ESSAYS ON INDIVIDUAL AND COLLECTIVE POWERS IN A VOTING BODY

SONALI ROY

INDIAN STATISTICAL INSTITUTE

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PREFACE

One of the most important concepts of political science is **power**. While power is a multi-faceted phenomenon, in this thesis we will deal with the issue of power in a collective decision making procedure modeled as a voting game. This thesis embodies the fruit of my intellectual perambulation in the Economic Research Unit of the Indian Statistical Institute (ISI) during the years 2001-2004. The plan of this thesis is as follows. In Chapter 1 we present a brief survey of the literature on voting power indices. In this chapter we also outline the background material and the definitions required for the analysis. In Chapter 2 we investigate the relationship between Coleman's preventive and initiative power indices and also study the properties that they satisfy in details. This chapter is based on **Barua, Chakravarty and Roy (2004):** "**On the Coleman indices of voting power**", ***European Journal of Operational Research*, (forthcoming)**. In Chapter 3 we provide an alternative characterization of the non-normalized Banzhaf index using a set of four independent axioms that have been drawn from different contributions to the literature. This chapter is based on **Barua, Chakravarty and Roy (2004a):** "**A new characterization of the Banzhaf index of power**", ***International Game Theory Review*, (forthcoming)**. In Chapter 4 we characterize the Banzhaf-Coleman-Dubey-Shapley index of sensitivity using a set of independent axioms. We also derive a bound on this index for a very general class of games. This paper is based on **Barua, Chakravarty, Roy and Sarkar (2004):** "**A characterization and some properties of the Banzhaf-Coleman-Dubey-Shapley sensitivity index**", ***Games and Economic Behavior* (2004), 49, 31-48.** Chapter 5 studies the Carreras-Coleman decisiveness index. This paper is based on **Barua, Chakravarty and Roy (2004b): "A note on the Carreras-Coleman decisiveness index".** An earlier version of this paper was presented at the International Conference on Game Theory and its Applications, January 2003, held at Mumbai, India. Chapter 6 is a numerical illustration of how the methodology of power indices can be used to study the distribution of power in real life voting bodies. For this purpose, we have used the example of the Indian

Lok Sabha (the lower house of Indian Parliament), which is the most important legislative body in India.

I cannot express in words my sense of gratitude towards my thesis supervisor Professor Satya R. Chakravarty. I have been very lucky to have got the opportunity to work under his guidance. His clarity of thought and unique approach to a problem has always inspired me and I hope that whatever little I have been able to learn from him will help me immensely in my future endeavours. I must also express my sincerest gratitude towards Professor Rana Barua of Statistics-Mathematics Unit, ISI for his immense help. I must also thank Professor Chakravarty, Professor Barua and Professor Palash Sarkar of Applied Statistics Unit, ISI for allowing me to include our joint work in this thesis. I am extremely grateful to two anonymous referees for their helpful comments and suggestions.

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Kolkata, August 2005 *Sonali Roy*

CONTENTS

CHAPTER 1

GENERAL INTRODUCTION

1.1 Motivation

The issue of measurement of voting power is a very important topic of discussion in social science these days. The concept of voting power concerns any collective decision making body (or, equivalently, a collectivity) which makes ‘yes’ or ‘no’ decisions on any issue, by the process of voting. Examples of such bodies abound in today’s world. The United Nations Security Council, The Council of Ministers in the European Union, the Parliament of the republic of India, the board room of any corporate house etc., are all examples of such decision making bodies.

The voting process of each of these bodies is governed by its own constitution, which lays down the decision making rule for the collectivity. This decision rule in turn aggregates individual votes to determine the decision of the voting body as whole. Typically, when a proposal suggesting a certain course of action is presented before such a body, its members are asked to vote either for the bill (‘yes’) or against it (‘no’). The decision rule then transforms these individual votes into a collective decision of the voting body. As an example, consider a board of directors of a company consisting of five members. Let the decision rule, as laid down by the constitution of the board be ‘simple majority’, i.e., at least three members of the board have to vote ‘yes’ in order that the board collectively passes the bill. So in a situation in which only two members of the board vote ‘yes’ and the remaining three vote ‘no’, the decision rule spells out that the bill is rejected and the course of action as suggested by the bill cannot be taken by the board (in spite of two members wanting it).

In this framework, by individual voting power we mean an individual voter’s ability to change the outcome of the voting procedure by changing his stand on the bill. It is a rough measure of the extent of control that an individual voter has over the collective action of the voting body. As the ability of a voter to influence the outcome of the voting process by changing his vote is determined by the decision rule, it can be said that the decision rule determines how the formal control over the actions of the collectivity is shared among its members. Often there arises the need to assess *decision rules* for its capacity in ensuring *fairness* in sharing of control over the collectivity's action among the members (according to some given definition of *fairness*, which might seem relevant in the given context). For this purpose, the use of some kind of a measure of individual power becomes imperative.

To understand this point better let us consider a real life example, which is the topic of much research these days, the European Union (EU). Of all the decision making bodies of the European Union, the Council of Ministers is by far the most important. The direct voters in the council are themselves representatives of the electorate of the respective EU states. Thus the electorate of the individual EU states exercise indirect influence over the council's decisions. If the accepted notion of *fairness* is that of equitability (i.e., one person one vote), then the indirect influence of electorate in various constituent countries ought to be equal, irrespective of the difference in their population size. In other words, a citizen of Germany ought, in principle, to have just as much influence over a decision of the council as a citizen of (say) Greece[1]. Thus in order to evaluate whether the decision rule of the council is equitable in this sense, we need to first quantify this *amount of influence* (see Felsenthal and Machover (2000)).

There could be many other reasons why the evaluation of a decision rule is necessary. Consider a voting body, which requires unanimity among all the members to pass a resolution, i.e., every member has to vote for the resolution ('yes') in order that the voting body passes it. Then it obvious that here, the power of the decision making body to act is very small. In fact, even in a situation where only one member votes 'no', and all the remaining members vote for the bill, the body cannot translate the wishes of the majority of the members into actual

[1] The question of which decision making procedure is the best involves questions of *fairness*. The phrase 'one person one vote' encapsulates a core idea of procedural fairness. However, national governments are elected by a variety of rules-some of them are far away from proportionality of seats to votes.

collective action. Thus it might sometimes be important to evaluate the degree to which the decision making body as a whole, is empowered as a decision maker. Here it is obvious that what concerns us is not the individual voting power but collective voting power. Hence the need for an index that gives us a quantification of the extent to which the body is able to control the outcome of a division of it.

Having thus stated the need for a quantitative measure of both individual and collective voting power, we proceed to the remaining part of the chapter. In different subsections of section 1.2, we discuss in details the issue of individual voting power. In section 1.2.1, we introduce some preliminary definitions and in section 1.2.2, we formally define what we mean by an index of individual voting power, and discuss some well-known indices. Then in section 1.2.3, some postulates which an index of individual power are expected to satisfy (following Felsenthal and Machover (1995, 1998)) and the associated paradoxes are presented. In section 1.2.4, we discuss some characterizations of the well-known indices of power and in section 1.2.5 some alternative approaches for measuring voting power are introduced. Section 1.2.6 deals with voting power when voters have more than two alternatives to choose from. Section 1.3 presents some characteristics of the voting body as a whole and finally in section 1.4 we list some applications where these indices have been used.

1.2 Individual Voting Power

The measurement of individual voting power is not very straightforward. Consider a voting situation where there are three voters, namely a, b and c. The *weight* of their votes are (say) 8, 4, 1 respectively. Also suppose that the decision rule specifies that at least 10 votes must be cast in favour of the resolution in order to pass it (this is in fact a weighted voting scheme which we define formally in definition 1.12). Now, it might seem reasonable for some to state that the power of voter a is greater than the power of b, because the weight of a's vote is twice that of b. Also one would expect c to have positive power, since the weight attached to his vote is positive. However, a closer look at the situation reveals that both a and b must vote 'yes' jointly, if the voting body has to pass the resolution.

Even if both a and c vote in favour of the resolution, the voting body will be unable to pass it unless b too votes for it. Similarly, if both b and c vote 'yes' and a votes 'no', the resolution will not be passed. However, it does not matter at all which way c votes. Thus c has no control over the collective action of the body and has no power in spite of having a positive weight. Also, the presumption that the power of voter a is greater than the power of b is not true. In fact both of them enjoy equal voting power. The one thing that this example makes clear is that a proper scientific analysis is required for arriving at any measure of individual voting power.

Before we go in to the details of the analysis, we need to give some preliminary definitions

1.2.1 Some Notation and Preliminary Definitions

We begin by defining a very general class of mathematical structures (cooperative games with transferable utility), a special case of which is commonly used to model voting situations. Let $N = \{a_1, a_2, ...a_n\}$ be a set of players. The collection of all subsets of N is denoted by 2^N. Any member of 2^N is called a coalition. For any set *S*, $|S|$ will denote the number of elements in *S*.

Definition 1.1: A game is a pair $(N;V)$, where N is a finite set of n players, ($|N| = n$) and $V : 2^N \rightarrow \mathbf{R}$, where $\mathbf{R}$ is the real line, is the characteristic function that assigns a real number $V(S)$ to each $S \subseteq N$, with $V(f) = 0$. The game is

(i) monotonic if $V(S) \leq V(T)$ whenever $S \subseteq T$.

(ii) superadditive if $V(S \cup T) \geq V(S) + V(T)$ whenever $S \cap T = f$.

(iii) constant sum if $V(S) + V(N \setminus S) = V(N)$ for all $S \subseteq N$.

We will use the notation $\mathbf{G}$ to denote the set of all games. Let the set of all games on N be denoted by $\mathbf{G}_N$. Obviously, $\mathbf{G}_N \subset \mathbf{G}$.

Definition 1.2: A value is any function $\psi : \mathbf{G}_N \to \mathbf{R}^N$, that assigns to each game $G = (N;V) \in \mathbf{G}_N$, and each player $i \in N$, a real number $\psi_i(G)$, called the value for i of G (according to ψ). Note that $\mathbf{R}^N$ is the $|N|$ dimensional Euclidean space indexed by the players of N.

$\psi_i(G)$ can also be interpreted as the payoff that the player $i \in N$ receives by participating in the game G.

A value ψ is said to be efficient if $\sum_{i \in N} \psi_i(G) = V(N) \;\; \forall \; G = (N;V) \in \mathbf{G}_N$.

Ordinarily, a voting situation is modeled by a monotonic game, the range of whose characteristic function is restricted to {0,1}. We assign the value 1 to any coalition that can pass a bill and 0 to any coalition that cannot. In this context, a player is a voter and the set N is called the set of voters. A coalition S will be called winning or losing depending on whether it can or cannot pass a resolution. N is sometimes called the grand coalition.

Such a game is also referred to as a simple game. Formally,

Definition 1.3: Given a set of voters N, a voting game (or equivalently, a simple game) associated with N is a pair $(N;V)$, where $V : 2^N \to \{0,1\}$ satisfies the following conditions:

(i) $V(\phi) = 0$,

(ii) $V(N) = 1$ and

(iii) if $S \subseteq T$, $S, T \in 2^N$, then $V(S) \le V(T)$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. The decision making rule for the committee is embodied in the characteristic function V. A decision-making committee can have any decision rule provided it satisfies very intuitively appealing conditions laid down in the above definition. If all voters unanimously vote against the bill, the committee should reject the bill i.e., an empty coalition should be losing (condition (i)). If all the voters unanimously vote for the bill, the committee should pass the bill, i.e., the grand coalition N should be a winning one (condition (ii)). Condition (iii) can be paraphrased as stipulating that increased support for a bill cannot hurt a bill. So if a coalition S can pass a bill, then any superset T of S can pass it as well.

A voting game $G=(N;V)$ is called proper if $V(S)=V(T)=1$ implies that $S \cap T \neq \phi$. Note that a superadditive game becomes a proper game in the context of simple games. According to this condition two winning coalitions cannot be disjoint. On the other hand a voting game is called improper if there exists at least two winning coalitions which are disjoint. Some authors feel that in the context of voting situations, improper (simple voting) games are quite out of place because they do not correspond to any coherent rule for decision making (Felsenthal and Machover (1995)).

For any $G=(N;V) \in \mathbf{SG}$, we write $\mathbf{W}(G)$ $(\mathbf{L}(G))$ for the set of all winning (losing) coalitions associated with G. Thus, for any $S \subseteq N$, $V(S)=1(0)$ is equivalent to the condition that $S \in \mathbf{W}(G)(\mathbf{L}(G))$. It is obvious that any voting game G is fully represented by the set of its winning coalitions $\mathbf{W}(G)$. The set of all simple voting games will be denoted by $\mathbf{SG}$. The set of all simple voting games on N will be denoted by $\mathbf{SG}_N$. Obviously, $\mathbf{SG}_N \subset \mathbf{SG}$.

Next we introduce the notion of compound games, which is often used in characterizing power indices.

Definition 1.4: Consider the games $G_1=(M_1;W_1)$, $G_2=(M_2;W_2),\ldots,$ $G_k=(M_k;W_k)$, $G_V=(N;V) \in \mathbf{SG}$ such that

(i) $|N|=k$

(ii) $M_1, M_2, \ldots, M_k$ are all disjoint.

Let $\alpha:\{1,\ldots\ldots,k\} \to N$ be a bijection. Then the game $G=(M^*,U) \in \mathbf{SG}$, where $M^*=\bigcup_{j=1}^{k} M_j$ is said to be the compounding of V with $W_1,\ldots\ldots,W_k$ via α (or, alternatively the $V-$composition of $W_1, W_2, \ldots, W_k$), if

$U(S) = V(\{\boldsymbol{a}(j) : W_j(S \cap M_j) = 1\}), \ \forall S \subseteq M^*$.

The composite simple game G defined above is used to represent a two-tier voting process. Such a system involves decision making in two stages. In the first stage there is simultaneous vote among the citizens of k constituencies. The set of voters in constituency j $(j = 1,...,k)$ is given by M_j, and the decision rule W_j determines the outcome of the vote in constituency j. Thus the voting games corresponding to the first stage of the decision making process are given by $G_j = (M_j; W_j)$, $(j = 1,...,k)$. The game $G_V = (N;V)$ represents the second stage of the decision making process. In this stage, the decisions of the k bottom tier constituencies are fed as k respective votes to the game G_V. The set of players in the game G_V is N, and the number of players in N is equal to the number of M_j s. We can imagine the k members of N as delegates, one from each of the bottom-tier constituencies, instructed to vote according to the decisions made by the respective constituencies. The delegate from constituency j is identified with $\boldsymbol{a}(j) \in N$. The decision rule in the second stage is given by V, which collates the k bottom tier decisions into a final decision. The voting game $G = (M^*, U)$ models the two tier voting process described above, as a whole. Thus the players in G are the voters of k constituencies put together, and the decision rule U is a compounding of the second stage decision rule V with the first stage decision rules W_j $(j = 1,...,k)$.

Definition 1.5: Let $G_1 = (N_1; V_1), G_2 = (N_2; V_2) \in$ **SG** be two voting (simple) games. We define $G_1 \vee G_2$ as the game with the set of voters $N_1 \cup N_2$, where a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ or $V_2(S \cap N_2) = 1$. (Also see Holler and Packel (1983) for an allied concept of ‘mergeability of games’.)

Definition 1.6: Given $G_1 = (N_1; V_1), G_2 = (N_2; V_2) \in \mathbf{SG}$, we define $G_1 \wedge G_2$ as the game with the set of voters $N_1 \cup N_2$, where a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ and $V_2(S \cap N_2) = 1$.

Thus, in order to win in $G_1 \vee G_2$ a coalition must win in either G_1 or G_2, whereas to win in $G_1 \wedge G_2$ it has to win in both G_1 and G_2. Clearly, given that G_1 and G_2 are simple games, $G_1 \vee G_2$ and $G_1 \wedge G_2$ are also simple games.

Definition 1.7: A voting game $G = (N; V) \in \mathbf{SG}$ with the voter set N, is called decisive if for all $S \in 2^N$, $V(S) + V(N - S) = 1$. It is obvious that a constant sum game is called a decisive game in the context of simple games.

Definition 1.8: Let $G = (N; V) \in \mathbf{SG}$ be a voting game.

(i) For any coalition $S \in 2^N$, we say that $i \in N$ is swing in S if $V(S) = 1$ but $V(S - \{i\}) = 0$.

(ii) For any coalition $S \in 2^N$, $i \in N$ is said to be swing outside S if $V(S) = 0$ but $V(S \cup \{i\}) = 1$.

(iii) A coalition $S \in 2^N$, is said to be minimal winning if $V(S) = 1$ but there does not exist $T \subset S$ such that $V(T) = 1$. The set of minimal winning coalitions in the game G will be denoted by $\mathbf{MW}(G)$.

Thus, voter i is swing, also called pivotal, key or critical, in the winning coalition S if his deletion from S makes the resulting coalition $S - \{i\}$ losing. Similarly, voter i is swing outside the losing coalition S if his addition to S makes the resulting coalition $S \cup \{i\}$ winning. For any voter i, the number of winning coalitions in which he is swing is same as the number of losing coalitions outside which he is swing (Burgin and Shapley (2001), Corollary 4.1). For any game $G = (N; V)$ and $i \in N$, we write $m_i(G)$ to denote this common number.

Equivalently, $m_i(G)$ is the number of coalitions for which voter i is swing in G. It is often said that $m_i(G)$ is the number of swings of voter i.

Definition 1.9: For a voting game $G = (N;V) \in \mathbf{SG}$ with the set of voters N, a voter $i \in N$ is called a dictator if $\{i\}$ is the sole minimal winning coalition of the game.

A dictator in a game is unique. If a game has a dictator, then he is the only swing voter in the game.

Definition 1.10: For a voting game $G = (N;V) \in \mathbf{SG}$ with the set of voters N, a voter $i \in N$ is called a blocker if i is a member of every minimal winning coalition of the game. Note that by definition a dictator is a blocker, but a blocker may not necessarily be a dictator.

Definition 1.11: Given a game $G = (N;V) \in \mathbf{G}$, a player $i \in N$ is called

(i) a dummy player in the game if $V(S \cup \{i\}) = V(S) + V(\{i\})$ $\forall S \subseteq N \setminus \{i\}$.

(ii) a null player in the game if $V(S \cup \{i\}) = V(S)$ $\forall S \subseteq N \setminus \{i\}$.

The term 'dummy' follows from the observation that such a player has no strategic role in the game. Whatever be the situation, he contributes precisely $V(\{i\})$, the value of the coalition consisting only of itself. If $V(\{i\})$=0, then a dummy player is called a null player. Thus a null player is one who contributes nothing to the game. On the domain of simple games, $\mathbf{SG}$, a dummy player is either a dictator or a null player. Thus in the context of simple games, a null player is defined as a voter who is never swing in the game.[2] A voter $i \in N$ is called a non-dummy (non-null) in $(N;V)$ if he is not a dummy (null) player (in $(N;V)$.

[2] However, many authors refer to a player who is never swing in a simple game as a dummy player (see for e.g., Felsenthal and Machover (1995, 1998), Owen (1978, 1995), Dubey and Shapley (1979)).

A very important voting game is a weighted majority game.

Definition 1.12: For a set of voters $N = \{1, 2, \ldots, n\}$, a weighted majority game is a quadruplet $G = (N; V; \mathbf{w}; q)$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is the vector of nonnegative weights of the $n = |N|$ voters in N, q is a nonnegative real number quota such that $q \leq \sum_{i=1}^{n} w_i$ and for any $S \in 2^N$,

$$V(S) = 1 \quad \text{if} \quad \sum_{i \in S} w_i \geq q$$

$$= 0 \text{ otherwise.}$$

That is, the i^{th} voter casts w_i votes and q is the quota of votes needed to pass a bill. Note that a weighted majority game satisfies condition (i) - (iii) of definition 1.3. A weighted majority game $G = (N; V; \mathbf{w}; q)$ will be proper if $\sum_{i=1}^{n} w_i < 2q$. For an improper game we have $\sum_{i=1}^{n} w_i \geq 2q$.

Another important concept that is used in our analysis is that of partitioned sets.

Definition 1.13: Given a non-empty set X, a $t-$partition of X is a collection of coalitions $\mathbf{X} = (X_1, X_2, \ldots, X_t)$, where

1. $X_1, X_2, \ldots, X_t \subseteq X$

2. $X_i \cap X_j = f$, $i, j = 1,2,\ldots,t$; $i \neq j$

3. $X_1 \cup X_2 \cup \ldots \cup X_t = X$.

If $t = 2$, then X is said to be bipartitioned.

Definition 1.14: Given $G = (N;V) \in \mathbf{SG}$, a yes-no bipartition B is a map from N to $\{-1,1\}$. A player is assigned the value 1 if he votes 'yes' and –1 if he votes 'no'. The 'yes' voting camp is referred to as B^+, and the 'no' voting camp is denoted by B^-.

Definition 1.15: Given $G = (N;V) \in \mathbf{SG}$, a voter $i \in N$ is said to agree with the outcome of a yes-no bipartition B in the game G, if either of the following two conditions hold:

1. $B(i) = 1$ and $B^+ \in \mathbf{W}(G)$.
2. $B(i) = -1$ and $B^+ \notin \mathbf{W}(G)$.

The statement that i agrees with the outcome of a bipartition means that the decision goes i's way: i votes 'yes' and the bill is passed or i votes 'no' and the bill is rejected.

For further discussion on these definitions see Shapley (1962), Dubey and Shapley (1979), Owen (1978), Felsenthal and Machover (1995, 1998), Dubey, Einy and Haimanko (2004).

After these preliminary definitions, we next define what is actually meant by a voting power index and discuss some of the properties that any index of voting power is expected to satisfy.

1.2.2 Some Indices of Individual Voting Power

Roughly speaking, as already mentioned before, by an index of individual power we mean a quantification of the amount of influence that a voter has on the outcome of the voting process. At the very outset we must mention that most of the well-known indices that have been suggested in the literature, measure a-priori voting power of individual voters. What these indices intend to quantify is the power that a voter has solely by the virtue of the decision rule itself, in a state of a-priori ignorance about some real life factors, like the nature of the bills to be voted on, the voters' actual interests, persuasive skills, mutual affinities, disaffinities etc. (see for e.g., Felsenthal and Machover (1998, 2000), Braham and Holler (2005)). Thus the set of information that is required in finding an index of individual power in a simple game G is wholly contained in the set $\mathbf{W}(G)$. No information that is exogenous to the rule itself is included while calculating such an index. Though there have been criticisms about a-priori indices being useless in evaluating the power distribution in real decision making bodies, nonetheless, a-priori voting power is an important analytical tool even for studying actual voting power (see, among others, Braham and Holler (2005), Felsenthal and Machover (1998)). Also an important point to note before we proceed to discuss the different indices of power is that the widely used tool to analyze voting power is that of simple games, which is essentially binary. This is because it offers each voter to choose from 'yes' or 'no'. Though, in real life situations there are other options besides them available to the voter, the mainstream literature has largely neglected this.

We will discuss the case in which there are more than two alternatives present in section 1.2.6. Otherwise, we assume throughout that the voter has the option of voting 'yes' or 'no' only.

Definition 1.16 By a (a-priori) index of voting power of a player i, we mean a mapping $\varphi_i : \mathbf{SG} \to \mathbf{R}_+$, i.e., a nonnegative real valued function defined on the set of simple (voting) games.

Felsenthal and Machover (1998) proposed the following three conditions as the minimal set of properties which a reasonable index of individual voting power should satisfy.

(i) Iso-invariance (**INV**): Let $G = (N; V)$ and $G' = (N'; V') \in \mathbf{SG}$ be two isomorphic games, that is, there exists a bijection h of N onto N' such that for all $S \subseteq N$, $V(S) = 1$ if and only if $V'(h(S)) = 1$, where $h(S) = \{h(x) : x \in S\}$. Then $\varphi_i(G) = \varphi_{h(i)}(G')$.

(ii) Ignoring null voters (**IGN**): For any $G = (N;\ V) \in \mathbf{SG}$ and for any null voter $d \in N$, $\varphi_i(G) = \varphi_i(G_{-d})$ for all $i \in N - \{d\}$, where G_{-d} is the game obtained from G by excluding d. Similarly, $\varphi_i(G) = \varphi_i(G_{+d})$, where G_{+d} is the game obtained from G by including $d \notin N$ as a null voter.

(iii) Vanishing just for null voter (**VJN**): For any $G = (N;V) \in \mathbf{SG}$, $\varphi_i(G) = 0$ if and only if $i \in N$ is a null voter.

By definition, a power index is always nonnegative. **VJN** shows that the necessary and sufficient condition that the power index attains its lower bound, zero, is that the concerned voter is a null voter. If a voter is a null voter, then he has no influence over the final outcome of the voting process. In no situation can he change the outcome by changing his vote. Since the essence of power of a voter lies in his capability of being a pivotal voter, a voter's power should be minimal (zero) if he is a null voter (see also Dubey (1975), Dubey and Shapley (1979), Taylor (1995) and Burgin and Shapley (2001)). A similar argument applies from the reverse direction.

INV is an anonymity condition. It says that any reordering of the voters does not change the power enjoyed by a voter. Influence of a voter over the outcome does not depend on the irrelevant characteristics of the voter, like his name or place of residence etc. Even if those characteristics change (e.g. he swaps his place of residence with another voter), his influence remains unaltered.

Since a null voter can never affect the outcome of voting it is natural to expect that if a he is excluded from a voting game, the powers of the remaining voters remain unaltered. Likewise, inclusion of a null voter in the game will not change the powers of the existing voters. This is essentially what **IGN** says.

We can also formulate a relative version of **IGN**.

Relative Null Voter Ignoring Principle (RNP): Let G and G_{-d} be the games as given in **IGN**. Then, for any $i, j \in N - \{d\}$, $\dfrac{\varphi_i(G_{-d})}{\varphi_j(G_{-d})} = \dfrac{\varphi_i(G)}{\varphi_j(G)}$,

where φ_j's are assumed to be positive.

RNP says that the power of voter i relative to another voter j remains unaltered if a null voter is excluded from the game. Clearly, **IGN** implies **RNP** but the converse is not true. For instance, $m_i(G)$ satisfies **RNP** but not **IGN**[3].

However, what any index of individual power satisfying the above three conditions gives is essentially an absolute measure of power. But if we are interested in a relative index of power, which gives an idea of how the control over the collective action of a voting body is shared by all the voters, we would require normalization postulate (Felsenthal and Machover (1995)). Hence we have an added condition,

(iv) Normalization (**NOM**): For any $G = (N;\ V) \in \mathbf{SG}$, $\sum_{i \in N} \boldsymbol{j}_i(G) = 1$.

The justification for including **NOM** has been questioned by some authors. Laruelle and Valenciano (1999, 2001) claim that **NOM** has no compelling interpretation as an a-priori requirement on an index of power. They argue that in a simple superadditive game, the requirement that the power index components sum up to 1, cannot be considered as a simple normalization. It is in fact the efficiency condition ($\sum_{i \in N} \boldsymbol{j}_i(G) = V(N)$), which is taken as the required criterion. Usually power indices are used to compare different games, and are axiomatically grounded on assumptions involving power in different games. Thus when different games are involved, **NOM** requires that the sum of power index components is identical in all the games. This makes the condition very demanding. An important use of voting power analysis is to study the dynamics of changing voting structures. Such studies do not require **NOM**. Also we do not require **NOM** while studying the relationship between voting weight and power. Further, normalization is not needed to study the relative power of different players or the relative power of the same player in different games.

The conditions **IGN**, **VJN** and **INV** suggested by Felsenthal and Machover (1998) are necessary but by no means sufficient for making any

[3] For relevance of **INV**, **VJN** and **IGN** in characterizations of different power indices, see section 1.2.4.

measure an acceptable index of voting power. There are some other intuitively appealing properties that any measure of individual power are expected to satisfy. However some indices suggested in the literature do not satisfy many of them, thus giving birth to some well-known paradoxes. These properties along with the paradoxes are discussed in section 1.2.3.

An important interpretation of power indices is that of a restricted notion of semivalues on the set of simple games, $\mathbf{SG}$. Following Weber's (1988) axiomatic description, a value y is a semivalue if and only if it satisfies linearity, positivity, dummy player property and iso-invariance[4]. We have already introduced iso-invariance in the context of power indices in the discussion following definition 1.16. The other properties are formally stated below:

a. Linearity: $y(G+G')=y(G)+y(G')$ and $y(lG)= l\,y(G)$, for all $G=(N;V)$, $G'=(N;V')\in \mathbf{G}$ and $l>0$. The game $(G+G')$ = $(N;V+V')$, where $(V+V')(S)=V(S)+V'(S)\forall S\subseteq N$, and the game $lG=(N;lV)$,where $(lV)(S)= l.V(S) \quad \forall S\subseteq N$.

This condition means that y_i is a linear function in $\mathbf{G}$.

b. Positivity: If $G\in \mathbf{G}$ is monotonic (i.e., satisfies condition (i) of definition 1.1), then $y(G)\geq 0$.

c. Dummy Player Property: If i is a dummy in the game $G=(N;V)\in \mathbf{G}$, i.e., $V(S\cup\{i\})=V(S)+V(\{i\}) \quad \forall S\subseteq N\setminus\{i\}$ (see definition 1.11), then $y_i(G)= V(\{i\})$.

Important examples of semivalues are the Shapley value and the Banzhaf value, which are introduced below (see Carreras, Freixas and Puente (2003), Laruelle and Valenciano (2003) for detailed discussions on semivalues as power indices).

After having defined what we mean by an a-priori power index, we will now introduce some well-known power indices.

The first systematic and scientific approach to the issue of measuring power was initiated by Penrose (1946). His key idea was very simple: the more

[4] These properties have also been used in characterizing different power indices, as we shall see in section 1.2.4.

powerful a voter is, the more often will the outcome of the voting procedure go the way he/she votes. But his work lay buried for a long time and was independently rediscovered by Banzhaf (1965). This was later again rediscovered by Rae (1969) and Coleman (1971). However, the paper that is regarded as the seminal work on this issue, by the mainstream literature, is by Shapley and Shubik (1954).

The Shapley-Shubik Index (1954)**:** This index is in fact the restriction of the Shapley value (Shapley, 1953) to the class of simple voting games. Given a game $G=(N;V)\in \mathbf{G}$, the Shapley value of a player $i\in N$ is defined as $\frac{1}{|N|!}\sum_{X\subseteq N}(|X|-1)!(|N|-|X|)!\{V(X)-V(X\setminus\{i\})\}$.

When this is applied to the class of simple games, $\mathbf{SG}$, we get the Shapley-Shubik index of the power of voter i. Formally,

$$S\text{-}S_i(G)=\sum_{\substack{X\in \mathbf{W}(G)\\ i\in X}}\frac{(|X|-1)!(|N|-|X|)!}{|N|!}.$$

The number of orderings (of voters) in which voter i is pivotal is called the Shapley-Shubik score of i.

The idea behind the index is that voters line up in order to vote for a bill, with the most enthusiastic supporter voting first. As soon as a ‘majority’ (more generally, a minimal winning coalition) has voted for it, the bill is declared passed. Given an ordering of voters, the swing voter for this ordering is the person whose deletion from the coalition of voters of which he is the last member in the given order, transforms this contracting coalition from a winning to a losing one. The Shapley-Shubik index for voter i is the fraction of the orderings in which i is the swing voter.

This index also has a nice probabilistic interpretation. Let the probability, p_i, that i will vote in favour of a bill be chosen from the uniform distribution on [0,1]. If each i approves or rejects a bill with the same probability, i.e., $p_i = p\forall i\in N$

(assumption of homogeneity), then the probability that i's vote will affect the outcome of the bill is given by the Shapley-Shubik index (Straffin (1977, 1988)).

While the Shapley value measures the contribution of a player to the grand coalition, Akimov and Kerby (2000) introduced a coalition supporting value and a coalition suppressing value for each player, and showed that the sum of these values gives the Shapley value for that player. Assuming that players commit themselves in some given order, they measured a player's 'passing power' or coalition supporting value, by counting the number of times he is swing in winning coalitions, for all orders and coalitions. Similarly they calculated a player's blocking power or coalition suppressing value by counting the number of times he is swing outside losing coalitions.

The Banzhaf Index (1965)**:** Given a game $G = (N;V) \in \mathbf{SG}$, while the Shapley-Shubik index is concerned with the order in which a winning coalition may form, the Banzhaf index examines any winning coalition, irrespective of the order in which it may be formed and considers any voter to have power from having a swing in it. The index, which Banzhaf actually defined and used in his work, was the swing number $m_i(G)$. This is often referred to in the literature as the 'raw' Banzhaf index (Dubey and Shapley (1979)). $m_i(G)$ is also called the Banzhaf score of voter i.

However, the forms of the Banzhaf index that are commonly used in the literature are the following:

The Absolute Banzhaf index: The Banzhaf absolute or non-normalized index of player i is defined as the number of winning coalitions in which i is pivotal, divided by the maximal value that this number can take. Formally,

$BZNN_i(G) = \frac{m_i(G)}{2^{|N|-1}}$**.** (See Dubey and Shapley, (1979).)

This index too has a nice probabilistic interpretation due to (Straffin (1977, 1988)).

If the probability p_i that voter i will vote in favour of the bill is chosen from the uniform distribution on [0,1], and if the decision of i has nothing to do with the decision of another voter j (assumption of independence), then the probability that i's vote will affect the outcome of the bill is given by the non-normalized Banzhaf index. However, Leech (1990) has shown that we do not need the assumption of uniform distribution. The only thing that we require is that the probabilities are selected independently at random from any distribution which has an expectation 0.5.

Banzhaf normalized index: The Banzhaf normalized index of player i is the ratio between his power, as measured by the non-normalized Banzhaf index, and the sum of such indices across voters. Formally,

$$BZ_i(G) = \frac{m_i(G)}{\sum_{i=1}^{|N|} m_i(G)}.$$

The concept of the Banzhaf index has also been extended to the space of all games $\mathbf{G}$, giving the formula $\frac{1}{2^{|N|-1}} \sum_{\substack{S \subset N \\ i \notin S}} \left[V\left(S \cup \{i\}\right) - V\left(S\right) \right]$ for what is referred to as the Banzhaf value for player i (Bv_i) (Owen (1975), Dubey and Shapley (1979)).

The Coleman Indices (1971)**:** The power of an individual member of a voting body, when power is interpreted as 'influence' over the outcome of the voting process, can be exercised in two ways: the member can initiate an action or can stall an action from being taken. The difference between these two becomes obvious if one considers the case of a 'vetoer' or a blocker. By the definition of a blocker, his 'yes' vote is necessary but not sufficient to obtain the passage of a bill. So while the blocker can stall the passage of a bill by individual action (without reference to how others vote), he cannot pass a bill by individual action. For this he needs to consider how others vote.

To capture these two aspects of power, Coleman suggested two different power indices for individual voters- an index to measure the power to prevent action and an index to measure the power to initiate action.

Coleman Index of the Power to Prevent Action

The Coleman index of the power to prevent action for voter i is defined as the number of winning coalitions in which i is decisive, divided by the total number of winning coalitions in the game. Formally, in a game $G = (N;V) \in \mathbf{SG}$, where $m_i(G)$ is the number of winning coalitions in which i is critical, voter i's power to block action is calculated as

$$P_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \in S}} \left[V(S) - V(S \setminus \{i\})\right]}{\sum_{S \subseteq N} V(S)} = \frac{m_i(G)}{|\mathbf{W}(G)|}.$$

The index can be interpreted as voter i 's probability to block a bill. $|\mathbf{W}(G)|$ is the number of possible situations which lead to the bill being passed. Since voter i 's 'yes' vote is pivotal in m_i of these situations, given that other voters do not change their vote, i can block the bill by changing his vote to 'no' only in these situations. So the probability that voter i can block a bill is $m_i(G)/|\mathbf{W}(G)|$.

Laruelle and Valenciano (2002a) show that the Coleman index of the power to prevent action gives voter i's probability of being decisive (or swing), conditional to the proposal being accepted, if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $1/2$ for each and all the voters vote independently.

Coleman Index of the Power to Initiate Action

The Coleman index of the power to initiate action for voter i is defined as the number of losing coalitions outside which i is critical divided by the number of losing coalitions in the game. Formally, voter i's power to initiate action is calculated as

$$I_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \notin S}} \left[V\left(S \cup \{i\}\right) - V(S)\right]}{\sum_{S \subseteq N} [1 - V(S)]} = \frac{m_i(G)}{|\mathbf{L}(G)|} = \frac{m_i(G)}{2^{|N|} - |\mathbf{W}(G)|}.$$

The index can be interpreted as voter i's probability to initiate action. While in the Coleman index of the power to prevent action, swings of a voter i are regarded as measuring his ability to destroy a winning coalition, in the Coleman index of the power to initiate action, swings are thought of as measuring a voter's ability to turn an otherwise losing coalition into a wining one.

Laruelle and Valenciano (2002a) show that the Coleman index of the power to initiate action gives voter i's probability of being decisive (or swing), conditional to the proposal being rejected, if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $1/2$ for each and all the voters vote independently.

Brams and Affuso (1976) pointed out that the two indices proposed by Coleman are proportional to the Banzhaf index and to each other. Dubey and Shapley (1979) showed that the harmonic mean of these two indices becomes the Banzhaf index.

However, often these two indices are clubbed with the non-normalized Banzhaf index and are jointly referred to as the Banzhaf-Coleman index (see for e.g., Owen (1978)). In Chapter 2 of this thesis, we study the properties of these two indices in details.

The Deegan Packel Index (1978)**:** According to Deegan and Packel only minimal winning coalitions should be considered in determining the power of a voter. They suggested an index under the assumptions that all minimal winning coalitions are equiprobable and that players in a victorious minimal winning coalition will divide the prize of victory which is available to the winning camp, equally. Thus any two voters belonging to the same minimal winning coalitions should enjoy the same power. Given $G = (N;V) \in \mathbf{SG}$, the Deegan-Packel index for a player $i \in N$ is given by,

$$DP_i(G) = \frac{1}{|\mathbf{MW}(G)|} \sum_{S \in \mathbf{MW}_i(G)} \frac{1}{|S|},$$

where $\mathbf{MW}_i(G)$ is the set of all minimal winning coalitions in the game G to which i belongs. For each $S \in \mathbf{MW}_i(G)$, the term $\frac{1}{|S|}$ suggests that player i shares the spoils of victory equally with the other $|S|-1$ players in the same minimal winning coalition *S*.

