Stability and largeness of core for symmetric games

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Abstract. Largeness of the core is sufficient for stability of the core. In general the necessity is not known. In this paper we answer affirmatively the necessity for symmetric games. We also prove its equivalence to n specified vectors being imputations and also to the convexity of the lower boundary of the set of all acceptable pay-off vectors of the game. In this paper we establish the equivalence of a condition given by Shapley to the newly evolved condition, thereby give an alternate proof to Shapley's condition.

Key words: Symmetric game, stable core, lower boundary, specified vectors

1. Introduction

In 1944 von Neumann and Morgenstern introduced a theory of solutions for n-person games in characteristic function form in which cooperation and coalition formation is a crucial aspect. The primary mathematical concern regarding this model is the existence of solutions. In 1968 Lucas described a ten person game which has no solution. However researchers have gone on to identify properties of such solutions when they exist and their relationship with other known concepts, in particular, the core. Muto (1978, 1982a, 1982b, 1983) and Heijmans (1986), studied extensively these aspects of von Neumann and Morgenstern solution concepts for symmetric games and also a special class of symmetric games known as (n, k) games. Sharkey (1982) defined and studied the concept of largeness of the core which arose while he was studying an economic problem involving cost allocation. He showed that largeness of the core is a sufficient condition for the stability of the core.

The purpose of this paper is to identify a subclass where largeness of the core turns out to be also necessary and leads to few other interesting and easy to check equivalent conditions for stability of the core in symmetric games. The convexity of the set of all lower boundary points of the set of all acceptable vectors is shown to play an important role in the largeness and stability of the core in this subclass of games.

We need the following definitions in the sequel.

Definition 1. *Given a finite and non-empty player set N, and a real valued function v on the set 2^N of coalitions of $N[v : 2^N \to R]$, the ordered pair (N, v) is called a cooperative game, which assigns 0 to the empty coalition ϕ. The function v is called the* **characteristic function** *of the game. For a coalition $S \subseteq N$ the worth $v(S)$ is interpreted as the savings that can be obtained by players in S in case they decide to cooperate.*

Definition 2. The 0–1 normalization

For a game (N, v) the 0-normalization v_0 of v is defined by

$$v_0(S) = v(S) - \sum_{j \in S} v(\{j\}) \quad \text{for each } S \subseteq N.$$

For a game v with $v_0(N) > 0$, i.e., $v(N) > \sum_{j \in N} v(\{j\})$, we call the game v_{01} with

$$v_{01}(S) = \frac{v_0(S)}{v_0(N)} \quad \text{for each } S \subseteq N$$

the $0 - 1$ normalization of v.

If v is a $0 - 1$ normalized game then $v(\{i\}) = 0$ for each $i \in N$ and $v(N) = 1$.

Definition 3. *I is called the set of all imputations x if*

$$I = \left\{ x \in R^n : x_i \geq v(\{i\}), \sum_{i=1}^{n} x_i = v(N) \right\}.$$

Definition 4. *A is called the set of all acceptable vectors y if*

$$A = \{ y \in R^n : y(S) \geq v(S) \text{ for all } S \subseteq N \}, \text{ where}$$

$$y(S) = \sum_{i \in S} y_i \quad \text{for each } S \subseteq N.$$

Definition 5. *The lower boundary $L(A)$ of A is defined by*

$$L(A) = \{ y \in A : \text{if } y' \in A \text{ and } y' \leq y \text{ then } y' = y \}.$$

Alternatively call x a lower boundary point of A, if $Q_x \cap A = \{x\}$, where $Q_x = \{ y \in R^n : y_i \leq x_i, \text{ for all } i = 1, 2, \ldots, n \}$. Then $L(A)$ is precisely the set of all lower boundary points of A.

Definition 6. *C is called the core of the game and is given by*

$$C = \{x \in R^n : x(S) \geq v(S) \text{ for all } S \subset N;\ x(N) = v(N)\};$$

When the core of a game is non-empty the game is called balanced.

Definition 7. *Let x and y be two imputations, and let S be a coalition. We say that x dominates y via S [notation : $x \succ y$] if*

1. $x_i > y_i$ *for all* $i \in S$.
2. $x(S) \leq v(S)$.

Definition 8. *A set K is called internally stable if for any* $x, y \in K, x \not\succ y$.

