Properties of Some Matrix Classes in Linear Complementarity Theory

Arup Kumar Das

Indian Statistical Institute
203, B. T. Road
Kolkata: 700 108
India

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Arup Kumar Das

Indian Statistical Institute

Kolkata, India

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Preface

This dissertation is devoted to a study of the properties of some matrix classes in linear complementarity theory, algorithms and an application of linear complementarity in stochastic games. The linear complementarity problem is the problem of finding a complementary pair of nonnegative vectors in a finite dimensional real vector space that satisfies a given system of inequalities. In particular, given a square matrix A of order n with real entries and an n dimensional vector q, the linear complementarity problem (LCP(q, A)) is to find an n dimensional vector z such that $Az + q \geq 0,\ z \geq 0$ and $z^t(Az + q) = 0$ or to show that no such vector z exists. Since several problems in optimization and engineering can be posed as LCPs, the theory of LCP has a wide range of applications in applied science and technology. The mathematical structure of the LCP has inspired several researchers to study the matrix properties and algorithms for its solution. A brief outline of the contents are presented in a chapterwise summary.

Chapter 1 is introductory in nature. Here we present the required definitions and introduce the notations used in this dissertation. We also include a survey of the results from the literature that will be used in our work.

In Chapter 2, we consider positive subdefinite matrices (PSBD) and pseudomonotone matrices studied in Martos [46], Crouzeix, Hassouni, Lahlou and Schaible [10] and Gowda [30]. We show that linear complementarity problems with positive subdefinite matrices of rank ≥ 2 are processable by Lemke's algorithm and that a copositive PSBD matrix (or a copositive pseudomonotone matrix) of rank ≥ 2 belongs to the class of sufficient matrices introduced by Cottle, Pang and Venkateswaran [8]. We also show that if a matrix A can be written as a sum of a copositive-plus merely positive subdefinite matrix and a copositive matrix and if it satisfies a feasibility condition then Lemke's algorithm applied to solve LCP(q, A) will terminate with a solution. This extends the results of

Jones [38] and Evers [18]. The results presented in this chapter have appeared in *Linear Algebra and its Applications,* 338 (2001) 275-285.

In Chapter 3, we consider the class of generalized positive subdefinite (GPSBD) matrices, an interesting matrix class introduced by Crouzeix and Komlósi [11]. In this chapter, we obtain some properties of GPSBD matrices. We show that copositive GPSBD matrices are P_0 and a merely generalized positive subdefinite (MGPSBD) matrix with some additional conditions belongs to the class of row sufficient matrices. Further, it is shown that for a subclass of GPSBD matrices, the solution set of a linear complementarity problem is same as the set of KKT-stationary points of the corresponding quadratic programming problem. We provide a counterexample to show that a copositive GPSBD matrix need not be sufficient in general. Finally, we show that if a matrix A can be written as a sum of a copositive-plus MGPSBD matrix with an additional condition and a copositive matrix and if it satisfies a feasibility condition then Lemke's algorithm can solve LCP(q, A). This further extends the applicability of Lemke's algorithm obtained in Chapter 2 and a result of Jones [38] and Evers [18]. The results presented in Chapter 3 will appear in *SIAM Journal on Matrix Analysis and Applications.*

Chapter 4 deals with the class of fully copositive matrices (C_0^f) and the class of fully semimonotone matrices (E_0^f). Murthy and Parthasarathy [63] proved that $C_0^f \cap Q_0$ matrices are sufficient. We show that this result is a consequence of a result proved by Cottle and Guu [5]. Further it is shown that if $A, A^t \in C_0^f$ with positive diagonal entries, then A is a completely Q_0-matrix. Murthy and Parthasarathy [61] proved that if $A \in R^{2\times 2} \cap C_0^f \cap Q_0$, then A is positive semidefinite and conjectured that this will be true for all $n \times n$ matrices. We present a counterexample to settle this conjecture. We finally consider the class of E_0^f-matrices introduced by Cottle and Stone [9] and partially address Stone's conjecture that $E_0^f \cap Q_0 \subseteq P_0$ by showing that $E_0^f \cap D^c \subseteq P_0$ where D^c is

Doverspike's class of matrices. The results presented in Chapter 4 have appeared in *Linear Algebra and its Applications,* 323 (2001) 87-97.

The concept of an *almost type* class was introduced by Väliaho and thoroughly investigated the class of almost copositive matrices in [102], [103]. Pye [82] studied the class of *almost P_0-matrices* of order n whose determinant is negative and all proper principal minors are nonnegative. In Chapter 5, we introduce a new matrix class called almost $\bar{N}$ (a subclass of almost N_0-matrices which are obtained as a limit of a sequence of almost N-matrices). We obtain a sufficient condition for a matrix of this class to possess the Q-property (or to be a Q-matrix). We produce a counterexample to show that an almost $\bar{N} \cap Q$ matrix need not be an R_0-matrix. We also introduce another two new limiting matrix classes, namely $\bar{N}$-matrix of exact order 2, $\bar{E}(d)$ for a positive vector d and prove sufficient conditions for these classes to satisfy the Q-property. Murthy et al. [62] showed that Pang's conjecture ($E_0 \cap Q \subset R_0$) is not true even when E_0 is replaced by C_0. We show that Pang's conjecture is true if E_0 is replaced by almost C_0. Finally, we present a game theoretic proof of necessary and sufficient condition of an almost P_0-matrix satisfying the Q-property. The results presented in Chapter 5 have appeared in *Linear and Multilinear Algebra,* 53 (2005) 243-257.

The results in Chapter 6 are concerned with the class of matrices for which PPTs are either in C_0 (E_0) or almost C_0 with at least one PPT almost C_0. The almost classes studied in this chapter have algorithmic significance and if these classes are also in Q_0 then these classes are processable by Lemke's algorithm. We also consider the problem of characterizing a class of matrices whose member possess at least one PPT that is a Z-matrix. The results presented in Chapter 6 have appeared in *Linear Algebra and its Applications,* 400 (2005) 243-252.

As an application of the LCP, Chapter 7 deals with two classes of structured stochastic games, namely, undiscounted zero-sum switching controller stochastic games and undiscounted zero-sum additive reward and additive transitions

(ARAT) games. The problem of computing the value vector and optimal stationary strategies is formulated as a linear complementarity problem for these two classes of undiscounted zero-sum games. This provides an alternative proof of the orderfield property for these two classes of games. The results presented in this chapter have appeared in an edited refereed volume, titled *Operations Research with Economic and Industrial Applications: Emerging Trends,* eds: S. R. Mohan and S. K. Neogy, Anamaya Publishers, New Delhi, India (2005) 156-180.

Numbering

For internal referencing, Section j in Chapter i is denoted by $i.j$ and $i.j.k$ is used to refer Item k of Section j in Chapter i. For example, the triple 2.3.5 refers to Item 5 in Section 3 of Chapter 2. All items (e.g., Lemma, Theorem, Example, Remark etc.) are identified in this fashion. Equation ($i.j.k$) is used to refer Equation k in Section j of Chapter i. We use brackets [ ] for a bibliographical reference.