The Johnston Index (1978)**:** Johnston proposed his index in answer to Laver's (1978) criticism of the Banzhaf index, that it registers one point every time a voter can destroy a coalition, regardless of how many other voters can do the same thing. Under this new index, instead of i gaining one point from each coalition S in which i is pivotal, i now gains only $1/piv(S)$ th of a point, where $piv(S)$ is the number of voters who are pivotal in S. Given $G = (N;V) \in \mathbf{SG}$,

let $\boldsymbol{c}_i(G) = \sum \{piv(S)^{-1} : S \in \mathbf{W}(G)$ and $S - \{i\} \notin \mathbf{W}(G)\}$. This is also referred to as the Johnston score of i in G.

Then the Johnston index of power of a player $i \in N$ is given by,

$$JN_i(G) = \frac{\boldsymbol{c}_i(G)}{\sum_{j \in N} \boldsymbol{c}_j(G)}.$$

The Holler-Packel Index (Holler (1982); Holler and Packel (1983))**:** The Holler-Packel index or what is alternatively referred as the Public Good index is also based on minimal winning coalitions, though the rationale for considering them is different from that of the Deegan-Packel index. The public good index is based upon the essential characteristic of a public good: non-rivalry in consumption and non-excludability in access. If the outcome of a game is the provision of a public good, each member of the winning coalition will receive the undivided value of the coalition. Only minimal winning coalitions are taken into account because when it comes to the provision of a public good, winning coalitions with excess players will form by sheer ‘luck’ because of the potential for free riding. Given $G=(N;V)\in \mathbf{SG}$, and assuming that all minimal winning coalitions are equally likely, the public good index for voter i is given by

$$PGI_i(G)=\frac{|\mathbf{MW}_i(G)|}{\sum_{j\in N}|\mathbf{MW}_j(G)|} \ .$$

The non-normalized or the absolute public good index for voter i is given by

$$PGI'_i(G)=\frac{|\mathbf{MW}_i(G)|}{|\mathbf{MW}(G)|}.$$

Brueckner (2001) has shown that the assumption of independence in combination with counting only minimal winning coalitions gives the non-normalized or absolute Public Good index.

All the indices of power listed above satisfy **INV**, **IGN** and **VJN**. While S-S, BZ, DP, JN and PGI satisfy **NOM**, $BZNN$, P and I are not normalized indices of power (Felsenthal and Machover (1998), Braham and Steffen (2002)).

Before proceeding to the next section, we must take note of a type of distinction that is made among the indices discussed above. Coleman (1971) pointed out that the problem of decision making in collectivities is not about bargaining or a “battle over the division of spoils, as assumed by the Shapley value…”. Rather, it a problem of controlling the actions of the collectivity and the actions generally have their own consequences and distribution of spoils. This distribution cannot be altered at will, i.e., the spoils cannot be split up among the members of the winning coalition.

Coleman (1971) also pointed out that the notion of voting power quantified by the Shapley-Shubik index is not the power to affect the outcome of voting body in the usual sense, that is, whether a resolution is passed or blocked. Rather, it is the power of the voter to appropriate a share in the fixed prize of victory, available only to the winning camp. This is because the origin of the Shapley-Shubik index is the Shapley value, which was adapted as an index of power by setting the total value of a game as 1, and “determining that a coalition received the value of the game” if the coalition was winning. This approach therefore entails that the problem of power in a collectivity involves a division of a fixed puse (which is normalized at 1) among the members of the winning coalition. However, according to Coleman, the issue of voting involved problems of controlling the action of a collectivity rather than bargaining over spoils (also see Leech (2002d) for further discussion along this line).

Based on Coleman's arguments, Felsenthal and Machover (1998) introduced the concept of I-power and P-power to distinguish between two different motivations of voting behaviour- policy seeking and office seeking. Indices of I-power measure individual voting power when it is interpreted as the voter's ability to change the outcome of the voting process by changing his stand on the bill. Here 'I' stands for 'influence'. On the other hand, from the rival office-seeking viewpoint, the real outcome of voting is the distribution of a fixed purse among the victors. Thus, here power is regarded as a prize. Indices of P-power (here 'P' stands for 'prize') give a measure of individual power when power is interpreted as a voter's estimated share in the fixed prize. Thus, in measuring P-power, the primary concept is a relative measure. By suitable choice of units, this fixed purse which is to be divided among the members of the winning camp can be taken as 1. This makes **NOM** an appropriate requirement of indices that measure P-power. However, when voting is concerned with collective action, rather than the problems of division of spoils, **NOM** becomes irrelevant. Felsenthal and Machover (1998) note that from the perspective of I-power, the meaningful concept is that of absolute power. Thus, in measuring I-power, **NOM** is not a relevant requirement. However, the postulates of **IGN**, **VJN** and **INV** are relevant concepts in measuring both I-power and P-power.

The indices suggested by Shapley-Shubik (1954), Deegan and Packel (1978), Johnston (1978) measure P-power, while the Banzhaf and Coleman indices are measures of I-power (see Felsenthal and Machover (1998)). Since the outcome of a voting game in the story behind Holler's Public Good index is a public good, this index is based on the characteristics of a public good: non-rivalry in consumption and non-excludability in access. Thus each member of the winning coalition receives the undivided value of the coalition and there is no concept of sharing of spoils in the Holler index.

Thus having introduced the most well known indices in the literature, we next turn to the discussion of the some other postulates that an index of voting power are expected to satisfy and the paradoxes that result from some of the indices not satisfying them.

1.2.3 Postulates and Paradoxes

Felsenthal and Machover (1998) laid down **INV**, **IGN**, **VJN** and **NOM** (discussed in section 1.2.2) as the very "minimal adequacy postulates" that any reasonable index of voting power must satisfy. However, apart from them, Felsenthal and Machover (1995, 1998) have also proposed some other postulates that ought to be imposed on indices of power. Violation of these postulates by some of the indices has given birth to some well-known paradoxes. Those postulates and the related paradoxes are discussed below.

Superadditivity[5] and the Paradox of Size: This paradox is due to Shapley (1973) and Brams (1975). To explain this paradox, we first need to know what we mean by superadditivity. Let i and j be two separate voters belonging to a simple game G. Suppose they now decide to form a bloc, and start operating as a single voter ij. It is clear this results in a new game, whose assembly is obtained from the assembly of G by deleting both i and j, and introducing a new voter ij. Formally,

Definition 1.17: Given $G=(N;V)\in \mathbf{SG}$, suppose that the voters $i, j\in N$ are amalgamated into one voter ij. Then the post-merger voting game is the pair $G'=(N';V')\in \mathbf{SG}$, where

$N'=N-\{i, j\}\cup\{ij\}$ and

$V'(S)=V(S)$ if $S\subseteq N'-\{ij\}$,

[5] Note that while the condition of superadditivity as discussed in definition 1.1 applies to a game $G\in \mathbf{G}$, the superadditivity postulate as discussed in this section applies to an index of power.

$= V\left(\left(S - \{ij\}\right) \cup \{i, j\}\right)$ if $ij \in S$.

An index of power φ is said to satisfy the postulate of superadditivity if

$\varphi_{ij}(G') \geq \varphi_i(G) + \varphi_j(G)$.

According to Brams (1975), this postulate is suggested by the conventional wisdom that the whole is at least as large as the sum of its parts. Felsenthal and Machover (1995, 1998) give a stronger argument in favour of the postulate of superadditivity. According to them, it seems natural to expect that if two voters form a bloc, this should not result in an increase in the relative power of any rival voter. However, power being a relative concept here, their argument applies only if φ is a normalized index of power (i.e., satisfies **NOM**).

Shapley (1973) showed that any normalized index of power, which satisfies **INV**, will violate the superadditivity postulate. The failure of superadditivity to hold in general is known as the paradox of size. Hence $S\text{-}S$, BZ, DP, JN and PGI display this paradox. Felsenthal and Machover (1995) show that $BZNN$, which satisfies **INV**, but is not normalized, not only satisfies superadditivity, but also the stricter postulate of additivity[6], i.e., $BZNN_{ij}(G') = BZNN_i(G) + BZNN_j(G)$.

Before we go on to discuss the next postulate, it may be worthwhile to note how the number of winning coalitions in which the merged identity ij is swing in the game G', compares with the number of coalitions in which its members were swing in the original game G. The following proposition gives the number of swings of the merged voter ij in a general voting game.

<u>**Proposition 1.1**</u>: **Let** $G = (N;V) \in$ **SG. Then for any two voters** $i, j \in N$**,** $m_i(G) + m_j(G)$ **is a nonnegative even integer. Moreover, if the voters** $i, j \in N$ **are merged into one voter** ij**, then the number of swings of the bloc voter** ij **in the post-merger game** $G' = (N';V') \in$ **SG is** $\left(m_i(G) + m_j(G)\right)/2$ **.**

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[6] Both the postulates of superadditivity and additivity can be thought of as having a similar spirit as the bloc postulate (**BOP**), because they all are formulated in terms of amalgamation of voters. We discuss **BOP** later in this section.

<u>Proof</u>: $m_i(G) + m_j(G) = \sum_{S \subseteq N-\{i\}} [V(S \cup \{i\}) - V(S)] + \sum_{S \subseteq N-\{j\}} [V(S \cup \{j\}) - V(S)]$

$$= \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i\}) - V(S)] + \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S \cup \{j\})] +$$

$$\sum_{S \subseteq N-\{i,j\}} [V(S \cup \{j\}) - V(S)] \; + \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S \cup \{i\})]$$

$$= 2 \sum_{S \subseteq N-\{i,j\}} [V(S \cup \{i,j\}) - V(S)]$$

$$= 2 \sum_{S \subseteq N-\{ij\}} [V'(S \cup \{ij\}) - V'(S)]$$

$$= 2 \; m_{ij}(G'),$$

where $m_{ij}(G')$ is the number of swings of the voter ij in the merged game G'. Since $m_{ij}(G') \geq 0$ is an integer, $m_i(G) + m_j(G)$ is an even integer. From above we have, $m_{ij}(G') = {(m_i(G) + m_j(G))}/{2}$. This completes the proof of the proposition.

The next postulate deals with dominance of power.

Dominance and Monotonicity[7]**:** Consider a game $G = (N;V) \in \mathbf{SG}$. Then a

player $j \in N$ is said to dominate another player $i \in N$, i.e., $j \succeq i$ if and only if $S \cup \{i\} \in \mathbf{W}(G) \Rightarrow S \cup \{j\} \in \mathbf{W}(G) \forall S \subseteq N \setminus \{j\}$. If $j \succeq i$ but not $i \succeq j$, then j is said to strictly dominate i $(j \succ i)$. Thus the dominance relation $(\succeq)$ is a preordering (i.e., it is transitive and reflexive) of players in a simple voting game. In a weighted voting game, dominance is a total relation while in an unweighted simple voting game, this may not be so (Felsenthal and Machover (1998)).

The Dominance Postulate (DOM): An index $\boldsymbol{j}$ is said to respect dominance if whenever $j \succeq i$, $\boldsymbol{j}_j(G) \geq \boldsymbol{j}_i(G)$.

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[7] Note that the monotonicity postulate is distinct from monotonicity (condition (i) of definition 1.1). While the condition of monotonicity applies to game $G \in \mathbf{G}$, the monotonicity postulate (**MON**) applies to an index of power. Thus if a game is said to be monotonic, then it must be understood to satisfy condition (i) of definition 1.1. But if an index of power is said to be monotonic, then it must be understood to satisfy **MON**.

This means that if j dominates i then any contribution that i can make to the victory of a coalition should not be higher than that of j. So intuitively j must be at least as powerful as i.

A special case of the dominance postulate is monotonicity, which demands that in a weighted majority game, if a voter j has at least as much voting weight as another voter i, then j cannot have less power than i (Felsenthal and Machover (1995, 1998), Freixas and Gambarelli (1997)).

> **The Monotonicity Postulate (MON):** An index $\boldsymbol{j}$ is said to be monotonic if in a voting game $G = (N;\mathbf{w};q)$, whenever $w_j \geq w_i$, $\boldsymbol{j}_j(G) \geq \boldsymbol{j}_i(G)$.

It is obvious that monotonicity is much weaker than dominance. While S-S, $BZNN$, BZ and JN respect dominance, and hence monotonicity, DP does not even satisfy the weaker condition of monotonicity (Deegan and Packel (1982), Felsenthal and Machover (1995, 1998)). PGI too violates monotonicity. However, under certain constraints on the number of non-null players, and on the distribution of votes, PGI satisfies **MON**. Consider a weighted majority game $G = (N;V;\mathbf{w};q)$, where $\mathbf{w} = (w_1, w_2, ...w_{|N|})$ such that $w_i > 0 \forall i \in N$ and $\sum_{k \in N} w_k = 1$. Holler, Ono and Steffen (2001) show that the Public Good index satisfies **MON** for all such weighted majority games with a simple majority rule $\left(q = \frac{1}{2}\right)$ and with $(|N| - g)$ null players for $g \leq 4$. Further, the Public Good index has been shown to satisfy (partial) **MON** for the players j and i, i.e., $\boldsymbol{j}_j(G) \geq \boldsymbol{j}_i(G)$, for all proper weighted majority games, if $w_j > w_i$ and $w_k = w'$ for all other players $k \neq i, j$. Note that nothing is said here about the power relationship between i and k, and j and k.

Freixas and Gambarelli (1997) and Felsenthal and Machover (1998) have taken the stand that dominance (monotonicity) is such an intuitively appealing postulate that any index of that violates it cannot be regarded as a reasonable yardstick of voting power. Felsenthal and Machover (1998) note that any index of a-priori voting power, be it I-power or P-power must respect dominance. On the other hand Deegan and Packel (1978, 1982), Brams and Fishburn (1995), Holler (1997, 1998), Brams and Steffen (2002) argue that such a strong position in favour of dominance (monotonicity) as a postulate for a-priori index of power, is unwarranted for They take the position that if the story of an index is reasonable then one must accept that power may not be monotone. Felsenthal and Machover (1998) have argued that 'any reasonable measure of a priori power… must respect dominance'. However, if we assume that characteristics like ideological affinities can play an important role in determining the influence a voter can have on the voting outcome, then notions like dominance are narrowly defined (see Holler and Napel (2003), Braham and Steffen (2002), Laruelle and Valenciano (2002e)). An example where the dominance postulate in not appropriate, is the Owen-Shapley (1989) spatial power index, which modifies the Shapley-Shubik index, and takes into account the ideological proximity among voters (Holler and Napel (2003)). The same reasoning applies to the Public Good index, which is derived under the supposition that coalitional values are public goods.

The Transfer Postulate and the Paradoxes of Redistribution and Donation: First we will discuss the paradox of redistribution, which is due to Fischer and Schotter (1978). This paradox concerns only weighted voting games. Let $G_1 = (N; \mathbf{w}; q)$ and $G_2 = (N; \mathbf{u}; q)$ be two weighted majority games such that $\sum_{i \in N} w_i = \sum_{i \in N} u_i$. In this configuration a voter $i \in N$ is said to be a donor if $w_i > u_i$ and a recipient if $w_i < u_i$. Intuitively, G_1 represents the initial distribution of weights, and G_2 is obtained from G_1 by a redistribution of weights, whereby some donor(s) donate some weight to the recipient(s). In this configuration, an index $\boldsymbol{j}$ is said to display the redistribution paradox if there is a donor j, who in spite of having lost some weight, actually gains power, i.e., $\boldsymbol{j}_j(G_2) > \boldsymbol{j}_j(G_1)$. Schotter (1982) considers the Shapley-Shubik index and the Banzhaf indices in connection with the paradox of redistribution (also see Dreyer and Schotter (1980)). Felsenthal and Machover (1995, 1998) provide examples to show that any normalized index of voting power satisfying **INV**, will display this paradox. They also show that $BZNN$, which is non-normalized, also displays this paradox.

Felsenthal and Machover (1995) propose the donation paradox as an amended version of the paradox of redistribution. This paradox also involves only weighted majority games, but here there is only one donor. We define the donation paradox formally below.

The Donation Paradox: Let $G_1 = (N; \mathbf{w}_1; q)$ and $G_2 = (N; \mathbf{w}_2; q)$ be two weighted voting games, where $\mathbf{w}_1 = \{w_{11}, w_{12}, ...w_{1i}, ..., w_{1j}, ..., w_{1n}\}$,

$\mathbf{w}_2 = \{w_{21}, w_{22}, ...w_{2i}, ..., w_{2j}, ..., w_{1n}\}$

$w_{2i} = w_{1i} + \boldsymbol{t}$, $w_{2j} = w_{1j} - \boldsymbol{t}$, $w_{1k} = w_{2k}$ $\forall k \in N$, $k \neq i, j$,

and $0 < \boldsymbol{t} \leq w_{1j}$.

Under this configuration an index $\boldsymbol{j}$ is said to display the donation paradox, if

$\boldsymbol{j}_i(G_2) < \boldsymbol{j}_i(G_1)$.

Felsenthal and Machover (1995, 1998) show that while S-S and $BZNN$ are immune to the donation paradox, BZ , DP and JN display it.

Felsenthal and Machover (1995) also propose the transfer postulate as a reasonable postulate that prevents a power index from displaying the donation paradox. Formally,

The Transfer Postulate (TRP): Let G_1 and G_2 be two different simple voting games with the same assembly N , and let $i, j \in N$ ($i \neq j$), such that the following conditions hold:

T1. Whenever i and j are on the same side of a yes-no bipartition B, the outcome of B is identical in G_1 and G_2.

T2. Whenever i and j are on opposite sides of a yes-no bipartition B and i agrees with the outcome of B in G_1 then i also agrees with the outcome of B in G_2.

T3. There exists at least one yes-no bipartition B such that i agrees with the outcome of B in G_2 but not in G_1.

Then we shall say that G_2 arises from G_1 by a transfer from j to i.

We say that an index φ satisfies **TRP** if whenever the above conditions hold, $\varphi_i(G_2) \geq \varphi_i(G_1)$. Likewise, $\varphi_j\left(G_2\right) \leq \varphi_j\left(G_1\right)$.

The meaning of the above postulate is as follows:

Suppose that when we go from the game G_1, to another game G_2, a voter j loses some pivotal roles and another voter i gains some new pivotal roles, while the coalitions that contain both i and j do not change their status. Here we say that G_2 arises from G_1 by a transfer of voting right from j to i. Then, **TRP** demands that power of voter j should not increase under a transfer of a part of his voting right to another voter i. Likewise the power of voter i should not reduce under the transfer.

In the case of weighted majority games **TRP** means that voter j cannot gain power by distributing some of his voting weight to another voter i. Similarly, voting power of i cannot reduce when he receives some voting weight from another voter j. Certainly, if i is a null player and remains a null player with the additional weight, then i's power should not decrease. But if the additional weight transforms i from a null player to a non- null player or if he was already non-null before receiving the additional weight, then too i's power should not reduce after the transfer. Thus, several possibilities regarding change of statuses of power of i and j may arise in going from G_1 to G_2. Clearly, j may lose some swing roles and i may gain some new swing roles. It is also likely that the transfer does not change their swing positions at all. Turnovec, Mercik and Mazurkiewicz (2005) used a concept of global monotonicity, which says that if the weight of one voter is increasing and the weights of all other voters are decreasing or staying the same, then the power of the voter with "growing weight" should at least not decrease. Let $G_1 = \{N; \mathbf{w}_1, q\}$ $\left(i, j \in N\right)$ be a weighted majority game. Suppose G_2 is obtained from G_1 when j donates some positive weight to i, the weights of the other players remaining the same. In this case, satisfaction of global monotonicity requires that $\varphi_i(G_2) \geq \varphi_i(G_1)$. Thus clearly global monotonicity is similar in spirit to **TRP** in weighted majority games.

Felsenthal and Machover (1998) suggest **TRP** as a postulate that indices of both I-power and P-power should satisfy.

It is clear that a power index that satisfies **TRP** cannot display the donation paradox.

While **TRP** is formulated in terms of power of either the donor or the recipient of the voting right, we can have a relative version of **TRP,** which involves powers of both j and i, the donor and the recipient of the voting right.

Relative Transfers Principle (**RTP**): Let G_1 and G_2 be the games as given in **TRP**. Then

$$\frac{\varphi_j(G_2)}{\varphi_i(G_2)} \leq \frac{\varphi_j(G_1)}{\varphi_i(G_1)},$$

where φ_i's are assumed to be positive.

Clearly, **TRP** implies **RTP**. But the converse is not true. For instance, the normalized Banzhaf index BZ satisfies **RTP** but not **TRP**.

The Bloc Postulate and The Bloc Paradox: In order to explain this paradox, we need to first define the bloc postulate.

> **The Bloc Postulate** (**BOP**)**:** Suppose i and j are two separate voters belonging to a simple game G, and let j be a non-null voter. Then an index of power φ is said to satisfy the bloc postulate if $\varphi_{ij}(G') \geq \varphi_i(G)$, where G' is the game defined in definition 1.17.

The bloc postulate is regarded as a compelling requirement that indices of both I-power and P-power should satisfy.

By the bloc paradox we mean any violation of the bloc postulate. Felsenthal and Machover justify this postulate by saying that the bloc ij can be regarded as a result of a takeover in which i annexes j's voting rights and now trades under the new name ij. Thus, it seems intuitively appealing that i should gain power by 'swallowing' a non-dummy voter j.

Felsenthal and Machover (1995, 1998) show that while $S\text{-}S$ and $BZNN$ satisfy the bloc postulate, BZ, DP and JN violate it and hence display the bloc paradox.

They also show that if an index that satisfies **INV**, respects the strict version of the bloc postulate, which requires $\varphi_{ij}(G') > \varphi_i(G)$ in **BOP**, then it must automatically satisfy the non-null postulate[8] (Felsenthal and Machover (1995, Theorem 5.10). The non-null postulate is formally stated below:

The Non-Null Postulate (NNP): If j is a non-null voter in a game $G \in \mathbf{SG}$, then $\varphi_j(G) > 0$.

BZ, DP and JN violate the bloc postulate but satisfy the non-null postulate.

Any index of power that satisfies **INV** and **IGN**, and also respects **TRP** will satisfy the bloc postulate (**BOP**) (Felsenthal and Machover (1995), Theorem 7.10). Moreover, any index satisfying **INV** and **TRP** will also satisfy the dominance postulate (Felsenthal and Machover (1995), Theorem 7.11). This implies that given **INV**, **TRP** is stronger than **DOM**. Thus, any power index satisfying **TRP**, **INV** and **IGN** is immune to the paradoxes of donation, dominance and bloc. It has been shown that $S\text{-}S$ and $BZNN$ satisfy the transfer postulate (Felsenthal and Machover (1995, Theorem 11.2; 1998, Theorem 7.8.26).

BOP is important because it relies on merger of voters. Many authors have used axioms which state relationship between power of a merged entity and its individual components under amalgamation of voters. Thus, these axioms and **BOP** have a similar spirit (see Lehrer (1988), Nowak (1997), Albizuri (2001), Nowak and Radzik (2000)).

The Paradox of New Members: This paradox is due to Brams (1975), and concerns weighted voting games with a normalized weighting system (i.e., the

[8] Felsenthal and Machover (1995) use the name 'non-dummy postulate' instead of 'non-null postulate'.

sum of the voting weights of all the players equals 1). Given such a game, let a new member join the assembly, while the quota and the proportion of weights among the old voters are left unchanged. In this set up, an index $\boldsymbol{j}$ displays the paradox of new members if at least one of the old members has greater voting power in the new game. In other words the paradox appears when old voters share their voting weight with a newcomer, and other things being equal, at least one old voter actually benefits in the process. Brams (1975) discusses this paradox in connection with the Shapley-Shubik index and the Banzhaf indices, while Brams and Affuso (1976) consider the Coleman indices as well. Moreover, Brams and Affuso (1976) note that adding a new voter to a weighted voting game may cause a voter who was previously a null player, to become a non-null player. Thus, this paradox is displayed by any index of voting power (see Felsenthal and Machover (1998)).

The Quarrelling Paradox: Consider a game $G = (N;V) \in \mathbf{SG}$ and let $i, j \in N$, $i \neq j$. Let the game $\tilde{G}$ be obtained from G by removing from $\mathbf{W}(G)$ all winning coalitions in which both i and j are members. The idea is that i and j have quarreled and no longer collaborate. Hence the winning coalitions which contained both the players now cannot form. If the measure of power (according to $\boldsymbol{j}$) of either i or j or both in the game $\tilde{G}$ is greater than in G, then $\boldsymbol{j}$ exhibits the quarrelling paradox. A point to note here is that $\tilde{G}$ may not be a simple voting game. Since $\boldsymbol{j}$ is defined on $\mathbf{SG}$ (see definition 1.16), $\boldsymbol{j}_i(\tilde{G})$ or $\boldsymbol{j}_j(\tilde{G})$ need not be defined. This paradox owes its name to Brams (1975), who showed that S - S and BZ display this paradox. Kilgour (1974) showed that S - S exhibits this paradox, while Deegan and Packel (1982) showed that their index, DP, also displays this paradox.

The Blocker's Share and the Added Blocker Postulates: Felsenthal and Machover (1998) present these two postulates as requirements of an index of relative power (i.e., indices that satisfy **NOM**).

The Blocker's Share Postulate (BSP): Given $G = (N;V) \in \mathbf{SG}$, an index of relative power $\boldsymbol{x}$ is said to satisfy the blocker's share postulate if whenever $i \in N$ is a blocker and $S \in \mathbf{W}(G)$, $\boldsymbol{x}_i(G) \geq \frac{1}{|S|}$.

This postulate sets a lower bound to the relative power of a blocker. Since by definition a blocker is an indispensable member of every winning coalition, therefore given that a winning coalition S is being formed, he will not accept a smaller share of the fixed prize than any other member of S. So he must get at least $\frac{1}{|S|}$. **BSP** yields the sharpest result when S is a minimal winning coalition of the least size.

It is clear that the above argument for **BSP** applies to only P-power. Felsenthal and Machover note that **BSP** is compelling for an index of P-power. However, there is no reason why an index of I-power should satisfy **BSP**.

Felsenthal and Machover (1998, Theorem 7.9.4) show that while S-S satisfies the above postulate, BZ, DP and JN violate it.

Next, consider two simple games G_1 and G_2 where $\mathbf{W}(G_2) = \{S \cup \{k\} : S \in \mathbf{W}(G_1)\}$ and k is a not a voter of G_1. Thus G_2 is obtained from G_1 by adding a blocker to the game. In the given setup, the added blocker postulate is defined as follows:

The Added Blocker Postulate (ABP): An index of relative power, $\boldsymbol{x}$, is said to satisfy the added blocker postulate if whenever i and j are two non-null voters of G_1, $\frac{\boldsymbol{x}_i(G_1)}{\boldsymbol{x}_j(G_1)} = \frac{\boldsymbol{x}_i(G_2)}{\boldsymbol{x}_j(G_2)}$.

This postulate means that the introduction of a blocker in a game must not result in a greater relative advantage to some of the voters than to others. Since this postulate has been proposed for indices of relative power, it is appropriate in the context of P-power.

Felsenthal and Machover (1998) show that BZ satisfies the above postulate, S-S, DP and JN violate it.

In conclusion, we can say that the postulates listed above can be regarded as relevant requirements of an index of voting power, depending on what type of power (I-power or P-power) that particular index is intended to measure. While an index of I-power is expected to satisfy the postulates of **VJN**, **INV**, **IGN**, **DOM**, **TRP** and **BOP,** an index of P-power should satisfy **VJN**, **INV**, **IGN**, **NOM**, **BOP**, **DOM**, **TRP**, **BSP** and **ABP**. For further discussion on postulates and paradoxes of voting power indices, also see Laruelle and Valenciano (2002e), Felsenthal and Machover (1998a, 2002), Felsenthal, Machover and Zwicker (1998).

In the next section we summarize the different studies that have been made on the indices.

1.2.4 Characterizations and Interpretations of the Indices

In this section we discuss the different axioms that have been used to characterize the well-known indices of voting power, and also the different interpretations that various authors have provided to them.

Many examples can be found in the literature, which show that different indices often result in different rankings of players (in terms of their power) (see Saari and Sieberg (2000)). In this scenario, when power indices can lead to conflicting conclusions, it becomes important to identify situations in which it is best suited to apply a certain index vis a vis any other. Thus it becomes very important to specify set of axioms that uniquely identify a particular index. However, the axiomatic approach by itself is insufficient to settle the question of the choice of a power index (Laruelle (1999)). Our research strategy is also the axiomatic approach. We have tried to illustrate the importance of the axioms that we have used, to the best possible extent. Discussing the axioms from the point of view of trying to get more empirical content to the analysis is a quite relevant and important question. However, this is a separate issue and we have not gone into that in this thesis.

Majority of the work that has been done in this area concerns the Shapley-Shubik index and the Banzhaf index. Different authors have used different set of axioms to characterize the indices. However, there are some common axioms that have been used in majority of the exercises. They are **VJN**, **INV**[9], efficiency (see definition 1.2) and the dummy player property, which says that if $i \in N$ is a dummy player in a game $G = (N;V) \in \mathbf{G}$, then $y_i(G) = V(\{i\})$.

Shapley (1954) had emphasized the importance of restricting the domain to simple games only in order to achieve an axiomatic foundation for the concept of voting power. However, many authors have characterized these indices on the set of all games **G**. So, while discussing the axioms that have been used in characterizing these indices on **G**, we will use the notation y (which we had reserved for values (see definition 1.2)), instead of j.

Dubey (1975) provided an axiomatic characterization of the Shapley value, whereas Dubey and Shapley (1979) used a similar set of axioms to uniquely characterize the 'raw' Banzhaf index, $m_i(G)$, on the set of simple games, **SG**. The axioms that were employed in both the characterizations are **VJN**, **INV**, and the transfer principle[10]. The fourth axiom, which distinguishes between the Shapley value and the Banzhaf index, is that of efficiency. While the axiom of efficiency was used in characterizing the Shapley value, the axiom of Banzhaf total power was employed in the characterization of the raw Banzhaf index. Banzhaf total power is in fact a modified version of the efficiency criterion, and is given by $\sum_{i \in N} j_i(G) = \sum_{i \in N} m_i(G)$. The transfer principle that has been used in both the

[9] We may note here that we have discussed these postulates in section 1.2.2 in the context of indices, which have been defined on the set of simple games **SG** only (definition 1.16). However, it is possible to formulate these postulates in terms of values (definition 1.2), which are defined on the set of all games, **G**.

[10] Note that in spite of the similarity in their names, the transfer principle is quite distinct from the transfer postulate, discussed in section 1.2.3.

characterization exercises involves two games. Formally, the axiom can be stated as follows:

The Transfer Principle: Given two games $G_1 = (N;V_1)$, $G_2 = (N;V_2) \in \mathbf{SG}$, an index of power $\boldsymbol{j}$ is said to satisfy the transfer principle if $\boldsymbol{j}(G_1 \vee G_2) + \boldsymbol{j}(G_1 \wedge G_2) = \boldsymbol{j}(G_1) + \boldsymbol{j}(G_2)$, where the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ are as defined in definitions 1.5 and 1.6.

The name of this principle is motivated by the following observation: The game $G_1 \wedge G_2$ is obtained from G_1 when all those coalitions that win only in G_1 are made losing. On the other hand, $G_1 \vee G_2$ arises from the game G_2 when these same coalitions are made winning. Thus $G_1 \wedge G_2$ and $G_1 \vee G_2$ arise from G_1 and G_2 when winning coalitions are 'transferred' from one game to the other (see Weber (1988)).

Though this principle has been subsequently used by many authors in their characterization exercises, it has also been criticized by some authors as being somewhat obscure (see Roth (1977), Straffin (1982), Felsenthal and Machover (1995), Laruelle and Valenciano (2001)). Dubey, Einy and Haimanko (2004) provides an equivalent form of this axiom, which makes its meaning clearer. Suppose there are two pairs of games, G_1, G_1' and G_2, G_2', such that the transition from G_1 to G_1', and G_2 to G_2' entail adding the same set of winning coalitions ($\mathbf{W}(G_1') - \mathbf{W}(G_1) = \mathbf{W}(G_2') - \mathbf{W}(G_2)$). Then the transfer principle is equivalent to saying that the change in power essentially depends only on the change in the voting game. That is, $\boldsymbol{j}(G_1') - \boldsymbol{j}(G_1) = \boldsymbol{j}(G_2') - \boldsymbol{j}(G_2)$. (For further discussion along this line, see Feltkamp (1995).)

Laruelle and Valenciano (2001) proposed more transparent substitutes of the axioms used by Dubey and Shapley (1979), in order to characterize the Shapley-Shubik and the Banzhaf non-normalized indices on the set of all proper simple games. To state their axioms formally, let $\mathbf{SPG}$ denote the set of simple proper games, and G_S^* be the game that is obtained from $G = (N;V) \in \mathbf{SPG}$ after the deletion of the minimal winning coalition S ($S \in \mathbf{MW}(G)$, $S \neq N$). The axioms (other than **INV**, which they have retained for its clear meaning) that they have employed in characterizing the indices are stated below:

The Axiom of Null Player*: Given $G = (N;V) \in$ **SPG** , $i \in N$ is a null player in G if and only if for all $G' = (N;V') \in$ **SPG** , $\varphi_i(G) \leq \varphi_i(G')$.

This axiom essentially says that being a null player is the worst role any player can expect to play. The above axiom replaces **VJN**, which was used by Dubey and Shapley (1979). **VJN** says that the power of any null player in any game is zero. However, in order to reveal that a null player has the least power, in addition to **VJN** we need to state that the power of any player in any game is greater than or equal to zero. The axiom used by Laruelle and Valenciano (2001) on the other hand is "clear and compelling, and makes full sense by itself without requiring the company of any other".

The Transfer Principle*: For any $G = (N;V), G' = (N;V') \in$ **SPG** , and all $S \in \mathbf{MW}(G) \cap \mathbf{MW}(G')$, $S \neq N$,

$\varphi_i(G) - \varphi_i(G_S^*) = \varphi_i(G') - \varphi_i(G_S'^*)$ for all $i \in N$.

This principle postulates that the effect of removing a single minimal winning coalition from the set of winning coalitions on a player's power, is the same in any game in which this coalition is minimal winning. Laruelle and Valenciano have used this axiom in place of the transfer principle of Dubey and Shapley, which they regard as being rather obscure. They have also shown that this principle is actually equivalent to the transfer principle of Dubey and Shapley (1979) on the domain of simple proper games. In fact, they showed that under **INV**, the transfer principle can be replaced by the following axiom which is much easier to understand:

The Axiom of Symmetric Gain-Loss: For all $G = (N;V) \in$ **SPG** , all $S \in \mathbf{MW}(G)$, $S \neq N$ and all $i, j \in S$ (or, $i, j \in N \setminus S$),

$\varphi_i(G) - \varphi_i(G_S^*) = \varphi_j(G) - \varphi_j(G_S^*)$.

This axiom says that the effect of eliminating a minimal winning coalition from a game is the same for any two players belonging to it and for any two players outside it.

The axioms that separate out the Shapley-Shubik and the Banzhaf indices are the axioms of total gain-loss balance and the average gain-loss balance.

The total gain-loss balance says that if a minimal winning coalition S ($S \neq N$) is deleted from a game, the total loss of power of players in S, is equal to the total loss of power of players outside the deleted coalition. Formally,

The Axiom of Total Gain-Loss Balance: For all $G = (N;V) \in$ **SPG**, all $S \in \mathbf{MW}(G)$, $S \neq N$,

$$\sum_{i \in S} \left(\varphi_i(G) - \varphi_i(G_S^*) \right) = \sum_{i \in N \setminus S} \left(\varphi_i(G) - \varphi_i(G_S^*) \right).$$

This axiom, in combination with **INV**, axiom of null player and the axiom of symmetric gain-loss, is used to obtain the Shapley-Shubik index up to a symmetric affine transformation. Laruelle and Valenciano replace the axiom of efficiency used by Dubey (1975) by the above axiom because efficiency is not a compelling axiom, when power is interpreted as the a-priori ability to affect the outcome of a vote.

The total gain-loss balance is replaced by the **Average Gain-Loss Balance** axiom in order to obtain the non-normalized Banzhaf index up to a symmetric affine transformation.

The average gain-loss axiom says that if a minimal winning coalition S ($S \neq N$) is deleted from a game, the average loss of power of players in S, is equal to the average loss of power of players outside the deleted coalition. Formally,

The Axiom of Average Gain-Loss Balance: For all $G = (N;V) \in$ **SPG**, all $S \in \mathbf{MW}(G)$, $S \neq N$,

$$\frac{1}{|S|} \sum_{i \in S} \left(\varphi_i(G) - \varphi_i(G_S^*) \right) = \frac{1}{|N - S|} \sum_{i \in N \setminus S} \left(\varphi_i(G) - \varphi_i(G_S^*) \right).$$

This axiom replaces the axiom of Banzhaf total power of Dubey and Shapley (1979), because the Banzhaf total power axiom "has some unavoidable ad hoc flavor: The index it helps to characterize is partially within it."(Laruelle and Valenciano (2001))

Owen (1972, 1975) introduced the multilinear extension (MLE) of games. The multilinear extension of a game $G=(N;V)$ is a function of n real variables:

$$f(q_1, q_2, ... q_n) = \sum_{S \subset N} \prod_{j \in S} q_j \prod_{j \notin S} (1 - q_j) V(S) \ .$$

Though f is defined on all real variables, it is sometimes useful to consider only values of f on the n-dimensional unit cube (where $n = |N|$) $[0,1]^n$ (i.e., $0 \le q_k \le 1 \ \forall k \in N$). If $G \in \mathbf{SG}$ and q_j is interpreted as the probability that player j will join a (random) coalition S, then the MLE of G can be interpreted as the probability that a winning coalition will form in the game. Owen (1975, 1988) showed that integrating the gradient of the MLE along the main diagonal of the cube gives the Shapley-Shubik index, while evaluating the gradient at the fixed point $\left(\frac{1}{2}, \frac{1}{2} ....., \frac{1}{2}\right)$ gives the non-normalized Banzhaf index.