Definition 9. *A set K is called externally stable if for all* $y \notin K$, *there exists* $x \in K$ *such that* $x \succ y$.

Definition 10. *A set is called stable if it is internally stable and externally stable.*

Definition 11. *The core of a game* (N, v) *is large if for every*

$$y \in A, \quad \text{there exists } x \in C \text{ such that } x_i \leq y_i \text{ for all } i.$$

Definition 12. *A game v is called symmetric if the characteristic function depends only on the cardinality of the coalitions. In other words, v is simply a real valued function defined on* $\{0, 1, 2 \dots n\}$. *Worth of any coalition S with cardinality s is simply the value of* $v(s)$.

Definition 13. *The totally balanced cover of a symmetric game* (N, v) *is denoted by* $(N, \bar{v})$, *and is given by* $\bar{v}(k) = \mathrm{Max}_{0<s\leq k} v(s) \cdot \begin{pmatrix} k \\ s \end{pmatrix}$, *for all* $k = 1, 2, \dots, n$

Organisation of the paper is as follows: In section 1 we introduce the problem and define the necessary concepts. In section 2 we state the theorems and the lemmas required to prove the theorems. In section 3 we prove the theorems and in section 4 we conclude the paper with some topics for further studies.

2. Main theorems

Apart from the main theorems we also state a few useful lemmas and remarks. Unless otherwise specified we assume the game to be in a $0-1$ normalized form.

Since we are concerned with symmetric games throughout this paper, we now introduce some notations which will be useful for our discussions. First we note that set $W \subseteq R^n$ is called *symmetric* if $x \in W$ implies that all n-dimensional vectors obtained from x by permuting its coordinates are also contained in W. Let $R^n_{\leq} = \{x \in R^n : x_1 \leq x_2 \leq \cdots \leq x_n\}$, and for any $x \in R^n_{\leq}$, let $\pi(x)$ be the set of all n dimensional vectors obtained from x by permuting its coordinates. For any $W_< \subseteq R^n_<$, let $\pi(W_<) = \bigcup_{x\in W_<} \pi(x)$. For simplicity denote $\sum_{i=s}^{t} x_i$ by $x(s,t)$ for any $x \in R^n_<$, and use $x(t)$ to denote $x(1,t)$. Let $I_< = \{x \in R^n_< : x(n) = 1, x_1 \geq 0\}$. Then we have $I = \pi(I_<)$. $I_<$ is called an *ordered imputation set*. For any $x, y \in I_<$ and nonempty $S = \{i(1), \ldots, i(s)\} \subseteq N$ with $i(1) \leq \cdots \leq i(s)$ we say *x dominates y* via *S*, denoted by $x \succ_< y$ via S if $x_{i(j)} > y_j$ for all $j = 1, \ldots s$ and $\sum_{j=1}^{s} x_{i(j)} \leq v(s)$. The core C is given by $C = \pi(C_<)$ where $C_< = \{x \in I_< : x(s) \geq v(s)$ for all $s = 1, \ldots, n-1\}$. In what follows, our discussions will be proceeded exclusively on ordered imputation set $I_<$, and thus, to simplify notations, we will eliminate $_<$ and use $I, A, C, L(A)$ for $I_<, A_<, C_<, L(A_<)$

The following is a well known result, refer to Owen(1982) for a proof.

Lemma 1. *An n-person* $0-1$ *normalized symmetric game* (N, v) *with characteristic function* v *has a nonempty core if and only if* $v(s) \leq \frac{s}{n}$, *for all* s *such that* $1 \leq s \leq n$.

Definition 14. *For* $y \in A$ *define* $\mathscr{S}_y = \{S \subseteq N : y(S) = v(S)\}$.

Lemma 2. *Let* (N, v) *be a symmetric game with nonempty core. Then* $y \in L(A)$ *if and only if* $\bigcup_{S\in\mathscr{S}_y} S = N$. *If* $y \in L(A)\backslash C$ *then* $y_{n-1} = y_n$.

Remark 1: In general it is simple to prove that for any $y \in L(A)\backslash C$, the coordinates of y will be as follows: $y_1 \leq y_2 \leq \cdots \leq y_l = y_{l+1} = \cdots = y_n$

Remark 2: From the definition of A and C it is easy to see that $C = L(A)$ if and only if C is large. (Refer to Sharkey (1982) for a proof.)

