerate complementary cone or q lies in infinitely many complementary cones.*

Proof. Suppose the assertion is not true, then it implies that q lies in an at most finitely many nondegenerate complementary cones. Thus LCP(K, L, q) can have at most finitely many solutions negating our hypothesis. ■

THEOREM 2.2.2 *Let $\mathcal{C}_F$ be a complementary cone of L corresponding to the face F. Then the following statements hold.*

(i) *If $\mathcal{C}_F$ is nondegenerate, then $q \in \operatorname{ri} \mathcal{C}_F$ if and only if there exist $x \in \operatorname{ri} F$ and $y \in \operatorname{ri} F^{\triangle}$ such that $q = y - L(x)$.*

(ii) *Any face $\mathcal{G}$ of $\mathcal{C}_F$ can be represented as*

$$\mathcal{G} = \{y - L(x) : x \in H \text{ and } y \in H^{'}\},$$

where H is a face of F and $H^{'}$ is a face of $F^{\triangle}$. Moreover, if $\mathcal{C}_F$ is nondegenerate, then any set of the above form is a face of $\mathcal{C}_F$.

Proof. The proof of (i) is easy and is left to the reader. For the proof of (ii) let $\mathcal{G}$ be a face of $\mathcal{C}_F$ for some face F. Define the sets

$$\begin{aligned} H &:= \{x \in F : -L(x) \in \mathcal{G}\}, \\ H^{'} &:= \{y \in F^{\triangle} : y \in \mathcal{G}\}, \text{ and} \\ \mathcal{G}^{'} &:= \{y - L(x) : x \in H \text{ and } y \in H^{'}\}. \end{aligned}$$

As $\mathcal{G}$ is a convex cone, $\mathcal{G}^{'} \subseteq \mathcal{G}$. We shall show that $\mathcal{G} = \mathcal{G}^{'}$. Consider a point in $\mathcal{G}$. Since such a point is in $\mathcal{C}_F$, it can be expressed in the form $y - L(x)$ where $x \in F$ and $y \in F^{\triangle}$. Since F and $F^{\triangle}$ are closed convex cones, both contain 0. Thus, $\mathcal{C}_F$ contains both y and $-L(x)$. Since $y \in \mathcal{C}_F$, $(y - L(x)) - y = -L(x) \in \mathcal{C}_F$, and $y - L(x) \in \mathcal{G}$, we have $y \in \mathcal{G}$, so $y \in H^{'}$. Likewise $-L(x) \in \mathcal{G}$, so $x \in H$. Thus, $y - L(x) \in \mathcal{G}^{'}$ and, hence, $\mathcal{G}$ is contained in $\mathcal{G}^{'}$. Therefore, $\mathcal{G} = \mathcal{G}^{'}$.

We now need to show that $H \lhd F$, and $H^{'} \lhd F^{\triangle}$. Since $0 \in F \cap F^{\triangle}$, $0 \in \mathcal{C}_F$. Thus, as $\mathcal{C}_F$ is a cone, any face of $\mathcal{C}_F$ must contain 0, so $\mathcal{G}$ contains 0. By their definitions, it follows that $0 \in H \cap H^{'}$. It is also easy to check from their definitions that H and $H^{'}$ are convex cones. Now if $x \in F$, $z - x \in F$ and $z \in H$, then $-L(x) \in \mathcal{C}_F$, $-L(z - x) \in \mathcal{C}_F$ and $-L(z) \in \mathcal{G}$. Since $\mathcal{G} \lhd \mathcal{C}_F$ we get $-L(x) \in \mathcal{G}$, so $x \in H$ and $H \lhd F$. Similarly, we can show that $H^{'} \lhd F^{\triangle}$.

Conversely, let $\mathcal{C}_F$ be nondegenerate and $\mathcal{N}$ be defined as $\mathcal{N} := \{y - L(x) : x \in H,\ y \in H^{'}\}$, where $H \lhd F$ and $H^{'} \lhd F^{\triangle}$. Obviously, $\mathcal{N}$ is a convex cone and

$\phi \neq \mathcal{N} \subseteq \mathcal{C}_F$. Let $y - L(x) \in \mathcal{C}_F$, $(y_0 - y) - L(x_0 - x) \in \mathcal{C}_F$, and $y_0 - L(x_0) \in \mathcal{N}$, where $x_0 \in H$, $x \in F$, $y_0 \in H'$ and $y \in F^{\triangle}$. Since $\mathcal{C}_F$ is nondegenerate $x_0 - x \in F$ and $y_0 - y \in F^{\triangle}$. Thus,

$$x \in F,\ x_0 - x \in F, \text{ and } x_0 \in H;$$
$$y \in F^{\triangle},\ y_0 - y \in F^{\triangle}, \text{ and } y_0 \in H'.$$

Since H and H' are the faces of F and $F^{\triangle}$, respectively, we get $x \in H$ and $y \in H'$. Hence $y - L(x) \in \mathcal{N}$ and $\mathcal{N}$ is a face of $\mathcal{C}_F$. ■

Corollary 2.2.3 *Given a linear transformation $L : V \rightarrow V$ and $q \in V$, the LCP(K, L, q) has infinitely many distinct solutions if q lies in the relative interior of infinitely many nondegenerate complementary cones.*

Proof. Let $q \in \cap\, \mathrm{ri}\, \mathcal{C}_{F_\alpha}$, where F_α is a family of distinct faces of K indexed by α and $\mathcal{C}_{F_\alpha}$ is nondegenerate for each α. Then $q = y_\alpha - L(x_\alpha)$ for $x_\alpha \in \mathrm{ri}\, F_\alpha$ and $y_\alpha \in \mathrm{ri}\, F_\alpha^{\triangle}$. Since each $\mathcal{C}_{F_\alpha}$ is a nondegenerate complementary cone, $x_\alpha\ \forall\ \alpha$ are infinitely many distinct solutions to LCP(K, L, q). ■

2.3 Finiteness of the solution set of a cone LCP

As mentioned before, in the context of a LCP over R^n_+, nondegenerate matrices characterize the finiteness of the solution set of a LCP(R^n_+, M, q) for all $q \in R^n$, see [8]. A similar study is made by Gowda and Song [21] in the context of a semidefinite linear complementarity problem. They have shown that when $K = S^n_+$, nondegeneracy of a linear transformation L need not be a sufficient condition for the finiteness of a solution set of a SDLCP(L, Q) for all $Q \in S^n_+$. The example below throws more light on the preceding discussion.

EXAMPLE 2.3.1 Let $M : R^3 \to R^3$ be defined as $M(x) = -x$, $K = \{(x_0, x_1, x_2)^T : x_0 \geq 0,\ \frac{x_0^2}{4} \geq x_1^2 + x_2^2\}$ and $q := (\frac{5}{8}, 0, 0)^T$. It is easy to check that M is nondegenerate, K is closed and convex (but not self-dual) and any point $x = (x_0, x_1, x_2)^T$ lies on the boundary of K if and only if $x_0 \geq 0$ and $\frac{x_0^2}{4} = x_1^2 + x_2^2$. Any complementary cone corresponding to a face F is of the form $\mathcal{C}_F = \{y + x : x \in F,\ y \in F^{\triangle}\}$. Except two 3-dimensional complementary cones, namely K^* and K, every other complementary cone is of dimension 2. The infinite set of solutions to LCP(K, L, q) is given by $\{(\frac{1}{2}, \frac{x_1}{2}, \frac{x_2}{2})^T : x_1^2 + x_2^2 = \frac{1}{4}\}$.

DEFINITION 2.3.1 (a) A solution x_0 of LCP(K, L, q) is locally unique if it is the only solution in a neighborhood of x_0.

(b) A solution x_0 is *locally-star-like* if there exists a sphere $\mathcal{S}(x_0, r)$ such that

$$x \in \mathcal{S}(x_0, r) \cap \text{SOL}(K, L, q) \Rightarrow [x_0, x] \subseteq \text{SOL}(K, L, q).$$

The following theorem generalizes the earlier results on the finiteness of the solution set of a LCP over specialized cones, see [8, 21, 46, 52, 67], to LCP over a closed convex cone in V.

THEOREM 2.3.1 *Given a linear transformation $L : V \to V$ and a closed convex cone K in V, the following statements are equivalent.*

(i) *SOL(K, L, q) is finite for all $q \in V$.*

(ii) *Every solution of LCP(K, L, q) is locally unique for all $q \in V$.*

(iii) *L is nondegenerate, and for all $q \in V$, each solution of LCP(K, L, q) is locally-star-like.*

Proof. The assertion (i) $\Rightarrow$ (ii) is obvious. For the reverse implication, note that (ii) implies that L has the $\mathbf{R_0}$-property. Thus SOL(K, L, q) is compact for all q and hence (in view of (ii)) is finite for all q.

(ii) $\Rightarrow$ (iii): First we shall show that L is nondegenerate. Let $x \in V$ be nonzero such that $x \in \operatorname{span} F$, $L(x) \in \operatorname{span} F^{\triangle}$ for some face F of K. Since $x \in \operatorname{span} F$, we can write $x = x_1 - x_2$ with $x_1,\ x_2 \in F$. Similarly, $L(x) = L(x)_1 - L(x)_2$ with $L(x)_1,\ L(x)_2 \in F^{\triangle}$. Now taking $q := L(x)_1 - L(x_1) = L(x)_2 - L(x_2)$ it is observed that LCP(K, L, q) has two distinct solutions x_1 and x_2 with

$$\langle (tx_1 + (1-t)x_2), (tL(x)_1 + (1-t)L(x)_2) \rangle = 0 \ \forall\, t \in [0, 1],$$

i.e., $[x_1, x_2] \subseteq$ SOL(K, L, q) which contradicts (ii).

Also for any $q \in V$ since the solution $x_0 \in$ SOL(K, L, q) is locally unique, it is locally-star-like.

(iii) $\Rightarrow$ (ii): Let for some fixed $q \in V$, the solution x_0 of LCP(K, L, q) be not locally unique. Then there exist a sequence $\{x_k\} \subseteq$ SOL(K, L, q) converging to x_0 with $x_k \neq x_0$ for all k. By the locally-star-like property we have $[x_0, x_k] \subseteq$ SOL(K, L, q) for all large k. Let F_i be the smallest face of K containing x_i ($x_i \in \operatorname{ri} F_i$) where $i = 0,\ 1,\ 2,\ \ldots$ From the complementarity of solutions we have for all large k

$$x_0 \in \operatorname{ri} F_0 \text{ and } L(x_0) + q \in F_0^{\triangle};$$

$$x_k \in \operatorname{ri} F_k \text{ and } L(x_k) + q \in F_k^{\triangle}.$$

Also from the fact that $[x_0, x_k] \subseteq \mathrm{SOL}(K, L, q)$ for large k we get

$$\langle x_0, L(x_k) + q\rangle = 0 \text{ and } \langle x_k, L(x_0) + q\rangle = 0.$$

Since $x_0 \in \operatorname{ri} F_0$ and $x_k \in \operatorname{ri} F_k$ we get $L(x_k) + q \in F_0^{\triangle}$ and $L(x_0) + q \in F_k^{\triangle}$. Defining a face $G := F_0^{\triangle} \cap F_k^{\triangle}$ of K^* we get $x_0,\ x_k \in G^{\triangle}$ and $L(x_0) + q$, $L(x_k) + q \in G$. Thus there exists a face $F = G^{\triangle}$ of K such that a nonzero $x := x_0 - x_k \in \operatorname{span} F$ with $L(x) \in \operatorname{span} F^{\triangle}$, which contradicts our assumption that L is nondegenerate. ■

COROLLARY 2.3.1 *When K is polyhedral LCP(K, L, q) has a finite number of solutions for all $q \in V$ if and only if $\det L_{FF} \neq 0$ for all nonzero $F \lhd K$, or equivalently L is nondegenerate.*

PROPOSITION 2.3.1 *If L is a monotone linear transformation on V, then L is nondegenerate if and only if LCP(K, L, q) has a unique solution for all $q \in V$.*

Proof. Let x_1 and x_2 with $x_1 \neq x_2$ be the two solutions of $\mathrm{LCP}(K, L, q)$ for some $q \in V$. Let $x_1 \in \operatorname{ri} F_1$ and $x_2 \in \operatorname{ri} F_2$ where $F_1,\ F_2$ are the two faces of K. By the monotonicity of L we have

$$0 \leq \langle x_1 - x_2, L(x_1 - x_2)\rangle = \langle x_1 - x_2, y_1 - y_2\rangle = -\langle x_1, y_2\rangle - \langle x_2, y_1\rangle \leq 0,$$

where $y_i = L(x_i) + q$ for $i = \{1, 2\}$. Thus $\langle x_1, y_2\rangle = 0$ and $\langle x_2, y_1\rangle = 0$. Since $x_1 \in \operatorname{ri} F_1$ and $x_2 \in \operatorname{ri} F_2$, $y_1 \in F_2^{\triangle}$ and $y_2 \in F_1^{\triangle}$. Defining a face $G := F_1^{\triangle} \cap F_2^{\triangle}$ of K^* we get $x_1,\ x_2 \in G^{\triangle}$ and $L(x_1) + q$, $L(x_2) + q \in G$. Thus for a face $F = G^{\triangle}$ of K we have a nonzero $x := x_1 - x_2 \in \operatorname{span} F$, such that $L(x) \in \operatorname{span} F^{\triangle}$, which contradicts that L is nondegenerate. ■

PROPOSITION 2.3.2 *Let L be copositive on K. Then L is nondegenerate only if LCP(K, L, q) has a unique solution for all $q \in K^*$.*

The proof is similar to that of Proposition 2.3.1 above and is omitted.