Straffin (1977, 1988) used the MLE approach of Owen to frame a probability model that yields the Shapley-Shubik and the non-normalized Banzhaf indices under different assumptions on the degree of statistical independence among voters.

Let $0 \le q_k \le 1$ be the probability that voter k votes for the bill. Under the assumption of independence, whereby each q_k is chosen independently from the uniform distribution on $[0,1]$, the probability that the voter k's vote will affect the outcome of the bill is estimated by the non-normalized Banzhaf index. However, if a number q is chosen from the uniform distribution on $[0,1]$ and $q_k = q \forall k$, then the probability that the voter k's vote will affect the outcome of the bill is estimated by the Shapley-Shubik index. Brueckner (2001) has shown that the assumption of independence, in combination with counting only minimal winning coalitions gives the absolute Public Good index. Laruelle and Valenciano (2001a) also provide a synthesis of the probabilistic models. Laruelle and Valenciano, (2002a) show that the Coleman index of the power to prevent (initiate) action gives voter i's probability of being decisive or swing, conditional to the proposal being accepted (being rejected), if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $1/2$ for each and all the voters vote independently.

Owen (1978) provided a characterization of the Banzhaf index in the space of all constant sum games, by using a composition principle, which in combination with the standard axioms - **VJN**, **INV,** linearity led to the Banzhaf value. What the composition principle essentially says is that in two tier compound game (see definition 1.4), the power of a voter i in the compound game is equal to i's power in the first tier game in which he participates, multiplied by the power of i's delegate in the second tier game[11]. Recalling definition 1.4, this principle can be written as $y_i(G) = y_i(G_j)y_j(G_V)$. Note that the composition principle holds in a scenario in which the first tier game is decisive. However, this characterization is not able to single out the Banzhaf value. The null value ($y_i(G) = 0$, $\forall i, G \in \mathbf{G}$) and the dictatorial value ($y_i(G) = V(\{i\})$ $\forall i, G \in \mathbf{G}$) also satisfy the axioms. (See also Dubey, Einy and Haimanko (2004) for a more recent characterization of the Banzhaf index using the composition principle. A weakened version of this principle was used by Albizuri and Ruiz (2001) for characterizing the Banzhaf semivalue.)

As we have already noted in section 1.2.1, a compound game is used to model a two-tier decision making procedure. To decide on a bill, first each electoral district votes on it and arrives at its own decision, according to its own rule. Then each electoral district sends its delegate to the next tier of the decision making process. These delegates then vote according to the decisions made by the

[11] When the population sizes (of the constituencies) are large, the voting power of a citizen in constituency j is approximated by $\sqrt{2}(pn_j)^{-1}(2^{1-k}m_j(G_V))$, where n_j is the population in constituency j and $m_j(G_V)$ is the number of coalitions in the second tier game which are

electoral districts that they represent. One thing to note here is that, in the second tier of the compound game, each delegate is required to vote according to the decision of the electoral district that he represents, on every issue. But in a real life representative multiparty system, the elected legislators serve for a given period of time, and do not poll their districts on each individual issue. Muto (1999) proposes a modification of the Banzhaf index, which is used to evaluate power of a voter in the above scenario.

In order to characterize their index, Deegan and Packel (1978) used the standard axioms - **VJN**, **INV** and **NOM** along with a fourth axiom which says that, given two games $G_1 = (N;V_1), G_2 = (N;V_2) \in \mathbf{SG}$, power in the game $G_1 \vee G_2$ is a weighted mean of the powers of the component games, with the number of minimal winning coalitions in each component game providing the weights. Holler and Packel (1983) used a similar set of axioms to characterize the Public Good index on the domain of simple games.

Myerson (1980) introduced the balanced contributions axiom, which says that the amounts that each player would gain or lose by the other's withdrawal from the coalition should be equal. Formally, this axiom can be stated as follows:

The Balanced Contributions Axiom: A value ψ is said to satisfy the balanced contributions axiom if for any game $G = (N;V) \in \mathbf{G}$,

$\psi_i(S) - \psi_i(S \setminus \{j\}) = \psi_j(S) - \psi_j(S \setminus \{i\})$ $\forall \{i, j\} \subseteq S$ and $\forall$ $S \subseteq N$, $S \neq \phi$.

Both the Shapley and the Banzhaf values satisfy the balanced contributions axiom. Myerson (1980) proved that the balanced contributions axiom and efficiency characterize the Shapley value.

An interesting result using the balanced contributions axiom was shown by Sanchez (1997). Given $G = (N;V) \in \mathbf{G}$, and a coalition $S \subseteq N$, let $G_S = (S;V_S)$

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converted from losing to winning by the admission of delegate j . This is the famous square root rule of voting theory (see among others, Felsenthal and Machover (1998, chapter 3; 1999).

denote the game that is obtained from G, by restricting the domain of V to the subsets of S. Suppose now that the Banzhaf value, Bv, is used to pay a coalition S that has already formed. That is, now the coalition no longer receives $V(S)$, instead it receives $\sum_{i \in S} Bv_i(G_S)$. Then the Shapley value of this new game is equal to the Banzhaf value of the original game.

Hart and Mas-Collel (1988, 1989) provide an alternative characterization of the Shapley value using potentials. In general, the potential of a value y for a game $G = (N;V) \in \mathbf{G}$, is a function $Q : \mathbf{G} \to \mathbf{R}$, such that the components of the value equals the marginals of the potential, where the marginal contribution of player i according to potential function is defined by $Q(N;V) - Q(N \setminus \{i\};V)$.
That is, $Q(N;V) - Q(N \setminus \{i\};V) = y_i(N;V) \quad \forall i \in N$,
where the game $(N \setminus \{i\};V)$ is the restriction of $(N;V)$ to $N \setminus \{i\}$.
Hart and Mas-Collel show that there exists a unique real function on games called the potential function, with respect to which, the marginal contributions are always efficient. Moreover, these marginal contributions are the Shapley value. Since the Shapley value is always efficient, the potential of the Shapley value can be obtained from the apparently weaker condition $\sum_{i \in N}[Q(N;V) - Q(N \setminus \{i\};V)] = V(N)$. But the Banzhaf value does not satisfy the efficiency criterion. Replacing $V(N)$ in the above expression by $\sum_{i \in N} Bv_i(N;V)$, Dragan (1996) finds the potential of the Banzhaf value. He refers to $\sum_{i \in N} Bv_i(N;V)$ as the power of the game $(N;V)$.

Lehrer (1988) provided an axiomatic characterization of the Banzhaf value on the set of all games $\mathbf{G}$, using among other standard axioms like the dummy player property and linearity, an equal treatment axiom and an amalgamation axiom.
The equal treatment axiom is formally stated below:

The Equal Treatment Axiom: Given a game $G = (N;V) \in \mathbf{G}$, let $i, j \in N$, $i \neq j$. If $V(S \cup \{i\}) = V(S \cup \{j\})$ for all $S \subseteq N - \{i, j\}$, then $y_i(G) = y_j(G)$.

This axiom essentially means that if two players are substitutes in a game, then they must enjoy the same power.

The amalgamation axiom that Lehrer used, says that if two players are amalgamated into one (as in definition 1.17), the value of the new player is at least as much as the sum of the values of the original players. This axiom is in fact the superadditivity postulate, which we have already discussed at length in section 1.2.3. (Also see Malawski (2002).)

Haller (1994) has shown that the Banzhaf value is the unique solution concept in $\mathbf{G}$, that satisfies linearity, dummy player property, **INV** and the proxy agreement property, which is defined below.

The Proxy Agreement Property: Let a player $i \in N$ act as a proxy for another player $j \in N$ in the game $G = (N;V) \in \mathbf{G}$. That is, we now consider the game $G_{ij} = (N;V_{ij})$, given by

$V_{ij}(S) = V(S \cup \{j\})$ if $i \in S$

$= V(S \setminus \{j\})$ otherwise.

Then y is said to satisfy the proxy agreement property if for every game and $i, j \in N$, $y_i(N;V) + y_j(N;V) = y_i(N;V_{ij}) + y_j(N;V_{ij})$.

This property can be explained as follows. Suppose that in order to strengthen their positions, players i and j $(i \neq j)$ sign a proxy agreement before entering the game, stipulating that i acts as proxy for j and that j acts as a null player. Then instead of G, we consider the game G_{ij} as given above. The proxy agreement property says that such an arrangement does not change their collective power, i.e., the sum of the two players' values turns out to be immune against manipulation via proxy agreement. (Also see Malawski (2002) for a similar discussion.)

A modified version of the proxy agreement property, namely, the coalitional unanimity proxy property, has been used by Brink and Laan (1998), along with efficiency, **IGN**, additive game property and independence of irrelevant permutations, to characterize the Shapley value. The additive game property says that every player earns his own weight in a monotonic additive game. A monotonic game $G = (N;V) \in \mathbf{G}$ is additive if there exists a weight vector $l \in \mathbf{R}_{+}^{N}$, such that $V(S) = \sum_{i \in S} l_i \forall S \subseteq N$. Thus this property implies that $y_i(G) = l_i \quad \forall i \in N$. The axiom of independence of irrelevant permutations in brief, says that the value of players who do not change roles do not change.

Brink and Laan also provide an axiomatic characterization of the normalized Banzhaf value. The normalized Banzhaf value is an efficient solution concept that distributes the value $V(N)$ proportional to the Banzhaf values of the players. Given a game $G = (N;V) \in \mathbf{G}$, the normalized Banzhaf value for player i is formally defined as follows:

$$BZN_i = \frac{V(N)}{\sum_{i \in N} Bz_i(G)} Bz_i(G).$$

The axioms that they have used in this characterization exercise are the same as those used in characterizing the Shapley value, except the coalitional unanimity proxy property, which has been replaced by the proportional proxy agreement property. This property is another modified version of the Haller's (1994) proxy agreement property.

Nowak (1997) used dummy player property and equal treatment together with the axioms of 2-efficiency and marginal contributions to uniquely characterize the Banzhaf value on the set of all games $\mathbf{G}$. The axiom of marginal contributions, which was originally introduced by Young (1985), says that if the marginal contribution of a player, belonging to the player set of two different games is same in both the games, then his/her power should be the same in the two games. Formally,

The Marginal Contributions Axiom: Let $G = (N;V)$, $G' = (N;U) \in \mathbf{G}$ be two different games. If for some player $i \in N$, we have $V(S \cup \{i\}) - V(S) = U(S \cup \{i\}) - U(S)$, for all $S \subset N \setminus \{i\}$, then $\psi_i(G) = \psi_i(G')$.

The 2-efficiency axiom that Nowak used is in fact the condition of additivity, which we have mentioned in 1.2.3. (For a discussion along this line, also see Malawski (2002).) In Chapter 3 of this thesis, we also provide an alternative axiomatic characterization of the non-normalized Banzhaf index.

In the studies that have been discussed above, the power indices are interpreted either as a unique measure satisfying a set of characterizing axioms (e.g., Dubey and Shapley (1979), Laruelle and Valenciano (2001) etc.) or the power index of a voter i is interpreted as the probability that the player is critical in passing a decision that is to be made according to the voting rule modeled by that game (Straffin (1977, 1988) etc.). There is however a third interpretation due to Roth (1977, 1988), in which the power indices are interpreted as utility functions representing von Neumann-Morgenstern preferences over lotteries on 'roles' in voting procedures. Under this new interpretation, the value that an index attaches to a player i in a game G, can be used to compare the capability to influence outcomes in position i in the game G, with the capability attached to other positions in other voting games, or even random mixtures of them. Under the assumption of 'veil of ignorance', whereby one is uncertain with respect to both the position and the game to be played, Laruelle and Valenciano (2002) translate the axioms that they used in Laruelle and Valenciano (2001) into Roth's setting to characterize the preferences represented by the Banzhaf and the Shapley-Shubik indices.

Some authors have extended some of the indices that have been discussed above, and have suggested new indices of their own.

The Banzhaf index presupposes that the influence of a swing voter i, on a coalition S, is independent of other swing voters in that coalition. However, in real life it is also important whether this swing voter i is unique for S or not.

Burgin and Shapley (2001) modified the non-normalized Banzhaf index taking this into account. They introduced the **Enhanced Banzhaf Power Index**, which for a voter $i \in N$ is formally defined as follows:

Given a game $G = (N;V) \in \mathbf{SG}$, $EBP_i(G) = \sum_{S \subseteq N} c_S(i)$, where

$c_S(i) = 0$ if i is not a swing voter in S

$= 1/k$ if i is a swing voter in S, and k is total number of swing voters in S.

They also provided an axiomatic characterization of this index.

Napel and Widgrén (2001) introduced a strengthened version of **VJN**, based on a formalized notion of inferior players.

Given a game $G = (N;V) \in \mathbf{SG}$, a player $i \in N$ is said to be an inferior player in the game if and only if $\exists j \in N$, $i \neq j$, such that $\mathbf{C}_i(G) \subseteq \mathbf{C}_j(G)$, where $\mathbf{C}_k(G) = \{S \in \mathbf{W}^k(G) : S \setminus \{k\} \notin \mathbf{W}(G)\}$, $k \in N$ ($\mathbf{W}^k(G)$ being the set of winning coalitions containing k).

Thus i is an inferior player if there exists another player j who can veto all coalitions in which i makes a positive contribution but who can himself form a coalition in which he is critical without i having an opportunity to interfere. The strengthened version of **VJN** is the inferior player axiom, which says that if i is an inferior player in the game G, then $\boldsymbol{j}_i(G) = 0$.

Using the notion of inferior players, Napel and Widgrén also introduced the **Strict Power Index**, which is in fact a modified version of the non-normalized Banzhaf index. Given a game $G = (N;V) \in \mathbf{SG}$, the strict power index for player $i \in N$ is given by

$SPI_i(G) = \dfrac{\boldsymbol{h}_i(G)}{2^{|N|-1}}$, where

$\boldsymbol{h}_i(G) = m_i(G)$, if i is not an inferior player

$= 0$ otherwise.

They also provided an axiomatic characterization of their index on the set of simple games, using the standard axioms in the literature (like **INV** etc.), except the **VJN**, which they replace by the inferior player axiom.

Mercik (2000) defined a power index for a cabinet, which is nominated by a legislature, composed of many disjoint and cohesive subgroups. The Shapley-Shubik or the Banzhaf index approach to the evaluation of the power index in this case yields trivial results- 1 for majority cabinets and 0 for minority cabinets. Mercik's index of the power of a cabinet is a function of the sizes and cohesiveness of supporting groups, where, the cohesiveness of the subgroup is measured by the probability that a member votes the same way as the leader of the subgroups.

Owen (1982) defined the modified Banzhaf-Coleman index, which is a modification of the non-normalized Banzhaf index, when a-priori unions represented by coalition structures are considered. A coalition structure **B** is a finite partition $\{B_p\}_{p\in \mathbf{N}}$ (where **N** is the set of natural numbers) of the player set N of a game (see definition 1.13).

Given a game $G=(N;V)\in \mathbf{G}$ and a coalition structure **B**, the modified Banzhaf –Coleman index ($B-C$) for player $i\in B_q$ is given by

$$B-C_i^{\mathbf{B}}(G)=2^{2-|\mathbf{B}|-|B_q|}\sum_{\substack{\mathbf{T}\subseteq\mathbf{B}\\ B_q\notin\mathbf{T}}}\sum_{\substack{S\subseteq B_q\\ i\notin S}}\left[V(L\cup S\cup\{i\})-V(L\cup S)\right],\text{ where } L=\cup_{T\in\mathbf{T}}T\ .$$

Albizuri (2001) characterized this modified Banzhaf-Coleman index on the class of simple games using appropriate versions of dummy player property and **INV**, two amalgamation axioms similar to Lehrer (1988) and an additional axiom of amalgamation stability. The axiom of amalgamation stability says that the index of a player $i\in B_p$ is independent of the amalgamation of any two players that belong to any coalition, where $B_p\neq B_q$.

Amer, Carreras and Giménez (2002) also provided an axiomatic characterization of the modified Banzhaf value for games with coalition structure using among other axioms, the appropriate versions of linearity, dummy player property and **INV**.

In some applications of cooperative games, players are given some a-priori weights. The very first construction of the weighted value for an exogenously given family of weights is due to Shapley (1953a). Axiomatic characterizations of the weighted Shapley value have been provided by Kalai and Samet (1987) and Nowak and Radzik (1995). Nowak and Radzik (1995) uniquely determined the weighted Shapley value on the space of all games, using linearity, efficiency, dummy player property and a modified version of Lehrer's (1988) equal treatment axiom. Radzik, Nowak and Driessen (1997) used a similar set of axioms (excluding efficiency) as Nowak and Radzik (1995) to uniquely determine the weighted Banzhaf value on the space of all games, for an exogenously given system of weights. In this characterization, efficiency was replaced by the Banzhaf value sum property, which is formally given by

$$\sum_{i\in N} y_i(G) = \sum_{i\in N} \sum_{S\subseteq N\setminus\{i\}} \frac{1}{2^{|N|-1}}\left[V(S\cup\{i\}) - V(S)\right], \text{ where } G=(N;V)\in \mathbf{G}.$$

Drawing upon Radzik, Nowak and Driessen (1997), Nowak and Radzik (2000) provided an alternative characterization of the weighted Banzhaf value on $\mathbf{G}$. They retained all the other axioms of Radzik, Nowak and Driessen (1997), except the Banzhaf value sum property, which they replaced by **IGN** and a weighted analogue of the 2-efficiency axiom (see Lehrer (1988), Nowak (1997)).

In the next sub-section we discuss some of the other approaches to measuring individual voting power in a voting situation.

1.2.5 Other Approaches to Measuring Individual Voting Power

Often common power indices are criticized for not taking account of voter's preferences, are accused of being policy blind. The voting power indices in common use treat all coalitions as equally probable or all vote sequences as being equally probable. Voters are treated as anonymous and are treated symmetrically. However, when voters are spread over an ideological spectrum (say from left to right), the problem arises as to how to model this situation in a reasonably realistic way, as in this a case not all coalitions are possible. Perlinger (2000) defined a class of voting games, called spectrum games, in which only connected coalitions (i.e., there is no ideological gap between coalition members) are allowed to form. He then proposed the Markov-Polya index as a parameterized family of power indices. Bilal, Albuquerque and Hosli (2001) integrated the spatial theory of voting (in which voters are assumed to be distributed along a policy scale), with voting power analysis by developing spatial indices of voting power based on the policy preferences of various players.

The Hoede Bakker Index, due to Hoede and Bakker (1982), takes into account the preferences of the players, and also the social structure in which the players may influence each other. Here each player belonging to the set of players $N = \{1, 2, ..., n\}$, has an inclination either to say 'yes' (represented by '1') or 'no' (represented by '0') with respect to a certain bill. Let i denote the inclination vector (it is nothing but a sequence of 0 and 1). The set of all inclination vectors is denoted by IN. The players may or may not actually vote according to their inclination. Let b be the actual decision vector, that is, $b = TRi$, where TR transforms the inclination vector to a decision vector. The decision vector too is thus a sequence of 0 and 1. The group decision gd is a function that is defined on decision vectors and has a value 1, if the group decision is to pass the bill (yes) and –1 otherwise. That is,

$$gd : TR(IN) \rightarrow \{-1, 1\}.$$

TR and gd should satisfy two axioms: firstly, changing all inclinations should lead to the opposite group decision, and secondly a group decision 'yes' is not changed into 'no' if the set of players with inclination 'yes' is enlarged.

Given TR and gd (such that they satisfy the above mentioned axioms), the Hoede-Bakker index of a player $k \in N$ is given by

$$HB_k = \frac{1}{2^{|N|-1}} \sum_{i:i_k=1} gd(TRi).$$

Rusinowska and Swart (2002) have shown that the Hoede-Bakker index displays the paradoxes of redistribution, large size and new members. Also it does not satisfy monotonicity, donation and the bloc postulate. Rusinowska and Swart have also shown that in case all players are independent, the Hoede-Bakker index reduces to the non-normalized Banzhaf index.

While all the standard power indices discussed above assign real numbers to the players in a simple game as a quantitative measure of their influence in the voting situation represented by the game, Taylor and Zwicker (1997) introduced interval measures of power that assign intervals of real numbers to the players.

1.2.6 Voting Power in the Presence of More than Two Alternatives

As has been noted above, the mainstream literature on voting power has confined itself to the simple voting game model, which does not admit abstention. However, in real life situations, to abstain is different from voting 'no'. For example, take the United Nations Security Council. If a permanent member votes 'no', the proposal is rejected, no matter how the other members vote. This is because, the permanent members have a veto power over the Council's actions. However, if a permanent member abstains from voting, the proposal may be passed, provided the other criteria laid down by the decision making rule are fulfilled. Thus, it is important to study the voting power of individual members when they are required to choose one of r $(r \geq 2)$ alternatives.

Bolger (1993) presented a unique extension of the Shapley value to games with r alternatives. This value gives an a-priori evaluation for each player relative to each alternative. In other words, given an arbitrary game $(N;V;r)$, he finds an a-priori value q_i^j, for player i relative to alternative j, where q_i^j can be thought of player i's share of $V(N;j)$ ($V(N;j)$ being the worth of the grand coalition if it chooses alternative j). Also see Bolger (1986). Amer, Carreras and Magaña (1998) also introduced a closely related r-game and defined the Shapley-Shubik index for this type of games.

Felsenthal and Machover (1997, 1998) made an elaborate study of ternary voting rules, where the voters have the choice of abstaining, apart from voting 'yes' and 'no'. They extended the Banzhaf index to apply to ternary voting rules and suggest $\frac{m_i(G)}{3^{|N|-1}}$ as the modified Banzhaf non-normalized index of voter $i \in N$. However the meaning of $m_i(G)$ is slightly different from that in simple voting games. This is because now a player i is critical not only if he can change the outcome of the bill from by changing his vote from ‘yes’ to ‘abstain’, but also from ‘abstain’ to ‘no’. They also obtained an extension of the Shapley-Shubik index so as to apply it to ternary voting rules.

Drawing upon Bolger (1993), Ono (2000) presented a generalized Banzhaf value for multialternative games, which she refers to as the Banzhaf-like value.

Freixas and Zwicker (2003) introduced the weighted (j,k) simple games, where a voter is able to express different levels of support for a bill, ranging from enthusiasm to total opposition. The outcome set is also enlarged from the binary case of the bill being accepted or rejected, to k different levels of collective support for the bill. Standard simple games which allow voters to vote either ‘yes’ or ‘no’ are (2, 2) simple games, whereas (3, 2) simple games allow each voter the option to abstain from voting. Freixas (2005) provides an a-priori Shapley-Shubik index for (j,k) simple games. Linder (2002) also made a study of voting power in weighted (j,k) games.

Till now, we have discussed issues relating to power of individual members of a collective decision making body. The other aspect of the study of power concerns the decision making body as a whole. Though a decision making body is comprised of individual voters, the decision making body has a distinct existence by itself. So in the next section we discuss some global characteristics of the simple voting game.

1.3 Collective Power and Sensitivity

Coleman (1971) proposed a probabilistic measure of what he termed ‘the power of a collectivity to act’. According to him, the power of a collectivity to act, as provided by a set of decision making rules governing the collectivity, lies in the ease with which individual members’ interests in collective action can be translated into actual collective action. In a game $G=(N;V)\in\mathbf{SG}$, Coleman's measure of the power of a collectivity to act is given by,

$$CC(G)=\frac{\sum_{S\subseteq N}V(S)}{2^{|N|}}.$$

Since, in a voting game, $V(S)=1$, if S is a winning coalition, and 0 otherwise,

$$CC(G)=\frac{|\mathbf{W}(G)|}{2^{|N|}}.$$

Since $|\mathbf{W}(G)|$ is the total number of winning coalitions and $2^{|N|}$ is the total number of coalitions (including the empty one) in the game *G*, $CC(G)$ is the prior probability of a positive outcome, that is, the probability that a resolution will be adopted by the voting body.

Laruelle and Valenciano (2002b) used a simple probabilistic model consisting of a voting game $G=(N;V)$, and a probability distribution over the set of '*voting configurations*', $p:2^N\rightarrow\mathbf{R}_+$, $0\leq p(S)\leq 1$ for any $S\subseteq N$ and $\sum_{S\subseteq N}p(S)=1$, where $p(S)$ gives the probability that the voters in S vote 'for' the resolution and the voters outside S vote 'against' the resolution, to arrive at a generalization of Coleman's index of the power of a collectivity to act. This generalized index is given by

Prob {the resolution is accepted}= $\sum_{S:S\in\mathbf{W}(G)}p(S)$.

CC drops out as a special case of the above index when p assigns the same probability to all voting configurations. Laruelle and Valenciano (2002b) also showed that this generalized Coleman's power of a collectivity to act coincides with a generalization in the context of voting situations, of Hart and Mas-Colell's (1988,1989) notion of potential (also see Laruelle and Valenciano (2002c)).

Carreras (2004) developed a decisiveness index for simple games. The decisiveness index provides a measure of the agility of the collective decision making mechanism. This index coincides with the Coleman's index of the power of a collectivity to act. However, Carreras provided different axiomatic characterizations and systematically developed the index. He also provided a probabilistic model of voting which leads in a natural way to the definition of decisiveness index. If the independent probability that a player $i \in N$ will vote in favour of a bill is $1/2 \ \forall \ i \in N$, then the probability that the voting body will accept the bill is $\frac{|\mathbf{W}(G)|}{2^{|N|}}$. This is what Carreras defined as his decisiveness index $(\boldsymbol{d})$. Carreras also established that there is a unique potential function for the Banzhaf index on simple games, and it is given by $2\boldsymbol{d}$. In fact this coincides with the restriction of Dragan's (1996) potential function for the Banzhaf value, which we have already discussed in also see section 1.2.4, to the set of all simple games. In future, we refer to CC as the Carreras-Coleman decisiveness index. In Chapter 5 of this thesis, we also make an elaborate study of the different characterizing properties of the Carreras-Coleman decisiveness index.

Another important global characteristic of a simple voting game is sensitivity. Dubey and Shapley (1979) proposed the sum of the number of swings of all players, $\sum_{i \in N} m_i(G)$, as a kind of democratic participation index, which gives a measure of the decision rule's 'sensitivity' to the desires of the average voter. (Also see Holler and Li (1995), where the sum of the swingers of all minimum winning coalitions is defined as an expression of the total power in the system.) In other words, it is a measure of the ease with which a decision rule responds to the fluctuations in the voters' wishes. Based on some assumptions about the size and the number of winning coalitions in the game, and using Hart's (1976) results, they found some bounds of this 'swing total'. They showed that if $G = (N;V) \in \mathbf{SG}$ is decisive, then $\sum_{i \in N} m_i(G) \geq 2^{|N|-1}$. Felsenthal and Machover (1998) also studied the sensitivity of decision rules in great details. Their modified version of Dubey and Shapley's (1979) sensitivity index is given by:

$$SS(G) = \frac{\sum_{i \in N} m_i(G)}{2^{|N|-1}} \ .$$

Since $SS(G)$ is the sum of the non-normalized Banzhaf-Coleman index of different voters in a game, we will refer to $SS(G)$ as the Banzhaf-Coleman-Dubey-Shapley (BCDS) index of sensitivity. Chapter 4 of this thesis also studies the sensitivity of decision rules in details.

They also proposed a measure of relative sensitivity. Given a simple game $G = (N;V)$, their measure of relative sensitivity is given by:

$$RS(G) = \frac{\log\left(\sum_{i\in N} m_i(G)\right) - \log|N|}{\log\left(m^{|N|}C_m\right) - \log|N|}$$, where m is the least integer $> |N|/2$.

This is also sometimes referred to as the ‘responsiveness index’.

Felsenthal and Machover (1998) also introduced a resistance coefficient, which measures the opposite of complaisance in a voting rule. This index was suggested because they found that sometimes, simple voting games with similar sensitivity differ greatly in their propensities to pass a bill. They defined the resistance coefficient as

$$R(G) = \frac{2^{|N|-1} - |\mathbf{W}(G)|}{2^{|N|-1} - 1}.$$

It is obvious that this resistance coefficient is inversely related to the Carreras-Coleman decisiveness index, discussed above.

Now, in the next section, we turn our attention to the different applications of the indices that we discussed above.

1.4 Applications of the Indices

As we have already mentioned in the beginning, there are numerous voting bodies, where decisions are take by means of vote. Examples include the United Nations Security Council, the International Monetary Fund, the Council of Ministers in the European Union etc. Thus there lies vast scope of applying the power indices into practical use, to study the distribution of power in these bodies. Also, everybody admits the usefulness of a-priori voting models in addressing constitutional issues. There is therefore a wide literature where power indices have been applied to these real life voting bodies. These papers include among others, Felsenthal and Machover

(1997a, 2001), Laruelle and Valenciano (2002d), Nurmi, Meskanen and Pajala (2000), Bouissou (2001), Laruelle and Widgrén (1998, 2000), Brueckner (2000), Nurmi (1982, 1997) and Freixas (2004), Hosli (1993, 1995, 1996, 1998, 2000), Johnston (1995), Sutter (2000), Widgren (1994), Wagner and Höhne (2001), Nurmi and Meskanen (1999), Leech and Manjon (2002), Leech (2002, 2002a, 2002b, 2002c, 2002d, 2003). In Chapter 6 of this thesis, we use these indices to study the distribution of power among the different political parties in the Indian Parliament.

CHAPTER 2

ON THE COLEMAN INDICES OF INDIVIDUAL VOTING POWER*

2.1 Introduction

The power of an individual voter of a decision making body, when power is interpreted as 'influence' over the outcome of the voting process, can be exercised in two ways: the voter can either initiate an action or can prevent an action from being taken. To capture these two aspects of power, Coleman (1971) suggested two different power indices for an individual voter, namely, the Coleman index of the power to prevent action and the Coleman index of the power to initiate action. The former index, which measures the power of voter i to prevent action, is given by the number of coalitions in which voter i is swing divided by the number of winning coalitions in the game. The idea is that given that the voting body makes a positive decision, this index determines the conditional probability that voter i will be able to prevent the decision by changing side. The latter index, which measures the power of voter i to initiate action, is defined as the number of coalitions in which voter i is swing divided by the total number of losing coalitions in the game (also see section 1.2.2). In order to illustrate the behaviour of these indices, Felsenthal and Machover (1998, p.50-51) considered weighted majority games, where each voter has a non-negative weight (vote) and there is a positive real number quota of votes required to pass a resolution. By constructing examples of such games they demonstrated that if one game is obtained from another through an increase of the quota, then while the non-normalized Banzhaf index of a voter may reduce slightly, his loss of power to initiate action may be very considerable. In contrast, he may gain a lot of power to prevent action. Thus, they surmise that the two Coleman indices can give information that one cannot get from the non-normalized Banzhaf index alone. However, these indices have not received much explicit attention in the literature

* This chapter is based on Barua, Chakravarty and Roy (2004).

so far. In fact, for most purposes in the literature, these two indices have been clubbed with the non-normalized Banzhaf index and are sometimes jointly referred to as the Banzhaf-Coleman index (see for e.g., Owen (1978)). But recently these two indices have started gaining some attention. Particularly, in a recent paper Leech (2002a) used these indices to examine the system of Qualified Majority Voting, used by the Council of the European Union, from the perspective of enlargement of the Union. Leech (2002a) argued that there is a difference between the Banzhaf and the Coleman indices "where there is a supermajority decision rule, and they (the Coleman indices) are useful in enabling the analysis to focus on these two different aspects of members' voting power" (op. cit., p.445), that is, power to initiate and prevent action. Felsenthal and Machover (2004) also employ the Coleman indices of the power to prevent action and the power to initiate action to evaluate the qualified majority decision rules for the Council of Ministers of the European Union, that are included in the draft Constitution for Europe, proposed by the European Convention.

In this chapter of the thesis we examine the two Coleman indices in the light of different postulates discussed in sections 1.2.2 and 1.2.3. We also establish a formal relation between these two indices.

The chapter is arranged as follows. In section 2.2 we recall the definitions of these two power indices (we have already discussed them in section 1.2.2). We study the relationship between these two indices for both proper and improper voting games in section 2.3. In section 2.4 we investigate the properties of both the Coleman indices in details. Section 2.5 concludes this chapter.

2.2 Coleman Indices of the Power to Prevent Action and Initiate Action

Though we have already discussed the Coleman indices in some details in section 1.2.2, it will be worthwhile to recall their definitions.

Formally, given a voting game $G = (N;V) \in \mathbf{SG}$, Coleman index of the power to prevent action for a voter $i \in N$ is given by,

$$P_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \in S}} \left[V(S) - V(S \setminus \{i\})\right]}{\sum_{S \subseteq N} V(S)} = \frac{m_i(G)}{|\mathbf{W}(G)|}. \tag{2.1}$$

As already pointed out in section 1.2.2, this index gives voter i 's probability of being decisive (or swing), conditional to the proposal being accepted, if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $1/2$ for each and all the voters vote independently (Laruelle and Valenciano (2002a)).

Similarly, given a voting game $G = (N; V) \in \mathbf{SG}$, the Coleman index of the power to initiate action for a voter $i \in N$ is given by,

$$I_i(G) = \frac{\sum_{\substack{S \subseteq N \\ i \notin S}} \left[V(S \cup \{i\}) - V(S)\right]}{\sum_{S \subseteq N} [1 - V(S)]} = \frac{m_i(G)}{|\mathbf{L}(G)|} = \frac{m_i(G)}{2^{|N|} - |\mathbf{W}(G)|}. \tag{2.2}$$

As previously pointed out, this index gives voter i 's probability of being decisive (or swing), conditional to the proposal being rejected, if it is assumed that all coalitions are equiprobable, that is, the voters make yes-no decision with probability $1/2$ for each and all the voters vote independently. (Also see section 1.2.2.)

As stated in section 1.2.2, several authors have studied the relationship between these two indices, but always with reference to the non-normalized Banzhaf index. Given a game $G = (N; V) \in \mathbf{SG}$,

$\frac{1}{BZNN_i(G)} = \frac{1}{2}\left(\frac{1}{P_i(G)} + \frac{1}{I_i(G)}\right)$(Dubey and Shapley (1979)).

In other words, power of a voter, as measured by the non-normalized Banzhaf index, is the weighted average of the power to prevent action and the power to initiate action, the weights being the proportion of winning coalitions and losing coalitions respectively. More precisely,

$$\frac{m_i(G)}{2^{|N|-1}} = \left(\frac{|\mathbf{W}(G)|}{2^{|N|}}\right) \cdot \frac{m_i(G)}{|\mathbf{W}(G)|} + \left(\frac{|\mathbf{L}(G)|}{2^{|N|}}\right) \cdot \frac{m_i(G)}{|\mathbf{L}(G)|}.$$

It is obvious from the above relationship that $BZNN$ is equal to the Coleman index of the power to prevent action (P) and the Coleman index of the power to initiate action (I), if P and I are identical. This happens when $|\mathbf{W}(G)| = |\mathbf{L}(G)| = 2^{|N|-1}$. Since in a decisive game, $|\mathbf{W}(G)| = |\mathbf{L}(G)| = 2^{|N|-1}$, therefore it follows that P and I are identical, and equal to $BZNN$ when the game is decisive. (This result becomes relevant for Chapter 6.).

Having thus defined both the Coleman indices, we will now present some results that compare their relative strengths for an individual voter.

2.3 The Relationship between P and I

We know that a voting game $G = (N;V) \in \mathbf{SG}$ satisfying the conditions in definition 1.3 can either be proper or improper. The following proposition compares I_i and P_i for proper voting games.

__Proposition 2.1:__ If $G = (N;V) \in \mathbf{SG}$ is a proper game, then an individual's power to initiate action I_i is always less than or equal to the power to prevent action P_i.

__Proof:__ Let a coalition $S \in \mathbf{W}(G)$. Since the game is proper, the coalition $N - S$ must be losing. That is, $N - S \in \mathbf{L}(G)$. Since for every coalition $S \subseteq N$, its complement coalition $N - S$ is unique, we can define a one to one map from $\mathbf{W}(G)$ into $\mathbf{L}(G)$, whereby each element $S \in \mathbf{W}(G)$ is associated with a unique element $N - S \in \mathbf{L}(G)$. Hence $|\mathbf{W}(G)| \le |\mathbf{L}(G)|$.

Or, $\dfrac{1}{|\mathbf{L}(G)|} \le \dfrac{1}{|\mathbf{W}(G)|}$.

This implies that $\dfrac{m_i(G)}{|\mathbf{L}(G)|} \le \dfrac{m_i(G)}{|\mathbf{W}(G)|}$.

Hence $I_i(G) \le P_i(G)$.

The demonstration of the relationship between I_i and P_i for improper voting games relies on lemma 2.1.

<u>Lemma 2.1</u>: Let $G = (N;V;\mathbf{w};q)$ be an improper weighted majority game. Then there cannot exist any bipartition of the set of players N, such that both the coalitions are losing.

<u>Proof:</u> We prove this result by contradiction.

Suppose a bipartition of N exists such that both the coalitions are losing. Let (N_1, N_2) be such a bipartition. Then $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \phi$. Since N_1 and N_2 are both losing coalitions, we have the following set of inequalities:

$\sum_{i \in N_1} w_i < q$

and $\sum_{i \in N_2} w_i < q$.

Adding both sides of the above two inequalities, we get $\sum_{i \in N_1} w_i + \sum_{i \in N_2} w_i < 2q$.

Or, $\sum_{i \in N} w_i < 2q$.

This contradicts the improperness of $G = (N;V;\mathbf{w};q)$ (see definition 1.12). Hence the proof of lemma 2.1.

Lemma 2.1 underlines an important distinction between proper voting games and weighted improper games. While in proper games, there can be no bipartition of the set of players such that both the coalitions are winning, in weighted improper games, we can have no bipartition of the player set such that both the coalitions are losing.

We now use lemma 2.1 in order to prove lemma 2.2.

<u>Lemma 2.2</u>: Let $G = (N;V;\mathbf{w};q)$ be a weighted majority game. Then G is improper if and only if $|\mathbf{L}(G)| < |\mathbf{W}(G)|$.