List of Notations

The special notations pertaining to a particular chapter are provided in Section 2 of each chapter. The most frequently used notations are given below:

Spaces

R^n	real n-dimensional space
R	the real line
$R^{n\times n}$	the space of $n \times n$ real matrices
R^n_+	the nonnegative orthant of R^n

Sets

$\in$	element membership
$\notin$	not an element of
$\subseteq$	set inclusion
$\subset$	proper set inclusion
$\cup, \cap, \times$	union, intersection, Cartesian product
$\emptyset$	the empty set
$\bar{\alpha}$	complement of an index set α
$\lvert\alpha\rvert$	cardinality of a finite set α

Vectors

x^t	the transpose of a vector x
$x^t y$	the standard inner product of vectors in R^n
e	an n-dimensional vector of all ones
$x \geq y$	$x_i \geq y_i,\ i = 1, \ldots, n$
$x > y$	$x_i > y_i,\ i = 1, \ldots, n$
x^+	the vectors whose components are x_i^+ $(= \max\{x_i, 0\})$ for all i
x^-	the vectors whose components are x_i^- $(= \max\{-x_i, 0\})$ for all i
$y \in R^n$ is *unisigned*	if either $y \in R^n_+$ or $-y \in R^n_+$

Matrices

$A = [a_{ij}]$	a matrix with real entries a_{ij}
$\det(A)$	the determinant of a square matrix A
A^{-1}	the inverse of a matrix A
A^t	the transpose of a matrix A
$A \leq B$	$a_{ij} \leq b_{ij}$ for all i and j
$A < B$	$a_{ij} < b_{ij}$ for all i and j
I	the identity matrix
$A_{\alpha\beta}$	submatrix formed by the rows and columns of A whose indices are in α and β, respectively
$A_{\alpha\cdot}$	submatrix formed by the rows of A whose indices are in α
$A_{\cdot\alpha}$	submatrix formed by the columns of A whose indices are in α
$A_{\alpha\alpha}$	the principal submatrix of A
$\det(A_{\alpha\alpha})$	the principal minor of A
For a given integer k $(1 \leq k \leq n)$, $A_{\alpha\alpha}$ where $\alpha = \{1, \ldots, k\}$	the leading principal submatrix of A
Inertia $(\nu_+(S), \nu_-(S), \nu_0(S))$	the number of positive, negative and zero eigenvalues of the symmetric $n \times n$ matrix S

Miscellaneous Symbols

$\mathrm{LCP}(q, A)$	the LCP with data (q, A)
$\mathrm{F}(q, A)$	the feasible region of $\mathrm{LCP}(q, A)$
$\mathrm{S}(q, A)$	the solution set of $\mathrm{LCP}(q, A)$
$\mathrm{pos}(A)$	the convex cone generated by the matrix A
	$= \{x \mid \exists\, y \geq 0,\ x = Ay\}$
$K(A)$	the set of all q for which $S(q, A) \neq \emptyset$
$\mathcal{C}(A)$	the union of the strongly degenerate complementary cones of A
$\mathcal{K}(A)$	the union of all facets of all the complementary cones of A

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To my beloved daughter Abhilasha

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Kolkata, April 2005 *Arup Kumar Das*

Contents

Chapter 1

General Introduction and Some Basic Concepts

1.1 Introduction

The *linear complementarity problem* is a fundamental problem that arises in optimization, game theory, economics, and engineering. It can be stated as follows:

Given a square matrix A of order n with real entries and an n dimensional vector q, find n dimensional vectors w and z satisfying

$$w - Az = q, \;\; w \geq 0, \; z \geq 0 \tag{1.1.1}$$

$$w^t z = 0. \tag{1.1.2}$$

This problem is denoted as LCP(q, A). The name comes from the condition (1.1.2), the *complementarity condition* which requires that at least one variable in the pair (w_j, z_j) should be equal to 0 in the solution of the problem, for each $j = 1, 2, \ldots, n$. This pair is therefore known as the j^{th} *complementary pair* in the problem, and for each j, the variable w_j is known as the complement of z_j and vice versa. If a pair of vectors (w, z) satisfies (1.1.1), then the problem LCP(q, A) is said to have a *feasible solution*. A pair (w, z) of vectors satisfying (1.1.1) and (1.1.2) is called a *solution* to the LCP(q, A). The problem has undergone several name changes, from composite problem to complementary pivot problem. The current name *linear complementarity problem* was proposed by Cottle [7, p. 37]. The LCP is normally identified as a problem of mathematical programming and provides a unifying framework for several optimization problems like linear programming, linear fractional programming, convex quadratic programming and the bimatrix game problem. More specifically, the LCP models the optimality conditions of these problems. It is well studied in the literature on mathematical programming and a number of applications are reported in operations research [29], multiple objective programming problem [50], mathematical economics [78], geometry and engineering ([12], [26] and [79]). Some new applications of the linear complementarity problem have been reported in the area of stochastic games. For details, see the survey paper by Mohan, Neogy and Parthasarathy [53] and the references cited therein. This sort of applications and the potential for future applications have motivated the study of the LCP, especially the study of the algorithms for the LCP and the study of matrix classes. In fact, much of linear complementarity theory and algorithms are based on the assumption that the matrix A belongs to a particular class of matrices. The early motivation for studying the linear complementarity problem was that the KKT optimality conditions for linear and quadratic programs reduce to an LCP of the form given by (1.1.1) and (1.1.2). The algorithm presented by Lemke and Howson [42] to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke [41] (known as Lemke's algorithm) to solve an LCP(q, A), contributed significantly to the development of the linear complementarity theory. In fact, the study of the LCP really came into prominence only when Lemke and Howson [42] and Lemke [41] showed that the problem of computing a Nash equilibrium point of a bimatrix game can be posed as an LCP following the publication by Cottle [1]. However, Lemke's algorithm does not solve every instance of the linear complementarity problem, and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists. Extending the applicability of Lemke's algorithm to more matrix classes have been considered by many researchers like Eaves [17], Garcia [27] and Todd [99]. For recent books on the linear complementarity problem and its applications, see Cottle, Pang and Stone [7], Murty [67] and [19].

A number of generalizations of the linear complementarity problem have been proposed to accomodate more complicated real life problems as well as to diversify the field of applications. See Gowda and Sznajder [32], Sznajder and Gowda [91], Mohan, Neogy and Sridhar [55] and Cottle, Pang and Stone [7].

The most frequently used notations are given in page v. However, the notations pertaining to a particular chapter only are explained therein.

1.2 Some Preliminaries in Linear Complementarity Theory

In this section, we introduce some basic definitions and the required terminologies related to linear complementarity theory.

The idea of using complementary cones to study the LCP was considered by Samelson, Thrall and Wesler [89]. Later, Murty [66] studied the LCP through complementary cones extensively and obtained some remarkable results. For details on complementary cones see [7] and [67].