Chapter 3

Strictly Semimonotone and Completely-Q Transformations

In this chapter we consider the linear complementarity problem over the cone of squares (symmetric cone) in a Euclidean Jordan algebra, studied by Gowda et al. in [24]. The Jordan algebraic framework unifies various linear complementarity problems such as SDLCP and second-order cone linear complementarity problems. We study the facial structure of a symmetric cone and use it in generalizing the notion of a strictly semimonotone matrix (linear transformation) from the theory of LCP [8] (SDLCP [18, 47]) to a Euclidean Jordan algebraic setting. We shall relate the strict semimonotonicity to the uniqueness of a solution to LCP$(\mathcal{K}, L, q)$ for all $q \in \mathcal{K}$. We also show that for a self-adjoint linear transformation L on $\mathcal{A}$, strict semimonotonicity of L is equivalent to the strict copositivity of L on $\mathcal{K}$. Finally, motivated by the class of completely-Q-matrices and their connection with the strictly semimonotone matrices in the standard

LCP theory [7, 8], we study the **Q**-property of all principal subtransformations in the context of a cone LCP and connect it to the strict semimonotonicity when specialized in the setting of a Euclidean Jordan algebra.

We state below the existence theorem of Karamardian [38] in the context of a LCP over a proper cone in a finite dimensional space.

THEOREM 3.0.2 **(Karamardian's theorem)** *Let $L : V \to V$ be a linear transformation. Let K be a proper cone in V and $\tilde{q} \in \operatorname{int} K^*$. If the problems LCP$(K, L, 0)$ and LCP$(K, L, \tilde{q})$ have unique solutions (namely zero), then LCP(K, L, q) has a solution for all $q \in V$.*

3.1 Faces of the symmetric cone

In a Euclidean Jordan algebra $\mathcal{A}$ of rank n, fix a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ and define sets

$$\mathcal{A}_+^{(r)} := \{x \in \mathcal{K} : x \circ (e_1 + e_2 + \ldots + e_r) = x\}$$

and

$$\mathcal{A}^{(r)} := \{x \in \mathcal{A} : x \circ (e_1 + e_2 + \ldots + e_r) = x\}$$

for $1 \leq r \leq n$. Then we have the following lemma.

LEMMA 3.1.1 *If $x \in \mathcal{A}^{(r)}$, then x and $\sum_{i=1}^{r} e_i$ operator commute and the number of non-zero eigenvalues of x is less than or equal to r.*

Proof. We shall make use of the following identity in a Euclidean Jordan algebra (Proposition II.1.1 in [10]):

$$[L_u, L_{v^2}] + 2[L_v, L_{u \circ v}] = 0,$$

for all $u, v \in \mathcal{A}$, where $[A, B] := AB - BA$. Substituting $u = x$ and $v = \sum_{i=1}^{r} e_i$, and noting that $\sum_{i=1}^{r} {e_i}^2 = \sum_{i=1}^{r} e_i$ we get $L_x L_{\sum_{i=1}^{r} e_i} = L_{\sum_{i=1}^{r} e_i} L_x$. Since x and $\sum_{i=1}^{r} e_i$ operator commute there exists a Jordan frame $\{f_1, f_2, \dots, f_n\}$ and real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $x = \sum_{i=1}^{n} \lambda_i f_i$ and $\sum_{i=1}^{r} e_i = \sum_{i=1}^{r} f_i$. Substituting the value of x and $\sum_{i=1}^{r} e_i$ in the equality $x \circ \sum_{i=1}^{r} e_i = x$ we get $\lambda_i = 0$ for $i = r+1, \dots, n$. ■

Proposition 3.1.1 (i) *The set* $\mathcal{A}_+^{(r)}$ *defined above corresponding to the Jordan frame* $\{e_1, e_2, \dots, e_n\}$ *is the smallest face of* $\mathcal{K}$ *containing* $\sum_{i=1}^{r} e_i$.

(ii) *The set* $\mathcal{A}^{(r)}$ *is the smallest subspace of* $\mathcal{A}$ *containing* $\mathcal{A}_+^{(r)}$.

Proof. (i) First we shall show that $\mathcal{A}_+^{(r)}$ is a face of $\mathcal{K}$. Clearly $\mathcal{A}_+^{(r)}$ is a convex cone. Take an $x \in \mathcal{K}$, $y - x \in \mathcal{K}$ and $y \in \mathcal{A}_+^{(r)}$. Since $y \in \mathcal{A}_+^{(r)}$, $y \circ \sum_{i=r+1}^{n} e_i = 0$. By Proposition 1.3.1, $\langle y, \sum_{i=r+1}^{n} e_i \rangle = 0$. Thus,

$$\langle y - x, \textstyle\sum_{i=r+1}^{n} e_i \rangle = -\langle x, \textstyle\sum_{i=r+1}^{n} e_i \rangle \leq 0.$$

Since $y - x \in \mathcal{K}$, we get $\langle x, \sum_{i=r+1}^{n} e_i \rangle = 0$, which by Proposition 1.3.1 gives $x \circ \sum_{i=r+1}^{n} e_i = 0$, proving that $\mathcal{A}_+^{(r)}$ is a face of $\mathcal{K}$. Now in the light of Lemma 2.8 in [2], showing $\sum_{i=1}^{r} e_i \in \operatorname{ri} \mathcal{A}_+^{(r)}$ is equivalent to showing that for every $x \in \mathcal{A}_+^{(r)}$ there exists $\alpha > 0$ such that $\sum_{i=1}^{r} e_i - \alpha x \in \mathcal{K}$. Take an arbitrary $x \in \mathcal{A}_+^{(r)}$. Then

$$x \circ (\textstyle\sum_{i=1}^{r} e_i - \alpha x) = x - \alpha x^2 = x \circ (e - \alpha x).$$

We can choose $\alpha > 0$, sufficiently small, so that $e - \alpha x \in \mathcal{K}$. Since $x \in \mathcal{K}$, $e - \alpha x \in \mathcal{K}$ for the above $\alpha > 0$, and x and $e - \alpha x$ operator commute, we have $x \circ (e - \alpha x) = x \circ (\sum_{i=1}^{r} e_i - \alpha x) \in \mathcal{K}$. By Lemma 3.1.1, x and $\sum_{i=1}^{r} e_i$ operator commute and hence they have a common spectral decomposition, i.e, there exists a Jordan frame $\{f_1, f_2, \dots, f_n\}$ such that

$$x = \sum_{i=1}^{k} \lambda_i f_i \text{ and } \sum_{i=1}^{r} e_i - \alpha x = \sum_{i=1}^{r} f_i - \alpha \sum_{i=1}^{k} \lambda_i f_i$$

where $k \leq r \leq n$ and $\lambda_i > 0$ for $i = 1, \ldots, k$. Also,

$$\begin{aligned} x \circ \left(\sum_{i=1}^{r} e_i - \alpha x\right) = \sum_{i=1}^{k} \lambda_i (1 - \alpha\lambda_i) f_i \in \mathcal{K} \\ \Rightarrow \lambda_i(1 - \alpha\lambda_i) \geq 0 \\ \Rightarrow 1 - \alpha\lambda_i \geq 0 \end{aligned}$$

for $i = 1, \ldots, k$. Thus, $\sum_{i=1}^{r} e_i - \alpha x \in \mathcal{K}$ for the above chosen $\alpha > 0$ and since $x \in \mathcal{A}_+^{(r)}$ is arbitrary, we have proved our claim.

(ii) Since $\mathcal{A}^{(r)}$ is a subspace of $\mathcal{A}$ containing $\mathcal{A}_+^{(r)}$, $x - y \in \mathcal{A}^{(r)}$ for all $x, y \in \mathcal{A}_+^{(r)}$. Conversely, for any $x \in \mathcal{A}^{(r)}$ there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ such that $x = \sum_{i=1}^{r} \lambda_i f_i$ and $\sum_{i=1}^{r} e_i = \sum_{i=1}^{r} f_i$. Define

$$x^+ := \sum_{i=1}^{r} \lambda_i^+ f_i \text{ and } x^- := \sum_{i=1}^{r} \lambda_i^- f_i$$

where $\lambda^+ = \max\{\lambda_i, 0\}$ and $\lambda_i^- = \lambda_i^+ - \lambda_i$. Thus, we have $x = x^+ - x^-$ with x^+, x^- in $\mathcal{A}_+^{(r)}$, which proves our claim. ■

THEOREM 3.1.1 *Let $\mathcal{A}$ be a Euclidean Jordan algebra of rank n. Let $x \in \mathcal{K}$ be an element of $\mathcal{A}$ having exactly r nonzero eigenvalues. Then the smallest face of $\mathcal{K}$ containing x is of the form*

$$W_+^{(r)} := \{y \in \mathcal{K} : y \circ (f_1 + f_2 + \ldots + f_r) = y\},$$

where $\{f_1, f_2, \ldots, f_n\}$ constitutes a Jordan frame.

Proof. In view of the spectral theorem, there exists a Jordan frame $\{f_1, f_2, \ldots, f_n\}$ and positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ where $1 \leq r \leq n$ such that x can be written as $x = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_r f_r$. Consider the set

$$W_+^{(r)} = \{y \in \mathcal{K} : y \circ (f_1 + f_2 + \ldots + f_r) = y\}.$$

We shall show that the smallest face of $\mathcal{K}$ containing x is $W_+^{(r)}$. Since $\mathcal{K}$ is a convex cone, without loss of generality assume that $0 < \lambda_i < 1$ for all i and $\sum_{i=1}^r \lambda_i = 1$. We can choose an $\alpha \in (0,\ 1)$, say $\alpha = \max\{1-\lambda_i\}$, such that the point $z = \frac{x}{\alpha} - \frac{(1-\alpha)}{\alpha}\sum_{i=1}^r f_i$ lies in $W_+^{(r)}$. Since $\sum_{i=1}^r f_i$ lies in the relative interior of $W_+^{(r)}$, $z \in W_+^{(r)}$ and $x = (1-\alpha)\sum_{i=1}^r f_i + \alpha z$ with $0 < \alpha < 1$, it follows that x lies in the relative interior of $W_+^{(r)}$. ■

Note that the above theorem characterizes all the faces of a symmetric cone in the sense that if F is a face of a symmetric cone $\mathcal{K}$ then it is the smallest face of $\mathcal{K}$ containing a point in its relative interior. The following example illustrates all the faces of the positive semidefinite cone in S^n. See also Hill and Waters [30].

EXAMPLE 3.1.1 Consider the space S^n with its associated cone of squares S_+^n and a Jordan frame $\{UE_{11}U^T, UE_{22}U^T, \ldots, UE_{nn}U^T\}$, where E_{ii} is an $n \times n$ diagonal matrix with $(i,i)^{th}$ entry 1 and others 0, and U is an orthogonal matrix. Then the set

$$W_+^{(r)} := \{X \in S_+^n : X \circ (UE_{11}U^T + UE_{22}U^T + \ldots + UE_{rr}U^T) = X\},$$

where $1 \le r \le n$, is of the form $\left\{ U \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} U^T : Y \in S_+^r \right\}$. Hence every face of S_+^n can be represented in the above form.

PROPOSITION 3.1.2 *For any $F \lhd \mathcal{K}$ following statements hold:*

(i) $x \in \operatorname{span} F,\ y \in \operatorname{span} F^\triangle \Leftrightarrow x$ *and* y *operator commute and* $x \circ y = 0$.

(ii) $x \in \operatorname{span} F,\ y \in \operatorname{span} F \Rightarrow x \circ y \in \operatorname{span} F$.

Proof. (i): Since x and y can be represented as a linear combination of the elements of F and $F^\triangle$, respectively, from Proposition 1.3.1 x and y operator commute and $x \circ y = 0$. Conversely, by the Spectral theorem and the condition $x \circ y = 0$ we can write

$$x = \sum_{i=1}^{r} \lambda_i f_i \text{ and } y = \sum_{i=r+1}^{n} \mu_i f_i.$$

Define a subset F of $\mathcal{K}$ as

$$F := \{z \in \mathcal{K} : z \circ (f_1 + f_2 + \ldots + f_r) = z\}.$$

By Proposition 3.1.1, $F \lhd \mathcal{K}$ and $x \in \operatorname{span} F$. Also it is easy to observe that $\operatorname{span} F^{\triangle}$ will be of the form

$$\operatorname{span} F^{\triangle} = \{z \in \mathcal{A} : z \circ (f_{r+1} + f_{r+2} + \ldots + f_n) = z\}.$$

Thus $y \in \operatorname{span} F^{\triangle}$.

(ii): Let x, $y \in \operatorname{span} F$ for some $F \lhd \mathcal{K}$. By Proposition 3.1.1 and Theorem 3.1.1, $\operatorname{span} F$ can be represented as

$$\operatorname{span} F = \{z \in \mathcal{A} : z \circ (f_1 + f_2 + \ldots + f_r) = z\}.$$

By Lemma 3.1.1, x and $\sum_{i=1}^{r} f_i$ operator commute which gives

$$(x \circ y) \circ (\textstyle\sum_{i=1}^{r} f_i) = x \circ ((\sum_{i=1}^{r} f_i) \circ y) = x \circ y.$$

Thus $x \circ y \in \operatorname{span} F$. ■

REMARK 3.1.1 (i) A linear transformation $L : \mathcal{A} \to \mathcal{A}$ is nondegenerate if and only if

$$x \text{ and } L(x) \text{ operator commute, } x \circ L(x) = 0 \Rightarrow x = 0.$$

(ii) For every $F \lhd \mathcal{K}$, $\operatorname{span} F$ is a Euclidean Jordan algebra with F as its cone of squares. Thus, every face of a symmetric cone is symmetric in its linear span.