Theorem 1. *For an n-person* $(0-1)$ *normalized symmetric game the core is the unique stable set if and only if the core is large.*

Shapley (1973), and Menshikova (1977) have given an equivalent condition in terms of the characteristic function of the game for stability of the core for symmetric games. We state Shapley's theorem below. Menshikova's conditions are the same, and the authors version of an equivalent condition which is simpler and easy to check is available in Theorem 5.

Theorem 2 [Shapley(1973)]. *Suppose* $C \neq \phi$ *in an n-person symmetric game. Then* C *is a stable set if and only if* $\frac{v(n) - \bar{v}(k)}{n-k} \geq \frac{v(t) - \bar{v}(k)}{t-k}$ for all t, k *with* $0 \leq k < t < n$, *where* $\bar{v}$ *denotes the* totally balanced cover *of* v.

This theorem is known to many researchers through private communications to Shapley, and a proof is available in Shapley(1973).

In the following remark we characterize the largeness of the core of a symmetric game by the convexity of the set of lower boundary vectors.

Remark 3: For an n-person $(0-1)$ normalized symmetric game with nonempty core, $L(A)$ is convex if and only if the core is large.

The above is easy to prove by taking any $y \in L(A) \backslash C$ so that $y(n) > 1$. Consider y^c the convex combination of all the permutations of y with equal weights. Observe that for all i, $y_i^c = \frac{(n-1)!}{n!} \cdot \sum_1^n y_i = \frac{y(n)}{n} > \frac{1}{n}$

This contradicts the fact that $y^c \in L(A)$ as $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) \in C$.

Remark 4: The core of a symmetric game when exists is the maximal symmetric convex set within $L(A)$ i.e., If C is the core and if D is another symmetric convex set in $L(A)$ then $D \subseteq C$.

Remark 5: The core of a symmetric game, when nonempty, either has a nonempty interior or has only one element. Further a single point core can not be large in a symmetric game.

Theorem 3. *In the subclass of balanced games the following statements are equivalent:*

(i) *v has a large core.*
(ii) *$\bar{v}$ has a large core.*

Note: This is true for general cooperative games and to prove, observe that the definition of large core depends on two concepts; namely, C and $L(A)$. It is easy to prove that these two are equal for v and $\bar{v}$ using balanced collections and Shapley-Bondareva type conditions.

Lemma 3. *For an n-person $(0-1)$ normalized symmetric game (N, v), the following vector y^k for any given k, $1 \leq k \leq n$ is a lower boundary point of the set of all acceptable vectors, i.e., $y^k \in L(A)$. For $1 \leq i \leq k$, $y_i^k = \frac{\bar{v}(k)}{k}$ and for $n \geq i > k$, $y_i^k = Max\{\bar{v}(i) - y^k(i-1), y_{i-1}^k\}$.*

Lemma 4. *If $y^0 \in L(A)$ is such that $Max_{y \in L(A)} y(n) = y^0(n)$, then $y^0(n) = y^k(n)$ for some k, $1 \leq k \leq n$ where y^k is as in lemma 3.*

Theorem 4. *For the symmetric game (N, v), the core of the game (N, v) is large if and only if the following n vectors are imputations. For each k, $1 \leq k \leq n$, we define y^k as follows: for $1 \leq i \leq k$, $y_i^k = \frac{\bar{v}(k)}{k}$ and for $n \geq i > k$, $y_i^k = Max\{\bar{v}(i) - y^k(i-1), y_{i-1}^k\}$*

Note: The above n vectors will be referred to in the rest of this paper as '**The Specified Vectors for a Symmetric Game**'.