Definition 1.2.1 Given $A \in R^{n\times n}$ and $\alpha \subseteq \{1, 2, \ldots, n\}$, $C_A(\alpha)$ is called a *complementary matrix* of A with respect to α (or a *complementary submatrix* of $(I, -A)$ with respect to α) where $C_A(\alpha)_{\cdot j} = -A_{\cdot j}$ if $j \in \alpha$ and $C_A(\alpha)_{\cdot j} = I_{\cdot j}$ if $j \notin$

α. The associated cone *pos* $C_A(\alpha)$ is called the *complementary cone* relative to A with respect to α. If $\det(C_A(\alpha)) \neq 0$, then it is called a *complementary basis.*

DEFINITION 1.2.2 The complementary cone with respect to α is said to be *nondegenerate or full* if $\det(A_{\alpha\alpha}) \neq 0$. Otherwise it is said to be a *degenerate complementary cone.* A degenerate cone *pos* $C_A(\alpha)$ is said to be *strongly degenerate* if there exists $0 \neq x \in R^n_+$ such that $C_A(\alpha)x = 0$.

DEFINITION 1.2.3 Given $A \in R^{n\times n}$ and $\alpha \subseteq \{1, 2, \ldots, n\}$, the matrix A is said to be *nondegenerate* if $\det(A_{\alpha\alpha}) \neq 0$, $\forall\ \alpha \subseteq \{1, 2, \ldots, n\}$. Any solution (w, z) of LCP(q, A) is said to be *nondegenerate* if $w + z > 0$. Otherwise it is called a *degenerate solution.* A vector $q \in R^n$ is said to be *nondegenerate* with respect to A if every solution to LCP(q, A) is nondegenerate.

DEFINITION 1.2.4 A set $C \subseteq R^n$ is *connected* if there do not exist disjoint open sets $U, V \subseteq R^n$ such that $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$ and $C \subseteq U \cup V$. For any set $S \subseteq R^n$ and any $x \in S$, the *connected component* of a set S containing x is defined as the union of all connected sets C such that $x \in C \subseteq S$.

The concept of principal pivot transforms (PPTs) was introduced by Tucker [101]. Consider an LCP(q, A) where $q \in R^n$ and $A \in R^{n\times n}$. Let $\alpha \subseteq \{1, 2, \ldots, n\}$ and suppose that the principal submatrix $A_{\alpha\alpha}$ is nonsingular. By means of principal rearrangement, we may assume that $A_{\alpha\alpha}$ is a leading principal submatrix of A.

DEFINITION 1.2.5 The *principal pivot transform* (PPT) of A with respect to $\alpha \subseteq \{1, 2, \ldots, n\}$ is defined as the matrix given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$.

The PPT of LCP(q, A) with respect to α (obtained by pivoting on $A_{\alpha\alpha}$) is given by LCP(q', M) where $q'_{\alpha} = -A_{\alpha\alpha}^{-1} q_{\alpha}$ and $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha} A_{\alpha\alpha}^{-1} q_{\alpha}$.

Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details, see [3], [7] and [100] in this connection.

1.3 Matrix Games

The linear complementarity problem and the matrix game have some important connections. Some of the results of the LCP can be stated in terms of the value of a matrix game. In this connection Kaplansky's result [39] on matrix games is useful for deriving certain results. Raghavan [83] used von Neumann's [73] minimax theorem and Kaplansky's game theory results to derive several interesting properties of $Z \cap P$-matrices that arises in the LCP. A matrix game may be stated as follows:

There are two players, player I and player II and each player has a finite number of actions (called *pure strategies*). Let player I have m pure strategies and player II, n pure strategies. Suppose player I chooses to play a pure strategy i $(i = 1, 2, \ldots, m)$ and player II chooses a pure strategy j $(j = 1, 2, \ldots, n)$ simultaneously. Then player I pays player II an amount a_{ij} (which may be positive, negative or zero). Since player II's gain is player I's loss, the game is said to be zero-sum. A *mixed strategy* for player I is a probability vector $x \in R^m$ whose i^{th} component x_i represents the probability of choosing pure strategy i where $x_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, a mixed strategy for player II is a probability vector $y \in R^n$.

From von Neumann's fundamental minimax theorem we know that there exist mixed strategies x^*, y^* and a real number v such that

$$\sum_{i=1}^{m} x_i^* a_{ij} \leq v, \ \forall\, j = 1, 2, \ldots, n.$$
$$\sum_{j=1}^{n} y_j^* a_{ij} \geq v, \ \forall\, i = 1, 2, \ldots, m.$$

The mixed strategies (x^*, y^*) with $x^* \in R^m$ and $y^* \in R^n$ are said to be *optimal strategies* for player I and player II respectively and v is called *minimax value of game*. We write $v(A)$ to denote the value of the game corresponding to A. In the game described above, player I is the minimizer and player II is the maximizer. A mixed strategy is *completely mixed* if $x > 0$. The value of the game $v(A)$ is *positive (nonnegative)* if there exists a $0 \neq x \geq 0$ such that $Ax > 0$ $(Ax \geq 0)$. Similarly, $v(A)$ is *negative (nonpositive)* if there exists a $0 \neq y \geq 0$ such that $A^t y < 0$ $(A^t y \leq 0)$.

1.4 Matrix Classes in LCP Theory

Matrix classes play an important role in studying the theory and algorithms of the LCP. The study of special properties of the data matrix A has historically been an important part of the LCP research. A variety of classes of matrices are introduced in the context of the linear complementarity problem. Many of the matrix classes encountered in the context of the LCP are commonly found in several applications. Some of these matrix classes are of interest because they characterize certain properties of the LCP and they offer certain nice features from the view point of algorithms. Several algorithms have been designed for the solution of the linear complementarity problem. Many of these methods are matrix class dependent, i.e., they work only for LCPs with some special classes of matrices and can give no information otherwise. It is useful to review some matrix classes and their properties which will form the basis for further discussions.

Let A be a given $n \times n$ matrix, not necessarily symmetric.

Definition 1.4.1 We say that a matrix $A = [a_{ij}]$ of order n is a *Z-matrix* if $a_{ij} \leq 0, \forall\, i \neq j$.

The class of Z-matrices has been introduced by Fiedler and Pták [20].

Definition 1.4.2 We say that A is *positive semidefinite* (PSD) if $x^t Ax \geq 0 \;\forall\, x \in R^n$ and A is *positive definite* (PD) if $x^t Ax > 0 \;\forall\, 0 \neq x \in R^n$.

Definition 1.4.3 A is said to be a $P\,(P_0)$*-matrix* if all its principal minors are positive (nonnegative).

Definition 1.4.4 A is called a $N(N_0)$*-matrix* if all its principal minors are negative (nonpositive).

Definition 1.4.5 A is called *copositive* (C_0) (*strictly copositive* (C)) if $x^t Ax \geq 0 \;\forall\, x \geq 0$ $(x^t Ax > 0 \;\forall\, 0 \neq x \geq 0)$. $A \in R^{n \times n}$ is said to be *conegative* if $x^t Ax \leq 0 \;\forall\, x \geq 0$.