<u>Proof:</u> Suppose G is improper. By lemma 2.1, if a coalition $S \in \mathbf{L}(G)$, then $N - S \in \mathbf{W}(G)$. Then $S \to N - S$ defines a one to one map from $\mathbf{L}(G)$ into

$\mathbf{W}(G)$. Hence $|\mathbf{L}(G)| \le |\mathbf{W}(G)|$. Now, since G is improper, $\exists S^* \subseteq N$ such that both S^* and $N - S^*$ are in $\mathbf{W}(G)$. Hence, S^* is not the image of any coalition $S \in \mathbf{L}(G)$ under this map. Hence $|\mathbf{L}(G)| < |\mathbf{W}(G)|$.

Conversely, suppose that $|\mathbf{L}(G)| < |\mathbf{W}(G)|$, and the game G is a proper game. Then, since the game is proper, we cannot obtain any bipartition of the player set such that both the coalitions are winning. So, if a coalition $S \in \mathbf{W}(G)$, then it must be the case that $N - S \in \mathbf{L}(G)$. Thus, we can define a one to one mapping $S \to N - S$ from $\mathbf{W}(G)$ into $\mathbf{L}(G)$. Therefore, $|\mathbf{W}(G)| \le |\mathbf{L}(G)|$. This is a contradiction. Therefore, G is an improper game. Hence the proof of lemma 2.2.

The following result drops out as an interesting corollary to lemma 2.2.

Corollary 2.1: A weighted majority game G is proper if and only if $|\mathbf{W}(G)| \le |\mathbf{L}(G)|$.

We will now use lemma 2.2 in order to compare I_i and P_i for an individual player for improper voting games.

Theorem 2.1: Let $G = (N;V) \in \mathbf{SG}$ be an improper game.

(i) Then if G can be represented as a weighted majority game, we have $|\mathbf{W}(G)| > 2^{|N|-1}$. Consequently, a non-null voter's power to initiate action I_i is always greater than the power to prevent action P_i.

(ii) However, if G is not a weighted majority game, nothing definite can be said about the relative magnitudes of I_i and P_i. That is, we can find a game for each of the following conditions:

1. $I_i(G) < P_i(G) \;\; \forall i \in N$

2. $I_i(G) > P_i(G) \;\; \forall i \in N$

3. $I_i(G) = P_i(G) \;\; \forall i \in N$

Proof:

(i) Since the number of all possible coalitions of the set of players in a voting game $G = (N;V)$ is $2^{|N|}$, we have

$$|\mathbf{W}(G)| + |\mathbf{L}(G)| = 2^{|N|}. \tag{2.3}$$

By lemma 2.2 for an improper weighted majority game, $|\mathbf{L}(G)| < |\mathbf{W}(G)|$. Hence, we can say that $|\mathbf{W}(G)| > 2^{|N|-1}$. It then follows that

$$\frac{m_i(G)}{|\mathbf{L}(G)|} > \frac{m_i(G)}{|\mathbf{W}(G)|}, \text{ if } i \in N \text{ is non-null.}$$

Therefore, $I_i(G) > P_i(G)$.

(ii) To prove this part of the theorem we will present three examples of improper voting games that cannot be represented by a weighted voting scheme. In the first example $I_i < P_i$**,** in the second $I_i > P_i$ and in the third example $I_i = P_i$**.**

Let G be a voting game with $N = \{a,b,c,d\}$ and $\mathbf{W}(G)$ = $\{\{a,b\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$. Clearly, it is an improper voting game. Suppose it is a weighted majority game with quota q. Let weights of a, b, c, d be w_1, w_2, w_3, w_4 respectively. Since $\{a,b\}$ is winning and $\{b,c\}$ is losing in G, we must have $w_1 + w_2 \geq q$ and $w_2 + w_3 < q$. Hence,

$$w_3 < w_1. \tag{2.4}$$

Again, since $\{c,d\}$ is winning and $\{a,d\}$ is a losing coalition in G, we have $w_3 + w_4 \geq q$ and $w_1 + w_4 < q$. Hence we must have $w_3 > w_1$, which in turn contradicts (2.4). Thus it is not possible to find a system of non-negative weights and quota that will represent G. Hence, this game cannot be a weighted majority game. In this game we have $|\mathbf{W}(G)| = 7 < |\mathbf{L}(G)| = 9$. It therefore follows that in this game $I_i(G) < P_i(G)$ $\forall i \in N$.

Now consider a second voting game G' with $N = \{a,b,c,d\}$ and $\mathbf{W}(G')$ = $\{\{a,b\},\{b,c\},\{a,d\},\{c,d\},\{a,b,c\}, \{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$. Again, this is an improper game. Arguing as above, one can see that it is not a weighted majority game. In this game, we have $|\mathbf{W}(G')| = 9 > |\mathbf{L}(G')| = 7$ and hence $I_i(G') > P_i(G') \;\; \forall i \in N$.

Finally, consider a third voting game G'' with $N = \{a,b,c,d\}$ and $\mathbf{W}(G'')$ = $\{\{a,b\},\{b,c\},\{c,d\},\{a,b,c\}, \{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$. Clearly this is an improper game, and arguing as in the first example, we can show that it is not a weighted majority game. In this game, we have $|\mathbf{W}(G'')| = 8 = |\mathbf{L}(G'')| = 8$, and hence it follows that $I_i(G'') = P_i(G'') \;\; \forall i \in N$.

Hence the proof of theorem 2.1.

Remark: That the three games considered in the proof of part **(ii)** of theorem 2.1 are not weighted majority games can also be shown using the theorem of Taylor and Zwicker (1992).

In the next section, we will study the properties of these two indices.

2.4 Properties of the Coleman Indices

As we have already noted in the discussion towards the end of section 1.2.2, indices of individual voting power can be broadly categorized into two classes- indices of P-power and indices of I-power. Whether an index measures P-power or I-power depends upon which of the two different motivations of voting behaviour- policy seeking or office seeking, that particular index is based. Since the discussions in sections 2.1 and 2.2 reveal that the Coleman indices of the power to prevent action and the power to initiate action measure an individual's voting power, when 'power' is interpreted as the voter's ability to change the outcome of the voting process by changing his stand on the bill, they can be regarded as indices of I-power (also see section 1.2.2). As already noted in sections 1.2.2 and 1.2.3, Felsenthal and Machover (1998) have argued that an index of I-power should satisfy **INV**, **IGN**, **VJN**, **DOM**, **TRP** and **BOP**. In this section we investigate whether the Coleman indices really qualify as indices of I-power, by studying whether they satisfy the above named postulates. Though some authors have questioned some of the above mentioned postulates (see section 1.2.3), we consider these postulates because of several reasons. First, as Holler and Napel (2003) pointed out, one possible way of choosing a power index is to define properties which an appropriate index of power should satisfy and these postulates have been proposed as such properties. Certainly, a power index satisfying them is not meant to supplant an index, which may not fulfil some of them, because a particular index may be generated using a given concept and a specific property may not be relevant there. Thus, we assume context-dependence of the postulates and appeal for their fulfillment in appropriate situations. Second, the two most well known indices of power, the Shapley-Shubik and Banzhaf indices respect these postulates. Also these postulates have been widely used in the literature to characterize several power indices (see section 1.2.4).

We have already discussed the postulates **INV**, **IGN**, **VJN**, **DOM**, **TRP** and **BOP** in details in sections 1.2.2 and 1.2.3.

We are now in a position to study the Coleman indices in the light of the properties discussed in sections 1.2.2 and 1.2.3. Theorem 2.2 below discusses P_i, defined in (2.1), in terms of the postulates laid down by Felsenthal and Machover (1995, 1998).

<u>Theorem 2.2</u>:

(a) The Coleman index of the power to prevent action, P_i, satisfies VJN, INV, IGN, DOM, TRP and BOP for all voting games.

(b) The index P_i achieves its upper bound of 1 if and only if voter i is a blocker.

(c) If $\hat{G} = (\hat{N}, \hat{V}) \in$ SG is obtained from $G = (N, V) \in$ SG by adding $b \notin N$ as a blocker in $\hat{G}$, then for any two non-null voters $i, j \in N$, we have

$$\frac{P_i(G)}{P_j(G)} = \frac{P_i(\hat{G})}{P_j(\hat{G})}.$$

Proof: **(a)** Let $G = (N,V) \in \mathbf{SG}$. It is easy to see that $P_i(G)$ satisfies both **VJN** and **INV**.

To check verification of the **IGN**, note that we can write $\mathbf{W}(G)$ as:

$$\mathbf{W}(G) = \mathbf{W}^1(G) \cup \mathbf{W}^2(G),$$

where $\mathbf{W}^1(G) = \{S \in \mathbf{W}(G) : d \in S\}$ and $\mathbf{W}^2(G) = \{S \in \mathbf{W}(G) : d \notin S\}$, where $d \in N$ is a null player in the game.

Clearly, $\mathbf{W}^2(G)$ coincides with $\mathbf{W}(G_{-d})$, the collection of all winning coalitions of the game G_{-d}, where G_{-d} is the game obtained from G by excluding the null player d. Since $S \subseteq N$ is winning in G if and only if $S - \{d\}$ is winning in G_{-d}, it follows that the mapping $S \to S - \{d\}$ is a bijection of $\mathbf{W}^1(G)$ onto $\mathbf{W}^2(G)$. Hence

$$|\mathbf{W}(G)| = |\mathbf{W}^1(G)| + |\mathbf{W}^2(G)| = 2|\mathbf{W}^2(G)| = 2|\mathbf{W}(G_{-d})|.$$

Therefore, $|\mathbf{W}(G_{-d})| = \frac{|\mathbf{W}(G)|}{2}$. By a similar argument $m_i(G_{-d}) = \frac{m_i(G)}{2}$. Hence $P_i(G) = P_i(G_{-d})$. We can establish analogously that $P_i(G) = P_i(G_{+d})$. Thus P_i satisfies the **IGN**.

We will now demonstrate that P_i satisfies **TRP**. Now, as we have stated in **TRP** (see section 1.2.3), let G_1 and G_2 be two different simple voting games with the same assembly N, and let $i, j \in N$ $(i \neq j)$, such that the following conditions hold:

T1. Whenever i and j are on the same side of a yes-no bipartition B, the outcome of B is identical in G_1 and G_2.

T2. Whenever i and j are on opposite sides of a yes-no bipartition B and i agrees with the outcome of B in G_1 then i also agrees with the outcome of B in G_2.

T3. There exists at least one yes-no bipartition B such that i agrees with the outcome of B in G_2 but not in G_1.

Then we shall say that G_2 arises from G_1 by a transfer from j (donor) to i (recipient).

We say that P_i satisfies **TRP** if whenever the above conditions hold, $P_i(G_2) \geq P_i(G_1)$. Likewise $P_j(G_2) \leq P_j(G_1)$.

To understand the conditions **T1- T3**, we first define certain sets.

The set of all winning coalitions in the game G_1, $\mathbf{W}(G_1)$, can be partitioned into the following subsets:

$\mathbf{W}_i(G_1) = \{S \in \mathbf{W}(G_1) : i \in S, j \notin S\}$,

$\mathbf{W}_j(G_1) = \{S \in \mathbf{W}(G_1) : j \in S, i \notin S\}$,

$\mathbf{W}_{i,j}(G_1) = \{S \in \mathbf{W}(G_1) : i, j \in S\}$, and

$\mathbf{W}_{\sim(i,j)}(G_1) = \{S \in \mathbf{W}(G_1) : i, j \notin S\}$.

We can obtain a similar partition of the set $\mathbf{W}(G_2)$ into the subsets $\mathbf{W}_i(G_2)$, $\mathbf{W}_j(G_2)$, $\mathbf{W}_{i,j}(G_2)$ and $\mathbf{W}_{\sim(i,j)}(G_2)$. An analogous partition of $\mathbf{L}(G_1)$ and $\mathbf{L}(G_2)$ (the set of all losing coalitions in the games G_1 and G_2 respectively) can be obtained.

<u>Condition **T1**</u>:

This condition means that if i and j vote *together* in favour of the bill or against the bill, the outcome of the voting process in G_1 is same as in G_2. From this it follows that

$$\mathbf{W}_{i,j}(G_1) = \mathbf{W}_{i,j}(G_2) \text{ and } \mathbf{W}_{\sim(i,j)}(G_1) = \mathbf{W}_{\sim(i,j)}(G_2).$$

<u>Condition **T2**</u>:

Consider a coalition $S \subseteq N$. Let $i \in S$. Then $j \in N - S$. Then, condition **T2** says that if S is winning in G_1, it will also be winning in G_2, and if $N - S$ is losing in G_1, then it will also be losing in G_2. This means $\mathbf{W}_i(G_1) \subseteq \mathbf{W}_i(G_2)$ and $\mathbf{L}_j(G_1) \subseteq \mathbf{L}_j(G_2)$. Hence $|\mathbf{W}_i(G_1)| \leq |\mathbf{W}_i(G_2)|$ and $|\mathbf{W}_j(G_1)| \geq |\mathbf{W}_j(G_2)|$.

Condition **T3**:

Consider a coalition $S \subseteq N$. Let $i \in S$. Then $j \in N - S$. Condition **T3** says that there must exist at least one yes-no bipartition B such that

(i) If S votes 'yes' in B, the bill is passed in G_2 but not in G_1. That is, there must exist at least one coalition S, $i \in S$, $j \notin S$, such that S is losing in G_1 but becomes a winning coalition in G_2.

Or

(ii) If S votes 'no' in B, the bill is passed in G_1 but not in G_2. That is, there must exist at least one coalition S, $i \in S$, $j \notin S$, such that $N - S$ is winning in G_1 but becomes a losing coalition in G_2.

Let $A_1 = \{S \subseteq N : i \in S, j \notin S; S \notin \mathbf{W}_i(G_1), S \in \mathbf{W}_i(G_2)\}$ and

$A_2 = \{S \subseteq N : j \in S, i \notin S;\ S \in \mathbf{W}_j(G_1)$ but $S \notin \mathbf{W}_j(G_2)\}$.

Condition **T3** can be summarized as below:

$$\text{If } |A_1| = \alpha_1 \text{ and } |A_2| = \alpha_2 \text{, then } \alpha_1 + \alpha_2 \geq 1.$$

It is easy to note that

$$\Delta|\mathbf{W}_i(G_1)| = |\mathbf{W}_i(G_2)| - |\mathbf{W}_i(G_2)| = \alpha_1.$$

$$\Delta|\mathbf{W}_j(G_1)| = |\mathbf{W}_j(G_2)| - |\mathbf{W}_j(G_1)| = -\alpha_2.$$

Note that $\Delta|\mathbf{W}(G_1)| = \Delta|\mathbf{W}_i(G_1)| + \Delta|\mathbf{W}_j(G_1)| + \Delta|\mathbf{W}_{i,j}(G_1)| + \Delta|\mathbf{W}_{\sim(i,j)}(G_1)| = \alpha_1 - \alpha_2$.

Consider an element S in the set A_1. Then $S \notin \mathbf{W}_i(G_1)$ but $S \in \mathbf{W}_i(G_2)$. Now $S \notin \mathbf{W}_i(G_1)$ implies that $S \setminus \{i\}$ is also losing in G_1. Since condition **T1** says that $\mathbf{W}_{\sim(i,j)}(G_1) = \mathbf{W}_{\sim(i,j)}(G_2)$, this means that $S \setminus \{i\}$ is losing in G_2 as well. But since $(S \setminus \{i\}) \cup \{i\}$ is winning in G_2, it implies that i is a critical member of these coalitions in the game G_2 but not in G_1.

Now consider an element S in the set A_2. Then $S \in \mathbf{W}_j(G_1)$ but $S \notin \mathbf{W}_j(G_2)$. Now $S \notin \mathbf{W}_j(G_2)$ implies that $S \setminus \{j\}$ is losing in G_2. By condition **T1** we know that $S \setminus \{j\}$ must be losing in G_1 as well. But $(S \setminus \{j\}) \cup \{j\}$ is winning in G_1. So j is a critical member of these coalitions in the game G_1 but not in G_2.

Though the set $\mathbf{W}_{i,j}(G_1)$ is the same as the set $\mathbf{W}_{i,j}(G_2)$, i might become a critical player in some of these coalitions in the game, in which he was previously non-critical. Let the number of these coalitions be $\boldsymbol{a}_3$. Formally, let $A_3 = \{S \subseteq N:\ i, j \in S;\ S \in \mathbf{W}_{i,j}(G_1)$ but $S \setminus \{i\} \in \mathbf{W}_j(G_1)$, and $S \in \mathbf{W}_{i,j}(G_2)$ but $S \setminus \{i\} \notin \mathbf{W}_j(G_2)\}$. Then $|A_3| = \boldsymbol{a}_3$.

Again there might be some winning coalitions containing both i and j in which j is critical in the game G_1, but ceases to be critical in G_2. Let the number of these coalitions be $\boldsymbol{a}_4$. Formally, let

$A_4 = \{S \subseteq N: i, j \in S;\ S \in \mathbf{W}_{i,j}(G_1)$ but $S \setminus \{j\} \notin \mathbf{W}_i(G_1)\}$, and $S \in \mathbf{W}_{i,j}(G_2)$ but $S \setminus \{j\} \in \mathbf{W}_i(G_2)\}$. Then $|A_4| = \boldsymbol{a}_4$.

It is easy to note that there cannot exist any winning coalition S containing both i and j such that i is a critical member of S in G_1 but not in G_2. To see this, suppose that such a coalition $S \in \mathbf{W}_{i,j}(G_1)$ exists. Then $S \setminus \{i\} \notin \mathbf{W}_j(G_1)\}$ and $S \setminus \{i\} \in \mathbf{W}_j(G_2)$. This violates condition **T2**, which says that if a coalition containing j and not i is losing in G_1, then it must be losing in G_2 as well. By a similar reasoning we can note that there cannot exist any winning coalition S containing both i and j such that j is a non-critical member of S in the game G_1, but a critical member in the game G_2. Therefore, we have the following set of equations:

$$\Delta m_i = m_i(G_2) - m_i(G_1) = \boldsymbol{a}_1 + \boldsymbol{a}_3,$$

$$\Delta m_j = m_j(G_2) - m_j(G_1) = -(\boldsymbol{a}_2 + \boldsymbol{a}_4) \tag{2.5}$$

$\Delta|\mathbf{W}(G_1)| = |\mathbf{W}(G_2)| - |\mathbf{W}(G_1)| = \boldsymbol{a}_1 - \boldsymbol{a}_2$.

Note that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4 \geq 0$. It is obvious that $\Delta|\mathbf{W}(G_1)|$ could be non-negative or non-positive. Accordingly $\Delta|\mathbf{L}(G_1)|$ could be non-positive or non-negative.

Case 1: $\Delta|\mathbf{W}(G_1)| \leq 0$.

(i) Then since, $\Delta m_i \geq 0$, P_i will rise. That is, the power to prevent action of the recipient will not fall.

(ii) From the above discussion it is clear that $\Delta m_j \leq 0$. Since $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4 \geq 0$, it is obvious that $\boldsymbol{a}_2 + \boldsymbol{a}_4 \geq \boldsymbol{a}_2 - \boldsymbol{a}_1$. That is, $-\Delta m_j \geq -\Delta|\mathbf{W}(G_1)|$. Also $|\mathbf{W}(G_1)| \geq m_j(G_1)$. Therefore, we have

$$-\Delta m_j|\mathbf{W}(G_1)| \geq -m_j(G_1).\Delta|\mathbf{W}(G_1)|. \tag{2.6}$$

Therefore,

$$P_j(G_2) - P_j(G_1) = \frac{m_j(G_1) + \Delta m_j}{|\mathbf{W}(G_1)| + \Delta|\mathbf{W}(G_1)|} - \frac{m_j(G_1)}{|\mathbf{W}(G_1)|}$$

$$= \frac{\Delta m_j|\mathbf{W}(G_1)| - m_j(G_1)\Delta|\mathbf{W}(G_1)|}{(|\mathbf{W}(G_1)| + \Delta|\mathbf{W}(G_1)|)|\mathbf{W}(G_1)|} \leq 0 \text{ (Using (2.6).)}$$

That is, $P_j(G_2) \leq P_j(G_1)$. Thus, the power to prevent action of the donor will not rise.

Case 2: $\Delta|\mathbf{W}(G_1)| > 0$

(i) First note that since $\Delta m_i \geq \Delta|\mathbf{W}(G_1)|$ and $m_i(G_1) \leq |\mathbf{W}(G_1)|$,

$$\Delta m_i|\mathbf{W}(G_1)| \geq m_i(G_1)\Delta|\mathbf{W}(G_1)|. \tag{2.7}$$

Therefore,

$$P_i(G_2) - P_i(G_1) = \frac{m_i(G_1) + \Delta m_i}{|\mathbf{W}(G_1)| + \Delta|\mathbf{W}(G_1)|} - \frac{m_i(G_1)}{|\mathbf{W}(G_1)|}$$

$$= \frac{m_i(G_1)|\mathbf{W}(G_1)| + \Delta m_i|\mathbf{W}(G_1)| - m_i(G_1)|\mathbf{W}(G_1)| - m_i(G_1)\Delta|\mathbf{W}(G_1)|}{(|\mathbf{W}(G_1)| + \Delta|\mathbf{W}(G_1)|)|\mathbf{W}(G_1)|}$$

$$= \frac{\Delta m_i|\mathbf{W}(G_1)| - m_i(G_1)\Delta|\mathbf{W}(G_1)|}{(|\mathbf{W}(G_1)| + \Delta|\mathbf{W}(G_1)|)|\mathbf{W}(G_1)|} \geq 0 \text{ (Using (2.7).)}$$

Therefore, $P_i(G_2) \geq P_i(G_1)$.

Thus, the power to prevent action of the recipient does not fall after the transfer.

(ii) For the donor j, the proof is straightforward. Since $\Delta m_j \leq 0$ and $\Delta|\mathbf{W}(G_1)| \geq 0$, the power to prevent action of the donor can never rise.

Since P_i satisfies **INV** and **TRP**, by theorem 7.11 of Felsenthal and Machover (1995) we can conclude that it satisfies **DOM**. Satisfaction of **BOP** by P_i follows from the fact that it satisfies **TRP**, **INV** and **IGN** (Felsenthal and Machover, 1995, theorem 7.10).

(b) If voter i is a blocker in the game G, then by definition he can stall a bill by voting 'no', irrespective of how others vote. This means that he is a pivotal voter in every winning coalition in the game. Therefore, $m_i(G) = |\mathbf{W}(G)|$, or, $P_i(G) = 1$. Conversely, if $P_i(G) = 1$, it means that $m_i(G) = |\mathbf{W}(G)|$. That is, i is critical in every winning coalition of the game. So i has to be a blocker.

(c) Let us denote the set of winning coalitions in the game $\hat{G}$ by $\mathbf{W}(\hat{G})$. Now $\mathbf{W}(\hat{G}) = \{S \in 2^{\hat{N}} : b \in S, S \setminus \{b\} \in \mathbf{W}(G)\}$. Therefore, $|\mathbf{W}(\hat{G})| = |\mathbf{W}(G)|$. Also it is obvious from the definition of $\mathbf{W}(\hat{G})$ that the number of coalitions in which any player i is critical remains unaltered in the new game. Therefore, power to prevent action of any player remains unchanged and hence the desired result follows.

Hence the proof of theorem.

The fact that the Coleman index of the power to prevent action P_i satisfies **VJN, INV, IGN, DOM, TRP and BOP** implies that it can be used as a measure of the extent of influence that voter i enjoys over the outcome of the voting process. In other words, P_i can be regarded as a valid index of I-power. Felsenthal and Machover (1998) suggested property **(c)** of theorem 2.2 as a desirable postulate for an index of P-power (e.g., the Shapley-Shubik index) and referred to it as the *added blocker postulate* (**ABP**) (see section 1.2.3). Although P_i is regarded as an index of I-power, we note its satisfaction of the **ABP**. It also follows from theorem 2.2 that P_i is bounded between zero and one, where the lower and upper bounds are achieved when voter i is a null voter and a blocker respectively. Finally we note that since P_i satisfies **TRP** and **IGN**, it satisfies **RTP** and **RNP** as well (see sections 1.2.2 and 1.2.3).

In the next theorem we present a similar set of results for the Coleman index of the power to initiate action.

<u>Theorem 2.3:</u>

(a) The Coleman index of the power to initiate action, I_i, satisfies VJN, INV, IGN, DOM, TRP and BOP for all voting games.

(b) In a proper voting game, the index I_i achieves its upper bound of 1 if and only if the voter is a dictator.

(c) If $\hat{G} = (\hat{N}, \hat{V}) \in \mathbf{SG}$ is obtained from $G = (N, V) \in \mathbf{SG}$ by adding $b \notin N$ as a blocker in $\hat{G}$, then for any two non-null voters $i, j \in N$, we have

$$\frac{I_i(G)}{I_j(G)} = \frac{I_i(\hat{G})}{I_j(\hat{G})}.$$

That is, I_i satisfies ABP.

The proof of this theorem relies on the following two lemmas.

<u>Lemma 2.3:</u> Consider the voting games G_1 and G_2, and the voters j and i as described in **TRP**. Let $\mathbf{L}(G_1)$ $(\mathbf{L}(G_2))$ be the set of losing coalitions in $G_1\,(G_2)$. Then assuming that $\Delta|\mathbf{L}(G_1)| = |\mathbf{L}(G_2)| - |\mathbf{L}(G_1)| \leq 0$, we have $abs(\Delta m_j)$ $\geq$ $abs(\Delta|\mathbf{L}(G_1)|)$, where $abs(x) = -x$ if $x < 0$

$= x$ if $x \geq 0$, x is a scalar.

Proof:

We have noted in the proof of theorem 2.2 that $\Delta m_j = -(a_2 + a_4)$. Also $\Delta|\mathbf{W}(G_1)| = a_1 - a_2$, which means that $\Delta|\mathbf{L}(G_1)| = a_2 - a_1 \leq 0$, by hypothesis. Suppose, contrary to what the lemma says, we have $abs(\Delta|\mathbf{L}(G_1)|) > abs(\Delta m_j)$. Then we must have

$a_1 - a_2 > a_2 + a_4$,

or, $a_1 - a_4 > 2a_2$ (2.8)

Let us recall the definition of the sets A_1 and A_4 which have been considered in the proof of theorem 2.2.

$A_1 = \{S \subseteq N : i \in S, j \notin S; S \notin \mathbf{W}_i(G_1), S \in \mathbf{W}_i(G_2)\}$, $|A_1| = a_1$.

$A_4 = \{S \subseteq N : i, j \in S;\ S \in \mathbf{W}_{i,j}(G_1)$ but $S \setminus \{j\} \notin \mathbf{W}_i(G_1)$, and $S \in \mathbf{W}_{i,j}(G_2)$ but $S \setminus \{j\} \in \mathbf{W}_i(G_2)\}$, $|A_4| = a_4$.

Now, consider an element $S \in A_1$. $S \in \mathbf{W}_i(G_2)$ implies that $S \cup \{j\} \in \mathbf{W}_{i,j}(G_2)$ (since a super set of a winning coalition is also winning). By condition **T1** we know that $S \cup \{j\} \in \mathbf{W}_{i,j}(G_1)$. This combined with the fact that $S \notin \mathbf{W}_i(G_1)$ (since $S \in A_1$) tells us that player j is a critical member of the coalition $S \cup \{j\}$ in the game G_1 but not in G_2. That is, $S \cup \{j\} \in A_4$. Thus, for every coalition $S \in A_1$, we get a unique coalition $S \cup \{j\} \in A_4$.

Conversely, it can be checked that with every coalition $S \in A_4$, we can associate a unique coalition $S \setminus \{j\} \in A_1$. In other words, with every element in A_1 we can associate a unique element in A_4 and vice versa. This implies $|A_1| = |A_4|$.

Or, $a_1 = a_4$. (2.9)

Using (2.9) and (2.8) we know that if $abs(\Delta|\mathbf{L}(G_1)|) > abs(\Delta m_j)$, then $0 > a_2$, which is a contradiction, since α_2 is the cardinality of a set and hence cannot be negative. So $abs(\Delta m_j) \geq abs(\Delta|\mathbf{L}(G_1)|)$. Hence the proof.

Lemma 2.4: Consider the voting games G_1 and G_2, and the voters j and i as described in **TRP**. Let $\mathbf{L}(G_1)$ $(\mathbf{L}(G_2))$ be the set of losing coalitions in G_1 (G_2). Then assuming that $\Delta|\mathbf{L}(G_1)| = |\mathbf{L}(G_2)| - |\mathbf{L}(G_1)| \geq 0$, we have $\Delta m_i = m_i(G_2) - m_i(G_1) \geq \Delta|\mathbf{L}(G_1)|$.

Proof:

We have noted in the proof of theorem 2.2 that $\Delta m_i = \alpha_1 + \alpha_3$. Also $\Delta|\mathbf{W}(G_1)| = \alpha_1 - \alpha_2$, which means that $\Delta|\mathbf{L}(G_1)| = \alpha_2 - \alpha_1$. Suppose, contrary to what the lemma says, we have $\Delta|\mathbf{L}(G_1)| > \Delta m_i$. Then we must have

$\alpha_2 - \alpha_1 > \alpha_1 + \alpha_3$.

Or, $\alpha_2 - \alpha_3 > 2\alpha_1$. (2.10)

(Note that by assumption $\Delta|\mathbf{L}(G_1)| \geq 0$.)

Recall the definitions of the sets A_2 and A_3 in the proof of theorem 2.2. Consider an element $S \in A_2$. It is easy to note that $S \cup \{i\} \in \mathbf{W}_{i,j}(G_1)$ and hence by condition **T1** $S \cup \{i\} \in \mathbf{W}_{i,j}(G_2)$. This combined with the fact $S \in \mathbf{W}_j(G_1)$ and $S \notin \mathbf{W}_j(G_2)$ (since $S \in A_2$) says that i is non-critical member of the winning coalition $S \cup \{i\}$ in the game G_1 but becomes critical in the game G_2. Therefore $S \cup \{i\} \in A_3$. Thus, for each $S \in A_2$, $S \cup \{i\} \in A_3$. Conversely it can be checked that for each $S' \in A_3$, $S' - \{i\} \in A_2$. Hence the correspondence $S \rightarrow S \cup \{i\}$ is a bijection from A_2 onto A_3. In other words, with every element in A_2 we can associate a unique element in A_3 and vice versa. Therefore, we have

$\alpha_2 = \alpha_3$ (2.11)

Using (2.10) and (2.11) we know that if $\Delta|\mathbf{L}(G_1)| > \Delta m_i$, then $0 > a_1$, which is a contradiction, since a_1 is the cardinality of a set and hence cannot be negative.

So $\Delta m_i \geq \Delta|\mathbf{L}(G_1)|$. Hence the proof.

Now, we come to the proof of theorem 2.3.

<u>Proof of theorem 2.3</u>:

(a) The proof that I_i satisfies **VJN**, **INV**, **IGN** is similar to the proof in theorem 2.2. We now prove that I_i satisfies **TRP**. We have already noted above that $\Delta|\mathbf{W}(G_1)|$ could be non-negative or non-positive, and that $\Delta m_i \geq 0$ and $\Delta m_j \leq 0$.

Case 1: $\Delta|\mathbf{W}(G_1)| \geq 0$.

This means $\Delta|\mathbf{L}(G_1)| \leq 0$. That is,

(i) $|\mathbf{L}(G_2)| \leq |\mathbf{L}(G_1)|$

$$\therefore \frac{1}{|\mathbf{L}(G_2)|} \geq \frac{1}{|\mathbf{L}(G_1)|}$$

or, $\dfrac{m_i(G_1) + \Delta m_i}{|\mathbf{L}(G_2)|} \geq \dfrac{m_i(G_1)}{|\mathbf{L}(G_1)|}$ (since $\Delta m_i \geq 0$).

Thus, I_i or the power to initiate action of the recipient does not fall after the transfer.

(ii) To show that the power to initiate an action of the donor does not rise after the transfer, we first note that by lemma 2.3, $abs(\Delta m_j) \geq abs(\Delta|\mathbf{L}(G_1)|)$. Also we know that $|\mathbf{L}(G_1)| \geq m_j(G_1)$. So

$$\Delta m_j.|\mathbf{L}(G_1)| \leq m_j(G_1).\Delta|\mathbf{L}(G_1)| \quad \text{(since } \Delta m_j \leq 0 \text{ and } \Delta|\mathbf{L}(G_1)| \leq 0\text{)} \qquad (2.12)$$

$$I_j(G_2) - I_j(G_1) = \frac{m_j(G_1) + \Delta m_j}{|\mathbf{L}(G_1)| + \Delta|\mathbf{L}(G_1)|} - \frac{m_j(G_1)}{|\mathbf{L}(G_1)|}$$

$$= \frac{\Delta m_j.|\mathbf{L}(G_1)| - m_j(G_1).\Delta|\mathbf{L}(G_1)|}{(|\mathbf{L}(G_1)| + \Delta|\mathbf{L}(G_1)|)|\mathbf{L}(G_1)|} \leq 0. \text{ (Using (2.12))}$$

Thus, the power of the donor cannot rise after the transfer.

Case 2: $\Delta|\mathbf{W}(G_1)| \leq 0$.

This means $\Delta|\mathbf{L}(G_1)| \geq 0$.

(i) From lemma 2.4 and using the fact that $|\mathbf{L}(G_1)| \geq m_i(G_1)$, we can say that

$$\Delta m_i|\mathbf{L}(G_1)| \geq m_i(G_1).\Delta|\mathbf{L}(G_1)|. \tag{2.13}$$

Let us now evaluate the expression $I_i(G_2) - I_i(G_1)$.

$I_i(G_2) - I_i(G_1)$

$$= \frac{m_i(G_2)}{|\mathbf{L}(G_2)|} - \frac{m_i(G_1)}{|\mathbf{L}(G_1)|}$$

$$= \frac{m_i(G_1) + \Delta m_i}{(|\mathbf{L}(G_1)| + \Delta|\mathbf{L}(G_1)|)} - \frac{m_i(G_1)}{|\mathbf{L}(G_1)|}$$

$$= \frac{m_i(G_1).|\mathbf{L}(G_1)| + \Delta m_i|\mathbf{L}(G_1)| - m_i(G_1)|\mathbf{L}(G_1)| - m_i(G_1)\Delta|\mathbf{L}(G_1)|}{(|\mathbf{L}(G_1)| + \Delta|\mathbf{L}(G_1)|)|\mathbf{L}(G_1)|}$$

$$= \frac{\Delta m_i|\mathbf{L}(G_1)| - m_i(G_1).\Delta|\mathbf{L}(G_1)|}{(|\mathbf{L}(G_1)| + \Delta|\mathbf{L}(G_1)|)|\mathbf{L}(G_1)|}$$

≥ 0 (Using (2.13)).

Thus power to initiate action of player i does not decrease after i receives some voting right from another player.

(ii) For the donor j the proof is straightforward because $\Delta m_j \leq 0$ and $\Delta|\mathbf{L}(G_1)| \geq 0$.

Proof of satisfaction of **DOM** and **BOP** by I_i is similar to the proof in theorem 2.2.

(b) If the game G has a dictator i, then it becomes a proper game and $|\mathbf{W}(G)| = |\mathbf{L}(G)| = 2^{|N|-1}$. Also by the definition of a dictator, he is critical player in all the winning coalitions. Therefore $I_i(G) = 1$. Conversely, $I_i(G) = 1$ implies that $m_i(G) = |\mathbf{L}(G)|$. Since $m_i(G)$ is also the number of losing coalitions outside which i is critical, $m_i(G) = |\mathbf{L}(G)|$ implies that i is critical outside every losing coalition in the game. Since the game is proper, $m_i(G) \leq |\mathbf{W}(G)| \leq |\mathbf{L}(G)| = m_i(G)$. Therefore, $m_i(G) = |\mathbf{W}(G)| = |\mathbf{L}(G)|$. So i is a critical member of each winning coalition of the game. This means i's 'yes' vote is necessary to pass the bill. Following definition 1.3 we know that f is a losing coalition. Since i is critical outside every losing coalition in the game, therefore $\{i\}$ is a winning coalition. This means i's 'yes' vote is sufficient to pass the bill. So i is a dictator.

(c) The proof of this part of the theorem is similar to the proof of part **(c)** of theorem 2.2.

It is important to note that while P_i reaches its maximum in the case of an ordinary blocker, I_i is maximum if the blocker is a dictator. In fact the difference between these two aspects of exercise of power is emphasized by the presence of a blocker in the game (also see the discussion on Coleman indices in section 1.2.2).

The previous two theorems show that both the Coleman index of the power to prevent action and the power to initiate action can be used to get an idea of the extent of influence that an individual voter enjoys over the outcome of the decision making body. Thus they can be regarded as reasonable indices of I-power.

2.5 Conclusion

An I-index of voting power is a measure of the extent to which a voter is able to influence the passage or defeat of a resolution. Coleman (1971) suggested two such indices to measure the power to prevent an action (P) and the power to initiate an action (I). The former gives an indication of the chance a voter has to block a bill and the latter is concerned with the voter's probability to initiate action. Incidentally there has not been much discussion of these indices in the literature on voting power. One of the possible reasons why these indices have not received separate attention could be that, these two indices have an exact relationship with the Banzhaf indices. As we have already noted, Dubey and Shapley (1979) have shown that the non-normalized Banzhaf index ($BZNN$) is the harmonic mean of P and I. (Also see Brams and Affuso (1976).) Further, by normalizing either P or I to make the values of the indices for all voters add up to 1, the equivalence of these indices with the normalized Banzhaf index (BZ) can be established. More precisely, given a voting game $G = (N;V)$, $BZ_i(G) = \frac{m_i(G)}{\sum_{i \in N} m_i(G)} = \frac{P_i(G)}{\sum_{i \in N} P_i(G)} = \frac{I_i(G)}{\sum_{i \in N} I_i(G)}$ (see Leech (2002d)). Thus P and I are also regarded as different ways of arriving at BZ. Because of these relationships with the Banzhaf indices, most authors regard P and I as mere different modifications of the Banzhaf indices. Consequently they are often clubbed with the Banzhaf index. Another possible reason why there has not been much discussion about these indices, could be that, since "the most common voting rule used in voting bodies is the simple majority" (Holler, Ono and Steffen (2001)), the Coleman indices are unable to shed any additional light on the voter's powers in most cases. This is because the Coleman indices are indistinguishable and effectively the same as the non-normalized Banzhaf index, when the decision rule is 'simple majority' (also see chapter 6). However, as some authors have noted, (see for e.g., Felsenthal and Machover (1998)) there are situations where these two indices can give information that one cannot get from the non-normalized Banzhaf index alone. These indices are particularly useful when there are supermajority voting rules (see Leech (2002a)). This chapter therefore rigorously examines these indices in the light of different properties for an index of voting power suggested by Felsenthal and Machover (1995, 1998) and demonstrates their suitability in this context. A relationship between these two indices is also established in the chapter.