An Example: Consider the following 6 person symmetric game, where it can be checked that y^1, y^2, y^5, y^6 are imputations and y^3 and y^4 are not. Consequently the game does not have a large core. This example depicts that defining n vectors as has been done is a necessity. Given $k \leq n$, examples can be constructed so that y^k fails to be an imputation, as long as the game does not have a large core.

s	1	2	3	4	5	6
$v(s)$ $\bar{v}(s)$	0 0	0.12 0.12	0.22 0.22	0.28 0.2933	0.65 0.65	1.0 1.0

$y^1 = (0, .12, .12, .12, .29, .35)$
$y^2 = (.06, .06, .10, .10, .33, .35)$
$y^3 = (.0733, .0733, .0733, .0733, .3567, .3567)$
$y^4 = (.0733, .0733, .0733, .0733, .3567, .3567)$
$y^5 = (.13, .13, .13, .13, .13, .35)$
$y^6 = (.1667, .1667, .1667, .1667, .1667, .1667)$

Theorem 5. *The specified vectors of a symmetric game are extreme points of the ordered core of the game, when the core is large.*

Remark 6: All extreme points of the (unordered)core are not necessarily 'specified vectors'.

Remark 7: Consider a symmetric game (N, v) and the corresponding $(N, \bar{v})$. The set of specified vectors for (N, v) coincides with the set of specified vectors for $(N, \bar{v})$.

This is trivial as the vector is defined in terms of the totally balanced cover of the game $\bar{v}$ and not v which need not be totally balanced.

Corollary 1. *For a symmetric game (N, v) if y^k's are as defined in Theorem 4, then* $\frac{v(n) - \bar{v}(s)}{n - s} \geq \frac{v(t) - \bar{v}(s)}{t - s}$ *for all t, s with $0 \leq s < t < n$, if and only if y^k is an imputation for all $k : 1 \leq k \leq n$, where $\bar{v}$ denotes the* totally balanced cover *of v.*

Alternate proof to Shapley's Condition: Theorems 1,4 and Corollary 1 put together can be regarded as an alternate proof to Theorem 2.

The following Theorem sums up the new results of this paper.

Theorem 6. *In a symmetric game (N, v) if the core is nonempty then the following are equivalent.*

(a) *The core is large.*
(b) *The core is stable.*
(c) *The lower boundary of the set of all acceptable vectors is convex.*
(d) $\frac{v(n) - \bar{v}(k)}{n - k} \geq \frac{v(t) - \bar{v}(k)}{t - k}$ *for all t, k with $0 \leq k < t < n$*
(e) *The following vectors are all imputations.*

$$\text{For } i \leq k,\ y_i^k = \frac{\bar{v}(k)}{k} \text{ and for } i > k,\ y_i^k = \text{Max}\{\bar{v}(i) - y^k(i-1), y_{i-1}^k\}$$

Weber Vector: Consider a permutation $\pi = (\pi(1), \ldots, \pi(n))$. The marginal worth vector defined below for each permutation of the player set N is known as a Weber Vector. [Refer to Weber (1988)] $x^{\pi}_{\pi(j)} = v(\{\pi(1), \pi(2), \ldots, \pi(j)\}) - v(\{\pi(1), \pi(2), \ldots, \pi(j-1)\})$.

Note: A symmetric game has only one Weber vector, and if the game is convex then the Weber vector is an ordered core element, i.e. $(0, v(2), v(3) - v(2), v(4) - v(3), \ldots, v(n) - v(n-1))$ is an imputation and the core is large.

3. Proofs of the Theorems

Proof of Lemma 2: if part: Let $y \in L(A)$. If $y(N) = v(N)$, then $\bigcup_{S \in \mathscr{S}_y} S = N$.
Suppose $y(N) > v(N)$.
Let $\max\{i : y(i) = v(i)\} = k < n$

Claim: $y_k = y_{k+1} = \cdots = y_n$.

Proof: Suppose not. Without loss of generality(w.l.g.) let $y_n > y_k$.
Find ε such that $0 < \varepsilon < \min_{i>k}\{y(i) - v(i)\}$
Define $y'_n = y_n - \varepsilon$ and $y'_j = y_j$ for all $j \neq n$. Then $y' \in A$.
This contradicts the fact that $y \in L(A)$. Thus the claim holds.
Now from the definition of k, $y(k) = v(k)$. Therefore we obtain

$$\sum_{i-1}^{k-1} y_i + y_{k+1} = v(k)$$

$$\sum_{i-1}^{k-1} y_i + y_{k+2} = v(k)$$

$$\vdots \quad \vdots\ \vdots \quad \vdots\ \vdots$$

$$\sum_{i-1}^{k-1} y_i + y_n = v(k)$$

Hence $\bigcup_{S \in \mathscr{S}_y} S = N$.
only if part: Let $\bigcup_{S \in \mathscr{S}_y} S = N$. Suppose there is an $x \in A$ such that $x \leq y$ and $x_i < y_i$ for some i. Then for this i, there exists an S_i such that $y(S_i) = v(S_i)$. Thus $x(S_i) < v(S_i)$, which implies $x \notin A$ and this leads to a contradiction. The last part is clear. Hence the lemma holds.