A is said to be *copositive-plus* (C_0^+) if $A \in C_0$ and the following implication holds:

$$[x^t Ax = 0, x \geq 0] \Rightarrow (A + A^t)x = 0.$$

We say that $A \in R^{n \times n}$ is *copositive-star* (C_0^*) if $A \in C_0$ and the following implication holds:

$$[x^t Ax = 0, Ax \geq 0, x \geq 0] \Rightarrow A^t x \leq 0.$$

A is called *copositive (strictly copositive, copositive-plus, PSD, PD)* of order k, $0 \leq k \leq n$, if every principal submatrix of order k is copositive (strictly copositive, copositive-plus, PSD, PD).

DEFINITION 1.4.6 A is said to be *column sufficient* if for all $x \in R^n$ the following implication holds:

$$x_i(Ax)_i \leq 0 \; \forall \, i \;\; \Rightarrow \;\; x_i(Ax)_i = 0 \;\; \forall \, i.$$

A is said to be *row sufficient* if A^t is column sufficient.

A is *sufficient* if A and A^t are both column sufficient.

A matrix A is *sufficient of order* k if all its $k \times k$ principal submatrices are sufficient.

For details on sufficient matrices, see [5], [8] and [104].

DEFINITION 1.4.7 $A \in R^{n \times n}$ is called a *Q-matrix* (or a matrix satisfying *Q-property*) if for every $q \in R^n$, LCP(q, A) has a solution.

We say that A is a *Q_0-matrix* (or a matrix satisfying *Q_0-property*) if $F(q, A) \neq \emptyset$ implies $S(q, A) \neq \emptyset$.

A is said to be a *completely Q (Q_0)-matrix* if all its principal submatrices are Q (Q_0)-matrices.

DEFINITION 1.4.8 $A \in R^{n \times n}$ is said to be an *E_0-matrix* if for every $0 \neq y \geq 0$, $y \in R^n$, $\exists$ an i such that $y_i > 0$ and $(Ay)_i \geq 0$. The class of such matrices is called the class of *semimonotone matrices.*

DEFINITION 1.4.9 A is said to be an *R-matrix* if for all $t \geq 0$, LCP(te, A) has only the trivial solution. A is said to be an *R_0-matrix* if LCP$(0, A)$ has only the trivial solution.

DEFINITION 1.4.10 A matrix $A \in R^{n \times n}$ is said to be an *L_2-matrix* if for each $0 \neq \xi \geq 0$, $\xi \in R^n$ satisfying $\eta = A\xi \geq 0$ and $\eta^t \xi = 0$, $\exists$ a $0 \neq \hat{\xi} \geq 0$ such that $\xi \geq \hat{\xi}$ and $\eta \geq \hat{\eta} \geq 0$, where $\hat{\eta} = -A^t \hat{\xi}$.

$A \in R^{n \times n}$ is said to be an *L-matrix* if it is in both E_0 and L_2. This class was introduced by Eaves ([17]) who showed that Lemke's algorithm processes LCP(q, A) (see Section 1.4.2) when $A \in L$ and hence $L \subseteq Q_0$.

DEFINITION 1.4.11 We say that a square matrix A is in the class $E(d)$ where $d \in R^n$ if $(\bar{w}, \bar{z}),\ \bar{z} \neq 0$ is a solution for the LCP(d, A) implies that $\exists$ a $0 \neq x \geq 0$ such that $y = -A^t x \geq 0, \quad x \leq \bar{z},\ y \leq \bar{w}$.

DEFINITION 1.4.12 We say that a square matrix A is in the class $E^*(d)$ for a $d \in R^n$ if $(\bar{w}, \bar{z})$ is a solution to the LCP(d, A) implies that $\bar{w} = d,\ \bar{z} = 0$.

Note that $E(d) = E^*(d)$ for any $d > 0$ or $d < 0$, $E(0) = L_2$ of [17] and $L(d) = E(d) \cap E(0)$. So, for $d > 0$, $A \in E(d)$ if LCP(d, A) has only the trivial solution $w = d,\ z = 0$.

Todd [99] defines larger classes $E_1(d)$ and $L_1(d)$ by extending the classes $E(d)$ and $L(d)$ of Garcia [27] as follows:

Let (w, z) solve LCP(d, A) for some $d \in R^n$. Consider the following conditions on a given $A \in R^{n \times n}$.

(a) For all α with $\{j \mid z_j > 0\} \subseteq \alpha \subseteq \{j \mid w_j = 0\}$, the principal submatrix of A corresponding to α has positive determinant.

(b) There is $0 \neq x \geq 0$ with $y = -A^t x \geq 0$ and $x \leq z,\ y \leq w$.

Todd defines the classes $E_1(d) = \{A \mid \text{Either condition (a) or (b) is satisfied}\}$ and $L_1(d) = E_1(d) \cap E_1(0)$. Note that $L(d) \subseteq Q_0$ [27] and $L_1(d) \subseteq Q_0$ [99] if $d > 0$. We refer to $L(d)$ as *Garcia's class* and to $L_1(d)$ as *Todd's class*.

DEFINITION 1.4.13 We say that A satisfies *Doverspike's condition* [13], if all the strongly degenerate complementary cones of $(I, -A)$ lie on the boundary of $pos(I, -A)$. We denote the class of matrices satisfying Doverspike's condition by D^c.

Using Lemke's algorithm, Doverspike [13] proved constructively that $E_0 \cap D^c \subset Q_0$.

1.4.1 Some Results in LCP Theory

We state a few results which will be useful for further discussions. For proofs and more details, we refer the reader to the excellent book of Cottle, Pang, and Stone [7].

THEOREM 1.4.1 *((3,3) Theorem in [20, p. 385] and 4.2 Theorem [66, p.75]) Let $A \in R^{n\times n}$. The following statements are equivalent:*

(i) A is a P-matrix.

(ii) A reverses the sign of no nonzero vector, i.e., $z_i(Az)_i \leq 0 \ \ \forall\, i$ implies $z = 0$.

(iii) All real eigenvalues of A and its principal submatrices are positive.

(iv) LCP(q, A) has a unique solution for every $q \in R^n$.

The following result gives some characterizations for a P_0-matrix.

THEOREM 1.4.2 *([7]) Let $A \in R^{n\times n}$. The following statements are equivalent:*

(i) A is a P_0-matrix.

(ii) For each vector $z \neq 0$ there exists an index k such that $z_k \neq 0$ and $z_k(Az)_k \geq 0$.

(iii) All real eigenvalues of A and its principal submatrices are nonnegative.

(iv) For each $\epsilon > 0$, $A + \epsilon I$ is a P-matrix.

Murthy et al. [63] presented a game theoretic proof of the following known results (see [7, pp. 185-187]).

THEOREM 1.4.3 *([63]) Let $A \in R^{n\times n}$. The following statements are equivalent:*

(i) $A \in E_0$.