CHAPTER 3

AN ALTERNATIVE CHARACTERIZATION OF THE NON-NORMALIZED BANZHAF INDEX[§]

3.1 Introduction

Among the indices of individual power that we have discussed in section 1.2.2, the most well known indices are the Shapley-Shubik (1954) and the Banzhaf (1965) indices.

The Banzhaf index is based on the number of coalitions in which the concerned voter is swing. Banzhaf (1965) had actually proposed the 'swing numbers' m_i as an index of individual voting power. This number is also referred to as the raw Banzhaf index (also see section 1.2.2). But this index is, for technical reasons, quite refractory. However, these technical reasons disappear if one rescales this index by multiplying it with an appropriate factor. The non-normalized Banzhaf index is such a rescaling of the Banzhaf index. Dubey and Shapley (1979) suggest that such a rescaling is in many respects more natural than the Banzhaf index itself.

As we have already noted in sections 1.2.2 and 2.2, the non-normalized Banzhaf index is a weighted average of the Coleman indices of the power to prevent action and the power to initiate action. Thus, while the Coleman indices reflect two different aspects of individual voting power- the ability of a voter to initiate action and the ability of a voter to prevent action from taking place, their (harmonic) mean, the non-normalized Banzhaf index, can be regarded as a measure of the overall influence that a voter has over the decision making process. Many authors have also advocated this variant of the raw Banzhaf index as a measure of absolute power of a voter.

In this chapter we provide a characterization of the non-normalized Banzhaf index on the set of simple games **SG**, using four axioms. We have already briefly discussed the numerous characterizations of the non-normalized Banzhaf index in

♣ This chapter is based on Barua, Chakravarty and Roy (2004a).

section 1.2.4. The axioms that we have used in our characterization exercise are taken directly from four different contributions to the area. Thus, our characterization shows the importance of this set of existing axioms from a new perspective. A very attractive feature of this characterization is independence of the axioms. By independence we mean that if one of these axioms is dropped, then there will be a power index other than the non-normalized Banzhaf index that will satisfy the remaining three axioms, but not the dropped one. That is, independence says that none of the axioms implies or is implied by another.

The chapter is arranged as follows. In the section 3.2 we first recall what we mean by the non-normalized Banzhaf index. We also discuss the axioms that have been used in the characterization exercise in this section. Section 3.3 presents the main results. That is, in this section we uniquely characterize the non-normalized Banzhaf index using a set of independent axioms. Finally in section 3.4 we conclude this chapter.

3.2 The Banzhaf Non-normalized Index

Though we have already discussed the non-normalized Banzhaf index in some details in section 1.2.2, it will be worthwhile to recall its definition.

Formally, given a voting game $G = (N;V) \in \mathbf{SG}$, the non-normalized Banzhaf index for a voter $i \in N$ is given by,

$$BZNN_i(G) = \frac{m_i(G)}{2^{|N|-1}}. \qquad (3.1)$$

It is actually the number of winning coalitions in which player i is pivotal in the game, divided by the maximal value that this number can take. For any $G = (N;V) \in \mathbf{SG}$, for any $i \in N$, $BZNN_i$ achieves its minimum value, zero, if and only if i is a null player. It remains invariant under any permutation of the voters. If a null player is excluded from the game, $BZNN_i$ does not change. Similarly, it remains unaltered if a null player is included in the game (see Felsenthal and Machover (1998)). If in a voting game, each voter $i's$ probability p_i of voting 'yes' or ‘no’ on a bill is chosen independently from the uniform distribution [0,1], then the power of the voter i is estimated by $BZNN_i$ (Straffin (1977, 1988)). However, Leech (1990) has shown that the assumption of uniform distribution is not needed. The only thing that we require is that the probabilities are selected independently at random from any distribution which has an expectation 0.5. Since $BZNN_i$ does not involve numbers of coalitions in which voters other than i are swing, Felsenthal and Machover (1998) regarded it as an absolute index of voter i's power.

Before we introduce the axioms that have been used in our characterization exercise, we need to define a special type of game, called the unanimity game, which we will require in our analysis.

Definition 3.1: For any $S \subseteq N$, the unanimity game, denoted by $(N;U_S)$ is the simple game whose characteristic function is given by:

$U_S(T) = 1$ if $T \supseteq S$

$= 0$ otherwise.

Thus the winning coalitions in this game are S together with all supersets of S.

We will now present four axioms on a power index $\boldsymbol{j}_i : \mathbf{SG} \to \mathbf{R}_+$ (see definition 1.16) that uniquely determines the $BZNN$.

The first axiom is taken from Dubey (1975) (see also Dubey and Shapley (1979)). It shows that the sum of powers of voter i in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ (see definitions 1.5 and 1.6) is equal to the sum of his powers in G_1 and G_2.

Axiom A3.1 (Transfer Principle): For $G_1 = (N_1;V_1), G_2 = (N_2;V_2) \in \mathbf{SG}$,

$$\boldsymbol{j}_i(G_1 \vee G_2) + \boldsymbol{j}_i(G_1 \wedge G_2) = \boldsymbol{j}_i(G_1) + \boldsymbol{j}_i(G_2). \quad (3.2)$$

Note that we have already discussed the above principle in section 1.2.4.

The idea of the next axiom, which makes a specification about a dictator's power, is taken from Felsenthal and Machover (1998). A dictator, if there is one, should possess maximum power in the game since he can be characterized as the only non-null voter.

Axiom A3.2 (Maximal Power Specification): For any game $G=(N;V)\in \mathbf{SG}$, if i is a dictator in the game, then

$$\boldsymbol{j}_i(G)=1. \tag{3.3}$$

The third axiom, which is formulated in terms of substitutability between two voters, is taken from Lehrer (1988). Two voters in a game are said to be substitutes if the worth of an arbitrary coalition in the game becomes the same when they join the coalition separately (Shapley (1953)). Therefore, it is reasonable to expect that their powers are the same (also see section 1.2.4). More precisely, we have the following axiom.

Axiom A3.3 (Equal Treatment): Let two non-null voters i and j be substitutes in the game $G=(N;V)\in \mathbf{SG}$, that is, $V(S\cup\{i\})=V(S\cup\{j\})$ for all $S\subseteq N-\{i,j\}$. Then,

$$\boldsymbol{j}_i(G)=\boldsymbol{j}_j(G). \tag{3.4}$$

The next axiom shows the relationship between the power of a bloc and its constituents in a unanimity game. It is similar to the 2- efficiency axiom used by Nowak (1997) in characterizing the Banzhaf value on the set of all games $\mathbf{G}$ (see also axiom A5 of Nowak and Radzik (2000), Lehrer (1988)). The 2- efficiency axiom says that if two voters i and j in any game belonging to $\mathbf{G}$, decide to merge to form a bloc ij, then the power of the merged entity in the post-merger game (see definition 1.17) is equal to the sum of the power of its components in the original game (also see section 1.2.4). We modify this axiom by stating it in terms of unanimity games only.

Axiom A3.4 (Two-Voter Bloc Principle): Let $G'=(N',U_{S'})\in \mathbf{SG}$ be the $(|N|-1)$-person game obtained from the game $G=(N;U_S)$ when the voters $i,j\in S$ form a bloc ij. Then,

$$\boldsymbol{j}_{ij}(G')=\boldsymbol{j}_i(G)+\boldsymbol{j}_j(G). \tag{3.5}$$

In the next section, we present the main results of the chapter.

3.3 The Characterization Exercise

In the following theorem we show that the above four axioms characterize the non-normalized Banzhaf index uniquely on the set of all simple games **SG**.

<u>Theorem 3.1</u>: A power index j_i satisfies axioms A3.1-A3.4 if and only if j_i is the index $BZNN_i$ in (3.1).

<u>Proof:</u> We will first show that $BZNN_i$ satisfies all the axioms **A3.1** through **A3.4**.

To show that **A3.1** is satisfied by $BZNN_i$, consider two voting games $G_1 = (N_1;V_1)$, $G_2 = (N_2;V_2)$ $\in$ **SG**. Assuming $N_1 - N_2 \neq f$, first let $i \in N_1 - N_2$. Now, any subset S' of $N_2 - N_1$ can be appended to a swing coalition $S \subseteq N_1$ for i in G_1 to obtain a swing coalition $S \cup S'$ for i in $G_1 \vee G_2$ unless $(S \cup S') \cap N_2$ is winning in G_2. Hence the number of swings for voter i in $G_1 \vee G_2$ is

$m_i(G_1 \vee G_2) = m_i(G_1)2^{|N_2 - N_1|} - m_i(G_1 \wedge G_2)$,

where $m_i(G_1 \wedge G_2)$ is the number of swings of i in $G_1 \wedge G_2$. Since for $i \in N_1 - N_2$, $m_i(G_2)$=0, we rewrite $m_i(G_1 \vee G_2)$ as

$m_i(G_1 \vee G_2) = m_i(G_1)2^{|N_2 - N_1|} + m_i(G_2)2^{|N_1 - N_2|} - m_i(G_1 \wedge G_2)$

The same expression for $m_i(G_1 \vee G_2)$ will be obtained if $i \in N_1 \cap N_2$ or $i \in N_2 - N_1$.

Therefore,

$$BZNN_i(G_1 \vee G_2) = \frac{m_i(G_1 \vee G_2)}{2^{|N_1 \cup N_2|-1}} = \frac{m_i(G_1)}{2^{|N_1|-1}} + \frac{m_i(G_2)}{2^{|N_2|-1}} - \frac{m_i(G_1 \wedge G_2)}{2^{|N_1 \cup N_2|-1}}, \qquad (3.6)$$

which in turn gives

$$BZNN_i(G_1 \vee G_2) + BZNN_i(G_1 \wedge G_2) = BZNN_i(G_1) + BZNN_i(G_2).$$

This shows that $BZNN_i$ verifies **A3.1**.

To check satisfaction of **A3.2** by $BZNN_i$, note that if i is a dictator in the game $G=(N;V)$, then i is the only non-null voter in the game, that is, $m_i(G)$ is maximized, which means that $m_i(G)=2^{|N|-1}$ and $m_j(G)=0$ for all $j\in N$, $j\neq i$. Substituting this value of $m_i(G)$ in (3.1), we get $BZNN_i(G)=1$, which shows that $BZNN_i$ meets **A3.2**.

Next we verify fulfillment of **A3.3** by $BZNN_i$.

Let $\boldsymbol{z}=\{S\subseteq N-\{i\}\}$. Clearly, we can write $\boldsymbol{z}$ as $\boldsymbol{z}_1\cup\boldsymbol{z}_2$, where $\boldsymbol{z}_1=\{S\subseteq N-\{i,j\}\}$ and $\boldsymbol{z}_2=\{S\subseteq N-\{i\}$ and $j\in S\}$. We rewrite $S\in\boldsymbol{z}_2$ as $S'\cup\{j\}$, where $S'\subseteq N-\{i,j\}$. Then,

$$m_i(G)=\sum_{S\subseteq N-\{i\}}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]$$

$$=\sum_{S\in z}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]$$

$$=\sum_{S\in z_1}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]+\sum_{S\in z_2}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]$$

$$=\sum_{S\subseteq N-\{i,j\}}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]+\sum_{S'\subseteq N-\{i,j\}}\left[V\left(S'\cup\{i,j\}\right)-V\left(S'\cup\{j\}\right)\right]. \tag{3.7}$$

We can rewrite $m_i(G)$ in (3.7) as

$$m_i(G)=\sum_{S\subseteq N-\{i,j\}}\left[V\left(S\cup\{i\}\right)-V\left(S\right)\right]+\sum_{S\subseteq N-\{i,j\}}\left[V\left(S\cup\{i,j\}\right)-V\left(S\cup\{j\}\right)\right], \tag{3.8}$$

which on simplification becomes $m_i(G)=\sum_{S\subseteq N-\{i,j\}}\left[V\left(S\cup\{i,j\}\right)-V\left(S\right)\right]$, since by hypothesis $V\left(S\cup\{i\}\right)=V\left(S\cup\{j\}\right),\forall S\subseteq N-\{i,j\}$.

By a similar calculation we get $m_j(G)=\sum_{S\subseteq N-\{i,j\}}\left[V\left(S\cup\{i,j\}\right)-V\left(S\right)\right]$.

Hence $m_i(G)=m_j(G)$. Therefore, $BZNN_i(G)=m_i(G)/2^{|N|-1}=m_j(G)/2^{|N|-1}=BZNN_j(G)$, which shows that $BZNN_i$ meets **A3.3**.

Finally, take $G=(N;U_S)$. Let $G'=(N';U_{S'})$ be the $(|N|-1)$- person game when the voters $i,j\in S$ form a bloc ij. Then

$BZNN_{ij}(G') = {m_{ij}(G')}/{2^{|N'|-1}} = {2^{|N'-S'|}}/{2^{|N'|-1}} = {1}/{2^{|S'|-1}} = {1}/{2^{|S|-2}}$,

Also, $BZNN_i(G) + BZNN_j(G) = {m_i(G)}/{2^{|N|-1}} + {m_j(G)}/{2^{|N|-1}} =$

${2^{|N-S|}}/{2^{|N|-1}} + {2^{|N-S|}}/{2^{|N|-1}} = {1}/{2^{|S|-2}}$, since $G = (N;U_S)$. Thus, $BZNN_i$ satisfies **A3.4.**

We now show that if a power index $\boldsymbol{j}_i$ satisfies **A3.1-A3.4**, then it must be $BZNN_i$. First observe that any $\boldsymbol{j}_i$ is uniquely determined by its values on unanimity games. This is because, for any game $G = (N;V) \in \mathbf{SG}$, $G = G_{S_1} \vee G_{S_2} \vee .... \vee G_{S_k}$, where $S_1, S_2, ..., S_k$ are minimal winning coalitions of G and G_{S_l} is the unanimity game corresponding to $S_l, l = 1,2,...,k$. Thus, by **A3.1**, $\boldsymbol{j}_i(G)$ is determined if $\boldsymbol{j}_i(G_{S_1})$, $\boldsymbol{j}_i(G_{S_2} \vee G_{S_3} \vee ....G_{S_k})$ and $\boldsymbol{j}_i(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are known. But, $G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}) = G_{S_1 \cup S_2} \vee ... \vee G_{S_1 \cup S_k}$ and hence, by induction hypothesis both $\boldsymbol{j}_i(G_{S_2} \vee G_{S_3} \vee ....G_{S_k})$ and $\boldsymbol{j}_i(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are determined. So $\boldsymbol{j}_i(G)$ is determined.

In view of the above discussion, we can say that it is enough to determine $\boldsymbol{j}_i(N;U_S)$ for any unanimity game $(N;U_S)$. We shall prove by induction on $|S|$ that $\boldsymbol{j}_i(N;U_S) = {1}/{2^{|S|-1}}$. If $|S|=1$, then the game has a dictator and hence by **A3.2**, $\boldsymbol{j}_i(N;U_S) = 1 = {1}/{2^{|S|-1}}$. So assume $|S|>1$ and the result for all games $(N;U_{\bar{S}})$, where $|\bar{S}| < |S|$. Let $(N';U_{S'})$ be the game obtained from $(N;U_S)$ by merging two voters i and j ($i \neq j$) in S. By **A3.4,** we have,

$$\boldsymbol{j}_i(N;U_S) + \boldsymbol{j}_j(N;U_S) = \boldsymbol{j}_{ij}(N';U_{S'}). \tag{3.9}$$

By induction hypothesis,

$$\boldsymbol{j}_{ij}(N';U_{S'}) = \frac{1}{2^{|S'|-1}} = \frac{1}{2^{|S|-2}}. \tag{3.10}$$

Also by **A3.3**,

$$\varphi_i(N;U_S) = \varphi_j(N;U_S). \tag{3.11}$$

Hence by (3.9)-(3.11), we have

$$2\varphi_i(N;U_S) = \frac{1}{2^{|S|-2}},$$

which gives $\varphi_i(N;U_S) = \frac{1}{2^{|S|-1}}$ .Thus, the values of φ_i coincide with $BZNN_i$ on unanimity games and hence on all voting games. Hence the proof of the theorem.

In order to demonstrate how the power of a voter in a game can be calculated using his powers in the minimal winning coalitions of the game, we give the following example:

Consider the weighted majority game G with the voter set $N = \{a_1, a_2, a_3, a_4\}$, where $w_{a_1} = 4, w_{a_2} = 3, w_{a_3} = 2, w_{a_4} = 1$ and $q = 7$ (see Straffin (1994)). The minimal winning coalitions here are $S_1 = \{a_1, a_2\}$ and $S_2 = \{a_1, a_3, a_4\}$. Denoting the unanimity game for S_j by $G_{S_j} = (N;U_{S_j})$, ($j =$ 1, 2), we get

$BZNN_i(G) = BZNN_i(G_{S_1}) + BZNN_i(G_{S_2}) - BZNN_i(G_N)$, where $G_N = (N;U_N)$ is the unanimity game related to N. Suppose now that $i = a_1$. Then $BZNN_{a_1}(G) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$. Similarly we can calculate the powers of other voters.

Finally, in theorem 3.2 below, we demonstrate the independence of the axioms **A3.1**-**A3.4**. Independence means that the given set of axioms is minimal in the sense that none of its proper subset will characterize the non-normalized Banzhaf index.

Theorem 3.2: Axioms **A3.1**-**A3.4** are independent**.**

Proof:

(1) Since the index given by

$\varphi_{1i}(G) = 1$ if i is a dictator,

$$= \frac{\log_2(m_i(G))}{2^{|N|-1}} \quad \text{otherwise,}$$

where $|N| > 2$, is nonlinear in $m_i(G)$, it fails to satisfy **A3.1**, but it satisfies **A3.2-A3.4**.

(2) Because the index $\boldsymbol{j}_{2i} = \frac{m_i(G)}{2^{|N|}}$ is not appropriately normalized, it is a violator of **A3.2**. Nevertheless, it verifies **A3.1**, **A3.3** and **A3.4**.

(3) Consider the index $\boldsymbol{j}_{3i} = \frac{m_i(G)}{2^{|N|-1}} + \boldsymbol{a}_i$,

where $\boldsymbol{a}_i$ satisfies the following conditions:

(a) $\boldsymbol{a}_i = 0$ or < 0 according as i is a dictator or not,

(b) $\boldsymbol{a}_i \neq \boldsymbol{a}_j$ if $i \neq j$ and

(c) if two voters i and j form a bloc ij, then $\boldsymbol{a}_{ij} = \boldsymbol{a}_i + \boldsymbol{a}_j$.

Since $\boldsymbol{a}_i's$ are different across voters, **A3.3** is violated. However, **A3.1**, **A3.2** and **A3.4** are satisfied by $\boldsymbol{j}_{3i}$.

(4) Finally, the index given by $\boldsymbol{j}_{4i}(G) = \frac{m_i(G)}{2^{|N|}} + \frac{1}{2}$ does not meet **A3.4** because of the presence of $1/2$ on the right-hand side. However, it fulfils **A3.1**-**A3.3**.

3.4 Conclusion

Power of an individual voter depends on the chance he has of being critical to the passage or defeat of a resolution. The well-known Banzhaf non-normalized index for a voter depends on the number of coalitions in which the voter is in the critical position of making winning (losing) coalitions losing (winning). Several characterizations of this index have been proposed in the literature. In this chapter we provide a new characterization of the index using four axioms from four different contributions to the area. Independence of the axioms is also demonstrated.

CHAPTER 4

A CHARACTERIZATION AND SOME PROPERTIES OF THE BANZHAF-COLEMAN-DUBEY-SHAPLEY SENSITIVITY INDEX•

4.1 Introduction

As we have already noted in section 1.3, a sensitivity index is a measure of the extent of volatility in a decision rule (voting body). It is an indicator of the degree of ease with which it responds to the fluctuations in the wishes of the members of the voting body. It can as well be regarded as a democratic participation index measuring sensitivity to the desires of the voting body members.

Dubey and Shapley (1979) considered the sum of the numbers of swings of different voters in a voting game as a measure of the sensitivity of a decision rule to the desires of the average voter. Thus, this index gives the numbers of possibilities in which different voters are in the critical position of being able to change the voting outcome by changing their votes. Since a critical voter's exit from a winning coalition makes it losing, it gives an indication that even a single voter could tip the scales. A normalized version of the Dubey-Shapley index was considered by Felsenthal and Machover (1998) for measuring sensitivity. We refer to this normalized formula, which is the sum of one of the Banzhaf (1965)-Coleman (1971) indices of power (more precisely, the non-normalized Banzhaf index) of different voters in the game, as the Banzhaf-Coleman-Dubey-Shapley (BCDS) sensitivity index.

Dubey and Shapley (1979) investigated several properties of their index, including determination of lower and upper bounds. A feasible and desirable direction of research along this line is to study additional/alternative properties of the BCDS index and characterize it uniquely. This is the objective of this chapter.

• This chapter is based on Barua, Chakravarty, Roy and Sarkar (2004).

More precisely, we first discuss some properties and develop an axiomatic characterization of the BCDS sensitivity index. It is shown that the set of axioms used in the characterization theorem is minimal, that is, no proper subset of this set can characterize the index. Equivalently, we say that axioms belonging to this minimal set are independent. Then using Fourier transform analysis, we derive some additional properties and bounds for the BCDS index for a class of games, which is much more general than the class considered by Dubey and Shapley (1979).

In the next section of the chapter, we recall the definition of the BCDS index and discuss some of its properties. Section 4.3 derives the index axiomatically and demonstrates independence of the properties employed in the axiomatization exercise. In section 4.4 we discuss some additional properties of the index, including derivation of bounds, using Fourier transform. Finally, section 4.5 concludes the chapter.

4.2 The Banzhaf-Coleman-Dubey-Shapley Sensitivity Index

As we have already noted in section 1.3, Dubey and Shapley (1979) suggested the use of $S_D(G)=\sum_{i=1}^{|N|} m_i(G)$ as a sensitivity index, where $G=(N;V)\in \mathbf{SG}$ ($\mathbf{SG}$ being the set of all simple games) is arbitrary. Let us denote the total number of swings $\sum_{i=1}^{|N|} m_i(G)$ in G by $m(G)$.

Then Dubey-Shapley (1979) measure of sensitivity can be written as:

$$S_D(G)=m(G). \qquad (4.1)$$

This measure of sensitivity does not ignore null voters. If a null voter is added to the game, the value of $S_D(G)$ is doubled. Felsenthal and Machover (1998) made this measure independent of the effect of null voters by dividing it by $2^{|N|-1}$. Thus the Felsenthal –Machover (1998) version of this index is given by

$$SS(G)=\frac{m(G)}{2^{|N|-1}}. \qquad (4.2)$$

As we have already noted earlier in chapter 3, the maximum number of winning coalitions in which a voter i can be pivotal in a game $G=(N;V)\in \mathbf{SG}$ is $2^{|N|-1}$. The Banzhaf non-normalized index for a voter i is given by the number of winning coalitions in which i is pivotal $(m_i(G))$ divided by the maximal value that this number can take ($2^{|N|-1}$). The summation of this index across all voters gives us the sensitivity index $SS(G)$. Given the direct involvement of swings of different voters in the construction of the index, it shows the sensitivity of the decision rule to the wishes of individual voters in the sense that even an individual can tip the scales. Since $SS(G)$ is the sum of the non-normalized Banzhaf indices (which is also referred to as the Banzhaf-Coleman index in the literature (see Owen (1978))) of different voters in a game, we refer to $SS(G)$ as the Banzhaf-Coleman-Dubey-Shapley (BCDS) index of sensitivity. It 'reflects the ''volatility'' or degree of suspense in the voting body' (Dubey and Shapley (1979)). Suppose in a voting game each voter's probability of voting for or against a bill is selected independently at random from a distribution with expectation 0.5. Then $m_i(G)/2^{|N|-1}$ becomes the probability p_i that other voters will vote such that the bill will pass or fail according as i votes for or against it (Straffin (1977), Leech (1990)). The index $SS(G)$ is simply $\sum_{i=1}^{|N|} p_i$.

Here it would be interesting to note the relationship between the sensitivity index and the notion of potential function introduced by Hart and MasCollel (1988, 1989). Hart and MasCollel had used the potential approach to provide an axiomatic characterization of the Shapley value. Dragan (1996) had shown that there exists a unique potential function for the Banzhaf value on the set of all games **G**. Using Dragan's tools on the class of simple games **SG**, Carreras (2004) proposed the potential function for the Banzhaf index on simple games. A function $Q:\mathbf{SG}\rightarrow \mathbf{R}$ is a potential function for the Banzhaf index on simple games if

(a) Q is standard for one-person games, i.e., $Q(G(i)) = 1$ whenever i is a dictator, and 0 otherwise, and where $G(i)$ denotes the game $G = (\{i\}; V)$;

(b) $\sum_{i \in N} [Q(G) - Q(G_{-i})] = \sum_{i \in N} \frac{m_i(G)}{2^{|N|-1}}$ for all $G = (N;V) \in \mathbf{SG}$, where $Q(G) - Q(G_{-i})$ is the marginal contribution of player i (according to the potential function), in the game G, and G_{-i} = $(N \setminus \{i\}; V)$. Note that, given a game $G = (N;V)$, and a coalition $S \subset N$, $(S;V)$ is the subgame that is obtained by restricting V to the subsets of S. That is, the domain of V is restricted to 2^S.

Carreras showed that there is a unique potential function $Q : \mathbf{SG} \to \mathbf{R}$ for the Banzhaf index on simple games and it is given by

$$Q\ (G) = \frac{|\mathbf{W}(G)|}{2^{|N|-1}} \ \forall G = (N;V) \in \mathbf{SG}.$$

Moreover, $Q(G) - Q(G_{-i}) = \frac{m_i(G)}{2^{|N|-1}}$ $\forall G = (N;V) \in \mathbf{SG}$ and $\forall i \in N$. That is, the marginal of the potential of a player is equal to Banzhaf index of that player.

Thus it follows that sum of the marginals of the potential across all voters gives the sensitivity index. That is, $\sum_{i \in N} [Q(G) - Q(G_{-i})] = SS(G)$.

Next we note some of the properties of the $SS(G)$ index.

(a) Iso-Invariance: Let $G = (N;V)$ and $G' = (N';V') \in \mathbf{SG}$ be two isomorphic games. That is, there exists a bijection h of N onto N' such that for all $S \subseteq N$, $V(S) = 1$ if and only if $V'(h(S)) = 1$, where $h(S) = \{h(x) : x \in S\}$. Then $SS(G) = SS(G')$.

(b) Increasingness: Let $G = (N;V)$ and $\overline{G} = (\overline{N};\overline{V}) \in \mathbf{SG}$ be two games such that $N = \overline{N}$ and $m_i(G) \geq m_i(\overline{G})$ for all $i \in N$ with $>$ for at least one $i \in N$. Then $SS(G) > SS(\overline{G})$.

(c) Ignoring Null Voters: For any $G = (N;V) \in \mathbf{SG}$ and for any null voter $d \in N$, $SS(G) = SS(G_{-d})$, where G_{-d} is the game obtained from G by excluding d.

Likewise, $SS(G)=SS(G_{+d})$, where G_{+d} is the game obtained from $G \in \mathbf{SG}$ by including d as a null voter.

(d) Maximality: For any $G=(N;V)\in \mathbf{SG}$, $SS(G)$ attains its maximal value $\frac{r\binom{|N|}{r}}{2^{|N|-1}}$ if and only if all coalitions with more than $\frac{|N|}{2}$ voters win and all coalitions with less than $\frac{|N|}{2}$ voters lose, where $r=\left[\frac{|N|}{2}\right]+1$, with $[x]$ being the largest integer $\leq x$ (Dubey and Shapley (1979)).

(e) Duality: For any $G=(N;V)\in \mathbf{SG}$, let $G^{*}=(N;V^{*})$ be the dual of G, that is, $V^{*}(S)=V(N)-V(N-S)$ for all $S\in 2^{N}$. Then $SS(G)=SS(G^{*})$ (Dubey and Shapley (1979)).

Iso-invariance is an anonymity condition, which says that a reordering of the voters does not change the sensitivity index SS. Thus, all characteristics other than swings of the voters, e.g., their living conditions, are irrelevant to the measurement of sensitivity. Note that we have already discussed anonymity or iso-invariance in the context of desirable properties that an index of individual voting power must satisfy (see section 1.2.2). Here it is discussed as a property that an index that measures a global characteristic of a voting game satisfies.

Increasingness requires the index SS to be an increasing function of the number of swings, given that the voter set remains unaltered. To understand increasingness, let us consider the weighted majority game $\hat{G}_0=(N;V;1,2,2;4)$ obtained from $G_0=(N;V;1,2,2;3)$ by augmenting the quota from 3 to 4. Given that the set of voters $N=\{1,2,3\}$ is the same in the two games, we get $m_2(G_0)=m_2(\hat{G}_0)$ =2, $m_3(G_0)=m_3(\hat{G}_0)$ =2 and $m_1(G_0)=2>m_1(\hat{G}_0)=0$. We thus have $SS(G_0)>SS(\hat{G}_0)$.

Since a null voter is not able to influence the voting outcome, we can argue that SS should satisfy the ignoring null voters principle. Given that the non-normalized Banzhaf index (alternatively the Banzhaf-Coleman voting power index) $\frac{m_i(G)}{2^{|N|-1}}$ remains invariant under inclusion or exclusion of a null voter (Owen (1978), Felsenthal and Machover, (1995, 1998)), SS also satisfies this condition.

Maximality specifies the necessary and sufficient condition for SS to achieve the maximum value and duality shows that the values of SS for a voting game and its dual are the same.

Dubey and Shapley (1979) showed that for any $G=(N;V)\in \mathbf{SG}$,

$$SS(G)\geq \boldsymbol{q}\frac{[|N|-\log_2 \boldsymbol{q}]}{2^{|N|-1}} \tag{4.3}$$

where $\boldsymbol{q}$ is the minimum of the numbers of winning and losing coalitions in G. Hart (1976) suggested a stronger but more complicated lower bound for $SS(G)$. Dubey and Shapley (1979) also noted that if G is a decisive game, then a lower bound of $SS(G)$ is 1.

Examples of sensitivity indices other than $SS(G)$ which satisfy properties **(a)–(e)** are $(SS(G))^c$, $c>0$, $c\neq 1$ and $\exp(SS(G))$. However, because of its probabilistic interpretation, expositional and computational ease, $SS(G)$ appears to be more attractive than such indices. Furthermore, in the next section we show that a characterization of $SS(G)$ can be developed using a set of intuitively reasonable axioms. These therefore make $SS(G)$ a desirable index of sensitivity.

4.3 The Characterization Result

Before we go on to characterize the BCDS index axiomatically, we will introduce a special type of game that we will require in our analysis of the sensitivity index.

Definition 4.1: A voting game $G=(N;V)\in \mathbf{SG}$ is called balanced if $|\mathbf{W}(G)|=|\mathbf{L}(G)|=2^{|N|-1}$.

Clearly, a decisive game (see definition 1.7) is balanced.

Now we will present three axioms on a general sensitivity index $\boldsymbol{r} : \mathbf{SG} \to \mathbf{R}_+$, which is a nonnegative real valued function defined on the set of voting games, that will uniquely isolate the BCDS index given by (4.2). The first axiom is taken from Dubey (1975) (see also Dubey and Shapley, 1979). It shows how the sensitivity levels in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ (see definitions 1.5 and 1.6) are related to individual sensitivities in G_1 and G_2.

Axiom A4.1 (Transfer Principle): For any $G_1 = (N_1;V_1)$, $G_2 = (N_2;V_2) \in \mathbf{SG}$ **,**

$$\boldsymbol{r}(G_1 \vee G_2) + \boldsymbol{r}(G_1 \wedge G_2) = \boldsymbol{r}(G_1) + \boldsymbol{r}(G_2) . \tag{4.4}$$

This axiom, which is also referred to as union-intersection property in the literature (also see the discussion on transfer principle in section 1.2.4), is quite similar to the condition characterizing additive measures (in measure theoretic sense), such as probabilities. If a_1 and a_2 are two events in a probability space and $\vee$ and $\wedge$ are the disjunction and conjunction operations respectively, then $\boldsymbol{r}(a_1 \vee a_2) + \boldsymbol{r}(a_1 \wedge a_2) = \boldsymbol{r}(a_1) + \boldsymbol{r}(a_2)$, where $\boldsymbol{r}$ denotes probability.

The next axiom captures the change in sensitivity levels under a merger of any two voters in a unanimity game (see definition 3.1). In a voting game the power of a voter is determined by his swings only. Since the number of swings across voters in unanimity games is a constant, an important source of difference between the extents of sensitivity in two such games is the number of non-null voters. One way of reflecting this difference is to assume that the ratio between sensitivity levels in a unanimity game and a new game obtained by merging two voters in this game is proportional to the ratio of the numbers of non-null voters in them. The following axiom gives a formulation along this direction.

Axiom A4.2 (Proportionality Principle): Let $G' \in \mathbf{SG}$ **be the game obtained from** $G = (N;U_S) \in \mathbf{SG}$ **by merging two voters** $i, j \in S$ **as given in definition 1.17. Then,**

$$\frac{\boldsymbol{r}(G)}{\boldsymbol{r}(G')} = \frac{1}{2}\frac{|S|}{|S'|} . \tag{4.5}$$

The third axiom states the value of the index if the game has a dictator.

Axiom A4.3 (Dictatorship principle): If $G=(N;V)\in$ SG has a dictator, then $r(G)=1$.

Since the BCDS index is obtained directly from the non-normalized Banzhaf index, our axioms can be compared with some existing axiom systems that characterize the non-normalized Banzhaf index (see section 1.2.4).

We now have

<u>Theorem 4.1</u>: A sensitivity index r satisfies axioms A4.1-A4.3 if and only if it is the Banzhaf-Coleman-Dubey-Shapley sensitivity index SS given by (4.2).

<u>Proof:</u> We first demonstrate that SS satisfies **A4.1-A4.3**. Let $G_1=(N_1;V_1)$, $G_2=(N_2;V_2)\in$ **SG**. Assuming that $N_1-N_2\neq f$, take $i\in N_1-N_2$. Now, any coalition $S'\subseteq N_2-N_1$ can be appended to a swing coalition $S\subseteq N_1$ for $i\in N_1$ to obtain a swing coalition $S\cup S'$ for $i\in N_1\cup N_2$ unless $(S\cup S')\cap N_2$ is winning in G_2. Hence the number of swings of voter $i\in N_1-N_2$ is

$$m_i(G_1\vee G_2)=m_i(G_1)2^{|N_2-N_1|}-m_i(G_1\wedge G_2)$$

$$=m_i(G_1)2^{|N_2-N_1|}+m_i(G_2)2^{|N_1-N_2|}-m_i(G_1\wedge G_2), \tag{4.6}$$

since $m_i(G_2)=0$ for $i\notin N_2$. The same expression for $m_i(G_1\vee G_2)$ will be obtained if $i\in N_2-N_1$ and $i\in N_1\cap N_2$. Therefore,

$$SS(G_1\vee G_2)=\sum_{i=1}^{|N_1\cup N_2|}\frac{m_i(G_1\vee G_2)}{2^{|N_1\cup N_2|-1}}$$

$$=\sum_{i=1}^{|N_1\cup N_2|}\left(\frac{m_i(G_1)2^{|N_2-N_1|}}{2^{|N_1\cup N_2|-1}}+\frac{m_i(G_2)2^{|N_1-N_2|}}{2^{|N_1\cup N_2|-1}}-\frac{m_i(G_1\wedge G_2)}{2^{|N_1\cup N_2|-1}}\right)$$

$$=\sum_{i=1}^{|N_1\cup N_2|}\left(\frac{m_i(G_1)}{2^{|N_1|-1}}+\frac{m_i(G_2)}{2^{|N_2|-1}}-\frac{m_i(G_1\wedge G_2)}{2^{|N_1\cup N_2|-1}}\right)$$

$$=SS(G_1)+SS(G_2)-SS(G_1\wedge G_2). \tag{4.7}$$

Thus, SS satisfies **A4.1**.

To check satisfaction of **A4.2** by SS, consider the unanimity game $G_S = (N;U_S) \in \mathbf{SG}$. Let $G_{S'} = (N';U_{S'})$ be the game obtained from G_S by merging any two voters $i, j \in S$. Then,

$SS(G_S) = \dfrac{2^{|N-S|}|S|}{2^{|N|-1}} = \dfrac{|S|}{2^{|S|-1}}$ and $SS(G_{S'}) = \dfrac{|S'|}{2^{|S'|-1}}$, from which we have

$\dfrac{SS(G_S)}{SS(G_{S'})} = \dfrac{1}{2}\dfrac{|S|}{|S'|}$, since $|S'| = |S| - 1$.

Thus, SS verifies axiom **A4.2**.

If a game $G = (N;V) \in \mathbf{SG}$ has a dictator i, then i is the only swing voter in the game, that is, m_i is maximized, which means that $m_i = 2^{|N|-1}$ and $m_j = 0$ for all $j \neq i$. Hence,

$$SS(G) = \frac{m_i(G)}{2^{|N|-1}} = \frac{2^{|N|-1}}{2^{|N|-1}} = 1, \tag{4.8}$$

which shows that SS fulfils **A4.3**.