Proof of Lemma 3: From the definition of y^k it is clear that $y^k \in A$. So we need to prove that $\bigcup_{S \in \mathscr{S}_{y^k}} S = N$. If $y^k(n) = v(n)$, then we are done. Assume $y^k(n) > v(n)$, and let $l = \max\{i : y^k(i) = \bar{v}(i)\}$.

Claim 1: $l > k$

Proof: If $l = k$, $y^k_i = \dfrac{\bar{v}(k)}{k}$ for $i \leq k$. If $y^k_{k+1} > y^k_k$, then by definition $y^k_{k+1} = \bar{v}(k+1) - y^k(k)$. Hence $y^k(k+1) = \bar{v}(k+1)$. This contradicts the maximality of l. Thus the claim holds.

Claim 2: $y_l^k = y_{l+1}^k = \cdots = y_n^k$

Proof: This also follows from the maximality of l.

From claim 2, we know $y_1^k + \cdots + y_l^k = \bar{v}(l) = y_1^k + \cdots + y_{l-1}^k + y_{l+1}^k$ etc. This shows that $\bigcup_{S \in \mathcal{S}_{yk}} S = N$ with respect to v. Here we use the fact that $\bar{v}$ is symmetric since v is symmetric. That is, $y^k \in L(A_v)$ where $A_v = \{y : y(S) \geq v(S) \text{ for all } S \subseteq N\}$ and $L(A_v)$ is the lower boundary of A_v. Since $A_v = A$, the lemma holds.

Proof of Lemma 4: Suppose $y^0 \in L(A)$ is such that $\text{Max}\{y(n) : y \in L(A)\} = y^0(n)$. As $y^0 \in L(A)$ and $L(A) = L(A_v)$, there is an i such that $y^0(i) = \bar{v}(i)$. Find $j = \text{Max}\{i : y^0(i) = \bar{v}(i)\}$. From Lemma 2 we have $y_j^0 = y_{j+1}^0 = \cdots = y_n^0$.

Now consider

$$y^{j-1} = \left(\underbrace{\frac{\bar{v}(j-1)}{j-1}, \ldots, \frac{\bar{v}(j-1)}{j-1}}_{(j-1)\text{ terms}}, \max\left\{ \bar{v}(j) - \bar{v}(j-1), \frac{\bar{v}(j-1)}{j-1} \right\}, \ldots \right)$$

To avoid some confusion later, let us rename the vector y^{j-1} as y by dropping the superscript $j-1$. Note $y^0(j) = \bar{v}(j) = \bar{v}(j) - \bar{v}(j-1) + \bar{v}(j-1) \leq y_j + \bar{v}(j-1)$. That is, $y^0(j-1) + y_j^0 \leq y_j + \bar{v}(j-1)$ or $y_j^0 \leq y_j + \bar{v}(j-1) - y^0(j-1) \leq y_j$. Since $y_j^0 = y_{j+1}^0 = \cdots = y_n^0$, we have $y_k^0 \leq y_k$ for all $k \geq j$. Therefore

$$\begin{aligned}
y^0(n) &= \bar{v}(j) + \sum_{k=j+1}^{n} y_k^0 \\
&= \bar{v}(j-1) + \bar{v}(j) - \bar{v}(j-1) + \sum_{k=j+1}^{n} y_k^0 \\
&= y_1 + \cdots + y_{j-1} + \bar{v}(j) - \bar{v}(j-1) + \sum_{k=j+1}^{n} y_k^0 \\
&\leq y_1 + \cdots + y_{j-1} + y_j + \sum_{k=j+1}^{n} y_k^0 \\
&\leq y(n).
\end{aligned}$$

Since $y^0 \in L(A)$, we obtain $y^0(n) = y(n)$. This completes the proof of Lemma 4.

Before embarking on the proof of Theorem 1 we will explain the following procedure which is crucially required to prove the theorem.