(ii) The LCP(q, A) has a unique solution for every $q > 0$.

(iii) $v(A_{\alpha\alpha}) \geq 0$ for every index set $\alpha \subseteq \{1, 2, \ldots, n\}$.

(iv) $v(A^t_{\alpha\alpha}) \geq 0$ for every index set $\alpha \subseteq \{1, 2, \ldots, n\}$.

(iv) $A^t \in E_0$.

Murthy and Parthasarathy [60] have proved the following result on nonnegative matrices.

THEOREM 1.4.4 *([60, Theorem 2.5]) Let $A \geq 0$ be an $n \times n$ matrix. A is a Q_0-matrix if and only if $A_{i\cdot} \neq 0 \Rightarrow a_{ii} > 0 \ \forall\, 1 \leq i \leq n$.*

THEOREM 1.4.5 *([16, Lemma 2.3], [80]) If $A \in Q(Q_0)$ then every PPT of $A \in Q(Q_0)$.*

REMARK 1.4.1 *It is easy to show that $A \in Q$ if and only if $A \in Q_0$ with $v(A) > 0$. Also $A \in E_0$ if and only if $v(A_{\alpha\alpha}) \geq 0$ for all $\alpha \subseteq \{1, 2, \ldots, n\}$. See [63] in this connection.*

We state the following result in connection with the PPT.

THEOREM 1.4.6 *([7, Theorem 4.1.2]) Let M be the matrix obtained from the square matrix A by a principal pivot on the submatrix $A_{\alpha\alpha}$. Then, for any principal submatrix $M_{\beta\beta}$ of M:*

$$\det M_{\beta\beta} = \det A_{\gamma\gamma} / \det A_{\alpha\alpha}$$

where $\gamma = \alpha \triangle \beta$ (symmetric difference of sets α and β).

1.4.2 Lemke's Algorithm

The complementary pivot scheme due to Lemke [41] (also known as Lemke's algorithm) for solving (1.1.1) and (1.1.2) has stimulated a considerable amount of research in the classes of matrices A for which it can process LCP(q, A). The steps of the algorithm are given below.

The initial solution to (1.1.1) and (1.1.2) is taken as

$$w = q + d\,z_0$$

$$z = 0$$

where $d \in R^n$ is any given positive vector which is called *covering vector* and z_0 is an artificial variable which takes a large enough value so that $w > 0$. The ray is called *primary ray* [43].

Step 1: Decrease z_0 so that one of the variables w_i, $1 \leq i \leq n$, say w_r is reduced to zero. We now have a basic feasible solution with z_0 in place of w_r and with exactly one pair of complementary variables (w_r, z_r) being nonbasic.

Step 2: At each iteration, the complement of the variable which has been removed in the previous iteration is to be increased. In the second iteration, for instance, z_r will be increased.

Step 3: If the variable corresponding to the selected column in step 2 that enters the basis can be arbitrarily increased, then the procedure terminates in a *secondary ray.* If a new basic feasible solution is obtained with $z_0 = 0$, we have solved (1.1.1) and (1.1.2). If in the new basic feasible solution $z_0 > 0$, we have obtained a new basic pair of complementary variables (w_s, z_s). We repeat step 2.

Lemke's algorithm consists of the repeated applications of steps 2 and 3. If nondegeneracy is assumed, the procedure terminates either in a secondary ray or in a solution to (1.1.1) and (1.1.2). If degenerate almost complementary solutions are generated, then cycling can be avoided using the methods discussed by Eaves [17]. See [7] for a detailed discussion on Lemke's algorithm.

We say that an algorithm *processes* a problem if the algorithm can either compute a solution to it if one exists, or show that no solution exists. For $A \in L(d)$ where $d > 0$ the success of Lemke's algorithm applied to LCP(q, A) with d as the covering vector is guaranteed if it is feasible. Todd [99] proved that Lemke's algorithm with covering vector $d > 0$ processes LCP(q, A) for all matrices $A \in L_1(d)$. Also LCP(q, A) are processed by Lemke's algorithm when A is a row sufficient matrix. See [8, p. 239]. Ramamurthy [87] showed that Lemke's algorithm for the linear complementarity problem can be used to check whether a given Z-matrix is a P_0-matrix and it can also be used to analyze the structure of finite Markov chains.

1.5 Degree Theory

We use some concepts of degree theory in Chapter 4 and Chapter 5. For the concept and the properties of the degree we refer to Lloyd [44] and Ortega and Rheinboldt [76]. For the use of this concept in linear complementarity we refer to Cottle, Pang and Stone [7], Gowda [31], Howe and Stone [35] and Morris [59].

Let Ω be a bounded open set in R^n with boundary $\partial\Omega$ and closure $\bar{\Omega}$. Let $\operatorname{dist}(0, S) := \inf\{||s||,\, s \in S\}$ where $S \subseteq R^n$ and $0 \in R^n$ denote the distance between 0 and the set S. Let $f : \bar{\Omega} \to R^n$ be continuous such that $0 \notin f(\partial\Omega)$. Then the *degree of* f *at* 0 *relative to* Ω is defined and is an integer. This degree is denoted by $\deg(f, \Omega, 0)$. See [76, Definition 6.1.7]. We make use of the following properties of degree.

Property 1: (Existence property) If $\deg(f, \Omega, 0) \neq 0$, then the equation $f(z) = 0$ has a solution in Ω.

Property 2: (Homotopy invariance property) Suppose that $H : [0,1] \times \bar{\Omega} \rightarrow R^n$ is continuous and $0 \notin H(t, \partial\Omega)$ for all $t \in [0,1]$. Then $\deg(H(0,\cdot), \Omega, 0) = \deg(H(1,\cdot), \Omega, 0)$.
For convenience, we shall denote $H(t,\cdot)$ as $H_t(\cdot)$.
Property 3: (Nearness property) Suppose that $\deg(f, \Omega, 0)$ is defined. If g is a continuous function on $\bar{\Omega}$ such that

$$\sup_{x \in \Omega} ||g(x) - f(x)|| < \operatorname{dist}(0, f(\partial\Omega))$$

then $\deg(g, \Omega, 0)$ is defined and is equal to $\deg(f, \Omega, 0)$.
Property 4: (Domain decomposition property) Suppose $\Omega = \cup_{i=1}^m \Omega_i$ where Ω_i's are bounded open sets such that $\Omega_i \cap \Omega_j = \emptyset,\ i \neq j$. Also, suppose that $0 \notin \cup_{i=1}^m f(\partial\Omega_i)$. Then $\deg(f, \Omega, 0) = \sum_i \deg(f, \Omega_i, 0)$.
Property 5: (Excision property) Suppose that $\deg(f, \Omega, 0)$ is defined and K is a compact subset of $\bar{\Omega}$ such that there is no solution of $f(x) = 0$ in K. Then $\deg(f, \Omega, 0) = \deg(f, \Omega \backslash K, 0)$.
Property 6: Suppose f is differentiable at z^* where $z^* \in \Omega$ is the unique point satisfying $f(z^*) = 0$. Suppose also that the Jacobian matrix $f'(z^*)$ is nonsingular. Then,

$$\deg(f, \Omega, 0) = \operatorname{sgn}(\det f'(z^*))$$

where for any real number a, $\operatorname{sgn}(a) = +1$ if $a > 0$ and -1 if $a < 0$.