We will now demonstrate that if a sensitivity index $\boldsymbol{r}$ fulfils **A4.1**-**A4.3**, then it must be the BCDS index. Note that a sensitivity index $\boldsymbol{r}$ satisfying **A4.1** is uniquely determined on unanimity games. This is because for any game $G \in \mathbf{SG}$, $G = G_{S_1} \vee G_{S_2} \vee ... \vee G_{S_k}$, where $S_1, S_2, ...S_k$ are minimal winning coalitions of G and G_{S_i} is the unanimity game corresponding to S_i, $i = 1,2,...,k$. Thus, by **A4.1**, $\boldsymbol{r}$ is determined if $\boldsymbol{r}(G_{S_1})$, $\boldsymbol{r}(G_{S_2} \vee ... \vee G_{S_k})$ and $\boldsymbol{r}(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are known. But $G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k})$ = $G_{S_1 \cup S_2} \vee G_{S_1 \cup S_3} \vee ... \vee G_{S_1 \cup S_k}$ and hence by induction hypothesis on k, both $\boldsymbol{r}(G_{S_2} \vee ... \vee G_{S_k})$ and $\boldsymbol{r}(G_{S_1} \wedge (G_{S_2} \vee ... \vee G_{S_k}))$ are determined. So $\boldsymbol{r}(G)$ is determined.

In view of the above discussion we can say that it is enough to determine $\boldsymbol{r}(N;U_S)$ for any unanimity game $(N;U_S)$. We will now show by induction on $|S|$ that $\boldsymbol{r}(N;U_S) = \dfrac{|S|}{2^{|S|-1}}$. If $|S| = 1$, then $(N;U_S)$ has a dictator and hence by **A4.3**, $\boldsymbol{r}(N;U_S) = 1 = SS(N;U_S)$. Therefore assume $|S| > 1$ and the result for all games $(N;U_{\bar{S}})$, where $|\bar{S}|<|S|$. Let $(N';U_{S'})$ be the game obtained from $(N;U_S)$ by merging two voters i and j in S. Then by induction hypothesis, $\boldsymbol{r}(N';U_{S'})=\frac{|S'|}{2^{|S'|-1}}=SS(N',U_{S'})$. By Axiom **A4.2**, $SS(N;U_S)=\frac{1}{2}\frac{|S|}{|S'|}\frac{|S'|}{2^{|S'|-1}}$

$=\frac{|S|}{2^{|S|-1}}$, since $|S|=|S'|+1$.

This demonstrates that $\boldsymbol{r}$ coincides with SS on any unanimity game and hence on all games in **SG**.

Theorem 4.1 above specifies a set of necessary and sufficient conditions for identifying the BCDS index SS uniquely.

Now, in order to illustrate how the BCDS index SS can be calculated from minimal winning coalitions, let us consider the weighted majority game $\tilde{G}=(N;V;1,2,3;4)$ with the voter set $N=\{1,2,3\}$. The minimal winning coalitions in this game are $S_1=\{1,3\}$ and $S_2=\{2,3\}$. Hence $SS(\tilde{G})=SS(\tilde{G}_{S_1})+SS(\tilde{G}_{S_2})-SS(\tilde{G}_N)$, where $\tilde{G}_{S_i}$ is the unanimity game corresponding to S_i, $i=1,2$ and $\tilde{G}_N=(N;U_N)$. Then $SS(\tilde{G})=\frac{2}{2^{2-1}}+\frac{2}{2^{2-1}}-\frac{3}{2^{3-1}}=1.25$.

We will now show that axioms **A4.1-A4.3** are independent. Demonstration of independence requires that if one of these three axioms is dropped, then there will exist a sensitivity index that will satisfy the two remaining axioms but not the dropped one.

Theorem 4.2: Axioms A4.1-A4.3 are independent.

Proof: Let $G=(N;V)\in\mathbf{SG}$ be arbitrary. Then consider the sensitivity indices given by

$$\boldsymbol{r}_1(G)=\frac{\sum_{i=1}^{|N|}m_i(G)}{2^{|N|}}, \tag{4.9}$$

$$\boldsymbol{r}_2(G) = \frac{\sum_{i=1}^{|N|} m_i(G)}{2^{|N|}} + \frac{1}{2} \qquad (4.10)$$

$$\boldsymbol{r}_3(G) = \frac{\sum_{i \in \overline{D}} \left| \mathbf{W}^i(G) \right|}{2^{|N|-1}}, \qquad (4.11)$$

where $\mathbf{W}^i(G)$ is the set of winning coalitions containing i, and $\overline{D}$ is the set of non-null players in the game G.

It is easy to see that $\boldsymbol{r}_1$ verifies **A4.1** and **A4.2** but not **A4.3**, whereas $\boldsymbol{r}_2$ verifies **A4.1** and **A4.3** but not **A4.2**. One can also check that $\boldsymbol{r}_3$ fulfils **A4.2** and **A4.3** but not **A4.1**.

4.4 Fourier Transform Analysis of the Banzhaf - Coleman -Dubey - Shapley Sensitivity Index

In this section we analyze voting games using tools from Boolean function literature. Before embarking on the details of the analysis, we discuss the connection of games to Boolean functions and the main results that we obtain.

An n-variable Boolean function is a map $f:\{0,1\}^n \rightarrow \{0,1\}$, where $\{0,1\}^n$ is the n-fold Cartesian product of $\{0,1\}$. With the conventional identification of n- bit strings and subsets of N, we can also take the domain to be 2^N, where $N = \{1,2,...,n\}$. Therefore, Boolean functions can be regarded as indistinguishable from general games $(N;V)$, considered by Owen (1978), where the domain and the range of V are 2^N and $\{0,1\}$ respectively. We denote the set of all such games by $\mathbf{SG}^*$. Thus, if $G = (N;V) \in \mathbf{SG}^*$, then V is a Boolean function as well. Since $\mathbf{SG} \subset \mathbf{SG}^*$, our analysis is also applicable to any game in $\mathbf{SG}$.

Boolean functions have been studied quite extensively in other areas such as computer science and engineering. Several analytical tools have been developed for this purpose. The most important of these tools is the Walsh transform, which is essentially the Fourier transform of $(-1)^{f(x)}$. In this section we use the Walsh transform to obtain bounds on the index. Since we will be performing the analysis on a general Boolean function (or game) we first generalize the concept $m_i(G)$ in the following manner.

Definition 4.2: For any game $G=(N;V)\in \mathbf{SG}^*$, the associated complement game is $\overline{G}=(N;\overline{V})\in \mathbf{SG}^*$, where for any $S\subseteq N$, $V(S)=1$ if and only if $\overline{V}(S)=0$. Further, G is said to be balanced if the number of winning coalitions in G and $\overline{G}$ are equal.

For any $G=(N;V)\in \mathbf{SG}^*$ and $i\in N$, we write

$$M_i(G)=m_i(G)+m_i(\overline{G}). \tag{4.12}$$

Also we set $M(G)=\sum_{i=1}^{n} M_i(G)$. We first show that given a general game $G\in \mathbf{SG}^*$, it becomes a simple voting game (i.e. $G\in \mathbf{SG}$) if and only if $M(G)=m(G)$. Then we go on to obtain upper and lower bounds for $M(G)$ which immediately provide upper and lower bounds for $m(G)$. The main results that we obtain are the following.

1. If G in $\mathbf{SG}^*$ is a balanced *n*-player game, then $M(G)\geq 2^{n-1}$. Consequently, for any balanced *n*-player game G in $\mathbf{SG}$, we have $m(G)\geq 2^{n-1}$. Further, equality is attained if there is a dictator.
2. If $G\in \mathbf{SG}^*$ is an n- player game and $w=|\mathbf{W}(G)|$ is the number of winning coalitions, then

$$\frac{w(2^n-w)}{2^{n-1}}\leq M(G)\leq n\,\frac{w(2^n-w)}{2^{n-1}}.$$

Further, both the upper and lower bounds are attained.

Consequently, for any *n*-player game $G\in \mathbf{SG}$, we have $\frac{w(2^n-w)}{2^{n-1}}\leq m(G)\leq n\,\frac{w(2^n-w)}{2^{n-1}}$.

Remarks:

(a) From result (1) above, it follows that for a decisive voting game, a lower bound of $SS(G)$ is 1. As stated earlier, Dubey and Shapley (1979) derived 1 as the lower bound of SS for decisive voting games. Evidently, Corollary 4.5 presents a lower bound for a more general class of games viz., balanced voting games. Moreover, Dubey and Shapley's (1979, p.108) claim that the lower bound can only be derived by using Hart's (1976) bound does not appear to be true.

(b) It is known (Felsenthal and Machover, 1998, p.56) that for $G = (N;V) \in \mathbf{SG}$, $m(G) \geq n$. Result (2) above provides a lower bound on $m(G)$ for monotone games. This lower bound depends on the number of winning coalitions. Though this can be lower than n, in general it is going to be a sharper lower bound. In fact, our lower bound $\frac{w(2^n - w)}{2^{n-1}}$ is greater than n if $w > 2^{n-1} - 2^{n-1}\sqrt{1 - \frac{n}{2^{n-1}}} \cong \frac{n}{2}$.

To obtain these results, we first prove a relation (Lemma 4.1) between $M_i(G)$ and the autocorrelation function of the Boolean function associated with G. (See equation (4.17) below for the definition of the autocorrelation function.) Thus, autocorrelation function becomes a helpful technique in studying swings in general voting games. Further algebraic analysis is performed using the Walsh transform, which ultimately leads to the desired results. This in turn establishes the role of the Walsh transform in proving results in general voting games. Since the Walsh transform can be expressed in matrix form using the Hadamard matrix, this motivates the use of the Hadamard matrix.

For the sake of convenience, we divide this section into two subsections.

4.4.1 Basics of Fourier Transform Analysis

In this subsection, we present the mathematical preliminaries necessary for understanding the Fourier transform analysis of games.

Let F_2 be the field $\langle\{0,1\},\oplus,\cdot\rangle$, where $\oplus$ and $\cdot$ denote modulo 2 addition and multiplication. We thus consider the domain of a Boolean function to be the vector space $\langle F_2^n,\oplus\rangle$ over F_2, where, as stated, $\oplus$ is the addition operator on F_2 and also on F_2^n. The inner product of two vectors $u=(u_1,...u_n)$, $v=(v_1,...,v_n)\in F_2^n$ is $\sum_{i=1}^{n} u_i v_i$ and will be denoted by $\langle u,v\rangle$. The weight of an n-bit vector u is the number of ones in u and will be denoted by $wt(u)$.

The Fourier transform is the most widely used tool in the analysis of Boolean functions. In most cases it is convenient to apply Fourier transform to $(-1)^{f(x)}$ instead of $f(x)$. The resulting transform is called the Walsh transform of $f(x)$. More precisely, the Walsh transform of $f(x)$ is an integer-valued function $W_f:\{0,1\}^n \to [-2^n,2^n]$ defined by (see, for example, Ding, Xiao and Shan (1978)).

$$W_f(u)=\sum_{w\in F_2^n}(-1)^{f(w)\oplus\langle u,w\rangle} \quad . \qquad (4.13)$$

The Walsh transform is called the spectrum of f. Note that the spectrum measures the cross-correlations between a function and the set of linear functions. Another way of looking at the spectrum is via Hadamard matrices. Let H_n be the Hadamard matrix of order 2^n defined recursively as (see MacWilliams and Sloane (1977))

$$H_1=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad (4.14)$$

$H_n=H_1\otimes H_{n-1}$ for $n>1$,

where $\otimes$ denotes the Kronecker product of two matrices. For example,

$$H_2=\begin{bmatrix}H_1 & H_1\\ H_1 & -H_1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}.$$

Considering the rows and columns of H_n to be indexed by the elements of F_2^n, we obtain $[H_n]_{(u,v)} = (-1)^{\langle u,v \rangle}$. Using this fact, the Walsh transform can be written as

$$[(-1)^{f(0)},...,(-1)^{f(2^n-1)}]H_n = [W_f(0),...,W_f(2^n-1)], \tag{4.15}$$

where $u \in F_2^n$ is identified with an integer in $[0, 2^n - 1]$.

Since $H_n H_n = 2^n I_{2^n}$, post-multiplying both sides by H_n we get the inverse of Walsh transform.

$$(-1)^{f(u)} = \frac{1}{2^n} \sum_{w \in F_2^n} W_f(w)(-1)^{\langle u,w \rangle} \quad . \tag{4.16}$$

Another commonly used tool in Boolean function analysis is the auto-correlation function. The auto-correlation function is an integer-valued map $C_f : \{0,1\}^n \to [-2^n, 2^n]$ defined by (see MacWilliams and Sloane (1977), for a related concept called directional derivative)

$$C_f(u) = \sum_{w \in F_2^n} (-1)^{f(w) \oplus f(u \oplus w)} \quad . \tag{4.17}$$

It is clear that $C_f(0) = 2^n$. The auto-correlation is not a transform in the sense that it does not uniquely determine the function.

For the weighted majority game $\tilde{G} = (N; V; 1,2,3; 4)$ with minimum winning coalitions $\{1,3\}$ and $\{2,3\}$ the corresponding Boolean function f and Walsh transform W_f are given in Table 4.1 below. The variable x_i in the table represents player i.

Table 4.1: The Walsh Transform and Autocorrelation.

x_3	x_2	x_1	f	W_f	C_f
0	0	0	0	2	8
0	0	1	0	2	4
0	1	0	0	2	4
0	1	1	0	2	4
1	0	0	0	6	-4
1	0	1	1	-2	-4
1	1	0	1	-2	-4
1	1	1	1	-2	-4

The next result is called the Wiener-Khintchine Theorem in continuous analysis and has also been obtained for Boolean functions (see Carlet (1992); Preneel (1993) and Zhang and Zheng (1995)).

Theorem 4.3: Let f be an n-variable function. Then

$$[C_f(0),...,C_f(2^n-1)]\, H_n = [W_f^2(0),...,W_f^2(2^n-1)]. \qquad (4.18)$$

Applying the inverse transform gives $\sum_{u\in F_2^n} W_f^2(u) = 2^n C_f(0) = 2^{2n}$. This is a conservation law for the spectral values of f and is known as Parseval's Theorem (see, for example, Ding, Xiao and Shan (1978)).

The next result states a useful property of Walsh Transform (see Canteaut, Carlet, Charpin and Fontaine (2000), Proposition 5). For a vector space E, we define $E^{\perp}$ to be the vector space which is orthogonal to E, i.e., $E^{\perp}$ = $\{u : \langle u, v\rangle = 0, \forall v\in E\}$.

Theorem 4.4: Let f and g be n-variable functions and E be a subspace of F_2^n. Then

$$\sum_{w\in E} W_f^2(w) = |E| \sum_{u\in E^{\perp}} C_f(u) \tag{4.19}$$

See (Sarkar and Maitra (2002)) for a discussion of the above results in a more general setting.

In the next subsection, we present the main results of this chapter.

4.4.2 The Results

In this subsection, we present the results mentioned at the beginning of section 4.4 along with complete proofs. First we generalize the notion of swing. The notion of swing is quite general in the sense that we do not require monotonicity (condition (iii) in definition 1.3) for swing to be defined.

Definition 4.3: Given a game $G = (N;V) \in \mathbf{SG}^*$, and $i\in N$, number of negative swings of i is defined as

$$m_i^-(G) = \left|\{S \subseteq N - \{i\} : V(S) - V(S \cup \{i\}) = 1\}\right|.$$

For any $G = (N;V) \in \mathbf{SG}^*$, we write

$$m^-(G) = \sum_{i\in N} m_i^-(G). \tag{4.20}$$

The following proposition, whose proof is very easy, states the relationship between $m_i^-(G)$ and $m_i(\overline{G})$.

Proposition 4.1: Let $G = (N;V) \in \mathbf{SG}^*$ be arbitrary. Then for any $i \in N$, $m_i^-(G)$ = $m_i(\overline{G})$.

Proposition 4.2: Let $G = (N;V) \in \mathbf{SG}^*$. Then $m(\overline{G})$=0 if and only if G satisfies monotonicity, that is, condition (iii) in definition 1.3.

Proof: The sufficiency part of the proof is easy to verify. We therefore establish the necessity. If $m(\overline{G})$=0, then $m_i(\overline{G})$=0 for all $i \in N$. Let S and T be two coalitions in G such that $V(S)=1$ and $S \subseteq T$. Then we need to show that $V(T)=1$. This is shown by induction on $r = |T|-|S|$. For $r=0$, we have $T = S$ and the result follows trivially. Assume that the result is true for $r-1$. Let T' be such that $S \subseteq T' \subseteq T$ and $|T'|$ = $r-1$. By induction hypothesis $V(T')=1$. Let $j \in N$ be such that $T = T' \cup \{j\}$. If possible, let $V(T)=0$. Then $\overline{V}(T')=0$ and $\overline{V}(T)=1$, which in turn implies that $m_j(\overline{G}) \neq 0$. This contradicts the assumption that $m_i(\overline{G})=0$ for all $i \in N$. Therefore G will fulfil monotonicity.

Corollary 4.1: Let $G = (N;V) \in \mathbf{SG}^*$. Then $M(G) = \sum_{i \in N} M_i(G) = m(G)$ if and only if G is monotone.

Corollary 4.2: Let $G = (N;V) \in \mathbf{SG}^*$. Then $SS(G) + SS(\overline{G}) = SS(G)$, that is $SS(\overline{G})$ = 0 if and only if G meets monotonicity.

Remarks:

(a) Propositions 4.1 and 4.2 show that a game does not have negative swing if and only if it is monotone.

(b) Since $m(G) = M(G) - m(\overline{G})$ and $m(\overline{G}) \geq 0$, $m(G)$ is maximized if and only if G satisfies monotonicity.

(c) It is evident that $M(G)$ can be regarded as a sensitivity index on the set $\mathbf{SG}^*$.

Given $G=(N;V)\in \mathbf{SG}^*$, we now express $m_i(G)$ in terms of the autocorrelation values of V. For $i\in N$, let $\boldsymbol{e}_i$ be the $n-$vector, which has 1 in the i^{th} position and 0 elsewhere.

Lemma 4.1: For any $n-$player game $G=(N;V)\in \mathbf{SG}^*$ and $i\in N$, we have

$$M_i(G) = 2^{n-2}-\frac{1}{4}C_V(\boldsymbol{e}_i) \tag{4.21}$$

Proof: Let $m_i(V)=\left|\{S\subseteq N: V(S\Delta\{i\})\oplus V(S)=1\}\right|$, where for any two sets A and B, $A\Delta B=(A-B)\cup(B-A)$. Then it is easy to verify that

$m_i(G)+m_i(\overline{G})=\frac{1}{2}m_i(V)$. We now compute

$$C_V(\boldsymbol{e}_i)=\sum_{x\in F_2^n}(-1)^{V(x)\oplus V(x\oplus e_i)}$$

$$= \left|\{x: V(x)=V(x\oplus \boldsymbol{e}_i)\}\right| - \left|\{x: V(x)\neq V(x\oplus \boldsymbol{e}_i)\}\right|$$

$$= 2^n-2\left|\{x: V(x)\neq V(x\oplus \boldsymbol{e}_i)\}\right|$$

$$= 2^n-2\, m_i(V)$$

$$=2^n-4(m_i(G)+m_i(\overline{G})).$$

This gives us the desired result.

Corollary 4.3: For any $n-$player game $G=(N;V)\in \mathbf{SG}^*$, we have

$$M(G)=n2^{n-2}-\frac{1}{4}\sum_{i=1}^{n}C_V(\boldsymbol{e}_i). \tag{4.22}$$

Thus the problem reduces to computing $\sum_{i=1}^{n}C_V(\boldsymbol{e}_i)$. We use algebraic techniques to tackle this problem. The first two steps are the following.

For two $n-$bit vectors u and v we denote $u\le v$ if $u_i\le v_i$ for each $i\in N$. Also by $\overline{u}$ we denote the bitwise complement of u.

Lemma 4.2: For any $n-$player game $G=(N;V)\in \mathbf{SG}^*$, we have

$$\sum_{i=1}^{n} C_V(\boldsymbol{e}_i) = -n2^n + \frac{1}{2^{n-1}} \sum_{i=1}^{n} \sum_{u \le \overline{\boldsymbol{e}}_i} W_V^2(u). \tag{4.23}$$

Proof: For $1 \le i \le n$, let E_i be the subspace of F_2^n defined by $E_i = \{u \in F_2^n : u \le \overline{\boldsymbol{e}}_i\}$. Then $E_i^{\perp} = \{u \in F_2^n : u \le \boldsymbol{e}_i\} = \{0, \boldsymbol{e}_i\}$. It is easy to see that $|E_i| = 2^{n-1}$. We now apply Theorem 4.4 to get $\sum_{u \le e_i} C_V(u) = \frac{1}{2^{n-1}} \sum_{u \le \overline{e}_i} W_V^2(u)$.

Note that $\sum_{u \le e_i} C_V(u) = C_V(0) + C_V(\boldsymbol{e}_i) = 2^n + C_V(\boldsymbol{e}_i)$. Hence summing both the sides from 1 to n we obtain the desired result.

The next task is to simplify the right hand side of Equation (4.23).

Lemma 4.3: For any $n-$player game $G = (N;V) \in \mathbf{SG}^*$, we have

$$\sum_{i=1}^{n} \sum_{u \le \overline{e}_i} W_V^2(u) = n2^{2n} - \sum_{u \in F_2^n} wt(u) W_V^2(u). \tag{4.24}$$

Proof: Let $u \in F_2^n$ be arbitrary. The number of times $W_V^2(u)$ occurs in the left-hand side of Equation (4.24) is $(n - wt(u))$. Hence the left-hand side is equal to

$$\sum_{u \in F_2^n} (n - wt(u)) W_V^2(u) = n \sum_{u \in F_2^n} W_V^2(u) - \sum_{u \in F_2^n} wt(u) W_V^2(u).$$

Using Parseval's Theorem, we have $\sum_{u \in F_2^n} W_V^2(u) = 2^{2n}$. This gives us the desired result.

Let $(N;V)$ be a $n-$player game. For $0 \le i \le n$, we define

$$K_V(i) = \sum_{u \in F_2^n, wt(u) = i} \frac{W_V^2(u)}{2^{2n}}.$$

Note that using Parseval's Theorem, we have $\sum_{i=0}^{n} K_V(i) = 1$. We rewrite Lemma 4.3 in the following manner.

Lemma 4.4: $\sum_{i=1}^{n} \sum_{u \le \overline{e}_i} W_V^2(u) = n2^{2n} - 2^{2n} \sum_{i=0}^{n} i K_V(i)$ (4.25)

Combining Corollary 4.3, Lemma 4.2 and Lemma 4.4 we obtain the main result.

Theorem 4.5: Let $G = (N;V) \in \mathbf{SG}^*$ be an n- player game. Then

$$M(G) = 2^{n-1} \sum_{i=0}^{n} iK_V(i) \tag{4.26}$$

Recall that an n- player game $(N;V)$ is balanced if the number of winning coalitions (i.e., the weight) of V is 2^{n-1}.

Corollary 4.4: Let $G = (N;V) \in \mathbf{SG}^*$ be an n- player game. Assume further that G is balanced. Then $M(G) \geq 2^{n-1}$.

Proof: If G is balanced, then $W_V(0) = 0$ and consequently $K_V(0) = 0$. Thus $K_V(1) + ... + K_V(n) = 1$ and $M(G) = 2^{n-1} \sum_{i=0}^{n} iK_V(i) \geq 2^{n-1}$.

Corollary 4.5: Let $G = (N;V) \in \mathbf{SG}$ be an n-player game. Assume also that G is balanced and monotone. Then $m(G) \geq 2^{n-1}$. Further, equality is attained if there is a dictator.

A class of Boolean functions called resilient functions has been extensively studied for cryptographic applications. These were introduced by Siegenthaler (1984) and were characterized in terms of Walsh transform (see Xiao and Massey (1988)). An n-variable Boolean function f is called k- resilient if $W_f(u) = 0$ for all $0 \leq wt(u) \leq k$. We can prove improved lower bound for games corresponding to resilient functions. The proof is similar to that of Corollary 4.4.

Corollary 4.6: Let $G = (N;V) \in \mathbf{SG}^*$ be an n- player game which is k-resilient. Then $M(G) \geq (k+1)2^{n-1}$.

Let X be a random variable on $\{0,...,n\}$ such that $P[X=i]=K_V(i)$. Then $\sum_{i=0}^{n} iK_V(i)$ is the expected value of X. Bounds on this expected value provide bounds on $m(G)$.

Theorem 4.6: Let $G=(N;V)\in \mathbf{SG}^*$ be an n- player game and $w=|\mathbf{W}(G)|$ is the number of winning coalitions. Then

$$\frac{w(2^n-w)}{2^{n-1}} \le M(G)=m(G)+m(\overline{G}) \le n\,\frac{w(2^n-w)}{2^{n-1}}. \qquad (4.27)$$

Further, both the upper and lower bounds are attained.

Proof: We have

$$\sum_{i=1}^{n} K_V(i) \le \sum_{i=0}^{n} iK_V(i) \le n\sum_{i=1}^{n} K_V(i).$$

Using $\sum_{i=1}^{n} K_V(i) = 1-K_V(0)$ we obtain

$$1-K_V(0) \le \sum_{i=0}^{n} iK_V(i) \le n\,(1-K_V(0)). \qquad (4.28)$$

By definition,

$$K(0)=\frac{W_V^2(0)}{2^{2n}}=\frac{(2^n-2w)^2}{2^{2n}}.$$

Putting this value of $K(0)$ in inequality (4.28) and using (4.26) we obtain the desired result.

The lower bound is attained if any one player becomes the dictator. The upper bound is attained if G is the parity game, i.e., $V(x)\equiv wt(x) \bmod 2$ for all $x\in F_2^n$. (Note that for a parity game the number of swings of any player i in both G and $\overline{G}$ is 2^{n-2}. Therefore, for such a game $m(G)=m(\overline{G})=n\,2^{n-2}$.)

Corollary 4.7: If $G=(N;V)\in \mathbf{SG}^*$ is monotone, then

$$\frac{w(2^n-w)}{2^{n-1}} \le m(G) \le n\,\frac{w(2^n-w)}{2^{n-1}}.$$

4.5 Conclusion

Dubey and Shapley (1979) argued that in a voting situation the sum of the number of ways in which each voter can affect a 'swing' in the outcome is a measure of the sensitivity of the situation. Following Felsenthal and Machover (1998) we consider a normalized value of this sum and refer to it as the Banzhaf (1965)-Coleman (1971)-Dubey-Shapley (1979) sensitivity index. This chapter investigates some of its properties, the main topics being a characterization from a set of independent axioms and derivation of bounds for a very general class of games.

CHAPTER 5

ON THE CARRERAS-COLEMAN DECISIVENESS INDEX[#]

5.1 Introduction

A decisiveness index of a voting body, under a given decision rule, is a quantification of the extent to which the body is able to control the outcome of a division of it. This index measures the propensity of the voting body to a proposed resolution in an unambiguous way. Thus, it is a characteristic of the voting body itself, rather than of any particular member.

In a recent paper, Carreras (2004) suggested a decisiveness index for assessing the decision rule of a voting body. It equals the a-priori probability that the decision-making committee under consideration will accept a proposed resolution. As Carreras (2004) showed, it has a relationship with the non-normalized Banzhaf index and several interesting properties. Earlier, Coleman (1971) had suggested an index of the 'power of a collectivity to act', which can be regarded as the extent of deference of the concerned voting body to the passage of a resolution. Since it coincides with Carreras' (2004) decisiveness index, we will refer to this common index as the Carreras- Coleman (*CC*) index.

Carreras (2004) also developed characterizations of the *CC* index using different axioms, some of which were used in characterizations of several individual power indices by Dubey (1975), Dubey and Shapley (1979), Roth (1988), Feltkamp (1995), Laruelle and Valenciano (2001) and others. One axiom that has been used extensively for characterizing the Banzhaf index or its weighted form is about amalgamation of voters (see, for example, Lehrer (1988), Nowak (1997), Nowak and Radzik (2000), Albizuri (2001), Barua, Chakravarty and Roy (2004a)). In this chapter we consider a similar amalgamation axiom for a collective power index. It is then shown that three different sets of axioms, where each set contains this axiom and two other axioms considered by Carreras (2004) and others, become equivalent to one of the axiom sets of Carreras (2004) that

[#] This chapter is based on Barua, Chakravarty and Roy (2004b).

characterizes the *CC* index. It thus follows that each of the three sets considered by us characterizes the *CC* index. We also show that axioms considered in each of the three sets are independent, that is, each set is minimal in the sense that none of its proper subset can characterize the index.

The chapter is organized as follows. In section 5.2 we discuss the Carreras-Coleman decisiveness index. We also characterize this index using three different sets of independent axioms, each of which have been shown to be equivalent to a set of axiom set that Carreras (2004) used in his characterization exercise. Finally in section 5.3 we conclude.

5.2 The Carreras-Coleman Decisiveness Index

A decisiveness index is a nonnegative real valued function D defined on $\mathbf{SG}$, the set of all simple games, that is, $D:\mathbf{SG}\rightarrow\mathbf{R}_+$, where $\mathbf{R}_+$ is the nonnegative part of the real line. For any $G\in\mathbf{SG}$, $D(G)$ is a summary statistic of the inclination of the voting body towards the passage of the proposed act. It determines the degree of ease with which the interests of the body members in a division can be transformed into actual decisions. It is 'intended to measure the possibilities that some winning coalition forms in this game' (Carreras (2004), p. 5).

The Carreras-Coleman (*CC*) decisiveness index is given by

$$CC(G)=\frac{|\mathbf{W}(G)|}{2^{|N|}}, \tag{5.1}$$

where $G=(N,V)\in\mathbf{SG}$ is arbitrary. Since $|\mathbf{W}(G)|$ is the total number of winning coalitions and $2^{|N|}$ is the total number of coalitions (including the empty one) in the game *G*, $CC(G)$ is the prior probability of a positive outcome, that is, the probability that the voting body will adopt a resolution. It can also be interpreted as the probability of a random coalition to be winning when each voter has a probability $1/2$ to belong.

Carreras (2004) has established the following relationship between CC and the non-normalized Banzhaf index $BZNN_i$ (see equation (3.1)).

$$\frac{1}{2}BZNN_i\left(G\right)= CC\left(G\right)-CC\left(G_{-\{i\}}\right) \tag{5.2}$$

where $G_{-\{i\}}$ is the game obtained from G by deleting voter $i \in N$. That is, when voter i leaves, the resulting loss of decisiveness equals half the Banzhaf power of i. CC satisfies an anonymity condition in that it remains invariant under any reordering of voters (see Carreras (2004) for additional discussion on CC).

In order to study properties of the CC index, we will consider the following axioms:

(**A5.1**) **Transfer principle:** Consider two games $G_1 = (N_1;V_1)$, $G_2 = (N_2;V_2)$ $\in$ **SG**. Then $D(G_1 \vee G_2) = D(G_1) + D(G_2) - D(G_1 \wedge G_2)$, where the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ have been defined in definitions 1.5 and 1.6.

(**A5.2**) **Ignoring null voters:** Given $G = (N;V) \in$ **SG**, if voter $d \in N$ is null, then $D(G)= D(G_{-\{d\}})$, where $G_{-\{d\}}$ is the game obtained from G by deleting d.

(**A5.3**) **Unanimity property:** $D(N;U_S) = 1/2^{|S|}$, where the game $(N;U_S)$ is as defined in definition 3.1**.**

(**A5.4**) **Dictatorship property:** If G has a dictator, then $D(G) = 1/2$.

(**A5.5**) **Single voter property:** If G is a single-person voting game, then $D(G) = 1/2$.

(**A5.6**) **Merger property:** If $(N',U_{S'}) \in$ **SG** is the game obtained from $(N,U_S) \in$ **SG** by merging any two voters $i, j \in S$ (see definition 1.17), then $D(N',U_{S'}) = 2D(N,U_S)$.

(A5.1) says how total decisiveness from G_1 and G_2 can be transferred to the games $G_1 \vee G_2$ and $G_1 \wedge G_2$. Carreras (2004) considered a weaker form of **(A5.1)** with $N_1 = N_2$, which was introduced by Dubey (1975) (see also Dubey and

Shapley (1979)) (see section 1.2.4). More precisely, the transfer principle considered by Carreras (2004) is given by:

(A5.1*) Transfer principle: Consider $G_1 = (N_1; V_1)$, $G_2 = (N_2, V_2) \in \mathbf{SG}$, $G_1 \vee G_2$ and $G_1 \wedge G_2$, as in definitions 1.5 and 1.6, under the assumption that $N_1 = N_2$.

Then $D(G_1 \vee G_2) = D(G_1) + D(G_2) - D(G_1 \wedge G_2)$.

(**A5.2**), which was considered by Carreras (2004), means that decisiveness remains unaffected under entry or exit of a null voter from the game (see also Felsenthal and Machover (1998)).

(**A5.3**) measures decisiveness in a unanimity game (see definition 3.1). Carreras (2004) used the form of (**A5.3**) when $S = N$. More precisely, the Carreras version of (**A5.3**) is given by:

(**A5.3***) **Unanimity property:** $D(N; U_N) = 1/2^{|N|}$.

(**A5.4**) was suggested by Carreras (2004).

(**A5.5**) has a similar spirit as (**A5.4**). (**A5.3**)-(**A5.5**), which specify values of the decisiveness index in particular cases, can be viewed as giving 'initial conditions' to the decisiveness index.

The final axiom, which is similar in nature to an axiom of Nowak and Radzik (2000), is concerning the change of decisiveness in a unanimity game under merger of two voters (see also Lehrer (1988)). To understand **(A5.6)** consider two unanimity games with number of voters being 1 and k $(>> 1)$ respectively. Consider the specific situation $S = N$. Clearly**,** while in the former, one individual enjoys the capability of making the coalition wining, in the latter it is shared by many individuals. It is therefore reasonable to expect that the former demonstrates a higher extent of power to act than the latter **.**We can provide similar explanations for other situations. In view of this we can argue that decisiveness increases under a merger of two voters in a unanimity game and **(A5.6)** gives a formulation along this line. (See also Barua, Chakravarty, Roy and Sarkar (2004) for a similar discussion.).

Carreras (2004) showed that the only decisiveness index satisfying **(A5.1*)**, **(A5.2)** and **(A5.3*)** is the CC index. The following theorem demonstrates equivalence of **(A5.1*), (A5.2)** and **(A5.3*)** with several seemingly unrelated axioms including **(A5.6)**.

<u>**Theorem 5.1:**</u> Let $D : \mathbf{SG} \rightarrow \mathbf{R}_+$ be a decisiveness index. Then the following conditions are equivalent.

(a) D is the Carreras-Coleman index given by (5.1).

(b) D satisfies **(A5.1)**, **(A5.4)** and **(A5.6).**

(c) D satisfies **(A5.1*)**, **(A5.4)** and **(A5.6).**

(d) D satisfies **(A5.1)**, **(A5.5)** and **(A5.6).**

(e) D satisfies **(A5.1*)**, **(A5.3*)** and **(A5.2).**

<u>**Proof:**</u>

(a) $\Rightarrow$ **(b):** We first show that CC satisfies (**A5.1)**. Let $G_1 = (N_1;V_1), G_2 = (N_2;V_2) \in \mathbf{SG}$. By definition $S \subseteq N_1 \cup N_2$ is winning in $G_1 \vee G_2$ if and only if $S \cap N_1 \in \mathbf{W}(G_1)$ or $S \cap N_2 \in \mathbf{W}(G_2)$, where $\mathbf{W}(G_i)$ is the set of all winning coalitions in $G_i, i = 1,2$. Hence we can write $\mathbf{W}(G_1 \vee G_2)$, the family of all winning coalitions in $G_1 \vee G_2$, as

$$\mathbf{W}(G_1 \vee G_2) = \mathbf{W}_1 \cup \mathbf{W}_2 , \tag{5.3}$$

where, $\mathbf{W}_1 = \{S_1 \cup S_2 : S_1 \subseteq N_1, S_2 \subseteq N_2 - N_1 \text{ and } S_1 \in \mathbf{W}(G_1)\}$,

$\mathbf{W}_2 = \{S_1 \cup S_2 : S_1 \subseteq N_1 - N_2, S_2 \subseteq N_2 \text{ and } S_2 \in \mathbf{W}(G_2)\}$.

Clearly, $\mathbf{W}(G_1 \wedge G_2) = \mathbf{W}_1 \cap \mathbf{W}_2$. (5.4)

Hence, by Inclusion-Exclusion Principle,

$$|\mathbf{W}(G_1 \vee G_2)| = |\mathbf{W}_1| + |\mathbf{W}_2| - |\mathbf{W}_1 \cap \mathbf{W}_2|$$

$$= |\mathbf{W}(G_1)| 2^{|N_2 - N_1|} + |\mathbf{W}(G_2)| 2^{|N_1 - N_2|} - |\mathbf{W}(G_1 \wedge G_2)| . \tag{5.5}$$

Therefore,

$$\frac{|\mathbf{W}(G_1 \vee G_2)|}{2^{|N_1 \cup N_2|}} = \frac{|\mathbf{W}(G_1)|}{2^{|N_1|}} + \frac{|\mathbf{W}(G_2)|}{2^{|N_2|}} - \frac{|\mathbf{W}(G_1 \wedge G_2)|}{2^{|N_1 \cup N_2|}} ,$$

or, $CC(G_1 \vee G_2) = CC(G_1) + CC(G_2) - CC(G_1 \wedge G_2)$,

which on rearrangement gives

$$CC(G_1 \vee G_2) + CC(G_1 \wedge G_2) = CC(G_1) + CC(G_2). \qquad (5.6)$$

Thus, CC verifies (**A5.1**). Carreras (2004, Theorem 4.4) showed that CC meets (**A5.4).**

Next, let $G = (N, U_S)$ and let $G' = (N'; U_{S'})$ be the merged game obtained from G by merging any two voters $i, j \in S$, where G' is given by definition 1.17. Hence $CC(G) = \frac{2^{|N-S|}}{2^{|N|}} = \frac{1}{2^{|S|}}$ and $CC(G') = \frac{1}{2^{|S|-1}}$, so that $CC(G') = 2CC(G)$, which demonstrates fulfillment of (**A5.6**) by CC.

(b) $\Rightarrow$ **(c) :** Any decisiveness index satisfying **(A5.1)** will satisfy **(A5.1*)** as well.