Procedure: Given $y \in L(A) \backslash C$, y can be reduced to an imputation y' such that for $1 \leq i \leq k$, $y_i' = y_i$ and $y_k' \leq y_{k+1}' = \cdots = y_n'$.

From lemma 2 and Remark 1 we know that $y \in L(A) \setminus C$ is of the form: $(y_1 \le y_2 \le \cdots \le y_p < y_{p+1} = y_{p+2} = \cdots = y_n)$,

Step 1: Check if $y_{p+1} - y_p \ge \dfrac{y(n) - 1}{n - p}$

If the answer is 'yes', define $y'_{p+i} = y_{p+1} - \dfrac{y(n) - 1}{n - p}$ for $1 \le i \le n - p$ and $y'_i = y_i$ for $1 \le i \le p$. Then $y'(n)$ is the required imputation.

If the answer is 'no' define $y'_{p+i} = y_p$ for $0 \le i \le n - p$ and $y'_i = y_i$ for $1 \le i \le p - 1$. Then $y'(n) > 1$

Step 2: Rename y'(defined above) as y and repeat Step 1.

It is easy to see that this is a finite procedure and will terminate with an imputation.

Proof of Theorem 1: if part: It follows from Sharkey (1982).

only if part: Assume the core of the game v is the unique stable set. We will show that for any $y \in A$ there is $x \in C$ such that $x \le y$. For any $y \in A$, take $y' \in L(A)$ such that $y'_i \le y_i$ for all $i = 1, \ldots, n$. For convenience, let us call any such vector in $L(A)$ by the same name y.

Suppose $y \notin C$. Then $y(n) > v(n) = 1$. Then from the Procedure above, there exists an imputation x such that $x_i = y_i$ for $i = 1, \ldots, k$, $x_i < y_i$ for $i = k + 1, \ldots, n$ and $x_k \le x_{k+1} = \cdots = x_n$.

Observe $x \notin C$ for $y \in L(A)$ and $x \le y$ $x \ne y$.

Now we prove that x is an undominated imputation.

Suppose x is dominated by some member $z \in C$, w.l.g. via the minimal set $\{1, 2, \ldots, k, k + 1\}$ such that $v(s) > x(s)$. This implies $z_i > x_i$, for all $i \le k + 1$ and $z(k + 1) = v(k + 1)$.

Let $z_i = x_i + \varepsilon_i$ for $i = 1, 2, \ldots, k, k + 1$ and $z_i \ge x_{k+1} + \varepsilon_{k+1}$, for $i > k + 1$ where $\varepsilon_1, \ldots, \varepsilon_{k+1} > 0$. It follows $\sum_{i=1}^n z_i \ge \sum_{i=1}^n x_i + \varepsilon > 1$, where $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_k + (n - k)\varepsilon_{k+1} > 0$. This contradicts the fact that z is an imputation. This completes the proof of theorem 1.

Proof of Theorem 4: As has been observed already, the vectors $y^k \in L(A)$ for all $k, 1 \le k \le n$. So the *only if part* is obvious by Sharkey (1982) as $C = L(A)$ when the core is large.

if part: By Lemma 4 it is clear that $\mathrm{Max}\{y(n) : y \in L(A)\} = y^k(n)$ for some k.

As all the y^k's are imputations $L(A) = C$ and hence the core is large. Recall Remark 2.

Proof of Theorem 5: Take y^k and suppose x^1 and x^2 are two ordered core vectors such that $y^k = \lambda x^1 + (1 - \lambda)x^2$, $x^1 \ne y^k \ne x^2$ and $0 < \lambda < 1$.

If $x_1^1 < \dfrac{\bar{v}(k)}{k}$ then $x_1^2 > \dfrac{\bar{v}(k)}{k}$. Consequently $x^2(k) > \bar{v}(k)$, and $x^1(k) \ge \bar{v}(k)$ as $x^1 \in C(v)$. Hence a contradiction to the definition of y^k. Similarly we can show that $x_1^1 \not> \dfrac{\bar{v}(k)}{k}$. So $x_i^1 = x_i^2 = \dfrac{\bar{v}(k)}{k}$ for all $i \le k$.