A particular case of Property 6 occurs when f is a piecewise affine function of the form $f(z) = z \wedge (Az + q)$ where $\wedge$ denotes componentwise minimum.

Let $f_A : R^n \rightarrow R^n$, be the piecewise linear map for a matrix $A \in R^{n \times n}$ given by $f_A(x) = x^+ - Ax^-$ where $x_i^+ = \max(0, x_i)$ and $x_i^- = \max(0, -x_i)\ \forall\ i = 1, 2, \ldots, n$. Note that for any $x \in R^n$, $x = x^+ - x^-$. An LCP(q, A) is equivalent to finding an $x \in R^n$ such that $f_A(x) = q$. If x belongs to the interior of an orthant of R^n and $\det(A_{\alpha\alpha}) \neq 0$ where $\alpha = \{i \mid x_i < 0\}$, then the index of $f_A(x)$ at x is well defined and

$$\text{ind}\, f_A(q,x) = \text{sgn}\ \det(A_{\alpha\alpha}) = \frac{\det(A_{\alpha\alpha})}{|det(A_{\alpha\alpha})|}.$$

Let $f_A^{-1}(q)$ stand for the set of all vectors $x \in R^n$, such that $f_A(x) = q$. From the linear complementarity theory, it is clear that the cardinality of $f_A^{-1}(q)$ denotes the number of solutions of LCP(q, A). In particular, if q is nondegenerate with respect to A, each index of f_A is well defined and we can then define *local degree* of A at q, denoted by $\deg_A(q)$, to be equal to the local degree of f_A at q, i.e.,

$$\deg_A(q) = \sum_{x \in f_A^{-1}(q)} \text{ind}\, f_A(q,x) = \sum_{x \in f_A^{-1}(q)} \frac{\det(A_{\alpha\alpha})}{|det(A_{\alpha\alpha})|}$$

where the summation is taken over the index sets $\alpha \subseteq \{1, 2, \ldots, n\}$ such that $q \in posC_A(\alpha)$.

If $q, q' \in R^n \backslash \mathcal{K}(A)$ and lie in the same connected component of $R^n \backslash \mathcal{C}(A)$ then $\deg_A(q) = \deg_A(q')$. See Theorem 6.1.17 in [7, p. 515]. More specifically when $R^n \setminus \mathcal{C}(A)$ is made up of a single connected component, we have the degree of A at q defined and equal to the same constant for every $q \in R^n$, except possibly for a set of vectors which has measure zero. Such a scalar is called the *global degree* of A and is denoted by $\deg(A)$.

Suppose $A \in R_0$ and q be nondegenerate with respect to A. For an R_0-matrix A, the number $\sum_{z \in S(q,A)} \text{sgn}\ \det(A_{\alpha\alpha})$ where $\alpha = \{i \ : \ z_i \neq 0\}$ is the same for all vectors q such that LCP(q, A) has a finite number of solutions and we write

$$\deg(A) = \sum_{z \in S(q,A)} \text{sgn}\ \det(A_{\alpha\alpha}) = \sum_{z \in S(q,A)} \frac{\det(A_{\alpha\alpha})}{|\det(A_{\alpha\alpha})|}.$$

Local and global degrees of an LCP map are quite useful in identifying subclasses of Q and Q_0 matrices. Some of the well-known characterizations are given a newer perspective in terms of degree theory. For further details on degree theory, see [7, Chapter 6].

In Chapter 5, we consider various subclasses of Q-matrices that are not known in the literature and analyse their global degree. We consider R_0-matrices here. This is due to the fact that for matrices that do not belong to R_0, the global degree may not be defined. We state the following result for R_0-matrices.

THEOREM 1.5.1 *Let* $A \in R_0$. R^n *is a single connected component and* $\deg(A)$ *is well-defined. If* $\deg(A) \neq 0$, *then* $A \in Q$.

An interesting property of degree is that it carries over to the principal pivot transforms also. This is stated in the following theorem.

THEOREM 1.5.2 *([95, Theorem 2.2]) Let* $A \in R_0$. *If* $\deg(A) = r$ *and* M *is a PPT of* A, *then* $\deg(M) = \pm r$.

Chapter 2

Positive Subdefinite Matrices and The Linear Complementarity Problem

2.1 Introduction

The class of positive subdefinite matrices is a generalization of the class of positive semidefinite (PSD) matrices. The study of pseudoconvex and quasiconvex quadratic forms leads to this new class of matrices, and it is useful in the study of quadratic programming problem. We say that a real square matrix A of order n is positive subdefinite (PSBD) if for all $x \in R^n$

$$x^t A x < 0 \;\text{ implies either }\; A^t x \leq 0 \;\text{ or }\; A^t x \geq 0.$$

Martos [46] introduced the class of symmetric positive subdefinite matrices in connection with a characterization of a pseudoconvex function. Cottle and Ferland [4] followed the path set by Martos in [46] and among other things, obtained converses for some of Martos's results. Rao [86] obtained a characterization of merely positive subdefinite matrices which enabled the easy recognition of quasiconvex and pseudoconvex quadratic forms. He also studied this class with respect to generalized inverse (g-inverse). Since Martos was considering the Hessians of quadratic functions, he was concerned only about symmetric matrices. Later, Crouzeix et al. [10] studied nonsymmetric PSBD matrices in the context of generalized monotonicity and the linear complementarity problem.

The purpose of this chapter is to summarize the known results and to further the study of properties of PSBD matrices. It is not surprising that many properties of PSD matrices are lost through the generalization. In Section 2.2, we present the required definitions and state the relevant results used in this chapter. In Section 2.3, main results are proved. We show that the linear complementarity problems with positive subdefinite matrices of rank ≥ 2 are processable by Lemke's algorithm and that a copositive PSBD matrix of rank ≥ 2 belongs to the class of sufficient matrices introduced by Cottle et al. [8]. Further, we also show that if a matrix A which can be written as a sum of a copositive-plus merely positive subdefinite matrix and a copositive matrix and if it satisfies a feasibility condition then Lemke's algorithm applied to solve LCP(q, A) terminates with a solution. This extends the results of Jones [38] and Evers [18].