3.2 Strictly Semimonotone transformations

Strictly semimonotone matrices are well studied in the standard LCP theory [8]. They characterize the class of matrices for which the LCP(R^n_+, M, q) has a unique (zero) solution for all $q \in R^n_+$. Also, in the space of real symmetric matrices, the class of strictly semimonotone matrices is equivalent to the class of strictly copositive matrices on R^n_+. On the algorithmic side, strictly semimonotone matrices are useful in the study of the variable dimension algorithm of Ludo Van der Heyden [32] to solve LCP over R^n_+.

Gowda and Song [18] generalized the above notion in the semidefinite linear complementarity problems. They have the following characterization of the **SSM**-property of a Lyapunov transformation in terms of the positive stability of a matrix. Note that a matrix $A \in R^{n\times n}$ is *positive stable* if the real part of every eigenvalue of A is positive.

Theorem 3.2.1 *Given a matrix $A \in R^{n\times n}$. The Lyapunov transformation L_A has the* **SSM***-property if and only if A is positive stable.*

The relationship between the **SSM**-property and the strict copositivity of self-adjoint linear transformation on S^n has been studied in [47].

In this section, we shall define the concept of a strictly semimonotone linear transformation in a Euclidean Jordan algebraic setting. We study its relationship with the uniqueness of the solution to LCP($\mathcal{K}, L, q$) for $q \in \mathcal{K}$. Also, for a self-adjoint linear transformation L on $\mathcal{A}$, the **SSM**-property of all principal subtransformations is shown to be equivalent to the strict copositivity of L on $\mathcal{K}$. We shall begin with the following definition.

DEFINITION 3.2.1 For a linear transformation $L : \mathcal{A} \to \mathcal{A}$, we say that

(a) L is *strictly semimonotone* (**SSM**) if

$$x \in \mathcal{K},\ x \text{ and } L(x) \text{ operator commute, and } x \circ L(x) \in -\mathcal{K} \Rightarrow x = 0.$$

(b) L_{FF} is *strictly semimonotone* if

$$x \in F,\ x \text{ and } L_{FF}(x) \text{ operator commute, and } x \circ L_{FF}(x) \in -F \Rightarrow x = 0.$$

It is known that every principal submatrix of a strictly semimonotone matrix is strictly semimonotone in a LCP over R^n_+, [8]. However, the following example shows that the strict semimonotonicity of L need not imply that L_{FF} is **SSM** for all $F \lhd \mathcal{K}$.

EXAMPLE 3.2.1 Consider the Lyapunov transformation $L_A : S^2 \to S^2$ for

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that A is positive stable and hence from Theorem 3.2.1 L_A has the **SSM**-property. However, for a face F of S^2_+ defined as

$$F := \left\{ \alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : \alpha \geq 0 \right\},$$

$(L_A)_{FF}(X) = 0$, for all $X \in \operatorname{span} F$. Hence, $(L_A)_{FF}$ does not have the **SSM**-property.

PROPOSITION 3.2.1 *If $L : \mathcal{A} \to \mathcal{A}$ is strictly semimonotone then $SOL(\mathcal{K}, L, q) \neq \phi$ for all $q \in \mathcal{A}$.*

Proof. It is easy to observe that L has the $\mathbf{R_0}$-property. Also, if $x \in \mathrm{SOL}(\mathcal{K}, L, e)$ then, by the Spectral theorem, x and $L(x)$ operator commute and $x \circ L(x) = -x \in$

$-\mathcal{K}$. By the strict semimonotonicity of L we have $x = 0$. Thus, by Karamardian's theorem, LCP$(\mathcal{K}, L, q)$ has a solution for all $q \in \mathcal{A}$. ■

THEOREM 3.2.2 *For a linear transformation $L : \mathcal{A} \to \mathcal{A}$ consider the following statements.*

(i) *L is strictly copositive on $\mathcal{K}$.*

(ii) *LCP$(\mathcal{K}, L, q)$ has a unique (zero) solution for all $q \in \mathcal{K}$.*

(iii) *L is strictly semimonotone.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *in the above statements.*

Proof. (i) $\Rightarrow$ (ii) is easy. For (ii) $\Rightarrow$ (iii) suppose that there exists $x \in \mathcal{K}$ such that x and $L(x)$ operator commute and $x \circ L(x) \in -\mathcal{K}$. From the spectral decomposition theorem, x and $L(x)$ can be written as

$$x = \sum_{i=1}^{n} \lambda_i f_i \text{ and } L(x) = \sum_{i=1}^{n} \beta_i f_i \ ,$$

where $\{f_1, f_2, \ldots, f_n\}$ constitutes a Jordan frame. For a real number a define $a^+ := \max\{a, 0\}$ and $a^- := \min\{-a, 0\}$. ¿From the condition $x \circ L(x) \in -\mathcal{K}$ which is equivalent to $\lambda_i \beta_i \leq 0$ for all i, we have

$$x \circ (L(x))^+ = 0$$

where $(L(x))^+ := \sum_{i=1}^{n} \beta_i^+ f_i$. Now taking $q := (L(x))^+ - L(x) \in \mathcal{K}$ it is observed that x solves LCP$(\mathcal{K}, L, q)$. From (ii) it implies that $x = 0$ and hence, L is **SSM.**

■

Since every face of a symmetric cone is symmetric in its linear span, we have the following corollary.

Corollary 3.2.1 *Given $F \lhd \mathcal{K}$, LCP(F, L_{FF}, q) has a unique (zero) solution for all $q \in F$ implies that L_{FF} is strictly semimonotone.*

The following example from [18] shows that strict semimonotonicity of L need not imply that LCP$(\mathcal{K}, L, q)$ has a unique (zero) solution for all $q \in \mathcal{K}$.

Example 3.2.2 [18] Consider the SDLCP(L_A, Q) for

$$A = \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix} \text{ and a positive definite } Q = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

Since A is positive stable, L_A has the **SSM**-property. However,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is a nonzero solution to SDLCP(L_A, Q).

To study the relationship between strict copositivity and strict semimonotonicity of self-adjoint linear transformations on $\mathcal{A}$ we shall generalize Lemma 2, [47], to a Euclidean Jordan algebraic setting, whose proof is an extension of the proof for Theorem 1, [35].

Lemma 3.2.1 *Let $L : \mathcal{A} \to \mathcal{A}$ be a self adjoint linear transformation on a Euclidean Jordan algebra $\mathcal{A}$. Then L is strictly copositive on $\mathcal{K}$ if every principal subtransformation L_{FF} of L has no eigenvector $v \in \operatorname{ri} F$ with associated eigenvalue $\lambda \leq 0$.*

Proof. Suppose there exists a nonzero $x_0 \in \mathcal{K}$ with $\langle x_0, L(x_0)\rangle \leq 0$. Define $\mathcal{S} := \{x \in \mathcal{K} : x \neq 0,\ \langle x, L(x)\rangle \leq 0\}$. Let $m(x)$ denote the number of positive eigenvalues of x. Since we can choose an $x \in \mathcal{S}$ that has the least number of positive eigenvalues among all $x \in \mathcal{S}$, we can assume, without loss of generality, that $r = m(x_0) \leq m(x), \ \forall \ x \in \mathcal{S}$. We consider the case $r > 1$. (For $r = 1$ the proof follows easily). Let F be the smallest face of $\mathcal{K}$ containing x_0 and $\Im := \{y : y \in F, \ ||y|| = 1\}$. We can also assume without loss of generality that $||x_0|| = 1$. Consider the function $f : \operatorname{span} F \to R$ defined as $f(y) := \langle y, L(y) \rangle = \langle y, L_{FF}(y) \rangle$ restricted to the set $\Im$. Note that x_0 is in the relative interior of F and $\langle x_0, L(x_0) \rangle \leq 0$. Moreover, any x on the relative boundary of F with $||x|| = 1$will have less than r positive eigenvalues and hence for such a x, $\langle x, L(x) \rangle > 0$. It follows that $f(y)$ restricted to $\Im$ will attain its minimum at some point v in the relative interior of F and $f(v) \leq 0$. But then v would be an eigenvector of L_{FF} with a negative or zero eigenvalue, contradicting our hypothesis. Thus, L is strictly copositive on $\mathcal{K}$. ∎

Theorem 3.2.3 *Suppose $L : \mathcal{A} \to \mathcal{A}$ is self-adjoint. Then the following statements are equivalent.*

(i) *L is strictly copositive on $\mathcal{K}$.*

(ii) *For every $F \lhd \mathcal{K}$, LCP(F, L_{FF}, q) has a unique (zero) solution for all $q \in F$.*

(iii) *L_{FF} is strictly semimonotone for all $F \lhd \mathcal{K}$.*

Proof. If L is strictly copositive on $\mathcal{K}$, then every nonzero principal subtransformation L_{FF} is strictly copositive on $F \lhd \mathcal{K}$. Thus, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows from Theorem 3.2.2. To prove the implication (iii) $\Rightarrow$ (i) assume that L is not strictly copositive. By Lemma 3.2.1, there exists a nonzero face $F \lhd \mathcal{K}$ such that $L_{FF}(x) = \lambda x$ for some $x \in \operatorname{ri} F$ and $\lambda \leq 0$. Thus we have

$$x \in \operatorname{ri} F,\ x \text{ and } L_{FF}(x) \text{ operator commute, and } x \circ L_{FF}(x) \in -F,$$

which contradicts the fact that L_{FF} has the **SSM**-property. ■

3.3 Completely-Q transformations

In the literature on LCP(R^n_+, M, q) where $M \in R^{n\times n}$ and $q \in R^n$, suppose $\mathcal{Y}$ is a fixed class of matrices defined by some property. We say that M is completely-$\mathcal{Y}$ if every principal submatrix of M also belongs to $\mathcal{Y}$ (equivalently, share the same property that defines the class $\mathcal{Y}$, for the corresponding order of a submatrix). This concept is useful in pivoting algorithms where different principal submatrices are used to generate intermediate solutions.

In the literature on LCP over R^n_+, we say that a matrix M is completely-Q if M and every principal submatrix of M is also a Q-matrix. It is known that a Q-matrix in general need not be a completely-Q matrix, see section 3.10, [8]. The study of the concept of a completely-Q matrix has been motivated by the variable dimension algorithm of Ludo Van der Heyden [32] to solve LCP over R^n_+. It has been shown (see [7, 8]) that M is completely-Q if and only if M is strictly semimonotone.

In this section we shall study the **Q**-property of each principal subtransformation of a linear transformation on a finite dimensional real inner product space V associated with a proper cone K.

The following definition is motivated by a similar class of matrices studied in the LCP theory [8].

Definition 3.3.1 (i) A linear transformation $L : V \to V$ is said to have the

S-*property* if LCP(K, L, q) has a feasible solution (i.e, there exists an $x \in K$ such that $L(x) + q \in K$) for all $q \in V$.

(ii) $L : V \to V$ is said to be *completely*-**Q** (*completely*-**S**) if L_{FF} has the **Q**-property (**S**-property) for all $F \lhd K$.

The proof of the following lemma is similar to that of Proposition 3.1.5 in [8].

LEMMA 3.3.1 *Given a linear transformation $L : V \to V$ and a proper cone K, LCP(K, L, q) has a feasible solution for all $q \in V$ if and only if there exists an $x \in \operatorname{int} K$ such that $L(x) \in \operatorname{int} K^*$.*

The next lemma can be seen as a specialization of Theorems 3.5 and 3.6 in [3]. It can also be obtained from Theorem 4 of [56].

LEMMA 3.3.2 *Let $A \in R^{n\times n}$ and K be a proper cone in R^n. Then the following are equivalent:*

(i) *The system $Ax \in \operatorname{int} K^*$, $x \in \operatorname{int} K$ is consistent.*

(ii) *The following implication holds*

$$-y \in K,\ A^T y \in K^* \Rightarrow y = 0.$$

PROPOSITION 3.3.1 *Let $L : V \to V$ be given and K be proper in V. Then* (i) $\Rightarrow$ (ii) *in the following statements.*

(i) *For all $F \lhd K$, L^T_{FF} has the* **S**-*property.*

(ii) *For all $F \lhd K$, LCP(F, L_{FF}, q) has a unique (zero) solution for all $q \in F^d := \operatorname{span} F \cap F^*$.*

Proof. Since every principal subtransformation of L_{FF} with respect to $G \lhd F$ is also a principal subtransformation of L, we can assume without loss of generality that there exists a $q \in K^*$ such that $0 \neq y \in K$, $z := L(y) + q \in K^*$ and $\langle y, z \rangle = 0$. Let F be the smallest face of K containing y. Then $y \in F$ and $z \in F^{\triangle}$. Since $q \in K^*$ can be written as $q = q_F + q_{F^\perp}$, where $q_F := \mathrm{Proj}_{\mathrm{span}\, F}(q)$ and $q_{F^\perp} := \mathrm{Proj}_{F^\perp}(q)$, we have $q_F \in F^d$. Thus we have a nonzero $y \in F$ such that $L_{FF}(-y) = \mathrm{Proj}_{\mathrm{span}\, F} L(-y) = q_F \in F^d$, which by Lemma 3.3.2 implies that the system $L_{FF}^T(x) \in \mathrm{ri}\, F^d$, $x \in \mathrm{ri}\, F$ is inconsistent. This, by Lemma 3.3.1, contradicts the **S**-property of L_{FF}^T. ■

THEOREM 3.3.1 *Let $L : V \to V$ be linear and K be a proper cone in V. Then the following statements are equivalent.*

(i) L_{FF}^T *has the* **Q**-*property for all* $F \lhd K$.

(ii) L_{FF}^T *has the* **S**-*property for all* $F \lhd K$.