(c) $\Rightarrow$ **(d):** Clearly, **(A5.4)** implies **(A5.5)**. To check satisfaction of **(A5.1)**, consider the two games $G_1 = (N_1; V_1), G_2 = (N_2; V_2) \in$ **SG**. Let $N = N_1 \cup N_2$. Consider the game $G_i' = (N; V_i')$ defined by $V_i'(S) = V_i(N_i \cap S)$, $S \subseteq N$, $i = 1, 2$. Clearly, $G_1' \vee G_2' = G_1 \vee G_2$ and $G_1' \wedge G_2' = G_1 \wedge G_2$. Therefore,

$D(G_1 \vee G_2) = D(G_1' \vee G_2')$

$= D(G_1') + D(G_2') - D(G_1' \wedge G_2')$ (by (**A5.1***))

$= D(G_1') + D(G_2') - D(G_1 \wedge G_2)$.

To complete the proof, we need to show that $D(G_1') = D(G_1)$ and $D(G_2') = D(G_2)$. We first show that **(A5.3)** holds. Suppose $G = (N; U_S)$ is a unanimity game, where $|S| = k$. Under successive application of merger $(k-1)$ times, from G we generate a game $\hat{G}$ with a dictator. Then applying **(A5.6)** repeatedly we get $D(G) = \frac{1}{2^{k-1}} D(\hat{G})$, which by **(A5.4)** becomes

$D(G) = \frac{1}{2^{k-1}} \cdot \frac{1}{2} = \frac{1}{2^k} = \frac{1}{2^{|S|}}$, which is **(A5.3)**.

Now, suppose $G_1' = (N;U_{S_1}) \vee (N;U_{S_2}) \vee .... \vee (N;U_{S_l})$**,** where S_i's $(i = 1,2,...,l)$ are minimal winning coalitions in G_1'. Note that each $S_i \subseteq N_1$ and hence is minimal winning in G_1. Also note that $G_1 = (N_1;U_{S_1}) \vee (N_1;U_{S_2}) \vee .... \vee (N_1;U_{S_l})$**.** If $l = 1$, then $S_1 \subseteq N_1$ is a minimal winning coalition in G_1 and $G_1 = (N_1;U_{S_1})$. Therefore by **(A5.3)**, $D(G_1') = \frac{1}{2^{|S_1|}} = D(G_1)$. So assume $l > 1$ and the result for $l' < l$. Then

$$D(G_1') = D(N;U_{S_1}) + D[(N;U_{S_2}) \vee ... \vee (N;U_{S_l})] - D[(N;U_{S_1 \cup S_2}) \vee ... \vee (N;U_{S_1 \cup S_l})]$$

(by (**A5.1***))

$$= D(N_1;U_{S_1}) + D[(N_1;U_{S_2}) \vee ... \vee (N_1;U_{S_l})] - D[(N_1;U_{S_1 \cup S_2}) \vee ... \vee (N_1;U_{S_1 \cup S_l})]$$

(By induction hypothesis)

$$= D[(N_1;U_{S_1}) \vee (N_1;U_{S_2}) \vee .... \vee (N_1;U_{S_l})]$$ (by **(A5.1*))**

$$= D(G_1).$$

Similarly, we can show that $D(G_2') = D(G_2)$. Thus, we have **(A5.1).**

(d) $\Rightarrow$ **(e):** Clearly**, (A5.1)** implies **(A5.1***). To obtain **(A5.2)**, let $d \in N$ be a null player in the game $G = (N;V) \in \mathbf{SG}$. Let $G_1 = G_{-\{d\}}$ (the game obtained from G by eliminating the null player d from the game) and let another game G_2 be such that $\mathbf{W}(G_2) = \{S \cup \{d\} : S \in \mathbf{W}(G_1)\}$.

Evidently, $G_1 \vee G_2 = G$ and $G_1 \wedge G_2 = G_2$.

Therefore, $D(G) = D(G_1 \vee G_2) = D(G_1) + D(G_2) - D(G_1 \wedge G_2)$

$= D(G_{-\{d\}}) + D(G_2) - D(G_2) = D(G_{-\{d\}})$. Thus we have **(A5.2).**

We shall obtain **(A5.3*)** by induction on $|N|$. If $|N| = 1$, then by **(A5.5)** $D(N;U_N) = \frac{1}{2} = \frac{1}{2^{|N|}}$**.** Now, suppose $|N| > 1$ and let us assume the result for all unanimity games $(N;U_{\overline{N}})$, such that $|\overline{N}| < |N|$. Let $(N';U_{N'})$ be obtained from $(N;U_N)$ by merging two voters. Then by **(A5.6),** $D(N;U_N) = \frac{1}{2} D(N';U_{N'})$, which by induction hypothesis becomes $D(N;U_N)=\frac{1}{2}.\frac{1}{2^{|N'|}}=\frac{1}{2^{|N|}}$. Hence **(A5.3*)** holds.

(e) $\Rightarrow$ **(a):** This part of the proof follows from theorem 4.3 of Carreras (2004).

Since (**A5.1***), (**A5.3***) and (**A5.2**) characterize the CC index, theorem 5.1 says that each of the three axiom sets, as described by **(b)**, **(c)** and **(d)**, will also characterize the CC index.

Remark: If a game $G=(N;V)$ is decisive, then the Carreras-Coleman decisiveness index takes the value $\frac{1}{2}$ and the properties discussed above become very straightforward.

In the following theorem we show independence of (**A5.1**), (**A5.4**) and (**A5.6**). Independence says that none of these axioms implies or is implied by a second one. Demonstration of independence will require that if one of them is dropped, then there will exist a decisiveness index other than the CC index that will satisfy the remaining axioms but not the dropped one.

Theorem 5.2: Axioms (**A5.1**), (**A5.4**) and (**A5.6**) are independent.

Proof: Consider the decisiveness indices given by

$$D_1(G)=\frac{a|\mathbf{W}(G)|}{2^{|N|}}, \text{ where, } a>0,\ a\neq 1, \text{ is a constant,} \tag{5.7}$$

$$D_2(G)=\frac{|\mathbf{W}(G)|}{2^{|N|+1}}+\frac{1}{4}, \tag{5.8}$$

$$D_3(G)=\frac{1}{|Y|}\sum_{i\in Y}\frac{|\mathbf{W}^i(G)|}{2^{|N|}}, \tag{5.9}$$

where Y is the set of all non-null voters and $\mathbf{W}^i(G)$ is the set of all winning coalitions containing i.

D_1 is a violator of (**A5.4**) but not of (**A5.1**) and (**A5.6**). On the other hand, D_2 violates (**A5.6**), but not the other two axioms. It is easy to show that D_3 satisfies

(**A5.4**). To show that D_3 satisfies (**A5.6**), consider the game $G = (N, U_S)$. Let $G' = (N'; U_{S'})$ be the game obtained G by merging $i, j \in S$. Now, $D_3(G) = \sum_{i \in Y} \frac{|\mathbf{W}^i(G)|}{|Y| 2^{|N|}} = \sum_{i \in S} \frac{|\mathbf{W}^i(G)|}{|S| 2^{|N|}}$, since S is the set of non-null voters. For each $i \in S$, $|\mathbf{W}^i(G)| = 2^{|N-S|}$. Hence $D_3(G) = \sum_{i \in S} \frac{2^{|N-S|}}{|S| 2^{|N|}} = \sum_{i \in S} \frac{1}{|S| 2^{|S|}} = \frac{|S|}{|S| 2^{|S|}} = 1/2^{|S|}$.

Similarly, $D_3(G') = 1/2^{|S'|} = 2/2^{|S|} = 2D_3(G)$. Hence D_3 satisfies (**A5.6**). If D_3 satisfies (**A5.1**), then it coincides with CC on all unanimity games and hence on all voting games, as can be argued by induction on the number of minimal winning coalitions. Hence D_3 does not meet (**A5.1**).

Clearly, theorem 5.2 holds if we replace **(A5.1)** by **(A5.1*).** Hence (**A5.1***), (**A5.4**) and (**A5.6**) are independent.

Finally in the next theorem we show that the axioms (**A5.1**), (**A5.5**) and (**A5.6**) are independent.

Theorem 5.3: Axioms (**A5.1**), (**A5.5**) and (**A5.6**) are independent.

Proof: We note that D_1 in (5.7) violates (**A5.5**) and D_2 in (5.8) violates (**A5.6**) and each satisfies respective complement of the axiom that it violates from the universal set {(**A5.1**), (**A5.5**), (**A5.6**)}. It is easy to check that D_3 fulfils (**A5.5**). As already shown, D_3 does not satisfy (**A5.1**) but satisfies (**A5.6**).

5.3 Conclusion

The Carreras-Coleman decisiveness index is an indicator of collective decision making power in a voting game. This chapter shows that a set of axioms used by Carreras (2004) for characterization of the index is equivalent to three different sets of axioms and also establishes independence between the axioms in each of the three sets.

CHAPTER 6

DISTRIBUTION OF POWER IN THE INDIAN LOK SABHA

6.1 Introduction

In this chapter, we briefly examine, in terms of voting power, the results of the elections to the Lower House of the Indian Parliament (Lok Sabha), held between the years 1951 and 2004. More precisely, we investigate how the powers of different major national political parties have changed over the years.

India is a republic, which has adopted a system of multiparty parliamentary democracy after its freedom from the British rule in 1947. The Parliament consists of the President, the Lok Sabha (House of the People) and the Rajya Sabha (Council of States). The Lok Sabha is composed of representatives of the people chosen by direct election on the basis of adult suffrage and hence it is the most important legislative body in the country. The Constitution puts a limit on the size of the Lok Sabha at 550 elected members, apart from two members who can be nominated by the President to represent the Anglo-Indian community, if in his opinion that community is not adequately represented in the House. At present the country is divided into 543 parliamentary constituencies, the size and shape of which are determined by an independent Delimitation Commission, which aims to create constituencies that have more or less the same number of electorate, subject to geographical considerations and boundaries of the states and administrative areas. Each constituency returns one representative to the Lok Sabha. The members owe their allegiance to the various registered political parties. The party whose members occupy a simple majority of the total number of seats in the Lok Sabha forms the government at the Centre. However, in times when a single party fails to acquire a majority, some like minded parties may form a post-poll alliance and can stake claim to form a government if their total number of seats fulfils the simple majority criterion. Thus, the governing parties may be a minority, when considered separately, in the Lok Sabha, but they can govern as long as they have the support of a majority of the members of the Lok Sabha. At present the number of members in the Lok Sabha is 543.

The passage of most bills in the Lok Sabha requires a simple majority. So the voting process in the Lok Sabha can be viewed as a weighted majority game (see definition 1.12), where the decision rule is 'simple majority' and the players are the different political parties with the number of seats they occupy in the Lok Sabha, as their respective weights. Using this framework, we find how the a-priori power of major political parties has changed over the years. Here, the a-priori power of a voter is indexed by the a-priori probability that the voter is in a position to change the outcome of the voting process by changing his vote. The term "a-priori" has been used to indicate that what we get by this methodology is not the actual power but just an indication of what the actual power could be. The reason is that in this methodology we neglect any affinities or animosity between voters etc. Here all possible coalitions of voters are regarded as equi-probable, whereas had we considered the prior affinities between voters, some of these coalitions might have very low probability of being formed and others high. The indices that we have used to measure the a-priori power of the different parties represented in the Lok Sabha are the Coleman indices of the power to prevent action and initiate action (Coleman (1971)), the Banzhaf normalized and non-normalized indices (Banzhaf, (1965)) (see section 1.2.2). There has been no ad hoc reason for avoiding the Shapley-Shubik index. The Shapley-Shubik index has been widely used in many application works in the literature, and it could have been calculated in this case as well. But, because throughout the thesis we have concentrated on the Banzhaf and the Coleman indices, so for the sake of continuity, we have avoided using the Shapley-Shubik index. The construction of all the above indices depends upon the concept of the swing or pivotal voter (see definition 1.8). We have also studied how the decisiveness for the Lok Sabha has changed after each Lok Sabha election. For this we have employed the Carreras-Coleman decisiveness index (Coleman (1971), Carreras (2004)), which is a quantification of the extent to which the Lok Sabha is able to control the outcome of a division of it (see section 1.3). To calculate all the above power indices we have used the program 'ipgenf' available on Dennis Leech's web site, http://www.warwick.ac.uk/~ecaae/#Progam_List.

The chapter is arranged as follows. In section 6.2 we present some definitions, which we will require in our analysis (apart from those introduced in section 1.2.1). We also briefly discuss the various power indices that we have used, in this section. In section 6.3 we present the results of the analysis. Section 6.4 presents some theoretical results that justify some of the findings. Finally section 6.5 concludes the chapter.

6.2 Some more definitions

In this section we will introduce some definitions that are required to carry out our analysis.

In definition 1.12, in section 1.2.1, we introduced a special type of voting game called the weighted majority game. In that definition the weights were required to be non-negative. But in our present analysis, since the weights represent the number of seats that each political party wins in the Lok Sabha, we modify the definition of a weighted majority game to suit our context, by imposing a restriction that all weights are strictly positive integers. We rule out zero weights because a party that fails to win a single seat has no representation in the Lok Sabha, has no direct influence on the decision making process and hence is a null player. Thus we have,

Definition 6.1: For a set of voters $N = \{1, 2, ..., n\}$, a weighted majority game with integral weights is a quadruplet $G = (N; V; \mathbf{w}; q)$, where $\mathbf{w} = (w_1, w_2, ..., w_n)$ is the vector of strictly positive integer weights of the $n = |N|$ voters in N, q is a strictly positive integer quota such that $q \leq \sum_{i=1}^{n} w_i$ and for any $S \in 2^N$,

$$V(S) = 1 \quad \text{if} \quad \sum_{i \in S} w_i \geq q$$

$$= 0 \text{ otherwise.}$$

As in the earlier case, $G=(N;V;\mathbf{w};q)$ will be proper or improper according as $\sum_{i=1}^{n} w_i < 2q$ or $\sum_{i=1}^{n} w_i \geq 2q$.

Let us use the notation w to denote $\sum_{i=1}^{n} w_i$.

Definition 6.2: A weighted majority game is said to be governed by the decision rule "simple majority", if the quota $q=\overline{w}$, where we define $\overline{w}$ as the smallest integer strictly greater than $\frac{w}{2}$.

Note that a game, which is governed by simple majority rule, is always proper (see the discussion following definition 1.3).

After these definitions, it would be proper to briefly discuss the indices that we use in this analysis.

1. **Banzhaf non-normalized index (*BZNN*)** (Banzhaf, 1965)**:** The Banzhaf non-normalized index of player i is defined as the number of winning coalitions in which i is pivotal, divided by the maximal value that this number can take. Formally,

   $BZNN_i = \frac{m_i(G)}{2^{|N|-1}}$. (Also see sections 1.2.2 and 3.2 for detailed discussion.)

2. **Banzhaf normalized index (*BZ*):** The Banzhaf normalized index of player i is defined as,

$$BZ_i = \frac{m_i(G)}{\sum_{i=1}^{|N|} m_i(G)}.$$

It is a derivative of the non-normalized Banzhaf index, obtained by rescaling $BZNN$ such that the sum of this index across the voters is 1. (Also see section 1.2.2.)

3. Coleman index of the power to prevent action (*P*) (Coleman 1971)**:** The Coleman preventive power index for voter i is defined as the number of winning coalitions in which i is pivotal, divided by the total number of winning coalitions in the game. Formally, in a game G, voter i's power to prevent action is calculated as

$$P_i = \frac{\sum_{\substack{S\subseteq N \\ i\in S}} \left[V(S) - V(S\setminus\{i\})\right]}{\sum_{S\subseteq N} V(S)} = \frac{m_i(G)}{\left|\mathbf{W}(G)\right|}.$$

(Also see sections 1.2.2 and 2.2 for detailed discussion.)

4. Coleman index of the power to initiate action (*I*) (Coleman, 1971)**:** The Coleman initiative power index for voter i is defined as the number of losing coalitions outside which i is critical divided by the number of losing coalitions in the game. Formally, voter i's power to initiate action is calculated as

$$I_i = \frac{\sum_{\substack{S\subseteq N \\ i\notin S}} \left[V(S\cup\{i\}) - V(S)\right]}{\sum_{S\subseteq N} [1 - V(S)]} = \frac{m_i(G)}{\left|\mathbf{L}(G)\right|} = \frac{m_i(G)}{2^{|N|} - \left|\mathbf{W}(G)\right|}.$$

(Also see sections 1.2.2 and 2.2 for detailed discussion.)

5. Carreras-Coleman decisiveness index (*CC*) (Coleman (1971), Carreras (2004))**:** The Carreras-Coleman decisiveness index for a voting body under a given decision rule (specified in the game G) is given by

$$CC(G) = \frac{\left|\mathbf{W}(G)\right|}{2^{|N|}}.$$

(Also see sections 1.3 and 5.2 for detailed discussion.)

After having recalled the power indices that we use in our analysis, we will now formally present our results.

6.3 The Results for the Lok Sabha elections (1989-2004)

In this section, we first briefly describe the Indian political scene, and then formally present the result of our analysis.

6.3.1 The Indian Political Scene

The current Indian political scenario is highly interesting and full of action, as every other day some new party is born, some existing parties split, while some others merge to form a new identity. The oldest national party in India is the Indian National Congress (INC). It was established in 1885 as a pro-British Indian organization. Later on, it became the main voice of India's freedom struggle. After India's independence in 1947, the British passed the administration of India to the leaders of the Indian National Congress. The first election to the Lok Sabha was held in 1951, in which the INC expectedly won a majority of the seats. Until 1966 the Congress was a stable party. In 1966 Indira Gandhi became the leader of the Congress and Prime Minister of India. However, it was from this period that the Congress started losing its stability. In 1969 the Congress split and some Congress leaders established a new party. But still INC remained the most significant party of India. It won a majority of the seats in the Lok Sabha in all the elections till 1971. INC lost the 1977 elections to the Janata Party. Janata Party, which was in fact a conglomeration of different parties, was the first political party in India to establish a non-Congress government when it won the 1977 elections. However, this government lost its majority after 30 months and later on it almost vanished from the political arena. Different factions of the Janata Party broke away from it and established their own parties. Among these parties were Jan Sangh which later on was renamed Bhartiya Janata Party (BJP), a prominent player in the Indian political scene today. The Indian National Congress again won majority of the seats in the 1980 and 1984 elections. But the political scene started changing after the 1989 elections. The trend of a single party winning a majority of the seats in the Lok Sabha came to a halt. Small regional parties started to become important players in the national political scene, as they held the key as to who forms the government at the Centre.

6.3.2 The Results

In this section we want to examine numerically how the political scenario in India has changed over time. To simplify our analysis we assume that the representation of the different parties in the Lok Sabha reflects the political sentiments and priorities of the people of India. Under this presumption, in order to demonstrate how the political loyalties of the population of India have changed over the years, we study how the a priori power of the different political parties, as measured by the voting power indices, has changed over the years. As we have already noted above, in India, a new political party is born almost every day. Therefore, the list of parties represented in the Lok Sabha changes after almost every election. So a study of the change in power of all the different political parties is a near impossible task. We therefore restrict our focus on a few parties (like the INC and the BJP) that make it to the Lok Sabha every time.

The nature of political parties in India is very diverse. While some parties are truly 'national' parties, with nationwide network, and a presence in almost every state in India (for e.g., INC), others are 'local' parties, whose presence is restricted to a certain geographical area and which mostly represent the concerns of a certain region (for e.g., TDP etc). In the years following independence (except 1977), INC always had a majority in the Lok Sabha. Its 'yes' vote was necessary as well as sufficient to pass a bill in the Lok Sabha. Thus it held a monopoly or a 'dictatorial' position in Indian national politics. The values of all the indices that we have studied, viz. the non-normalized Banzhaf index, the normalized Banzhaf index and the Coleman indices of the power to prevent and initiate action were identical and equal to 1. This is obvious since here the analysis was same as in the case when there is a dictator in the game. The other political parties were just null players. However, this changed after 1989. Small regional parties with local concerns started gaining importance, and 'national' parties could not do without them. Though big 'national' parties still won a lot of seats, their power in national politics (as reflected by their a priori power in the Lok Sabha) started decreasing. The tables 6.2 to 6.7 below give the detailed results for the years 1989 to 2004, indicating the values of the different power indices for the parties represented in the Lok Sabha. The values of the non-normalized Banzhaf index $(BZNN)$ and the Banzhaf normalized index (BZ) have been listed for all the years from 1989-2004. However, as we show later in section 6.4, since the number of seats in the Lok Sabha for the years 1989, 1996, 1998, 1999 and 2004 are odd numbers, the voting games for all these years turn out to be decisive. Therefore the Coleman indices of the power to prevent (P) and initiate (I) action become identical to each other and also equal to $BZNN$ for each party for these years (see section 2.2). Since, there is no distinction between a voter's overall power and the power to prevent and initiate action, the Coleman indices are unable to reveal any additional information regarding the power of the voters for these years. For this reason we have not presented the values of P and I for the years 1989, 1996, 1998, 1999 and 2004. However, in the year 1991, the number of seats in the Lok Sabha was even and consequently the voting game corresponding to this year was not decisive. The power to prevent action was greater than the power to initiate action, and values of $BZNN$ differed marginally from the values of both P and I for all players. Therefore, we have listed the values of P and I along with $BZNN$ and BZ for 1991 (table 6.3). The fact that P and I coincide with $BZNN$ for most years, or differ from it only marginally (as in the year 1991) is quite expected because as Leech (2002d) had pointed out, Coleman indices are effectively equal to each other and equivalent to the non-normalized Banzhaf index when the decision rule is simple majority. However, Leech has also pointed out that the Coleman indices reveal much more information about the power of voters when there is a supermajority voting rule.

Coming back to the tables, we know that the value of $BZNN$ for a particular party, and for a particular election year, gives us the a-priori absolute power that the party has in the Lok Sabha during that tenure. We can therefore use this figure to get an idea of the hold that the party has on the Indian political scene during that period of time. Tracing this value for the INC from 1951 to 2004, we find that just after independence (except 1977), INC's control over Indian politics was more or less absolute, as $BZNN = 1$. But in 1989, the value of $BZNN$ dropped to 0.634765. It declined further to 0.392047 in 1996, 0.232289 in 1998 and 0.151826 in 1999. This shows that INC's monopoly over Indian politics has been slowly but steadily declining. Between 1989 and 1996, the value of

BZNN increased to 0.970046 in 1991. But this was just after Rajiv Gandhi's assassination, and possibly there was a sympathy wave in the entire country. However, after the recently concluded 2004 elections, INC has recovered some lost ground (*BZNN* =0.536821), and has again emerged as the largest party in the Lok Sabha. In contrast, another major player in the Indian political scene since 1989, the BJP has been increasing its influence in the country. From being a null player before 1989, the value of *BZNN* for BJP stood at 0.365235 in 1989. Leaving out 1991, we find that this value has been increasing from 0.607953 in 1996 to 0.767711 in 1998. Their hold on Indian politics was at its peak during 1999 (*BZNN* =0.848174). However, after the 2004 elections, its influence has declined (*BZNN* =0.463179) with the reemergence of INC. Figure 6.1 shows how the values of *BZNN* have changed for INC and BJP over time.

Since *BZ* is a normalized index whereby the sum of values of the index for all voters is 1, we have used the values of *BZ* for a voter as a measure of the voter's share in the cake. Studying the values of *BZ* for INC, we get an idea of the changing priorities of the people. Since INC had its roots in the pre-independence era and was the main voice of India's freedom struggle, peoples' loyalties lay with it in the years following independence. The values of *BZ* for INC was 1 for all elections till before 1989. That means INC enjoyed the entire cake. However, INC's share started falling steadily from 1989. The value of *BZ* for INC in 1989 stood at 0.341694. In the election just following Rajiv Gandhi's assassination (1991), its share of the control over the proceedings of the Lok Sabha reached a high of 80% (*BZ* =0.800964). But since then there has been a steady decline. In 1996 their share in the total power was 17% (*BZ* =0.172309). It fell to 10% in 1998 (*BZ* =0.103392), and 7.9% in 1999 (*BZ* =0.079746). However after the 2004 elections, its share in the total power stood at near 22% (*BZ* =0.221686). BJP has been one of the main parties who have been eating into INC's share of the cake. Their share in the total power has been rising ever since 1989. In 1989 their share in the total power was 19.66% (*BZ* = 0.196606). It rose to 26.7% (*BZ* =0.267203) in 1996, 34% in 1998 (*BZ* =0.341709) and 44.5% in 1999 (*BZ* =0.445499). However after the last elections in 2004, this share dropped to 19% (BZ =0.191275). Figure 6.2 shows how the values of BZ have changed for INC and BJP over the time period 1989-2004. A significant thing to note here is that small regional parties like TDP, which had near zero percent share on the control over the outcomes of the Lok Sabha in the years preceding 1989, are making their presence felt in the national political scenario. In fact in 1999, TDP's share in the total power stood at 5.5% (BZ =0.055735), which is a big number, given the fact that TDP has presence in only one or two states. Thus people are showing some preference towards local parties. INC, which has once again emerged as the largest party in the Lok Sabha after 2004 polls, enjoys only 22% of the total power as against 100% in the years preceding 1989. Thus one thing that is clear from the tables is that the era where one single party formed the government is gone and a new era of 'coalitional politics' has come in, where even small regional parties enjoy control over decisions that effect the entire county's population.

Table 1: Names of parties and their abbreviations

Abbreviation	Party Name
ABLTC	Akhil Bhartiya Lok Tantrik Congress
AC	Arunachal Congress
ADMK[1]	All India Anna Dravida Munnetra Kazhagam
AGP	Asom Gana Parishad
AIFB[2]	All India Forward Bloc
AIIC(S)	All India Indira Congress (Secular)
AIIC(T)	All India Indira Congress (Tiwari)
AIMIM[3]	All India Majlis-E-Ittehadul Muslimeen
AIRJP	All India Rashtriya Janata Party
AITC	All India Trinamool Congress
ASDC[4]	Autonomous State Demand Committee
BBM	Bharipa Bahujan Mahasangha
BJD	Biju Janata Dal
BJP	Bharatiya Janata Party
BNP	Bharatiya Navshakti party
BSP	Bahujan Samaj Party
CPI	Communist Party of India
CPI(ML)(L)	Communist Party of India (Marxist-Leninist) (Liberation)
CPM	Communist Party of India(Marxist)
DMK	Dravida Munnetra Kazhagam
GNLF	Gorkha National Liberation Front
HLD(R)	Haryana Lok Dal (Rastriya)
HMS	Akhil Bhartiya Hindu Mahasabha
HVC	Himachal Vikas Congress
HVP	Haryana Vikas Party
ICS(SCS)	Indian Congress(Socialist- Sarat Chandra Sinha)
IFDP	Indian Federal Democratic Party
INC	Indian National Congress
IND	Independent candidates
INLD	Indian National Lok Dal
IPF	Indian Peoples Front
JD	Janata Dal
JD(G)	Janata Dal (Gujrat)
JD(S)	Janata Dal (Secular)
JD(U)	Janata Dal (United)
JKN	Jammu and Kashmir National Conference
JKPDP	Jammu & Kashmir Peoples Democratic Party
JMM	Jharkhand Mukti Morcha
JP	Janata Party
KCM	Kerala Congress(M)
KCP	Karnataka Congress Party
KEC	Kerala Congress
KEC(M)	Kerala Congress (M)
LJNSP	Lok Jan Shakti Party
LS	Lok Shakti
MADMK	M.G.R.Anna D.M.Kazhagam
MAG	Maharashtrawadi Gomantak

M-COR	Marxist (Co-ordination)
MDMK	Marumalarchi Dravida Munnetra Kazhagam
MNF	Mizo National Front
MPVC	Madhya Pradesh Vikas Congress
MRP	Manipur Peoples Party
MSCP	Manipur State Congress Party
MUL	Muslim League Kerala State Committee
NCP	Nationalist Congress Party
NLP	National Loktantrik Party
NPC	Nagaland Peoples Council
NPF	Nagaland Peoples Front
PMK	Pattali Makkal Katchi
PWPI	Peasants And Workers Party of India
RJD	Rashtriya Janata Dal
RLD	Rashtriya Lok Dal
RPI	Republican Party of India
RPI(A)	Republican Party of India(A)
RSP	Revolutionary Socialist Party
SAD	Shiromani Akali Dal
SAD(M)	Shiromani Akali Dal (Simranjit Singh Mann)
SAP	Samata Party
SDF	Sikkim Democratic Front
SHS	Shivsena
SJP(R)	Samajwadi Janata Party(Rashtriya)
SP	Samajwadi Party
SSP	Sikkim Sangram Parishad
TDP	Telugu Desam
TMC(M)	Tamil Maanila Congress (Moopanar)
TRS	Telangana Rashtra Samithi
UGDP	United Goans Democratic Party
UMFA	United Minorities Front, Assam
WBTC	West Bengal Trinamool Congress

Source: Election Commission of India
(http://www.eci.gov.in)

1 In the election years prior to 1998, ADMK was known as ADK.
2 In the election years prior to 2004, AIFB was known as FBL.
3 In the election years prior to 1996, AIMIM was known as MIM.
4 In the election years prior to 1996, ASDC was known as ADC.

Table 6.2: Values of Power Indices (1989)

Year of election:1989
Total Number of Seats: 529 Quota: 265

Party Name		**No. of seats**		*BZNN*	*BZ*
BJP		**85**		**0.365235**	**0.196606**
CPI		12		0.064772	0.034867
CPM		33		0.134765	0.072544
ICS(SCS)		1		0.004868	0.002621
INC		**197**		**0.634765**	**0.341694**
JD		143		0.365235	0.196606
ADK		11		0.056096	0.030197
BSP		3		0.014722	0.007925
FBL		3		0.014722	0.007925
GNLF		1		0.004868	0.002621
HMS		1		0.004868	0.002621
IPF		1		0.004868	0.002621
JKN		3		0.014722	0.007925
JMM		3		0.014722	0.007925
KCM		1		0.004868	0.002621
M-COR		1		0.004868	0.002621
MAG		1		0.004868	0.002621
MIM		1		0.004868	0.002621
MUL		2		0.009765	0.005257
RSP		4		0.019769	0.010642
SAD(M)		6		0.031544	0.01698
SHS		1		0.004868	0.002621
SSP		1		0.004868	0.002621
TDP		2		0.009765	0.005257
IND		12		0.004868	0.002621

CC= 0.5
Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 1989, Election Commission of India.

Table 6.3: Values of Power Indices (1991)

Year of election: 1991

Total Number of Seats: 534 Quota: 268

Party Name	**No. of seats**	*BZNN*	*BZ*	*P*	*I*
BJP	**120**	**0.029954**	**0.024733**	**0.030014**	**0.029895**
CPI	14	0.027623	0.022808	0.027678	0.027568
CPM	35	0.029954	0.024733	0.030014	0.029895
ICS(SCS)	1	0.001986	0.00164	0.00199	0.001982
INC	**244**	**0.970046**	**0.800964**	**0.971976**	**0.968123**
JD	59	0.029954	0.024733	0.030014	0.029895
JP	5	0.00965	0.007968	0.009669	0.009631
ADK	11	0.022955	0.018954	0.023001	0.02291
AGP	1	0.001986	0.00164	0.00199	0.001982
BSP	3	0.005912	0.004882	0.005924	0.005901
FBL	3	0.005912	0.004882	0.005924	0.005901
JMM	6	0.011208	0.009254	0.01123	0.011186
KCM	1	0.001986	0.00164	0.00199	0.001982
MRP	1	0.001986	0.00164	0.00199	0.001982
MUL	2	0.003968	0.003277	0.003976	0.00396
NPC	1	0.001986	0.00164	0.00199	0.001982
RSP	4	0.007834	0.006469	0.00785	0.007819
SHS	4	0.007834	0.006469	0.00785	0.007819
SSP	1	0.001986	0.00164	0.00199	0.001982
TDP	13	0.026445	0.021835	0.026497	0.026392
ADC	1	0.001986	0.00164	0.00199	0.001982
HVP	1	0.001986	0.00164	0.00199	0.001982
JD(G)	1	0.001986	0.00164	0.00199	0.001982
MIM	1	0.001986	0.00164	0.00199	0.001982
IND	1	0.001986	0.00164	0.00199	0.001982

CC=.998

Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 1991, Election Commission of India.

Table 6.4: Values of Power Indices (1996)

Year of election: 1996

Total Number of Seats: 543 Quota: 272

Party Name		**No. of seats**		*BZNN*	*BZ*
BJP		**161**		**0.607953**	**0.267203**
AIIC(T)		4		0.020125	0.008845
CPI		12		0.060728	0.026691
CPM		32		0.167061	0.073425
INC		**140**		**0.392047**	**0.172309**
JD		46		0.276368	0.121467
SAP		8		0.040338	0.017729
TDP		16		0.081442	0.035795
AIMIM		1		0.005028	0.00221
ASDC		1		0.005028	0.00221
AGP		5		0.025167	0.011061
JMM		1		0.005028	0.00221
MAG		1		0.005028	0.00221
UGDP		1		0.005028	0.00221
HVP		3		0.01509	0.006632
KCP		1		0.005028	0.00221
MUL		2		0.010058	0.004421
KEC(M)		1		0.005028	0.00221
MPVC		1		0.005028	0.00221
SHS		15		0.076236	0.033507
SAD		8		0.040338	0.017729
BSP		11		0.055617	0.024444
SDF		1		0.005028	0.00221
DMK		17		0.086675	0.038095
TMC(M)		20		0.102568	0.04508
SP		17		0.086675	0.038095
FBL		3		0.01509	0.006632
RSP		5		0.025167	0.011061
IND		9		0.005028	0.00221

$CC = 0.5$

Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 1996, Election Commission of India.

Table 6.5: Values of Power Indices (1998)

Year of election: 1998

Total Number of Seats: 543 Quota: 272

Party Name	No. of seats	*BZNN*	*BZ*
BJP	**182**	**0.767711**	**0.341709**
BSP	5	0.027949	0.01244
CPI	9	0.050486	0.022471
CPM	32	0.18738	0.083403
INC	**141**	**0.232289**	**0.103392**
JD	6	0.033562	0.014939
SAP	12	0.067591	0.030085
AIMIM	1	0.005582	0.002484
TDP	12	0.067591	0.030085
AC	2	0.011165	0.00497
UMFA	1	0.005582	0.002484
ASDC	1	0.005582	0.002484
RJD	17	0.096774	0.043074
AIRJP	1	0.005582	0.002484
HLD(R)	4	0.022347	0.009947
HVP	1	0.005582	0.002484
JKN	3	0.016753	0.007457
LS	3	0.016753	0.007457
RSP	5	0.027949	0.01244
MUL	2	0.011165	0.00497
KEC(M)	1	0.005582	0.002484
RPI	4	0.022347	0.009947
PWPI	1	0.005582	0.002484
SHS	6	0.033562	0.014939
MSCP	1	0.005582	0.002484
SAD	8	0.044828	0.019953
AIIC(S)	1	0.005582	0.002484
SDF	1	0.005582	0.002484
MDMK	3	0.016753	0.007457
ADMK	18	0.102733	0.045727
PMK	4	0.022347	0.009947
TMC(M)	3	0.016753	0.007457
JP	1	0.005582	0.002484
DMK	6	0.033562	0.014939
SP	20	0.115022	0.051196
SJP(R)	1	0.005582	0.002484
WBTC	7	0.039187	0.017442
FBL	2	0.011165	0.00497
BJD	9	0.050486	0.022471
IND	6	0.005582	0.002484

CC=0.5

Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 1998, Election Commission of India.

Table 6.6: Values of Power Indices (1999)

Year of election: 1999

Total Number of Seats: 543 Quota: 272

Party name	**No of seats**	*BZNN*	*BZ*
BJP	**182**	**0.848174**	**0.445499**
BSP	14	0.051462	0.02703
CPI	4	0.014669	0.007705
CPM	33	0.119071	0.062542
INC	**114**	**0.151826**	**0.079746**
JD(S)	1	0.003667	0.001926
JD(U)	21	0.076909	0.040396
ABLTC	2	0.007334	0.003852
ADMK	10	0.036706	0.01928
AIMIM	1	0.003667	0.001926
AITC	8	0.029353	0.015418
BBM	1	0.003667	0.001926
BJD	10	0.036706	0.01928
CPI(ML)(L)	1	0.003667	0.001926
DMK	12	0.044069	0.023147
FBL	2	0.007334	0.003852
HVC	1	0.003667	0.001926
INLD	5	0.018338	0.009632
JKN	4	0.014669	0.007705
KEC	1	0.003667	0.001926
KEC(M)	1	0.003667	0.001926
MADMK	1	0.003667	0.001926
MDMK	4	0.014669	0.007705
MSCP	1	0.003667	0.001926
MUL	2	0.007334	0.003852
NCP	8	0.029353	0.015418
PMK	5	0.018338	0.009632
PWPI	1	0.003667	0.001926
RJD	7	0.02568	0.013488
RLD	2	0.007334	0.003852
RSP	3	0.011001	0.005778
SAD	2	0.007334	0.003852
SAD(M)	1	0.003667	0.001926
SDF	1	0.003667	0.001926
SHS	15	0.055196	0.028991
SJP(R)	1	0.003667	0.001926
SP	26	0.095237	0.050023
TDP	29	0.106112	0.055735
IND	6	0.003667	0.001926

CC **=**0.5

Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 1999, Election Commission of India.

Table 6.7: Values of Power Indices (2004)

Year of election: 2004

Total Number of Seats: 543   Quota: 272

Party name	**No of seats**	*BZNN*	*BZ*
INC	**145**	**0.536821**	**0.221686**
BJP	**138**	**0.463179**	**0.191275**
CPM	43	0.260121	0.10742
SP	36	0.200753	0.082903
RJD	24	0.131935	0.054484
BSP	19	0.101567	0.041943
DMK	16	0.085207	0.035187
SHS	12	0.063461	0.026207
BJD	11	0.058096	0.023991
CPI	10	0.052751	0.021784
NCP	9	0.047425	0.019585
JD(U)	8	0.042116	0.017392
SAD	8	0.042116	0.017392
PMK	6	0.031538	0.013024
TDP	5	0.026266	0.010847
TRS	5	0.026266	0.010847
JMM	5	0.026266	0.010847
LJNSP	4	0.021002	0.008673
MDMK	4	0.021002	0.008673
JD(S)	3	0.015746	0.006502
RLD	3	0.015746	0.006502
AIFB	3	0.015746	0.006502
RSP	3	0.015746	0.006502
JKN	2	0.010494	0.004334
AITC	2	0.010494	0.004334
AGP	2	0.010494	0.004334
AIMIM	1	0.005246	0.002167
BNP	1	0.005246	0.002167
JKPDP	1	0.005246	0.002167
IFDP	1	0.005246	0.002167
RPI(A)	1	0.005246	0.002167
MUL	1	0.005246	0.002167
MNF	1	0.005246	0.002167
KEC	1	0.005246	0.002167
NLP	1	0.005246	0.002167
NPF	1	0.005246	0.002167
SDF	1	0.005246	0.002167
SJP(R)	1	0.005246	0.002167
IND	5	0.005246	0.002167

CC=0.5

Source: The data on seat distribution has been collected from The Statistical Report on General Elections, 2004, Election Commission of India.