2.2 Preliminaries

The class of PSD matrices is a subclass of PSBD matrices. A matrix A is said to be *merely positive subdefinite* (MPSBD) if A is a PSBD matrix but not positive semidefinite (PSD). The concept of PSBD matrices leads to a study of pseudomonotone matrices. Crouzeix et al. [10] have obtained new characterizations for generalized monotone affine maps on R^n_+ using PSBD matrices. Given a matrix $A \in R^{n\times n}$ and a vector $q \in R^n$, an affine map $\mathcal{F}(x) = Ax + q$ is said to be *pseudomonotone* on R^n_+ if

$$(y-z)^t(Az+q) \geq 0,\ y \geq 0, z \geq 0 \Rightarrow (y-z)^t(Ay+q) \geq 0.$$

A matrix $A \in R^{n\times n}$ is said to be pseudomonotone if $\mathcal{F}(x) = Ax$ is pseudomonotone on the nonnegative orthant. Gowda [30] establishes a connection between affine pseudomonotone mapping and the linear complementarity problem. It is also shown that for an affine pseudomonotone mapping, the feasibility of the LCP implies its solvability. A result of this type was proved earlier by Karamardian [40]. Crouzeix et al. ([10]) proved that an affine map $\mathcal{F}(x) = Ax+q$ where $A \in R^{n\times n}$ and $q \in R^n$ is pseudomonotone if and only if

$$z \in R^n,\ \ z^tAz < 0 \Rightarrow \begin{cases} A^tz \geq 0 \text{ and } z^tq \geq 0 \text{ or} \\ A^tz \leq 0,\ z^tq \leq 0 \text{ and } z^t(Az^- + q) < 0. \end{cases}$$

We require the following theorems in the next section. For the proof of these results, see [10].

Theorem 2.2.1 *([10, Proposition 2.1]) Let $A = ab^t$ where $a \neq b,\ a, b \in R^n$. A is PSBD if and only if one of the following conditions holds:*

(i) $\exists$ *a* $t > 0$ *such that* $b = ta$;

(ii) *for all* $t > 0$, $b \neq ta$ *and either* $b \geq 0$ *or* $b \leq 0$.

Further suppose that $A \in MPSBD$. Then $A \in C_0$ if and only if either ($a \geq 0$ and $b \geq 0$) or ($a \leq 0$ and $b \leq 0$) and $A \in C_0^*$ if and only if A is copositive and $a_i = 0$ whenever $b_i = 0$.

Combining Theorem 2.1 and Proposition 2.5 in [10], we get:

Theorem 2.2.2 *([10, Theorem 2.1, Proposition 2.5]) Suppose $A \in R^{n\times n}$ is PSBD and rank$(A) \geq 2$. Then A^t is PSBD and at least one of the following conditions holds:*

(i) *A is PSD;*

(ii) $(A + A^t) \leq 0$;

(iii) *A is C_0^*.*

THEOREM 2.2.3 *([10, Proposition 2.2]) Assume that $A \in R^{n \times n}$ is MPSBD and rank$(A) \geq 2$. Then*

(a) $\nu_-(A + A^t) = 1$,

(b) $(A + A^t)z = 0 \;\; \Leftrightarrow \;\; Az = A^t z = 0.$

THEOREM 2.2.4 *([10, Theorem 3.3]) A matrix $A \in R^{n \times n}$ is pseudomonotone if and only if A is PSBD and copositive with the additional condition in case $A = ab^t$, that $b_i = 0 \Rightarrow a_i = 0$.*

In fact, the class of psedomonotone matrices coincides with the class of matrices which are both PSBD and copositive-star.

THEOREM 2.2.5 *([30, Corollary 4]) If A is pseudomonotone, then A is a row sufficient matrix.*

2.3 Main Results on PSBD and MPSBD Matrices

Since a PSBD matrix is a natural generalization of a PSD matrix, it is of interest to determine which of the properties of a PSD matrix also holds for a PSBD matrix. In particular, we may ask whether
(i) A is PSBD if and only if $(A + A^t)$ is PSBD and
(ii) any PPT (Principal Pivot Transform) of a PSBD matrix is a PSBD matrix.

The following examples show that these statements are false.

EXAMPLE 2.3.1 Let $A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x^tAx = x_1x_2 < 0$ implies x_1 and x_2 are of opposite sign. Clearly $A \in$ PSBD since $x^tAx < 0$ and $A^tx = \begin{bmatrix} -x_2 \\ 2x_1 \end{bmatrix}$ imply either $A^tx \leq 0$ or $A^tx \geq 0$.

Also it is easy to see that $A + A^t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not a PSBD matrix.

Similarly, let $A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ so that $A + A^t = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. It is easy to verify that $A + A^t$ is PSBD but A is not a PSBD matrix.

EXAMPLE 2.3.2 Let us consider the matrix $A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ in Example 2.3.1. Note that $A \in$ PSBD but it is easy to see that $A^{-1} = \begin{bmatrix} 0 & -1 \\ 0.5 & 0 \end{bmatrix}$ is not a PSBD matrix.

Since A^{-1} is a PPT of A therefore any PPT of a PSBD matrix is not a PSBD matrix.

The following theorem says that PSBD is a complete class in the sense of [7, 3.9.5].

THEOREM 2.3.1 *Suppose $A \in R^{n\times n}$ is a PSBD matrix. Then $A_{\alpha\alpha} \in$ PSBD where $\alpha \subseteq \{1, \ldots, n\}$.*

Proof. Let $A \in$ PSBD and $\alpha \subseteq \{1, \ldots, n\}$. Let $x_\alpha \in R^{|\alpha|}$ and

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}.$$

Suppose that $x_\alpha^t A_{\alpha\alpha} x_\alpha < 0$. Now define $z \in R^n$ by taking $z_\alpha = x_\alpha$ and $z_{\bar{\alpha}} = 0$. Then $z^tAz = x_\alpha^t A_{\alpha\alpha} x_\alpha$. Since A is a PSBD matrix, $z^tAz = x_\alpha^t A_{\alpha\alpha} x_\alpha <$

$0 \Rightarrow$ either $A^t z \geq 0$ which implies that $A^t_{\alpha\alpha} x_\alpha \geq 0$ or $A^t z \leq 0$ (which implies $A^t_{\alpha\alpha} x_\alpha \leq 0$). Hence $A_{\alpha\alpha} \in$ PSBD. As α is arbitrary, it follows that every principal submatrix of A is a PSBD matrix. ∎

THEOREM 2.3.2 *Suppose $A \in R^{n\times n}$ is a PSBD matrix. Let $D \in R^{n\times n}$ be a positive diagonal matrix. Then $A \in$ PSBD if and only if $DAD^t \in$ PSBD.*

Proof : Let $A \in$ PSBD. For any $x \in R^n$, let $y = D^t x$. Note that $x^t DAD^t x = y^t Ay < 0 \Rightarrow A^t y = A^t D^t x \leq 0$ or $A^t y = A^t D^t x \geq 0$. This implies that either $DA^t D^t x \leq 0$ or $DA^t D^t x \geq 0$ since D is a positive diagonal matrix. Thus $DAD^t \in$ PSBD. The converse follows from the fact that D^{-1} is a positive diagonal matrix and $A = D^{-1}(DAD^t)(D^{-1})^t$. ∎

THEOREM 2.3.3 *PSBD matrices are invariant under principal rearrangement, i.e., if $A \in R^{n\times n}$ is a PSBD matrix and $P \in R^{n\times n}$ is any permutation matrix, then $PAP^t \in$ PSBD.*

Proof. Let $A \in$ PSBD and let $P \in R^{n\times n}$ be any permutation matrix. For any $x \in R^n$, let $y = P^t x$. Note that $x^t PAP^t x = y^t Ay < 0 \Rightarrow A^t y = A^t P^t x \leq 0$ or $A^t y = A^t P^t x \geq 0$. This implies that either $PA^t P^t x \leq 0$ or $PA^t P^t x \geq 0$ since P is just a permutation matrix. It follows that PAP^t is a PSBD matrix. ∎

Now we settle the question whether PSBD $\subseteq Q_0$ and Lemke's algorithm processes PSBD matrices. In this connection, we rewrite Theorem 2.2.1 as follows.