(iii) *For every* $F \lhd K$, *LCP*(F, L_{FF}, q) *over* F *has a unique (zero) solution for all* $q \in F^d$.

(iv) L_{FF} *has the* **Q**-*property for all* $F \lhd K$.

(v) L_{FF} *has the* **S**-*property for all* $F \lhd K$.

(vi) *For every* $F \lhd K$, *LCP*(F, L_{FF}^T, q) *over* F *has a unique (zero) solution for all* $q \in F^d$.

Proof. (i)$\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) follows from Proposition 3.3.1. (iii) $\Rightarrow$ (iv) follows from Karamardian's theorem or Theorem 2.5.10 in [9]. Since $(L_{FF}^T)^T = L_{FF}$, other implications follow. This completes the proof. ■

COROLLARY 3.3.1 *For any linear transformation* $L : V \to V$ *the following implication holds.*

L_{FF} *has the* **S**-*property for all* $F \lhd K \Rightarrow L$ *has the* $\mathbf{R_0}$-*property (implies that every complementary cone of* L *is closed).*

COROLLARY 3.3.2 *If* $L(K) \subseteq K^*$, *then all the statements of the above theorem are equivalent to the* $\mathbf{R_0}$-*property of* L.

Proof. It is evident that if L satisfies Theorem 3.3.1 (iii), then L is $\mathbf{R_0}$. Conversely, the conditions L is $\mathbf{R_0}$ and $L(K) \subseteq K^*$ imply that $\langle x, L(x)\rangle > 0$ for every $0 \neq x \in K$. Thus L is strictly copositive on K. It is easy to observe that any linear transformation L is strictly copositive on K implies that LCP(K, L, q) has a unique (zero) solution for all $q \in K^*$. Since L is strictly copositive implies that L_{FF} is strictly copositive on F for all $F \lhd K$, we have proved our claim. ■

REMARK 3.3.1 In the setting of a Euclidean Jordan algebra we know that (see, Example 3.2.2) strict semimonotonicity of L need not imply that LCP$(\mathcal{K}, L, q)$ has a unique (zero) solution for all $q \in \mathcal{K}$. However, by Proposition 3.2.1, L_{FF} is strictly semimonotone for all $F \lhd \mathcal{K}$ is equivalent to the statements (i)-(vi) of Theorem 3.3.1.

The following lemma is a generalization of Theorem 3.8.3, [8].

LEMMA 3.3.3 *Let* $L : V \to V$ *be self-adjoint and* K *be proper in* V. *Then* (i) $\Leftrightarrow$ (ii) *in the following statements.*

(i) *For every nonzero* $F \lhd K$, *the system*

$$L_{FF}(x) \in \operatorname{ri} F^d,\ x \in \operatorname{ri} F$$

is consistent.

(ii) *L is strictly copositive on K.*

Proof. The proof is by induction on the dimension of V. If $\dim V = 1$, it is easy to see that the implication holds. Now suppose (i) $\Rightarrow$ (ii) holds for all V of dimension less than n. Let $\dim V = n$ and F be a nontrivial face of K. Since $G \lhd F \lhd K \Rightarrow G \lhd K$ and $(L_{FF})_{GG} = L_{GG}$ for all $G \lhd F$, we have from condition (i) and the induction hypothesis that L_{FF} is strictly copositive on F. Since F is arbitrary, L_{FF} is strictly copositive on F for all nontrivial faces of K. Now, from condition (i), let $\bar{x} \in \operatorname{int} K$ be such that $L(\bar{x}) \in \operatorname{int} K^*$. Obviously, $\langle \bar{x}, L(\bar{x}) \rangle > 0$. For any other nonzero $x \in K$ there exists $\alpha \geq 0$ such that $0 \neq x - \alpha\bar{x} \in \operatorname{bd} K$. Let F be the smallest face containing $0 \neq x - \alpha\bar{x}$. Write

$$\langle x, L(x) \rangle = \langle x - \alpha\bar{x}, L(x - \alpha\bar{x}) \rangle + 2\alpha \langle x - \alpha\bar{x}, L(\bar{x}) \rangle + \alpha^2 \langle \bar{x}, L(\bar{x}) \rangle.$$

Thus we have,

$$\begin{aligned} \langle x, L(x) \rangle &\geq \langle x - \alpha\bar{x}, L(x - \alpha\bar{x}) \rangle \\ &= \langle x - \alpha\bar{x}, L_{FF}(x - \alpha\bar{x}) \rangle \\ &> 0, \end{aligned}$$

where the last inequality follows from the strict copositivity of L_{FF}. Hence, L is strictly copositive on K.

Conversely, note that by Lemma 3.3.1, (i) is equivalent to saying that L is completely-**S**. Since L is strictly copositive on K implies that for every $F \lhd K$ LCP(F, L_{FF}, q) over F has a unique (zero) solution for all $q \in F^d$, by Theorem 3.3.1, L is completely-**S**.

■

THEOREM 3.3.2 *Suppose L is self-adjoint on V. Then all the statements of Theorem 3.3.1 are equivalent to the strict copositivity of L on K.*

Proof. By Lemma 3.3.3, for a self-adjoint L, L_{FF} has the **S**-property for all $F \lhd K$ implies that L is strictly copositive on K. Thus for every $F \lhd K$, LCP(F, L_{FF}, q) has a unique (zero) solution for all $q \in F^d$. This completes our proof. ■

REMARK 3.3.2 In view of Remark 3.3.1, Theorem 3.2.3 is a restatement of the above theorem in a Euclidean Jordan algebra. However, in Theorem 3.2.3 the strict copositivity of a self-adjoint L is obtained independently of showing that L_{FF} has the **Q**-property (or, **S**-property) for all $F \lhd \mathcal{K}$.

Chapter 4

Q and $\mathbf{R_0}$ Properties of a Quadratic Representation in SOCLCP

Given a $n \times n$ real square matrix M and a vector q in R^n the *second-order cone linear complementarity problem* (SOCLCP(M, q)) is to find a vector x in Λ_+^n such that $Mx + q$ is in Λ_+^n and $\langle x, Mx + q\rangle = x^T(Mx + q) = 0$. It is well known that the second-order cone is the cone of squares of its associated Euclidean Jordan algebra. In this regard the complementarity problem SOCLCP is a special case of the more general linear complementarity problem studied in the setting of a Euclidean Jordan algebra. However, the important feature which makes the SOCLCP interesting and draws a special attention is the nature of the faces of the second-order cone. Unlike the cone of symmetric positive semidefinite matrices, which is also studied in the setting of a Euclidean Jordan algebra, the only nontrivial faces of Λ^n_+ are its extreme rays and its only nonpolyhedral face is the cone Λ^n_+ itself, see [13].

Loewy and Schneider [43], have studied the closed convex cone of matrices which leave Λ^n_+ invariant, denoted by $\Pi(\Lambda^n_+)$, and characterized the extreme rays of $\Pi(\Lambda^n_+)$. Our focus in this chapter is on the SOCLCP(M, q) where the matrix $M \in \Pi(\Lambda^n_+)$. Our study is motivated by a result proved by Murty [52] in the context of a LCP over R^n_+ with a nonnegative square matrix. It states that a nonnegative square matrix M is a Q-matrix if and only if the diagonal entries of M are positive, which is equivalent to saying that M is a R_0-matrix. Though we do not have a complete generalization of Murty's result to a second-order cone, in this chapter we shall show that for a *quadratic representation* P_a of Λ^n for $a \in \Lambda^n$, defined as $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$, see [10], SOCLCP$(P_a, q)$ has a solution for all $q \in \Lambda^n$ if and only if a or $-a$ lies in the interior of Λ^n_+. An important feature being exploited in showing the above equivalence is the property of the faces of the second-order cone. Note that the quadratic representation $P_a \in \Pi(\Lambda^n_+)$ for all $a \in R^n$ and $P_a(\Lambda^n_+) = \Lambda^n_+$ when a is invertible. The quadratic representation plays a fundamental role in the study of Euclidean Jordan algebras. On the space of real symmetric matrices the quadratic representation is seen to be the map $X \to AXA$ where A is a real symmetric matrix. The solvability of semidefinite linear complementarity problem SDLCP(L, Q) with $L(X) = AXA$, where A is real symmetric, has been characterized in terms of A being positive or negative definite in [55, 59].

We shall begin with a brief survey of some Jordan algebraic properties of the second-order cone in section 4.1. In section 4.2 we present our main results.

4.1 Second-order cone and its Jordan algebra

In this chapter we shall confine our attention to the space R^n, whose elements $x = (x_0, \bar{x}^T)^T$ are indexed from zero, equipped with the usual inner product and the Jordan product defined as

$$x \circ y = (\langle x, y\rangle, x_0\bar{y}^T + y_0\bar{x}^T)^T.$$

Then R^n is a Euclidean Jordan algebra, denoted by Λ^n, with the cone of squares as second-order cone Λ^n_+. The interior of Λ^n_+ is the cone given by $\operatorname{int}\Lambda^n_+ = \{x \in R^n : ||\bar{x}|| < x_0\}$, see [1]. The identity element in this algebra is given by $e = (1, 0, \ldots, 0)^T$. Also the spectral decomposition of any x with $\bar{x} \neq 0$ is given by $x = \lambda_1 f_1 + \lambda_2 f_2$ with

$$\lambda_1 := x_0 + ||\bar{x}||,\ \lambda_2 := x_0 - ||\bar{x}||,$$
$$f_1 := \tfrac{1}{2}(1, \bar{x}^T/||\bar{x}||)^T, \text{ and } f_2 := \tfrac{1}{2}(1, -\bar{x}^T/||\bar{x}||)^T$$

where $\{f_1, f_2\}$ constitutes a Jordan frame. From the above decomposition $\det(x) = x_0^2 - ||\bar{x}||^2$. The rank of Λ^n is always 2 and it can be shown that all Jordan frames are of the above form. Also $x \in \Lambda^n_+$ $(\operatorname{int}\Lambda^n_+)$ if and only if both λ_1 and λ_2 are nonnegative (positive), see [1, 13].

The quadratic representation of Λ^n, denoted by P_a for $a \in \Lambda^n$, is the matrix

$$P_a := 2L_a^2 - L_{a^2} = 2aa^T - \det(a)\mathcal{J}_n,$$

where $\mathcal{J}_n$ is the $n \times n$ matrix defined as

$$\mathcal{J}_n := \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$

OBSERVATION 4.1.1 *For $a \in \Lambda^n$, $P_a \in \Pi(\Lambda^n_+)$.*

Proof. *Case 1:* When $\det(a) \neq 0$, $P_a(\Lambda^n_+) = \Lambda^n_+$ and $P_a(\operatorname{int}\Lambda^n_+) = \operatorname{int}\Lambda^n_+$, see

[1].

Case 2: When $\det(a) = 0$, $a_0^2 = ||\bar{a}||^2$ and $P_a(x) = 2a^T x a$ for $x \in \Lambda_+^n$. If $a_0 = ||\bar{a}||$, then $a \in \Lambda_+^n$ and $2a^T x a_0 - 2|a^T x| a_0 = 0$, because $a^T x \geq 0$. Again if $a_0 = -||\bar{a}||$, then $2a^T x a_0 + 2|a^T x| a_0 = 0$, because $a^T x \leq 0$. ■

Below we shall state some of the important properties of a quadratic representation.

PROPOSITION 4.1.1 ([1]) *Let α be a real number and $x \in \Lambda^n$. For $a \in \Lambda^n$ with the spectral decomposition $a = \lambda_1 f_1 + \lambda_2 f_2$ we have the following properties.*

(i) $\lambda_1 = a_0 + ||\bar{a}||$ *and* $\lambda_2 = a_0 - ||\bar{a}||$ *are the eigenvalues of* L_a. *Moreover, if* $\lambda_1 \neq \lambda_2$ *then each one has multiplicity one with corresponding eigenvectors* f_1 *and* f_2, *respectively. Also* a_0 *is an eigenvalue of* L_a *with multiplicity* $n-2$ *when* $a \neq 0$.

(ii) $\lambda_1^2 = (a_0 + ||\bar{a}||)^2$ *and* $\lambda_2^2 = (a_0 - ||\bar{a}||)^2$ *are eigenvalues of* P_a. *Moreover, if* $\lambda_1 \neq \lambda_2$ *then each one has multiplicity one with corresponding eigenvectors* f_1 *and* f_2, *respectively. Also* $\det(a) = a_0^2 - ||\bar{a}||^2$ *is an eigenvalue of* P_a *with multiplicity* $n-2$ *when* a *is invertible and* $\lambda_1 \neq \lambda_2$.

(iii) P_a *is an invertible matrix iff* a *is invertible.*

(iv) $P_{\alpha a} = \alpha^2 P_a$.

(v) $P_{P_a(x)} = P_a P_x P_a$.

(vi) $\det P_a(x) = \det^2(a)\det(x)$.

(vii) *If* a *is invertible, then* $P_a(a^{-1}) = a$ *and* $P_{a^{-1}} = P_a^{-1}$.