Figure 6.1

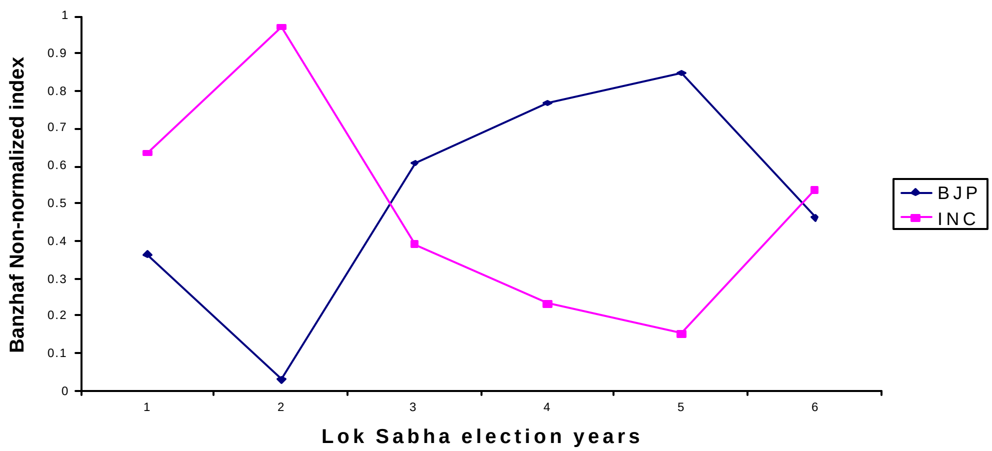
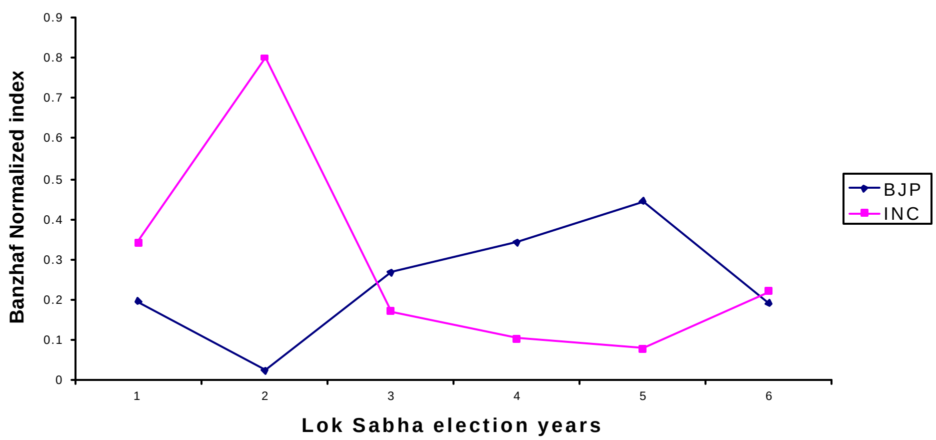


Figure 6.2



On X- axis, we have plotted the Lok Sabha election years from 1989 to 2004.

1 (on the x-axis) represents the year 1989.
2 (on the x-axis) represents the year 1991.
3 (on the x-axis) represents the year 1996.
4 (on the x-axis) represents the year 1998.
5(on the x-axis) represents the year 1999
6 (on the x-axis) represents the year 2004.

6.4. Some theoretical findings

One finding of this analysis is that, the value of the decisiveness index for the Lok Sabha has remained unchanged at 0.5 after all the parliamentary elections except 1991. Note that the maximum value of this index under proper games is 0.5. The result is expected for the years 1951 to 1984-85 because in all these Lok Sabha elections, one party always had the majority seats. In all the Lok Sabha elections held between the years 1951 and 1984-85, INC (barring the 1977 elections) had a simple majority of the seats. Thus, the analysis becomes similar to the case when there is a dictator, and the decisiveness index value is 0.5. However, in the years 1989, 1996, 1998 and 2004 when no single party had a majority in the Lok Sabha, the index value has remained unchanged at 0.5. This result has a very simple explanation which is provided below.

Proposition 6.1: Let $G=(N;V;\mathbf{w};q)$ be a weighted majority game as given in definition 6.1. If the decision rule is "simple majority", and the total number of seats, $\boldsymbol{w}$, is an odd number, then, whatever be the distribution of weights among the players, the game will always be decisive.

Proof: Let N_1 be any subset of N. Consider the bipartition (N_1, N_2) of the player set N. We know that $\sum_{i\in N_1} w_i + \sum_{i\in N_2} w_i = \boldsymbol{w}$. If $\sum_{i\in N_1} w_i > \frac{\boldsymbol{w}}{2}$, then we must have $\sum_{i\in N_2} w_i < \frac{\boldsymbol{w}}{2}$. Similarly, if $\sum_{i\in N_1} w_i < \frac{\boldsymbol{w}}{2}$, then we must have $\sum_{i\in N_2} w_i > \frac{\boldsymbol{w}}{2}$. The situation that both $\sum_{i\in N_1} w_i$ and $\sum_{i\in N_2} w_i$ are equal can never arise. This is because $\frac{\boldsymbol{w}}{2}$ is not an integer, but since w_i's are integers for all $i\in N$, both $\sum_{i\in N_1} w_i$ and $\sum_{i\in N_2} w_i$ are integer numbers. Thus, either $\sum_{i\in N_2} w_i \geq \overline{\boldsymbol{w}} = q$ or $\sum_{i\in N_1} w_i \geq \overline{\boldsymbol{w}} = q$. In other words for the arbitrary bipartition (N_1, N_2) of N, either N_1 or N_2 has to be a winning coalition. Thus for all $S \in 2^N$, $V(S)+V(N-S)=1$. Hence the game is decisive.

We have already noted in chapter 5 that if a voting game is decisive, $|\mathbf{W}(G)| = |\mathbf{L}(G)| = 2^{|N|-1}$, and the Carreras-Coleman decisiveness index takes the value 0.5. Therefore, since in the years 1989, 1996, 1998 and 2004, the number of seats in the Lok Sabha was odd, the value for the decisiveness index remained 0.5, in spite of no single party getting an absolute majority in the Lok Sabha during the years.

In the year 1991 however, the number of seats in the Lok Sabha was even and the value of the decisiveness index was less than 0.5. This is because if w is even, then the game may not be decisive. In this case it is difficult to say anything a-priori about the value of the decisiveness index, unless one single party has an absolute majority. But it is sometimes possible to predict a range in which the index value will lie.

<u>Proposition 6.2</u>: Let $G = (N;V;\mathbf{w};q)$ be a weighted majority game that is governed by the decision rule "simple majority". Let w and the number of players, n, both be even. Then if w is exactly divisible by n, then $0.5 - \frac{{}^{n}C_{n/2}}{2^{n+1}} \leq CC \leq 0.5$.

<u>Proof</u>:

At the outset, let us obtain a partition of the power set of N, according to the cardinalities of the subsets. Thus $2^N = N_0 \cup N_1 \cup ... \cup N_n$, where $N_i = \{S \subseteq N : |S| = i\}$, $(i = 0,1,....,n)$. Obviously, $|N_i| = {}^{n}C_i$.

Let $\mathbf{G}(w,n)$ be the entire class of weighted majority games, governed by the decision rule "simple majority", and which have n as the number of players and w as the sum of weights of all the players.

Now let us begin the proof by considering a game $G_1 = \{N;V;\mathbf{w}_1,q\} \in \mathbf{G}(w,n)$, in which each player has an equal share of the total weights. Thus, $\mathbf{w}_1 = \{w/n, w/n, ..., w/n\}$. It is easy to verify that since the quota is $\frac{w}{2}+1$,

$S(\subseteq N) \in \mathbf{W}(G_1)$ if and only if $|S| \geq n/2 + 1$. (6.1)

Therefore, the total number of winning coalitions in the game G_1 is

$$|\mathbf{W}(G_1)| = \sum_{i=n/2+1}^{n} {}^nC_i .$$

Recalling a very well known result in algebra, $\sum_{i=0}^{n} {}^nC_i = 2^n$, we can easily see that $\sum_{i=n/2+1}^{n} {}^nC_i = 2^{n-1} - \frac{{}^nC_{n/2}}{2}$. Thus the Carreras-Coleman decisiveness index in the game G_1 is

$$\frac{|\mathbf{W}(G_1)|}{2^n} = \frac{2^{n-1} - \frac{{}^nC_{n/2}}{2}}{2^n} = 0.5 - \frac{{}^nC_{n/2}}{2^{n+1}}.$$

Now consider another game $G_2 = \{N; V; \mathbf{w}_2, q\} \in \mathbf{G}(w, n)$, which is derived from the game G_1 by some kind of a redistribution of weights such that w remains unchanged.

We will compare the number of winning coalitions in this game, $|\mathbf{W}(G_2)|$, with $|\mathbf{W}(G_1)|$ and show that $|\mathbf{W}(G_2)| \geq |\mathbf{W}(G_1)|$.

Let $\Delta\mathbf{W}(G_1)$ ($\Delta\mathbf{L}(G_1)$) be the set of coalitions which are winning (losing) in G_1 but losing (winning) in G_2.

Note that we can write the following:

$|\mathbf{W}(G_2)| = |\mathbf{W}(G_1)| - |\Delta\mathbf{W}(G_1)| + |\Delta\mathbf{L}(G_1)|$, where $|\Delta\mathbf{W}(G_1)|$ ($|\Delta\mathbf{L}(G_1)|$) is the number of coalitions which are winning(losing) in G_1 but losing(winning) in G_2.

To show that $|\mathbf{W}(G_2)| \geq |\mathbf{W}(G_1)|$, it will be suffice to show that $|\Delta\mathbf{W}(G_1)| \leq |\Delta\mathbf{L}(G_1)|$.

We shall define a 1-1 map $\Phi : \Delta\mathbf{W}(G_1) \to \Delta\mathbf{L}(G_1)$ as follows. Suppose $S \in \Delta\mathbf{W}(G_1)$. First note that $\sum_{i \in S} w_{2i} \leq \frac{w}{2}$.

If $\sum_{i\in S} w_{2i} < \frac{w}{2}$, then we define $\Phi(S) = N - S$. Since $\sum_{i\in N-S} w_{2i} \geq \frac{w}{2} + 1 = q$, $N - S \in \mathbf{W}(G_2)$. Also since $S \in \mathbf{W}(G_1)$, by (6.1), $|S| \geq \frac{n}{2} + 1$. Hence $|N - S| < \frac{n}{2}$, which by (6.1) implies $N - S \in \mathbf{L}(G_1)$. Thus, $N - S \in \Delta\mathbf{L}(G_1)$, that is, $\Phi(S) \in \Delta\mathbf{L}(G_1)$.

Next, consider the case $\sum_{i\in S} w_{2i} = \frac{w}{2}$. As above, we have, $|S| \geq \frac{n}{2} + 1$. Choose (by some fixed ordering) $S^* \subset S$, such that $|S^*| = \frac{n}{2}$. It is obvious that since the weights are positive integers, $\sum_{i\in S^*} w_{2i} < \frac{w}{2}$.

We now define $\Phi(S) = N - S^*$. Since $|N - S^*| = \frac{n}{2}$, $N - S^* \in \mathbf{L}(G_1)$. However, $\sum_{i\in S^*} w_{2i} < \frac{w}{2}$ implies that $\sum_{i\in N-S^*} w_{2i} > \frac{w}{2}$. Since the weights are positive integers, this in turn implies that $\sum_{i\in N-S^*} w_{2i} \geq \frac{w}{2} + 1 = q$. Therefore, $N - S^* \in \mathbf{W}(G_2)$. Hence $N - S^* \in \Delta\mathbf{L}(G_1)$. That is, $\Phi(S) \in \Delta\mathbf{L}(G_1)$.

It is easy to check that Φ is 1-1. Thus, $\Phi : \Delta\mathbf{W}(G_1) \rightarrow \Delta\mathbf{L}(G_1)$ is 1-1 and so $|\Delta\mathbf{W}(G_1)| \leq |\Delta\mathbf{L}(G_1)|$.

The upper bound of 0.5 is the maximum possible under proper games, since in a proper game the number of winning coalitions is always less than or equal to 2^{n-1}.

6.5 Conclusion

In this chapter, we have examined the results of the elections to the Lower House of the Indian Parliament (Lok Sabha), held between the years 1951 and 2004. More precisely, we have used the methodology of power indices to evaluate voting power and the relative influence of the parties represented in the Lok

Sabha, over the decision making process. We have also studied the decisiveness or the power of the Lok Sabha to act. The power indices that have been analyzed for studying individual voting power are the Coleman index of the power to initiate action, Coleman index of the power to prevent action, Banzhaf normalized and the non-normalized indices. The question of which voting power index is most suitable here is a different issue and needs to be addressed separately. This chapter is more of a numerical illustration of how the indices are actually used in real life.

List of abbreviations

ABP: The Added Blocker Postulate

BOP: The Bloc Postulate

BSP: The Blocker's Share Postulate

DOM: The Dominance Postulate

IGN: Ignoring Null Voters

INV: Iso-invariance

MON: The Monotonicity Postulate

NNP: The Non-Null Postulate

NOM: Normalization

RNP: Relative Null Voter Ignoring Principle

RTP: Relative Transfers Principle

TRP: The Transfer Postulate

VJN: Vanishing just for Null Voter

REFERENCES

1. Akimov, V. and W. Kerby (2000): "An Average Value Function for Cooperative Games", *Homo Oeconomicus* XVII (1/2), 155-164.
2. Albizuri, M. J. (2001): "An axiomatization of the modified Banzhaf-Coleman index", International Journal of Game Theory, 30(2), 167-176.
3. Albizuri, M. J. and L. Ruiz (2001): "A new axiomatization of the Banzhaf semivalue", *Spanish Economic Review*, 3, 97-109.
4. Amer, R., Carreras, F. and A. Magaña (1998): "Extension of values to games with multiple alternatives", *Annals of Operation Research*, 84, 63-78.
5. Amer, R., Carreras, F. and J. M. Giménez (2002): "The modified Banzhaf value for games with coalition structure: an axiomatic characterization", *Mathematical Social Sciences*, 43, 45-54.
6. Banzhaf, J. F. (1965): "Weighted voting doesn't work: A mathematical analysis," *Rutgers Law Review*, 19, 317-343.
7. Barua, R., Chakravarty, S. R. and S. Roy (2004): "On the Coleman indices of voting power", *European Journal of Operational Research*, forthcoming.
8. Barua, R., Chakravarty, S. R. and S. Roy (2004a): "A new characterization of the Banzhaf index of power", *International Game Theory Review*, forthcoming.
9. Barua, R., Chakravarty, S. R. and S. Roy (2004b): A note on the Carreras-Coleman decisiveness index", mimeographed, Indian Statistical Institute, Kolkata.
10. Barua, R., Chakravarty, S. R., Roy, S. and P. Sarkar (2004): "A characterization and some properties of the Banzhaf-Coleman-Dubey-Shapley sensitivity index", *Games and Economic Behavior*, 49(1), 31-48.
11. Bilal, S., Albuquerque, P. and M. O. Hosli (2001): "The probability of coalition formation: spatial voting power indices", mimeographed, European Centre for Development Policy Management, London.

12. Bouissou, J-M. (2001): “Party factions and the politics of coalition: Japanese politics under the system of 1955”, *Electoral Studies*, 20, 581-602.
13. Bolger, E. M. (1986): “Power indices for multicandidate voting games”, *International Journal of Game Theory*, 14, 175-186.
14. Bolger, E. M. (1993): “A value for games with *n* players and *r* alternatives”, *International Journal of Game Theory*, 22, 319-334.
15. Braham, M. and F. Steffen (2002): “Local Monotonicity of Voting Power: A Conceptual Analysis”, Presented at the Workshop of Voting Power Analysis, London School of Economics and Political Science, August 2002.
16. Braham, M. and M. J. Holler (2005): “The impossibility of a preference-based power index”, *Journal of Theoretical Politics*, 17, 137-157.
17. Brams, S.J. (1975): Game Theory and Politics, Free Press, New York.
18. Brams, S.J. and P. J. Affuso (1976): “Power and size: A new paradox”, *Theory and Decision*, 7(1), 29-56.
19. Brams, S.J. and P. C. Fishburn (1995): “When is size a liability? Bargaining power in minimal winning coalitions”, *Journal of Theoretical Politics*, 7, 301-316.
20. Brink, R. and G. Laan (1998): “Axiomatizations of the normalized Banzhaf value and the Shapley value”, *Social Choice and Welfare*, 15(4), 567-528.
21. Brueckner, M. (2000): “Voting Power in the European Central Bank”, *Homo Oeconomicus* XVII (1/2), 57-66.
22. Brueckner, M. (2001): “Extended probabilistic characterization of power indices”, In: Holler, M. J. and G. Owen (Eds), Power Indices and Coalition Formation, Kluwer, Dordrecht. pp. 29-43.
23. Burgin, M. and Shapley, L.S. (2001): “Enhanced Banzhaf power index and its mathematical properties”, WP-797, Department of Mathematics, UCLA.

24. Canteaut, A., Carlet, C., Charpin, P. and C. Fontaine (2000): "Propagarion characteristics and correlation immunity of highly non-linear Boolean functions", In: Advances in Cryptology-Eurocrypt 2000, Lecture Notes in Computer Science, Springer-Verlag, New York, pp. 507-522.
25. Carlet, C. (1992): "Partially bent functions", In: Advances in Cryptology-CRYPTO'92, Lecture Notes in Computer Science, Springer-Verlag, New York, pp. 280-291.
26. Carreras, F. (2004): "A Decisiveness Index for simple games", *European Journal of Operational Research*, forthcoming.
27. Carreras, F., Freixas, J. and M. A. Puente (2003): "Semivalues as Power Indices", *European Journal of Operational Research*, 149, 676-687.
28. Coleman, J.S. (1971): "Control of collectives and the power of a collectivity to act", In: Lieberman, B. (Ed.), Social Choice, Gordon and Breach, New York, pp. 269-298.
29. Deegan, J. and E. W. Packel (1978): "A new index of power for simple *n*-person games", *International Journal of Game Theory*, 7, 113-123.
30. Deegan, J. and E. W. Packel (1982): "To the (minimal winning) victors go the (equally divide) spoils: a new power index for simple *n*-person games, In: Brams, S. J., Lucas, W. F. and P. D. Straffin (Eds), Political and Related Models, Springer Verlag, New York, pp. 239-255.
31. Ding, C., Xiao, G. and W. Shan (1978): "The stability theory of stream ciphers", Lecture Notes in Computer Science, Number 561, Springer-Verlag, New York.
32. Dragan, I. (1996): "New mathematical properties of the Banzhaf value", *European Journal of Operational Research*, 95, 451-463.
33. Dreyer, J. and A. Schotter (1980): "Power relationship in the International Monetary Fund: the consequences of quota change", *Review of Economics and Statistics*, 62, 97-106.
34. Dubey, P. (1975): "On the uniqueness of the Shapley value", *International Journal of Game Theory*, 4, 131-140.

35. Dubey, P. and L.S. Shapley (1979): "Mathematical properties of the Banzhaf power index", *Mathematics of Operations Research*, 4(2), 99-131.
36. Dubey, P., Einy, E. and O. Haimanko (2004): "Compound voting and the Banzhaf Index", *Games and Economic Behavior*, forthcoming.
37. Felsenthal, D. and M. Machover (1995): "Postulates and paradoxes of relative voting power – A critical reappraisal", *Theory and Decision*, 38(2), 195-229.
38. Felsenthal, D. and M. Machover (1997): Ternary voting games", *International Journal of Game Theory*, 26, 335-351.
39. Felsenthal, D. and M. Machover (1997a): "The weighted voting rule in the EU's Council of Ministers, 1985-95: intentions and outcomes", *Electoral Studies*, 16, 33-47.
40. Felsenthal, D. and M. Machover (1998): The Measurement of Voting Power, Edward Elgar, Cheltenham.
41. Felsenthal, D. and M. Machover (1998a): "The product paradox of voting power", *Public Choice*, 96, 81-92.
42. Felsenthal, D. and M. Machover (1999): "Minimizing the mean majority deficit: the second square root rule", *Mathematical Social Sciences*, 37, 25-37.
43. Felsenthal, D. and M. Machover (2000): Enlargement of the EU and weighted voting in the Council of Ministers, LSE Voting Power and Procedures Report VPP 01/00.
44. Felsenthal, D. and M. Machover (2001): "The Treaty of Nice and Qualified Majority Voting", *Social Choice and Welfare*, 18, 431-464.
45. Felsenthal, D. and M. Machover (2002): "Annexations and alliances: when are blocs advantageous a priori", *Social Choice and Welfare*, 19, 295-312.
46. Felsenthal, D. and M. Machover (2004): "Analysis of QM rules in the draft constitution for Europe proposed by the European Convention, 2003", *Social Choice and Welfare*, 23, 1-20.

47. Felsenthal, D., M. Machover and W. S. Zwicker (1998): “The bicameral postulates and indices of a priori voting power”, *Theory and Decision*, 44, 83-116.
48. Feltkamp, V. (1995): “Alternative axiomatic characterizations of the Shapley and Banzhaf values”, *International Journal of Game Theory*, 24, 179-186.
49. Fischer, D. and A. Schotter (1978): “The inevitability of the paradox of redistribution in the allocation of voting weights”, *Public Choice*, 33, 49-67.
50. Freixas, J. and G. Gambarelli (1997): “Common internal properties among power indices”, *Control and Cybernetics*, 26, 591-603.
51. Freixas, J. (2004): “The Dimension for the European Union Council under the Nice rules”, *European Journal of Operational Research*, 156, 415-419.
52. Freixas, J. (2005): “The Shapley-Shubik power index for games with several levels of approval in the input and output”, *Decision Support Systems*, 39, 185-195.
53. Freixas, J. and W. S. Zwicker (2003): “Weighted voting, abstention, and multiple levels of approval”, *Social Choice and Welfare*, 21, 399-431.
54. Haller, H. (1994): “Collusion properties of values”, *International Journal of Game Theory*, 23, 261-281.
55. Hart, S. (1976): “A note on the edges of the *n*-cube”, *Discrete Mathematics*, 14, 157-163.
56. Hart, S. and A. Mas-Colell (1988): “The potential of the Shapley value”, In: Roth, A.E. (Ed), The Shapley Value: Essays in honor of Lloyd S. Shapley, Cambridge University Press, pp.127-137.
57. Hart, S. and A. Mas-Colell (1989): “Potential, value and consistency”, *Econometrica*, 57, 589-614.
58. Hoede, C. and R. Bakker (1982): “A theory of decisional power”, *Journal of Mathematical Sociology*, 8, 309-322.

59. Holler, M.J. (1982): "Forming coalitions and measuring voting power", *Political Studies*, XXX (2), 262-271.

60. Holler, M. J. and E. W. Packel (1983): "Power, luck and the right index", *Journal of Economics*, 43(1), 21-29.

61. Holler, M. J. and X. Li (1995): "From public good index to public value: An axiomatic approach and generalization", *Control and Cybernetics*, 24, 257-270.

62. Holler, M.J. (1997): "Power, monotonicity and expectations", *Control and Cybernetics*, 26, 605-607.

63. Holler, M.J. (1998): "Two stories, one power index", *Journal of Theoretical Politics*, 10, 179-190.

64. Holler, M. J., Ono, R. and F. Steffen (2001): "Constrained monotonicity and the measurement of power", *Theory and Decision*, 50, 385-397.

65. Holler, M. J. and S. Napel (2003): "Monotonicity of power and power measures", mimeographed, University of Hamburg.

66. Hosli, M.O. (1993): "The admission of the European Free Trade Association States to the European Community: effects on voting power in the European Community Council of Ministers", *International Organization*, 47, 629-643.

67. Hosli, M.O. (1995): "The balance between small and large: effects of a double-majority system on voting power in the European Union", *International Studies Quarterly*, 39, 351-370.

68. Hosli, M.O. (1996): "Coalitions and power: effects of qualified majority voting in the Council of the European Union", *Journal of Common Market Studies*, 34, 255-273.

69. Hosli, M.O. (1998): "An institution's capacity to act: what are the effects of majority voting in the Council of the EU and European Parliament?", Current European Issues, European Institute of Public Administration, Maastricht,.

70. Hosli, M.O. (2000): "Coalition Formation and Voting in the Council of the European Union", *Homo Oeconomicus* XVII (1/2), 31-55.

71. Johnston, R.J. (1978): “On the measurement of power: some reactions to Laver”, *Environment and Planning A*, 10, 907-914.

72. Johnston, R.J. (1995): “The conflict over qualified majority voting in the European Union Council of Ministers: an analysis of the UK negotiating stance using power indices”, *British Journal of Political Science*, 25, 245-288.

73. Kalai, E. and D. Samet (1987): “On weighted Shapley values”, *International Journal of Game Theory*, 16, 205-222.

74. Kilgour, D. M. (1974): “A Shapley value for cooperative games with quarreling”, In: Rapoport, A. (Ed), Game Theory as a Theory of Conflict Resolution, Reidel, Boston, 193-206.

75. Laruelle, A. (1999): “On the choice of a power index”, IVIE Discussion paper WP-AD 99-10, Instituto Valenciano de Investigaciones Económicas, Valenciana, Spain.

76. Laruelle, A. and M. Widgrén (1998): “Is the allocation of voting power among the EU states fair?”, *Public Choice*, 94, 317-339.

77. Laruelle, A. and N. Widgrén (2000): “Voting Power in a Sequence of Cooperative Games: The Case of EU Procedures”, *Homo Oeconomicus* XVII (1/2), 67-84.

78. Laruelle, A. and F. Valenciano (1999): “Why Power Indices should be efficient”. Discussion Paper 1/1999, Departmento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.

79. Laruelle, A. and F. Valenciano (2001): “Shapley-Shubik and Banzhaf indices revisited”, *Mathematics of Operational Research*, 26(1), 89-104.

80. Laruelle, A. and F. Valenciano (2001a): ‘ A probabilistic reformulation of power measures”, Discussion Paper, Departmento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.

81. Laruelle, A. and F. Valenciano (2002): “Power indices and the veil of ignorance”, *International Journal of Game Theory*, 31, 331-339.

82. Laruelle, A. and F. Valenciano (2002a): “Assessment of voting situations: the probabilistic foundations”, Discussion Paper 26/2002, Departmento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.
83. Laruelle, A. and F. Valenciano (2002b): “Potential and power of a collectivity to act”, Mimeographed, Departmento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.
84. Laruelle, A. and F. Valenciano (2002c): “Potential, value and probability”, Discussion Paper 27/2002, Departmento de Economía Aplicada IV, Basque Country University, Bilbao, Spain.
85. Laruelle, A. and F. Valenciano (2002d): “Inequality among EU citizens in the EU’s Council decision procedure”, *European Journal of Operational Research*, 18, 475-498.
86. Laruelle, A. and F. Valenciano (2002e): “Some voting power postulates and paradoxes revisited”, Mimeographed, Universidad de Alicante and Universidad del País Vasco.
87. Laruelle, A. and F. Valenciano (2003): “Semivalues and Voting Power”, *International Game Theory Review*, 5(1), 41-61.
88. Leech, D. (1990): “Power indices and probabilistic voting assumptions”, *Public Choice*, 66, 293-299.
89. Leech, D. (2002): “An empirical comparison of the performance of classical power indices”, *Political Studies*, 50, 1-22.
90. Leech, D. (2002a): “Designing the voting system for the Council of the European Union”, *Public Choice*, 113, 437-464.
91. Leech, D. (2002b): “Voting power in the governance of the International Monetary Fund”, *Annals of Operations Research*, 109, 373-395.
92. Leech, D. (2002c): “Shareholder voting power and ownership control of companies”, *Homo Oeconomicus*, XIX (3), 345-373.
93. Leech, D. (2002d): “The use of Coleman’s power indices to inform the choice of voting rule with reference to the IMF governing body and EU Council of Ministers”, Warwick Economics Research Papers, No. 645.

94. Leech, D. (2003): “Power indices as an aid to institutional design: the general apportionment problem”, In: Holler, M. J., Kliemt, H., Schmidtchen, D. and M. Streit (Eds), Yearbook on New political Economy.

95. Leech, D. and M. Manjon (2002): “Corporate governance in Spain (with an application of the power indices approach)”, *European Journal of Law and Economics*, 13, 157-173.

96. Lehrer, E. (1988): “An axiomatization of the Banzhaf value”, *International Journal of Game Theory*, 17, 89-99.

97. Lindner, I. (2002): “Voting Power in Weighted (j,k) Games: A Limit Theorem and a Numerical Method”, Presented at the Workshop of Voting Power Analysis, London School of Economics and Political Science, August 2002.

98. MacWilliams, F.J. and N. J. A. Sloane (1977): The Theory of Error Correcting Codes, North Holland, Amsterdam.

99. Malawski, M. (2002): “Equal treatment, symmetry and Banzhaf value axiomatizations”, *International Journal of Game Theory*, 31, 47-67.

100. Mercik, J. W. (2000): “Index of Power of Cabinet”, *Homo Oeconomicus* XVII (1/2).

101. Muto, S. (1999): “The Banzhaf index in representative systems with multiple political parties”, *Games and Economic Behaviour*, 28, 73-104.

102. Myerson, R. B. (1980): “Conference structures and fair allocation rules”, *Journal of Game Theory*, 9(3), 169-182.

103. Napel, S. and M. Widgrén (2001): “Inferior players in simple games”, *International Journal of Game Theory*, 30, 209-220.

104. Nowak, A. S. and T. Radzik (1995): “On axiomatizations of the weighted Shapley values”, *Games and Economic Behavior*, 8, 389-405.

105. Nowak, A. S. (1997): “On an axiomatization of the Banzhaf value without the additivity axiom”, *International Journal of Game Theory*, 26, 137-141.

106.Nowak, A. S. and T. Radzik (2000): “An alternative characterization of the weighted Banzhaf value”, *International Journal of Game Theory*, 29, 127-132.

107.Nurmi, H. (1997): “The Representation of Voter Groups in the European Parliament: A Penrose-Banzhaf Index Analysis”, *Electoral Studies*, 16, No. 3, 317-327.

108.Nurmi, H. (1982): “The problem of the right distribution of voting power”, in Holler, M. J. (ed), Power, voting and voting power, Physica, Würzburg.

109.Nurmi, H. and T. Meskanen (1997): “A priori power measures and the institutions of the European Union”, *European Journal of Political Research,* 35, 161-179.

110.Nurmi, H., Meskanen, T. and A. Pajala (2000): “Calculus of Consent in the EU Council of Ministers”, *Homo Oeconomicus* XVII (1/2).

111.Ono, R. (2000): “Values for Multialternative Games and Multilinear Extensions”, *Homo Oeconomicus* XVII (1/2), 193-214.

112.Owen, G. (1972): “Multilinear extensions of games”, Management Science, 18, 64-79.

113.Owen, G. (1975): “Multilinear extensions and the Banzhaf value”, *Naval research Logistics Quarterly*, 22, 741-750.

114.Owen, G. (1978): “Characterization of the Banzhaf- Coleman index”, *SIAM Journal of Applied Mathematics*, 35, 315-327.

115.Owen, G. (1982): ‘Modification of the Banzhaf-Coleman index for games with a priori unions”, In: Holler, M. J. (Ed), Power, Voting and Voting Power, Physica-Verlag, Würzburg, pp. 232-238.

116.Owen, G. (1995): Game Theory, 3rd edition, Academic Press, New York.

117.Owen, G. (1988): “Multilinear extensions of games”, In: Roth, A.E. (Ed), The Shapley Value: Essays in honor of Lloyd S. Shapley, Cambridge University Press, pp.139-151.

118. Owen, G. and L. S. Shapley (1989): “Optimal location of candidates in ideological space”, *International Journal of Game Theory*, 18(3), 339-356.

119. Penrose, L. S. (1946): “The elementary statistics of majority voting”, *Journal of the Royal Statistical Society*, 109, 53-57.

120. Perlinger, T. (2000): “Voting power in an ideological spectrum-The Markov-Pólya index”, *Mathematical Social Sciences*, 40, 215-226.

121. Preneel, B. (1993): Analysis and design of cyptographic hash functions, Doctoral Dissertation, K.U. Leuven.

122. Radzik, T., Nowak, A. S. and Theo S. H. Driessen (1997): “Weighted Banzhaf values”, *Mathematical Methods of Operations Research*, 45, 109-118.

123. Rae, D. (1969): “Decision rules and individual values in constitutional choice”, *American Political Science Review*, 63, 40-56.

124. Roth, A. E. (1977): “Utility functions for simple games”, *Journal of Economic Theory*, 16, 481-489.

125. Roth, A. E. (1988): “The expected utility of playing a game”, In: Roth, A.E. (Ed), The Shapley Value: Essays in honor of Lloyd S. Shapley, Cambridge University Press, pp.51-70.

126. Rusinowska, A. and H. de Swart (2002): “On some properties of the Hoede-Bakker Index”, Presented at the Workshop of Voting Power Analysis, London School of Economics and Political Science, August 2002.

127. Saari, D. G. and K. K. Sieberg (2000): “Some surprising properties of power indices”, *Games and Economic Behavior*, 36, 241-263.

128. Sanchez S, F. (1997): “Balanced Contributions Axiom in the Solution of Cooperative Games”, *Games and Economic Behavior*, 20, 161-168.

129. Sarkar, P. and S. Maitra (2002): “Cross-correlation analysis of cryptographically useful boolean functions and S-boxes”, *Theory of Computing Systems*, 35, 39-57.

130. Schotter, A. (1982): “The paradox of redistribution: some theoretical and empirical results”, In: Holler, M. J. (Ed), Power, Voting and Voting Power, Physica, Würzburg, pp. 324-338.
131. Shapley, L.S. (1953): “A value for n-person games”, In: Kuhn, H. W. and A.W. Tucker (Eds.), Contributions to the theory of games II, Annals of Mathematics studies, Princeton University Press, Princeton, pp. 307-317.
132. Shapley, L.S. (1953a): “Additive and non-additive set functions”, Ph.D. thesis, Department of Mathematics, Princeton University.
133. Shapley, L.S. and M.J. Shubik (1954): “A method for evaluating the distribution of power in a committee system”, *American Political Science Review*, 48, 787-792.
134. Shapley, L.S. (1962): “Simple games: an outline of the descriptive theory”, *Behavioral Science*, 7, 59-66.
135. Shapley, L.S. (1973): “Political science: voting and bargaining games”, Selby, H. A. (Ed), Notes of Lectures on Mathematics in the Behavioral Sciences, Mathematical Association of America, pp. 37-92.
136. Siegenthaler, T. (1984): “Correlation-immunity of nonlinear combining functions for cryptographic applications”, *IEEE Transactions on Information Theory*, IT-30 (5), 776-780.
137. Straffin, P. D. (1977): “Homogeneity, independence and power indices”, *Public Choice*, 30, 107-118.
138. Straffin, P.D. (1982): “Power indices in politics” In: Brams, S. J., Lucas, W. F. and P. D. Straffin (Eds), Political and Related Models, Springer Verlag, New York, pp. 256-321.
139. Straffin, P.D. (1988): “The Shapley-Shubik and the Banzhaf power indices as probabilities”, In: Roth, A.E. (Ed), The Shapley Value: Essays in honor of Lloyd S. Shapley, Cambridge University Press, pp.71-81.
140. Straffin, P.D. (1994): “Power and stability in politics”, In: Aumann, R. J. and S. Hart (Eds), Handbook of game theory with economic applications, Vol 1, North Holland, Amsterdam, pp. 1128-1151.

141. Sutter, M. (2000): “Fair allocation and reweighting of votes and voting power in the EU before and after the next enlargement”, *Journal of Theoretical Politics*, 12, 433-449.
142. Taylor, A.D. and W. S. Zwicker (1992): “A characterization of weighted voting”, *Proceedings of the American Mathematical Society*, 115(4), 1089-1094.
143. Taylor, A.D. (1995): Mathematics and Politics: Strategy, Voting Power and Proof, Springer-Verlag, New York.
144. Taylor, A.D. and W. S. Zwicker (1997): “Interval measures of power”, *Mathematical Social Sciences*, 33, 23-74.
145. Turnovec, F., Mercik, J. W. and M. Mazurkiewicz (2005): “Power indices: a unified approach”, Unpublished.
146. Wagner, F. and N. Höhne (2001): “Influence of national governments for or against the entry into force of the Kyoto Protocol: a Banzhaf index analysis”, *Climate Policy*, 1, 517-520.
147. Weber, R. J. (1988): “Probabilistic values for games”, In: Roth, A.E. (Ed), The Shapley Value: Essays in honor of Lloyd S. Shapley, Cambridge University Press, pp.101-119.
148. Widgrén, M. (1994): “Voting power in the EU and the consequences of two different enlargements”, *European Economic Review*, 38, 1153-1170.
149. Xiao, G. and J. L. Massey (1988): “A spectral characterization of correlation-immune combining functions”, *IEEE Transactions on Information Theory*, 569-571.
150. Young, H. P. (1985): “Monotonic solutions of cooperative games”, *International Journal of Game Theory*, 14, 65-72.
151. Zhang, X. M. and Y. Zheng (1995): “GAC- the criterion for global avalanche characteristics of cryptographic functions”, *Journal for Universal Computer Science*, 1, 316-333.
152. Statistical report on General Elections, Election Commission of India: http://www.eci.gov.in.
153. Program ‘ipgenf”: http://www.warwick.ac.uk/~ecaae/#Progam_List