$y_{k+1}^k = \text{Max}\left\{\bar{v}(k+1) - \bar{v}(k), \frac{\bar{v}(k)}{k}\right\}$, Since x is ordered, $x_{k+1}^j \geq \frac{\bar{v}(k)}{k}$ by the conclusion above. Further $x_{k+1}^j \geq \bar{v}(k+1) - \bar{v}(k)$ as $x^j \in C$. Hence $x_{k+1}^j \geq y_{k+1}^k$ for $j = 1, 2$. Therefore $x_{k+1}^j = y_{k+1}^k$ for $j = 1, 2$.

The equality of other x_i^j s to the corresponding y_i^k follows sequentially in a similar manner. Hence the theorem holds.

Though Corollary 1 follows from Theorems 1, 2 and 4, a direct proof is interesting. In the following we give the simple direct proof.

Proof of Corollary 1: Let us first observe that because of *Remark 3*, Shapley's condition is true with $v(t)$ replacing $\bar{v}(t)$ in the right hand side of the inequality. In the following we make use of this fact and hence $\bar{v}(t)$ appears in place of $v(t)$.

if part: Let $0 \leq k < t < n$ and consider y^k. Observe that $y^k(t) \geq \bar{v}(t)$ for $k < t < n$.

$$\frac{v(n) - \bar{v}(k)}{n-k} \text{ is the average of } y_{k+1}^k, \ldots, y_n^k \qquad \cdots\ \cdots (1)$$

$$\frac{y^k(t) - \bar{v}(k)}{t-k} \text{ is the average of } y_{k+1}^k, \ldots, y_t^k \qquad \cdots\ \cdots (2)$$

Because of the ordered nature of the vector y^k it is easy to see that (1) is greater than or equal to (2).

$$\frac{v(n) - \bar{v}(k)}{n-k} \geq \frac{y^k(t) - \bar{v}(k)}{t-k} \geq \frac{\bar{v}(t) - \bar{v}(k)}{t-k} \geq \frac{v(t) - \bar{v}(k)}{t-k}.$$

This is the end of *if part*.

only if part:

$$\frac{v(n) - \bar{v}(k)}{n-k} \geq \frac{\bar{v}(t) - \bar{v}(k)}{t-k} \quad 0 \leq k < t < n \qquad \cdots\ \cdots (3)$$

Observe that Shapley's condition above is also equivalent to the following two conditions. [Refer Kikuta and Shapley (1986)]

$$\frac{v(n) - \bar{v}(t)}{n-t} \geq \frac{\bar{v}(t) - \bar{v}(k)}{t-k} \quad 0 \leq k < t < n \qquad \cdots\ \cdots (4)$$

$$\frac{v(n) - \bar{v}(t)}{n-t} \geq \frac{\bar{v}(n) - \bar{v}(k)}{n-k} \quad 0 \leq k < t < n \qquad \cdots\ \cdots (5)$$

Suppose if possible that a vector y^k defined in Theorem 4 is such that $y^k(n) > v(n)$. This implies that there exists l such that $y_l^k = y_{l+1}^k = \cdots = y_n^k$ and $\bar{v}(l) = y^k(l)$, i.e. $y_l^k = \bar{v}(l) - y^k(l-1)$.

Now put $t = l$ and $k = l - 1$ in (4) above, we get

$$\frac{\bar{v}(n) - \bar{v}(l)}{n - l} \geq \frac{\bar{v}(l) - \bar{v}(l-1)}{1} \geq \bar{v}(l) - y^k(l-1) = y_l^k.$$

Hence $\bar{v}(n) \geq \bar{v}(l) + (n - l)y_l^k = y^k(n)$.

This is contrary to the hypothesis that $y^k(n) > \bar{v}(n)$.

This completes the proof of *only if part*.

4. Concluding remarks

The primary interest in Game Theory has always been the question of existence of a solution or a stable set. Lucas (1968) showed that there are games for which a solution may not exist. Search is still on for subclasses where solution may always exist. Do Symmetric games form one such subclass? – remains the question! However Rabie (1985) showed that a solution need not necessarily exhibit the symmetry of the game. In view of these the question assumes greater significance. Do symmetric games always have a solution? – not necessarily a symmetric one! Sharkey has proved that a convex/subconvex game has a large core, but the authors had a feeling that these conditions are still too strict as sufficient conditions. One can ask whether exactness is sufficient. Biswas et al (1998) have answered this question in the affirmative for symmetric games.

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