In view of the above discussions the second-order cone Λ^n_+ can also be represented by $\Lambda^n_+ := \{x \in \Lambda^n : x^T \mathcal{J}_n x \geq 0, x_0 \geq 0\}$ and any $x \in \Lambda^n$ has $\det(x) = x^T \mathcal{J}_n x$. The elements on the boundary of Λ^n_+ are exactly those for which $x_0 = ||\bar{x}||$. Below we shall state some relations on the boundary structure of the cone Λ^n_+, which will be useful in this chapter.

$$x \in \Lambda^n_+ \cup (-\Lambda^n_+) \Rightarrow \mathcal{J}_n x \in \Lambda^n_+ \cup (-\Lambda^n_+).$$
$$x^T \mathcal{J}_n x > 0 \Rightarrow x \in \operatorname{int} \Lambda^n_+ \cup \operatorname{int} (-\Lambda^n_+).$$
$$x^T \mathcal{J}_n x \geq 0 \Rightarrow x \in \Lambda^n_+ \cup (-\Lambda^n_+).$$
$$x^T \mathcal{J}_n x = 0 \Rightarrow x \in \operatorname{bd} \Lambda^n_+ \cup \operatorname{bd} (-\Lambda^n_+).$$
$$x^T \mathcal{J}_n x < 0 \Rightarrow x \notin \Lambda^n_+ \cup (-\Lambda^n_+).$$

One of the important properties of automorphisms of Λ^n, which we shall make use in this chapter is that for any two Jordan frames $\{e_1, e_2\}$ and $\{f_1, f_2\}$ in Λ^n there exists an automorphism Ψ such that $\Psi f_1 = e_1$ and $\Psi f_2 = e_2$, see [10]. Also any automorphism Ψ of Λ^n can be written as

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

where U is an $(n-1) \times (n-1)$ orthogonal matrix, see [43].

Fix a canonical Jordan frame $\{e_1, e_2\}$ where

$$e_1 := (1/2, 1/2, 0, \ldots, 0)^T \text{ and } e_2 := (1/2, -1/2, 0, \ldots, 0)^T.$$

In case of a second-order cone with a fixed Jordan frame $\{e_1, e_2\}$ consider the following subspaces of Λ^n. (Similar subspaces have been considered in Theorem IV 2.1 in [10].)

$V_{11} = \{x \in \Lambda^n : x \circ e_1 = x\} = \{\lambda e_1 : \lambda \in R\}$,

$V_{22} = \{x \in \Lambda^n : x \circ e_2 = x\} = \{\beta e_2 : \beta \in R\}$, and

$V_{12} = \left\{x \in \Lambda^n : x \circ e_1 = \frac{1}{2}x = x \circ e_2\right\} = \{x \in \Lambda^n : x_0 = x_1 = 0\}$.

Thus given an $x \in \Lambda^n$ we can write

$$x = (x_0 + x_1)e_1 + (x_0 - x_1)e_2 + (0, 0, x_2, \ldots, x_{n-1})^T.$$

We shall designate $(x_0 + x_1)$ and $(x_0 - x_1)$ as the diagonal entries of a vector x with respect to the Jordan frame $\{e_1, e_2\}$.

Observation 4.1.2 For $a \in \Lambda^n_+$ we have the following relationship among its entries. $a \in \Lambda^n_+$ ($\text{int}\, \Lambda^n_+$) if and only if $(a_0 + a_1) \geq 0\,(> 0)$, $(a_0 - a_1) \geq 0\,(> 0)$, and $(a_0 + a_1)(a_0 - a_1) - ||\tilde{a}||^2 \geq 0\,(> 0)$ where $\tilde{a} = (a_2, \ldots, a_{n-1})^T$.

Similar to the notion of a diagonal matrix we introduce the notion of a diagonal vector $d \in \Lambda^n$ with respect to a canonical Jordan frame $\{e_1, e_2\}$ as

$$d = \lambda_1 e_1 + \lambda_2 e_2, \quad \lambda_1, \lambda_2 \in R.$$

4.2 Equivalence of Q and $\mathbf{R_0}$-property of a quadratic representation

Proposition 4.2.1 *Let $M \in \Pi(\Lambda^n_+)$. Then M has the $\mathbf{R_0}$-property if and only if $\langle x, M(x)\rangle > 0$ for all $0 \neq x \in \Lambda^n_+$.*

The proof of the above proposition follows easily from the definitions. ■

Remark 4.2.1 For a linear complementarity problem $\text{LCP}(R^n_+, M, q)$ with $M(R^n_+) \subseteq R^n_+$, we have $\text{LCP}(R^n_+, M, q)$ is solvable for all $q \in R^n$ iff $\text{LCP}(R^n_+, M, 0)$ has a unique solution zero, which is also equivalent to stating that M is strictly copositve on R^n_+, see [8].

THEOREM 4.2.1 *Let $M \in \Pi(\Lambda_+^2)$. Then M has the* **Q***-property if and only if M has the* $\mathbf{R_0}$*-property.*

Proof. Observing the fact that $\{e_1, e_2\}$ is the only Jordan frame of Λ_+^2 unique up to permutation, $\mathbf{R_0}$-property of the above matrix M is equivalent to the property that $\langle e_1, M(e_1)\rangle > 0$ and $\langle e_2, M(e_2)\rangle > 0$. Now suppose without loss of generality that $\langle e_1, M(e_1)\rangle = 0$. We shall show that M does not have the **Q**-property. Take $q = e_2 - e_1 = (0, -1)^T$. Let $x \in \Lambda_+^2$ be a nonzero solution to SOCLCP(M, q). Then we have

$$x = \lambda_1 e_1 + \lambda_2 e_2 \text{ and } M(x) + q = \beta_1 e_1 + \beta_2 e_2$$

such that $\lambda_1\beta_1 = 0$ and $\lambda_2\beta_2 = 0$. Since $M(x) \in \Lambda_+^2$ and $x \neq 0$, $\lambda_2 = \beta_1 = 0$. Thus $\langle e_1, M(x) + q\rangle = \langle e_1, \lambda_1 M(e_1) + e_2 - e_1\rangle = -\langle e_1, e_1\rangle < 0$, which contradicts the fact that $M(x) + q \in \Lambda_+^2$. The 'if part' is apparent from Karamardian's theorem.

■

Loewy and Schneider [43] proved the following result which characterizes the extreme matrices of the closed convex cone of square matrices which leave the cone Λ_+^n invariant.

THEOREM 4.2.2 ([43]) *Let M be an $n \times n$ real matrix, with $n \geq 3$. Then M is an extreme matrix of $\Pi(\Lambda_+^n)$ (generates a 1-dimensional face of $\Pi(\Lambda_+^n)$) if and only if either $M(\Lambda_+^n) = \Lambda_+^n$ or $M = uv^T$ for u, $v \in \operatorname{bd}\Lambda_+^n$.*

REMARK 4.2.2 Any $M : \Lambda^n \to \Lambda^n$ satisfying $M(\Lambda_+^n) = \Lambda_+^n$ can be written as $M = P_a\Psi$, where $a \in \operatorname{int}\Lambda_+^n$ and $\Psi \in Aut(\Lambda^n)$, see page 56 of [10].

PROPOSITION 4.2.2 *Let $M \in \Pi(\Lambda^n_+)$. Then M has the* **Q***-property if and only if $\Theta M \Theta^T$ has the* **Q***-property for all $\Theta : \Lambda^n \to \Lambda^n$ such that $\Theta(\Lambda^n_+) = \Lambda^n_+$.*

Proof. Take an arbitrary $q \in \Lambda^n$ and define $\tilde{q} = \Theta^{-1} q$. Then SOCLCP$(M, \tilde{q})$ has a solution $\tilde{x}$ such that

$$\tilde{x} \in \Lambda^n_+,\ M(\tilde{x}) + \tilde{q} \in \Lambda^n_+ \text{ and } \langle \tilde{x}, M(\tilde{x}) + \tilde{q} \rangle = 0.$$

Define $x = (\Theta^{-1})^T \tilde{x}$. Since both Θ^{-1}, $\Theta^T \in \Pi(\Lambda^n_+)$, we have

$$x \in \Lambda^n_+,\ M(\Theta^T x) + \Theta^{-1} q \in \Lambda^n_+ \text{ and } \langle \Theta^T x, M\Theta^T x + \Theta^{-1} q \rangle = 0,$$

equivalently,

$$x \in \Lambda^n_+,\ \Theta M \Theta^T x + q \in \Lambda^n_+ \text{ and } \langle x, \Theta M \Theta^T x + q \rangle = 0,$$

which implies that $\Theta M \Theta^T$ has the **Q**-property. ■

THEOREM 4.2.3 *Let $M = uv^T$ for $u, v \in \operatorname{bd} \Lambda^n_+$. Then M does not have the* **Q***-property.*

Proof. First we shall show that if $M = e_i e_j^T$, where $i, j \in \{1, 2\}$, then M does not have the **Q**-property. When $M = e_1 e_2^T$ or $M = e_1 e_1^T$ we can take $q = (0, 1, 0, \ldots, 0)^T$ and can show easily that SOCLCP(M, q) does not have a solution. Again when $M = e_2 e_1^T$ or $M = e_2 e_2^T$ we can take $q = (0, -1, 0, \ldots, 0)^T$ and can easily prove that SOCLCP(M, q) is not solvable. In fact, in both the above cases SOCLCP(M, q) is not even feasible. Now we consider the two cases.

Case 1: Suppose that u and v are linearly dependent. Then there exists an automorphism Ψ of Λ^n such that $\Psi u = e_1$. By Proposition 4.2.2, uv^T has the **Q**-property implies that $e_1 e_1^T$ has the **Q**-property. Since we know that $e_1 e_1^T$ is not **Q**, we have proved our claim.

Case 2: Suppose that u and v are linearly independent. Then by Lemma 3.7 in [43], there exists a $\Theta \in \Pi(\Lambda^n_+)$ with $\Theta(\Lambda^n_+) = \Lambda^n_+$ such that $\Theta u = e_1$ and $\Theta v = e_2$. Again by Proposition 4.2.2, uv^T has the **Q**-property implies that $e_1 e_2^T$ has the **Q**-property. Since we know that $e_1 e_2^T$ is not **Q**, uv^T does not have the **Q**-property. ∎

THEOREM 4.2.4 *For a quadratic representation P_a of Λ^n, $n \geq 3$, we have the following equivalence.*

(i) $a \in \operatorname{int} \Lambda^n_+$ *or* $-a \in \operatorname{int} \Lambda^n_+$.

(ii) P_a *is positive definite.*

(iii) *SOCLCP*(P_a, q) *has a unique solution for all* $q \in \Lambda^n$.

(iv) P_a *has the* $\mathbf{R_0}$*-property.*

Proof. The proof of (i) $\Rightarrow$ (ii) follows from Proposition 4.1.1 (ii), since all the eigenvalues of P_a are positive. From (ii), P_a is positive definite implies that SOCLCP(P_a, q) has at most one solution for all $q \in \Lambda^n$. Also from Karamardian's theorem, P_a has the **Q**-property. Thus (iii) follows. The proof of (iii) $\Rightarrow$ (iv) is obvious. To prove (iv) $\Rightarrow$ (i), let us suppose that neither $a \in \operatorname{int} \Lambda^n_+$ nor $-a \in \operatorname{int} \Lambda^n_+$. To complete the proof, it is sufficient to show that there exists some $z \in \operatorname{bd} \Lambda^n_+$ such that $\langle a, z \rangle = 0$. We consider the following two cases.

Case 1: When $a \in \operatorname{bd} \Lambda^n_+$ or $-a \in \operatorname{bd} \Lambda^n_+$ we can find a nonzero $z \in \operatorname{bd} \Lambda^n_+$ or $-z \in \operatorname{bd} \Lambda^n_+$ on a face complementary to which a or $-a$ lies.

Case 2: When $a \notin \operatorname{bd} \Lambda^n_+$ and $-a \notin \operatorname{bd} \Lambda^n_+$, there exist an $x \in \operatorname{bd} \Lambda^n_+$ and $y \in \operatorname{bd} \Lambda^n_+$ such that $\langle a, x \rangle < 0$ and $\langle a, y \rangle > 0$. Since the inner product $\langle a, \cdot \rangle$ is continuous on the line segment joining x and y denoted as $[x, y]$, there exists a $\tilde{z} = \alpha x + (1 - \alpha)y$ for $0 < \alpha < 1$ such that $\langle a, \tilde{z} \rangle = 0$. Since Λ^n_+ has faces of dimension 0, 1 and n only, either $\tilde{z}$ lies on a one dimensional face of Λ^n_+ in which case we are done otherwise $\tilde{z} \in \operatorname{int} \Lambda^n_+$. It means that $a^\perp \cap \operatorname{int} \Lambda^n_+$ is nonempty where $a^\perp$ is the orthogonal complement of the span of a in Λ^n. Note that $a^\perp$ is a $(n-1)$ dimensional subspace of Λ^n intersecting a full dimensional cone Λ^n_+ in its interior where $n \geq 3$. Hence from Lemma 3.6, [43], there exists an invertible linear transformation Γ on Λ^n such that $\Gamma(\Lambda^n_+) = \Lambda^n_+$ and $\Gamma(a^\perp) = \{x \in \Lambda^n : x_{n-1} = 0\}$. Thus there exists a nonzero $z \in a^\perp$ such that $\Gamma z = e_1$, which by Lemma 3.3, [43], lies on the boundary of Λ^n_+.

Now for the above chosen z we have $\langle z, P_a(z) \rangle = \langle z, (2aa^T - \det(a)\mathcal{J}_n)z \rangle = 0$, which contradicts that P_a has the $\mathbf{R_0}$-property. ■

LEMMA 4.2.1 *If the quadratic representation P_a has the $\mathbf{Q}$-property then a is invertible.*

Proof. Taking $-q \in \operatorname{int} \Lambda^n_+$ let $x \in \Lambda^n_+$ be a solution to SOCLCP(P_a, q). Since $P_a(x) + q \in \Lambda^n_+$ and $-q \in \operatorname{int} \Lambda^n_+$, $P_a(x) \in \operatorname{int} \Lambda^n_+$. From Proposition 4.1.1 (vi), $\det P_a(x) = \det^2(a)\det(x)$. Hence a is invertible. ■

The following lemma is analogous to Lemma 4.3.2 in [59].

LEMMA 4.2.2 *Let $d = d_1e_1 + d_2e_2$ be a diagonal vector with nonzero entries. Let $|d| = |d_1|e_1 + |d_2|e_2$. Write $d = |d| \circ s$ where $s = \pm e_1 \pm e_2$. The coefficients of e_1 and e_2 in the expression for s are determined by the signs of d_1 and d_2, respectively. If P_d has the $\mathbf{Q}$-property then P_s has the $\mathbf{Q}$-property.*

Proof. Fix an arbitrary $q \in \Lambda^n$ and define $\tilde{q} := P_{\sqrt{|d|}}(q)$ where $\sqrt{|d|} = \sqrt{|d_1|}e_1 + \sqrt{|d_2|}e_2$. Let $x \in \Lambda^n_+$ be a solution to SOCLCP$(P_d, \tilde{q})$. Then $z := P_d(x) + \tilde{q} \in \Lambda^n_+$ is such that $x \circ z = 0$. We have

$$\begin{aligned} z &= P_{|d|\circ s}(x) + \tilde{q} \\ &= P_{|d|}P_s(x) + P_{\sqrt{|d|}}(q) \\ &= P_{\sqrt{|d|}}P_{\sqrt{|d|}}P_s(x) + P_{\sqrt{|d|}}(q), \end{aligned}$$

where the last equality follows from Proposition 4.1.1 (v). Thus by Proposition 4.1.1 (vii), $P_{\sqrt{|d|}^{-1}}(z) = P_{\sqrt{|d|}}P_s(x) + q = P_sP_{\sqrt{|d|}}(x) + q$. Define $y = P_{\sqrt{|d|}}(x)$ and $w = P_{\sqrt{|d|}^{-1}}(z)$. Then $y \in \Lambda^n_+$, $w = P_s(y) + q \in \Lambda^n_+$ and $\langle y, w\rangle = \langle x, P_{\sqrt{|d|}}P_{\sqrt{|d|}^{-1}}(z)\rangle = 0$. Hence $y \in \Lambda^n_+$ solves SOCLCP(P_s, q). ∎

LEMMA 4.2.3 *Let s be a diagonal vector of the form $s = \pm e_1 \pm e_2$ and $s \neq \pm e$. Then P_s does not have the* **Q**-*property.*

Proof. Without loss of generality take $s = e_1 - e_2 = (0, 1, 0, \ldots, 0)^T$. Then the matrix P_s will be

$$P_s = 2ss^T - \det(s)J_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I \end{pmatrix}.$$

Consider a vector $q = (0, 0, -1, 0, \ldots, 0)$. If $y \in \Lambda^n$ is the solution of the SOCLCP(P_s, q) then

$$y \in \Lambda^n_+,\ P_s(y) + q \in \Lambda^n_+ \text{ and } \langle y, P_s(y) + q\rangle = 0.$$

The complementary condition $\langle y, P_s(y) + q\rangle = 0$ gives us $y_0^2 + y_1^2 - y_2 = \sum_{i=2}^{n-1} y_i^2$. Also from $y \in \Lambda^n_+$ and $P_s(y) + q \in \Lambda^n_+$ we have

$$y_0^2 \geq y_1^2 + \ldots + y_{n-1}^2 \text{ and } y_0^2 \geq y_1^2 + (1+y_2)^2 + y_3^2 + \ldots + y_{n-1}^2.$$

Substituting the value of $\sum_{i=2}^{n-1} y_i^2$ in the above two inequalities we get $2y_1^2 \leq y_2$ and $2y_1^2 + 1 + y_2 \leq 0$, which are inconsistent. ∎

THEOREM 4.2.5 *Let P_a be a quadratic representation of Λ^n, $n \geq 3$. Then P_a has the **Q**-property if and only if P_a has the $\mathbf{R_0}$-property.*

Proof. Since P_a has the **Q**-property we have by Lemma 4.2.1, a is invertible. By the spectral decomposition and the fact that for any two Jordan frames $\{e_1, e_2\}$ and $\{f_1, f_2\}$ in Λ^n there exists an automorphism Ψ such that $\Psi e_1 = f_1$ and $\Psi e_2 = f_2$ (Theorem IV.2.5, [10]), we can write $a = d_1 \Psi e_1 + d_2 \Psi e_2$, where $d_1, d_2 \neq 0$ and $\Psi \in Aut(\Lambda^n)$. Since $\Psi P_a \Psi^T = P_{\Psi a}$, by Proposition 4.2.2, P_a has the **Q**-property implies that $P_{\Psi^T a}$ has the **Q**-property, which is equivalent to the statement that P_d has **Q**-property where $d := d_1 e_1 + d_2 e_2$. By Lemma 4.2.2, P_s has the **Q**-property where s is the vector corresponding to d as defined above. Since P_s has the **Q**-property if and only if $s = \pm e$, either $d_1 > 0$ and $d_2 > 0$ or $d_1 < 0$ and $d_2 < 0$. It means that either $a \in \operatorname{int} \Lambda_+^n$ or $-a \in \operatorname{int} \Lambda_+^n$, which by Theorem 4.2.4 proves that P_a has the $\mathbf{R_0}$-property. Conversely, $\mathbf{R_0}$ implies **Q** is evident from the Karamardian's theorem. ∎

The above result can also be extended to linear complementarity problems over the direct product of second-order cones, which is defined as follows. Given a matrix M on R^n and $q \in R^n$ the linear complementarity problem over the cone K^n is the problem of finding an $x \in K^n$ such that $M(x) + q$ is in K^n and $\langle x, M(x) + q \rangle = 0$, where the cone K^n is defined as

$$K^n := \Lambda_+^{n_1} \times \ldots \times \Lambda_+^{n_m},$$

with $n = n_1 + \ldots + n_m$.

It should be noted that R^n is a Euclidean Jordan algebra with the usual inner product and the Jordan product defined as the componentwise Jordan product of the elements of Λ^{n_i}, $i = 1 \ldots m$. Formally, for $x := (x_1, \ldots, x_m)^T$ and $y := (y_1, \ldots, y_m)^T$, where each component vector x_i is written in a row vector form, the Jordan product is defined as

$$x \circ y := (x_1 \circ y_1, \ldots, x_m \circ y_m)^T.$$

The cone of squares with respect to the above Jordan product is the cone K^n. Also the quadratic representation P_a for $a \in R^n$ is a block diagonal matrix of the form

$$P_a = \begin{pmatrix} P_{a_1} & 0 & \ldots & 0 \\ 0 & P_{a_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & P_{a_m} \end{pmatrix},$$

where $a = (a_1, \ldots, a_m)^T$. For further information one can see the references [1, 10].

THEOREM 4.2.6 *In the space R^n with the cone of squares K^n and the Jordan product defined above the following statements are equivalent.*

(i) *For $a \in R^n$, P_a has the $\mathbf{R_0}$-property, that is, $x \in K^n$, $P_a(x) \in K^n$ and $\langle x, P_a(x) \rangle = 0$ implies $x = 0$.*

(ii) *The linear complementarity problem LCP (K^n, P_a, q) has the solution for all $q \in R^n$.*

Proof. The proof of the above theorem is apparent once we notice that P_a has the $\mathbf{R_0}$-property if and only if P_{a_i} has the $\mathbf{R_0}$-property for each component vector $a_i \in \Lambda^i$ for $i = 1, \ldots, m$. ∎

Chapter 5

Some Geometrical Aspects of a SDLCP

In the first section of this chapter we study Murty's result [52] (in the context of LCP over R^n_+, a nonnegative matrix is a Q-matrix if and only if it is an R_0-matrix) in the semidefinite setting and provide a necessary condition for the transformations of the type $L(S^n_+) \subseteq S^n_+$ to have the **Q**-property. In the second section, we discuss a question whether the matrix representation of a transformation L with the **P**-property, with respect to the canonical basis in S^n, is a P-matrix.

5.1 Q-property of positive semidefiniteness preserving transformations

In this section we study the transformation $L : S^n \to S^n$ for which $L(S^n_+) \subseteq S^n_+$. We call such transformations *semidefiniteness preserving*. Special cases of such transformations have earlier been studied by [19, 22, 55]. These transformations generalize a nonnegative matrix in the context of the linear complementarity problem.

We first note that transformations satisfying $L(S^n_+) = S^n_+$ can be represented as $L(X) = AXA^T$ for some invertible matrix A of order n, see [62]. However, there are semidefiniteness preserving transformations which cannot be represented as AXA^T. The following is an example.

Example 5.1.1 Consider the transformation $L : S^2 \to S^2$ given by

$$L(X) = \begin{pmatrix} x_{11} + x_{22} & 0 \\ 0 & x_{11} + x_{22} \end{pmatrix}, \text{ for all } X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}.$$

If we try to represent it in the form AXA^T we get inconsistent equations in the elements of the matrix A.

For a general L the semidefiniteness preserving property and in addition the $\mathbf{R_0}$-property, have the following interpretation in terms of the faces of S^n_+, which has been observed in the more general setting of a Euclidean Jordan algebra.

Observation 5.1.1 (i) *A linear transformation $L : \mathcal{A} \to \mathcal{A}$ has the property $L(\mathcal{K}) \subseteq \mathcal{K}$ if and only if for every pair of 1-dimensional faces F and G of the symmetric cone $\mathcal{K}$, $\langle y, L(x)\rangle \geq 0$, $\forall\, x \in F$ and $y \in G$.*

(ii) *Let $L(\mathcal{K}) \subseteq \mathcal{K}$. Then L has the $\mathbf{R_0}$-property if and only if $\langle x, L(x)\rangle > 0$ for every nonzero $x \in F$, where F is any 1-dimensional face of $\mathcal{K}$.*

The proof of the above can be easily obtained using Spectral Theorem 1.3.2.

REMARK 5.1.1 Note that the defining condition $\langle y, L(x)\rangle \geq 0$, for all $x \in F$ and $y \in G$, is a generalization of the condition that $e_i^T M e_j = m_{ij} \geq 0$, where M is a square matrix and e_i is a vector whose i^{th} entry is 1 and others 0.

Motivated by the characterization of a nonnegative Q-matrix (equivalently, nonnegative R_0-matrix) in terms of the positive diagonal entries of a matrix by Murty [52] in LCP over R^n_+, we introduce the following definition.

DEFINITION 5.1.1 L has the *positive diagonal property* if for every 1-dimensional face F of S^n_+, $\langle X, L(X)\rangle > 0$ for every nonzero $X \in F$.

The above observation shows that a semidefiniteness preserving transformation has the $\mathbf{R_0}$-property if and only if it has the positive diagonal property. At present we are unable to settle the question whether for such transformations $\mathbf{Q}$-property implies $\mathbf{R_0}$-property. However we have the following result.

THEOREM 5.1.1 *Let $L : S^n \rightarrow S^n$ satisfy $L(S^n_+) \subseteq S^n_+$. If L has the $\mathbf{Q}$-property then for every 1-dimensional face G there exists a 1-dimensional face F of S^n_+ such that $\langle Y, L(X)\rangle > 0$ for all nonzero $X \in F$ and $Y \in G$.*

Proof. Suppose the result is not true. Then without loss of generality we assume that $\langle E_{11}, U^T L(X) U\rangle = 0$ for all rank 1 matrices $X \succeq 0$ and some fixed orthogonal matrix U. Consider the matrix

$$Q = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $R = UQU^T$. Since $L(X)$ is positive semidefinite for all $X \succeq 0$, it follows that for all $X \succeq 0$ at least $(n-1)$ eigenvalues of $U^T(L(X)+R)U = U^TL(X)U + Q$ are positive. Since $X = 0$ cannot be a solution to SDLCP(L, R), it follows that if $\widetilde{X}$ is a solution to it, then the rank of $L(\widetilde{X}) + R$ must be $(n-1)$ and hence $\widetilde{X}$ must have rank 1. Now for any rank 1 matrix $X \succeq 0$,

$$\langle E_{11}, U^T(L(X)+R)U\rangle = \langle E_{11}, U^TL(X)U\rangle + \langle E_{11}, Q\rangle = -1,$$

which shows that any $X \succeq 0$ of rank 1 cannot be a solution to SDLCP(L, R). Since X is an arbitrary rank 1 matrix, it follows that there is no solution to SDLCP(L, R). This concludes the proof. ■

In the last result of this section we observe that a nonnegative matrix is a Q-matrix if and only if a related linear transformation on S^n, to be defined below, has the **Q**-property.

THEOREM 5.1.2 *Let M be a given nonnegative matrix and define the $n \times n$ diagonal matrix A_i by taking its jth diagonal entry as m_{ij}, the $(i,j)^{th}$ entry of M. Let the transformation L be defined by*

$$L(X) = \begin{pmatrix} \langle A_1, X\rangle & 0 & \dots & 0 \\ 0 & \langle A_2, X\rangle & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \langle A_n X\rangle \end{pmatrix}.$$

Then L has the **Q***-property if and only if M is a Q-matrix.*

Proof. Suppose L has the **Q**-property. Given any $q \in R^n$ let $\overline{Q}$ denote the diagonal matrix whose i^{th} diagonal entry is q_i. Note that SDLCP$(L, \overline{Q})$ has a solution X, since L has the **Q**-property. Define $x \in R^n$ as $x_i = x_{ii}$. Then it is easy to note that $x \geq 0$ and that x is a solution to the LCP(R^n_+, M, q) proving that M is a Q-matrix. Conversely, note that M is a Q-matrix if and only if all the diagonal entries of M are positive, see Murty ([52]). We shall show that L is $\mathbf{R_0}$ when M is a Q-matrix. Let $X \succeq 0$ solve SDLCP$(L, 0)$. Then $XL(X) = 0$ implies $x_{ii}(\sum_{i \neq j} m_{ij}x_{jj} + m_{ii}x_{ii}) = 0 \; \forall \; i$. Since $m_{ii} > 0 \; \forall \; i$ and $m_{ij} \geq 0 \; \forall \; i \neq j$ we get $x_{ii} = 0 \; \forall \; i$, which in turn gives $X = 0$. Also on observing the fact that for any $Q \succ 0$, SDLCP(L, Q) has the unique solution $X = 0$, it follows from Karamardian's theorem that L has the **Q**-property. ∎

5.2 Relationship between P-property and P-matrix property

DEFINITION 5.2.1 For a linear transformation $L : S^n \rightarrow S^n$ and $F \lhd S^n_+$, we say that L_{FF} has the $\mathbf{P}$-*property* if

$$X \in \operatorname{span} F,\ X \text{ and } L_{FF}(X) \text{ commute, and } XL_{FF}(X) \in -F \Rightarrow X = 0.$$

In connection with the SDLCP, following results on Lyapunov transformation [18, 23], Stein transformation [19], and the transformation of the type AXA^T for $A \in R^{n \times n}, X \in S^n$, see [22], will be referred to in the sequel.

THEOREM 5.2.1 *For any matrix $A \in R^{n \times n}$, the following statements hold.*

(i) *A is positive definite (i.e, $x^T Ax > 0 \forall\; 0 \neq x \in R^n$) if and only if $(L_A)_{FF}$ has the $\mathbf{P}$-property for all $F \lhd S^n_+$.*

(ii) *A is positive stable if and only if L_A has the $\mathbf{P}$-property.*

(iii) *S_A has the **P**-property if and only if $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A.*

(iv)) *For the transformation $L(X) = AXA^T$, A is positive definite implies that L_{FF} has the **P**-property for all $F \lhd S^n_+$.*

Given a linear transformation $L : S^n \to S^n$, we denote by $\mathcal{N}(L)$ the matrix of L of order $\frac{n(n+1)}{2}$ corresponding to the basis $\{E_{ij})$ where E_{ij}, for $i \neq j$, is the symmetric matrix whose ij^{th} and ji^{th} elements are $1/\sqrt{2}$ and other elements are 0, and E_{ii} is the symmetric matrix whose i^{th} diagonal entry is 1 and all other entries are equal to 0. The elements in a column of this matrix represent the matrix $L(E_{rs})$ as a linear combination of the basis elements E_{ij} taken in the order $\{E_{11}, E_{12}, E_{22}, E_{13}, E_{23}, E_{33}, E_{14}, \ldots, E_{nn}\}$. Note that each column will have $\frac{n(n+1)}{2}$ entries.

We say that L has the *P-matrix property* if $\mathcal{N}(L)$ is a P-matrix. The motivation for asking whether a L with **P**-property has the P-matrix property is partly the issue studied in Theorem 8 of [18] (also see [20]). Also, when L is self-adjoint we have the following equivalence:

L_{FF} has the **P**-property for all $F \Leftrightarrow L$ has **P**-property $\Leftrightarrow L$ is strictly monotone $\Leftrightarrow \mathcal{N}(L)$ is symmetric positive definite, (see Theorem 1 in [22]). In this section we shall study the relationship between the **P**-property of L and the P-matrix property of L.

The following example shows that $\mathcal{N}(L)$ may be a P-matrix, but L_{FF} does not have the **P**-property for all F.

EXAMPLE 5.2.1 For $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, the matrix of the Lyapunov transformation L_A is

$$\mathcal{N}(L_A) = \begin{pmatrix} 2 & -2\sqrt{2} & 0 \\ 0 & 2 & -2\sqrt{2} \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that $\mathcal{N}(L_A)$ is a P-matrix. However A is not positive definite, which implies from Theorem 5.2.1 (i) that not every principal subtransformation of L_A has the **P**-property.

The next example shows that **P**-property of L need not imply that $\mathcal{N}(L)$ is a P-matrix.

EXAMPLE 5.2.2 Consider a Lyapunov transformation $L_A : S^2 \to S^2$. Take $A = \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix}$. Note that A is positive stable and hence from Theoerm 5.2.1 (ii), L_A has the **P**-property. The matrix of L_A is $\mathcal{N}(L_A) = \begin{pmatrix} -2 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 4 \end{pmatrix}$ which is not a P-matrix.

Given a set of indices $\alpha = \{i_1 < i_2 < \ldots < i_k\}$ where $1 \leq i_1;\ i_k \leq n$, the canonical face F_α of S^n_+ corresponding to α is the face defined as

$$F_\alpha := P \begin{pmatrix} S_+^{|\alpha|} & 0 \\ 0 & 0 \end{pmatrix} P^T,$$

where P is a permutation matrix such that for any $X \in S^n$, $(P^TXP)_{\beta\beta} = X_{\alpha\alpha}$, where $\beta = \{1, 2, \ldots, |\alpha|\}$. As has been discussed earlier, due to the nonpolyhedral nature of S^n_+, $\operatorname{span} F + \operatorname{span} F^{\triangle}$ does not generate the whole space S^n. This motivates us to study a class of linear transformations for which

$$L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha + \operatorname{span} F_\alpha^{\triangle},$$

where F_α is a canonical face. In what follows we shall characterize these transformations and study the **P**-property and the P-matrix property for these transformations. We shall assume, without loss of generality, the following form of a linear transformation L.

$$L(X) = \begin{pmatrix} \langle A_{11}, X\rangle & \cdots & \langle A_{1n}, X\rangle \\ \vdots & \cdots & \vdots \\ \langle A_{n1}, X\rangle & \cdots & \langle A_{nn}, X\rangle \end{pmatrix} = ((\langle A_{ij}, X\rangle)), \tag{5.2.1}$$

where A_{ij} and A_{ji} are $n \times n$ symmetric matrices and $A_{ij} = A_{ji}$ for all $i, j \in \{1, \ldots, n\}$. We will use the notation a^{rs}_{ij} for the $(i, j)^{th}$ entry in the matrix A_{rs}.

THEOREM 5.2.2 *A linear transformation $L : S^n \to S^n$ written in the form (5.2.1) satisfies $L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha + \operatorname{span} F_\alpha^{\triangle}$ for all $\alpha \subseteq \{1, \ldots, n\}$ iff every entry other than the $(i, j)^{th}$ entry of the $n \times n$ symmetric matrix A_{ij} in (5.2.1) is zero for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.*

Proof. *If part:*

Let $\alpha \subseteq \{1, 2, \ldots, n\}$ and $L_{F_\alpha F_\alpha}$ be a principal subtransformation of L corresponding to F_α. We shall show that $L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha + \operatorname{span} F_\alpha^{\triangle}$. Without loss of generality assume that $\alpha = \{1, \ldots, k\}$, where $1 \leq k \leq n$. Consider an arbitrary $X \in \operatorname{span} F_\alpha$ and a matrix A_{ij} for any $(i, j) \in \alpha \times \beta$, where β is the complement of α in $\{1, 2, \ldots, n\}$. Then by our hypothesis $\langle A_{ij}, X\rangle = 0$, which immediately proves our claim.

Only if part:

Consider an $n \times n$ symmetric matrix A_{ij} for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. We shall show that every $(k, l)^{th}$ entry of A_{ij} is zero where $(k, l) \neq (i, j)$. Let F_{α_1} be a canonical face of S^n_+ corresponding to $\alpha_1 := \{1, 2, \ldots, n\} \setminus \{i\}$. Since $L(\operatorname{span} F_{\alpha_1}) \subseteq \operatorname{span} F_{\alpha_1} + \operatorname{span} F^{\triangle}_{\alpha_1}$, we have $\langle A_{ij}, X\rangle = 0\ \forall X \in \operatorname{span} F_{\alpha_1}$, which gives $(A_{ij})_{\alpha_1\alpha_1} = 0$. Thus all the $(k, l)^{th}$ entries of A_{ij} other than $k = i$ or $l = i$ are 0. Similarly $(A_{ij})_{\alpha_2\alpha_2} = 0$ for $\alpha_2 := \{1, 2, \ldots, n\} \setminus \{j\}$, which shows that every $(k, l)^{th}$ entry of A_{ij} other than $k = j$ or $l = j$ are 0. Thus every entry other than $(i, j)^{th}$ entry of A_{ij} is 0. ■

THEOREM 5.2.3 *Suppose $L : S^n \to S^n$ has the property that*

$$L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha + \operatorname{span} F^{\triangle}_\alpha$$

for all $\alpha \subseteq \{1, \ldots, n\}$ with $L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha$ for $\alpha = \{1, \ldots, r\}$, $1 \leq r \leq n$. Then (i)⇒(ii)⇔(iii) *in the following statements*

(i) *L has the **P**-property.*

(ii) *All the real eigenvalues of L and those of its canonical principal subtransformations are positive.*

(iii) *$\mathcal{N}(L)$ is a P-matrix.*

Proof. (i) ⇒ (ii)

Let F be a face of S^n_+ for which $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}$. Then for any $X \in \operatorname{span} F$, $XL(X) = XL_{FF}(X)$ and $L(X)X = L_{FF}(X)X$. This immediately gives us that L_{FF} has the **P**-property when L has the **P**-property. By using Theorem 1, [22], which states that if L has the **P**-property then all real eigenvalues of L are positive, the above implication follows.

(ii)$\Leftrightarrow$(iii)

We assume w.l.g that the given L is represented in the form (5.2.1). The proof is by induction on n. We first verify the theorem for $n = 2$. For $n = 2$, $\mathcal{N}(L)$ is given by:

$$\mathcal{N}(L) = \begin{pmatrix} a_{11}^{11} & \sqrt{2}a_{12}^{11} & a_{22}^{11} \\ 0 & 2a_{12}^{12} & 0 \\ 0 & \sqrt{2}a_{12}^{22} & a_{22}^{22} \end{pmatrix}.$$

The hypothesis that the real eigenvalues of the canonical principal subtransformations are positive shows that the diagonal entries a_{11}^{11} and a_{22}^{22}, and the determinant of the above matrix are positive. From the structure of the matrix it follows that $2a_{12}^{12}$ is also positive. Further note that any principal minor of the above matrix is a product of a subset of the diagonal entries and hence is positive. Thus the theorem holds for $n = 2$.

Induction hypothesis: The theorem is true when $n \leq k$.

We shall now show that the theorem holds when $n = k+1$. When $n = k+1$ the matrix $\mathcal{N}(L)$ of order $\frac{(k+1)(k+2)}{2}$ can be partitioned as follows: Let $\alpha = \{1, 2 \ldots k\}$. Now

$$\mathcal{N}(L) = \begin{pmatrix} A_{\alpha\alpha} & C \\ B & G \end{pmatrix},$$

where $A_{\alpha\alpha}$ is of order $\frac{k(k+1)}{2}$, B is of order $k+1 \times \frac{k(k+1)}{2}$, C is of order $\frac{k(k+1)}{2} \times k+1$ and G is of order $(k+1) \times (k+1)$. The matrix $A_{\alpha\alpha}$ is the same as $\mathcal{N}(L_{F_\alpha F_\alpha})$ where

F_α is the canonical face of S^n_+ corresponding to α. Since L and all its canonical principal subtransformations have the property that all their real eigenvalues are positive, it follows by the induction hypothesis that $A_{\alpha\alpha}$ is a P-matrix. We now note that $B = 0$. This is so since the column entries in the block B are the coefficients of $E_{l(k+1)}$ for $1 \leq l \leq k+1$ in the representation of $L(E_{ij}),\ 1 \leq i \leq k, 1 \leq j \leq k, i \leq j$, and by our hypothesis $L(E_{ij}) \in L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha$. Since B is zero, any principal minor of $\mathcal{N}(L)$ will either be a principal minor of $A_{\alpha\alpha}$ or a product of a principal minors of $A_{\alpha\alpha}$ and a principal minor of G. Note that G is given by

$$G = \begin{pmatrix} 2a^{1(k+1)}_{1(k+1)} & 0 & \dots & 0 \\ 0 & 2a^{2(k+1)}_{2(k+1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{2}a^{(k+1)(k+1)}_{1(k+1)} & \sqrt{2}a^{(k+1)(k+1)}_{2(k+1)} & \dots & a^{(k+1)(k+1)}_{(k+1)(k+1)} \end{pmatrix}.$$

Since G is lower triangular, it follows that any principal minor of G is a product of some of its diagonal entries. That these diagonal entries are positive follows by considering the canonical principal subtransformations $L_{F_{\{i,k+1\}}}$ of L and using our hypothesis about the eigenvalues of such canonical principal subtransformations. From these observations it follows that $\mathcal{N}(L)$ is a P-matrix. ■

Below we give an example to illustrate the above theorem.

EXAMPLE 5.2.3 Consider a Stein transformation $S_A(X) = X - AXA^T$ corresponding to $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$. It is easy to check that S_A satisfies the assumption made in Theorem 5.2.3. The matrix of S_A with respect to the basis $\{E_{11}, E_{12}, E_{22}\}$ is

$$\mathcal{N}(S_A) = \begin{pmatrix} 1 - a_{11}^2 & -\sqrt{2}a_{11}a_{12} & -a_{12}^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is not a self adjoint matrix. The eigenvalues of A are 0 and a_{11} and from Theorem 5.2.1 (iii) we have S_A has the **P**-property iff $|a_{11}| < 1$. Thus choosing $|a_{11}| < 1$ it is immediate that $\mathcal{N}(S_A)$ is a P-matrix.

PROPOSITION 5.2.1 *For $A \in R^{2\times 2}$ we have the following implications:*

(i) *$(L_A)_{FF}$ has the **P**-property for all $F \Rightarrow \mathcal{N}(L_A)$ is a P-matrix.*

(ii) *$\mathcal{N}(L_A)$ is a P-matrix $\Rightarrow L_A$ has the **P**-property.*

Proof. (i) For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we have $\mathcal{N}(L_A) = \begin{pmatrix} 2a_{11} & \sqrt{2}a_{12} & 0 \\ \sqrt{2}a_{21} & a_{11} + a_{22} & \sqrt{2}a_{12} \\ 0 & \sqrt{2}a_{21} & 2a_{22} \end{pmatrix}$.

Let $(L_A)_{FF}$ have the **P**-property for all F. Then from Theorem 5.2.1 (i) A is positive definite. From simple calculations we can easily see that all the principal minors of $\mathcal{N}(L_A)$ are positive. Hence $\mathcal{N}(L_A)$ is a P-matrix.

(ii) If $\mathcal{N}(L_A)$ is a P-matrix, then $\det A > 0$. The eigenvalues of A are given by $\lambda = \frac{Tr(A) \pm \sqrt{(Tr(A))^2 - 4det(A)}}{2}$. Since $\det(A) > 0$, the real parts of the eigenvalues of A are positive. Hence A is positive stable and L_A has the **P**-property. ∎

We do not know if the above proposition can be proved for any n.

We conclude this chapter by presenting an example, which shows that **P**-property of L_{FF}, for all F need not imply the P-matrix property of L.

EXAMPLE 5.2.4 Consider $L(X) = AXA^T$ with $A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$. Note that

$$\mathcal{N}(L) = \begin{pmatrix} 1 & -2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 & -6\sqrt{2} \\ 4 & 6\sqrt{2} & 9 \end{pmatrix}.$$

Since A is positive definite we can check easily that L_{FF} has the **P**-property for all F (see also Theorem 5.2.1 (iv)) but $\mathcal{N}(L)$ is not a P-matrix.

Chapter 6

Concluding Remarks and Open Problems

In this thesis we have studied various aspects of a linear complementarity problem in a finite dimensional real inner product space. Our study of LCP is made over a general as well as specialized closed convex cones. In this chapter we pose a few interesting problems, which had arisen during the course of this thesis.

(i) A question of interest, as it is relevant to the solvability of a LCP is: are the complementary cones corresponding to a transformation L with the **Q**-property closed? Except for an affirmative answer in some special cases like Lyapunov [18] and Stein transformations [19], where **Q**-property is equivalent to the **P**-property, this question remains open.

(ii) In chapter 4, we have shown that when the matrix $M \in R^{n\times n}$ is a quadratic representation, SOCLCP(M,q) has a solution for all $q \in R^n$ if and only if SOCLCP$(M, 0)$ has a unique solution zero. However, it is not known whether the above equivalence holds for all matrices $M \in \Pi(\Lambda_+^n)$ for $n \geq 3$.

Bibliography

[1] F. Alizadeh and D. Goldfarb, Second-order cone programming, Math. Program. Ser. B 95 (2003) 3-51.

[2] G. P. Barker, The lattice of faces of a finite dimensional cone, Linear Algebra Appl. 7 (1973) 71-82.

[3] A. Berman, Cones, Matrices and Mathematical Programming, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1972.

[4] A. Brondsted, An Introduction to Convex Polytopes, Springer, Berlin, 1983.

[5] M. K. Çamlibel, W. P. M. H. Heemels, A. J. van der Schaft and J. M. Schumacher, Switched networks and complementarity, IEEE Transactions on Circuits and Systems, I: Fundamental theory and applications, 50 (2003) 1036-1046.

[6] R. W. Cottle and G. B. Dantzig, Complementarity pivot theory of mathematical programming, Linear Algebra Appl. 1 (1968) 103-125.

[7] R. W. Cottle, Completely-Q matrices, Math. Program. 19 (1980) 347-351.

[8] R. W. Cottle, J.-S. Pang and R. E. Stone, The Linear Complementarity Problem, Academic press, Boston, 1992.

[9] F. Facchinei and J.-S. Pang, Finite-Dimensional Variational Inequalities and Complemetarity Problems, Springer, New York, 2003.

[10] J. Faraut and A. Korányi, Analysis on Symmetric Cones, Oxford University Press, Oxford, 1994.

[11] M. Fiedler and V. Pták, On matrices with non-positive off-diagonal elements and positive principle minors, Czechoslovak Math. J. 12 (1962) 382-400.

[12] M. C. Ferris and J.-S. Pang, Engineering and economic applications of complementarity problems, SIAM Rev. 39 (1997) 669-713.

[13] M. Fukushima, Z. Q. Luo, P. Tseng, Smoothing functions for second-order cone complementarity problems, SIAM J. Optim. 12 (2001) 436-460.

[14] M. S. Gowda, Complementarity problem over locally compact cones, SIAM J. Control Optim. 27 (1989) 836-841.

[15] M. S. Gowda and Thomas Seidman, Generalized linear complementarity problems, Math. Program. Ser. A 46 (1990) 329-340.

[16] M. S. Gowda and J.-S. Pang, On the boundedness and stability of solutions to the affine variational inequality problem, SIAM J. Control Optim. 32 (1994) 421-441.

[17] M. S. Gowda, An analysis of zero set and global error bound properties of a piecewise affine function via its recession function, SIAM J. Matrix Anal. Appl. 17 (1996) 594-609.

[18] M. S. Gowda and Y. Song, On semidefinite linear complementarity problems, Math. Program. 88 (2000) 575-587.

[19] M. S. Gowda and T. Parthasarathy, Complementarity forms of Theorems of Lyapunov and Stein and related results, Linear Algebra Appl. 320 (2000), 131-144.

[20] M. S. Gowda and Y. Song, Errata: On semidefinite linear complementarity problems, Math. Program. Ser. A 91 (2001) 199-200.

[21] M. S. Gowda and Y. Song, Some new results for the semidefinite linear complementarity problem, SIAM J. Matrix Anal. Appl. 24 (2002) 25-39.

[22] M. S. Gowda, Y. Song and G. Ravindran, On some interconnections between strict monotonicity, globally uniquely solvable, and **P**-properties in semidefinite linear complementarity problems, Linear Algebra Appl. 370 (2003) 355-368.

[23] M. S. Gowda, Y. Song and G. Ravindran, On some interconnections between strict monotonicity, GUS, and P-properties in semidefinite linear complementarity problems, Research Report TR 01-11, Department of Mathematics and Statistics, University of Maryland, Baltimore County, October 2001.

[24] M. S. Gowda, R. Sznajder and J. Tao, Some **P**-properties for linear transformations on Euclidean Jordan algebras, Linear Algbera Appl. 393 (2004) 203-232.

[25] G. J. Habetler and A. L. Price, Existence theory for generalized nonlinear complementarity problems, J. Optim. Theory Appl. 7 (1971) 223-239.

[26] S. Hayashi, T. Yamaguchi, N. Yamashita and M. Fukushima, A matrix splitting method for symmetric affine second-order cone complementarity problems, (to appear in J. Comput. Appl. Math.).

[27] S. Hayashi, N. Yamashita and M. Fukushima, A combined smoothing and regularization method for monotone second-order cone complementarity problems, (to appear in SIAM J. Optim.).

[28] S. Hayashi, N. Yamashita and M. Fukushima, Robust Nash equilibria and second-order cone complementarity problems, Journal of Nonlinear and Convex Analysis 6 (2005) 283-296.

[29] W. P. M. H. Heemels, M. K. Çamlibel, A. J. van der Schaft and J. M. Schumacher, Modelling, well-posedness, and stability of switched electrical networks In: O. Maler, A. Pnueli (eds.), Hybrid Systems: Computation and Control (Proc. 6th International Workshop, HSCC2003, Prague, April 2003), LNCS 2623, Springer, Berlin, 2003, 249-266.

[30] R. D. Hill and S. R. Waters, On the cone of positive semidefinite matrices, Linear Algebra Appl. 90 (1987) 81-88.

[31] R. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

[32] L. Van der Heyden, A variable dimension algorithm for the linear complementarity problem, Math. Program. 19 (1980) 328-346.

[33] G. Isac, Complementarity problems, Lecture notes in Mathematics; v. 1528, Springer-Verlag, Berlin, 1992.

[34] Y. Kanno, J. A. C. Martins and A. Pinto da Costa, Three-dimensional quasi-static frictional contact by using second-order cone linear complementarity problem, Int. J. Numer. Meth. Engng 65 (2006) 62 - 83.

[35] W. Kaplan, A test for copositive matrices, Linear Algebra Appl. 313 (2000) 203-206.

[36] S. Karamardian, Generalized complementarity problem, J. Optim. Theory Appl. 8 (1971) 161-168.

[37] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps, J. Optim. Theory Appl. 18 (1976) 445-454.

[38] S. Karamardian, An existence theorem for the complementarity problem, J. Optim. Theory Appl. 19 (1976) 227-232

[39] M. Kojima, S. Shindoh and S. Hara, Interior point methods for the monotone semidefinite linear complementarity problems in symmetric matrices, SIAM J. Optim. 7 (1997) 86-125.

[40] M. Kojima, M. Shida and S. Shindoh, Reduction of monotone linear complementarity problems over cones to linear programs over cones, Acta Math. Vietnam. 22 (1997) 147-157.

[41] C. E. Lemke, Bimatrix equilibrium points and mathematical programming, Management Science, 11 (1965) 681-689.

[42] C. E. Lemke and J. Howson, Equilibrium points of bimatrix games, SIAM J. Appl. Math. 12 (1964) 413-423.

[43] R. Loewy and H. Schnieder, Positive operators on the n-dimensional ice-cream cone, J. Math. Anal. Appl. 49 (1975) 375-392.

[44] M. Malik and S. R. Mohan, On the geometry of semidefinite linear complementarity problems, Discussion Paper Series No. DPS/SQCOR/Delhi/02-2003, Indian Statistical Institute, New Delhi.
http://www.isid.ac.in/ srm/trmay03.pdf

[45] M. Malik, On linear complementarity problems over symmetric cones In: S. R. Mohan, S.K. Neogy (Eds.), Operations Research with Economic and Industrial Applications: Emerging Trends, Anamaya Publishers, New Delhi, 2005.

[46] M. Malik and S. R. Mohan, On $\mathbf{Q}$ and $\mathbf{R_0}$ properties of a quadratic representation in linear complementarity problems over the second-order cone, Linear Algebra Appl. 397 (2005) 85-97.

[47] M. Malik and S. R. Mohan, Some geometrical aspects of semidefinite linear complementarity problems, Linear Multilinear Algebra 54 (2006) 55-70.

[48] M. Malik and S. R. Mohan, Cone complementarity problems with finite solution sets, OR Letters 34 (2006) 121-126.

[49] M. Mesbahi and G. P. Papavassilopoulos, A cone programming approach to bilinear matrix inequality problem and its geometry, Math. Program. 77 (1997) 247-272.

[50] M. Mesbahi, M. G. Safonov, and G. P. Papavassilopoulos, Bilinearity and complementarity in robust control . In "Advances in Linear Matrix Inequality Methods in Control", SIAM, 2000.

[51] R. D. C. Monteiro and T. Tsuchiya, Polynomiality of primal-dual algorithms for semidefinite linear complementarity problems based on the Kojima-Shindoh-Hara family of directions, Math. Program. 84 (1999) 39-54.

[52] K. G. Murty, On the number of solutions to the complementarity problem and spanning properties of complementary cones, Linear Algebra Appl. 5 (1972) 65-108.

[53] J.-S. Pang and D. Ralph, Piecewise smoothness, local invertibility, and parametric analysis of normal maps, Math. Oper. Res., 21 (1996) 401-426.

[54] J.-S. Pang, D. Sun and J. Sun, Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems, Math. Oper. Res., 28 (2003) 39-63.

[55] T. Parthasarathy, D. Sampangi Raman and B. Sriparna, Relationship between strong monotonicity, $\mathbf{P_2}$-property, and GUS property in semidefinite linear complementarity problems, Math. Oper. Res., 27 (2002) 326-331.

[56] G. Pataki, On the closednes of the linear image of a closed convex cone, Research Report TR-02-3, University of North Carolina, Chapel Hill. *http://www.or.unc.edu/ pataki/research.html*

[57] G. Pataki, The geometry of cone-LP's, H. Wolkowicz, L. Vandenberghe and R. Saigal, (eds.) The Handbook of Semidefinite Programming, Kluwer, 2000 *http://www.unc.edu/ pataki/research.html*

[58] L. Qi and H. Y. Jiang, On the range set of variational inequalities, J. Optim. Theory Appl. 83 (1994) 565-586.

[59] D. Sampangi Raman, Some contributions to semidefinite linear complementarity problem, PhD Thesis, Indian Statistical Institute, 2003.

[60] S. M. Robinson, Normal maps induced by linear transformations, Math. Oper. Res., 17 (1992) 691-714.

[61] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.

[62] H. Schneider, Positive operators and an inertia theorem, Numerische Mathematik, 7 (1965) 11-17.

[63] S. H. Schmieta and F. Alizadeh, Associative and Jordan algebras, and polynomial time interior-point algorithms for symmetric cones, Math. Oper. Res. 26 (2001) 543-564.

[64] S. H. Schmieta and F. Alizadeh, Extension of primal-dual interior point algorithms to symmetric cones, Math. Program. Ser. A 96 (2003) 409-438.

[65] J. M. Schumacher, Complementarity systems in optimization, Math. Program. Ser. B 101 (2004) 263-295.

[66] B. S. Tam, Private Communication, October 2, 2004.

[67] J. Tao, Some P-properties for linear transformations on the Lorentz cone, PhD Thesis, Department of Mathematics and Statistics, UMBC, 2004.