THEOREM 2.3.4 *Let $A = ab^t \in R^{n\times n}$, $a, b \in R^n$, $a, b \neq 0$ be a PSBD matrix. Suppose either $a \geq 0$ or $a \leq 0$ when $b \neq ta$ for any $t > 0$. Then $A \in Q_0$ if and only if one or more of the following conditions hold:*

(i) A is PSD;

(ii) a and b have opposite signs;

(iii) a and b have the same sign and

$$a_i = 0 \ \textit{whenever}\ b_i = 0 \quad \forall\ i = \{1, 2, \ldots, n\}. \tag{2.3.1}$$

Proof. We need to consider two cases.

Case 1: There exists a $t > 0$ so that $b = ta$. It is easy to see that A is PSD and hence $A \in Q_0$.

Case 2: For all $t > 0$, $b \neq ta$. In this case, it follows from Theorem 2.2.1 that either $b \geq 0$ or $b \leq 0$. Under our hypothesis about a, either $A \leq 0$ or $A \geq 0$. If $A \leq 0$ then $A \in Q_0$. But if $A \geq 0$ then from Theorem 1.4.4, it is easy to see that $A \in Q_0$ if and only if

$$a_i = 0 \ \text{whenever}\ b_i = 0 \quad \forall\ i = \{1, 2, \ldots, n\}.$$

■

REMARK 2.3.1 *Note that any PSBD matrix $A = ab^t \in R^{n \times n}$, $a, b \in R^n$, $a, b \neq 0$ is a sufficient matrix if $a_i = b_i = 0$ or $a_i b_i > 0$. See [104, Corollary 4.2].*

LEMMA 2.3.1 *Let $A \in R^{n \times n}$ be a PSBD matrix with rank$(A) \geq 2$ and let $A + A^t \leq 0$. We have*

(i) If $a_{ii} < 0$, then the column/row containing a_{ii} is nonpositive.

(ii) If A has a principal submatrix of the form

$$\begin{bmatrix} 0 & a_{ks} \\ a_{sk} & 0 \end{bmatrix}$$

with $(a_{ks} + a_{sk}) < 0$ then the s^{th} and k^{th} rows as well as s^{th} and k^{th} columns of A are nonpositive.

Proof. By Theorem 2.2.2, A^t is a PSBD matrix. By Theorem 2.3.1, every principal submatrix of A as well as A^t is also a PSBD matrix. To prove (i) we proceed as follows. Suppose the diagonal entry $a_{ii} < 0$. Let (assuming $i < k$) $\alpha = \{i, k\}$. Consider the 2×2 submatrix $A_{\alpha\alpha} = \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ki} & a_{kk} \end{bmatrix}$ which is a PSBD matrix. Now for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$, $x^t A_{\alpha\alpha} x = a_{ii}x_1^2 + x_1x_2(a_{ik} + a_{ki}) + a_{kk}x_2^2 < 0$ if x is nonnegative with $x_1 > 0$ since by hypothesis, $a_{kk} \leq 0$ and $a_{ik} + a_{ki} \leq 0$. Thus $(A_{\alpha\alpha})^t x$ is unisigned for any nonnegative x with $x_1 > 0$. Now by taking $x_2 = 0, x_1 > 0$ we conclude that $a_{ik} \leq 0$. Applying the same argument for A^t and $(A^t_{\alpha\alpha}) = (A_{\alpha\alpha})^t$, we conclude that $A_{\alpha\alpha}x$ is also unisigned and hence $a_{ki} \leq 0$. This completes the proof of (i).

To prove (ii) we proceed as follows: Note that for any $y \in R^n$,

$$y^t A y = \sum_{i=1}^{n} a_{ii}y_i^2 + \sum_{i<j}(a_{ij} + a_{ji})y_i y_j.$$

By our hypothesis a_{ii} and $a_{ij} + a_{ji}$ are nonpositive for all i and j. Suppose now $a_{kk} = a_{ss} = 0$ and $(a_{ks} + a_{sk}) < 0$. In this case, note that if $z \in R^n$ be any vector such that $z_i = 0, i \neq k, s,\ \ z_k > 0$ and $z_s > 0$, then $z^t A z = z_s z_k(a_{ks} + a_{sk}) < 0$. Therefore, it follows that for such a z, $A^t z$ is unisigned. Suppose now for some $r,\ r \neq s, k,\ a_{kr} > 0$. Choose $z_k = 1$. Let δ be a positive number such that $a_{kr} + a_{sr}\delta > 0$. It is easy to see that such a δ exists. Define the vector $\bar{z}$ by taking $\bar{z}_i = 0, i \neq k, s,\ \bar{z}_k = 1, \bar{z}_s = \delta$. Note that $A^t\bar{z}$ is not unisigned, a contradiction. This contradiction shows that $a_{kr} \leq 0,\ \forall\, r$. In a similar manner, it can be shown that a_{sr} is nonpositive for all r. From the fact that A^t is also a PSBD matrix, by a similar argument it follows that a_{rk} and a_{rs} are also nonpositive for all r. This completes the proof. ∎

Lemma 2.3.2 *Suppose $A \in R^{n\times n}$ is a PSBD matrix with rank$(A) \geq 2$ and $A + A^t \leq 0$. If A is not a skew-symmetric matrix, then $A \leq 0$.*

Proof. Let the index sets L_1, L_2 and L be defined as follows:

$$L_1 = \{i | a_{ii} < 0\};\ \ L_2 = \{i | a_{ii} = 0,\ \exists\, k, with\ a_{kk} = 0, a_{ik} + a_{ki} < 0\}.$$

Note that if $i \in L_2$, then L_2 will also contain the index k that satisfies the defining conditions of L_2 for i. Let $L = L_1 \cup L_2$. By the hypothesis of the lemma L is nonempty, for otherwise, A is skew symmetric. Consider the following partitioned form of A induced by the index set L.

$$PAP^t = \begin{bmatrix} A_{LL} & A_{L\bar{L}} \\ A_{\bar{L}L} & A_{\bar{L}\bar{L}} \end{bmatrix},$$

where $\bar{L}$ denotes the set of indices $\{1, 2, \ldots, n\} \setminus L$ and P is the appropriate permutation matrix. (In what follows we will simply use the symbol A to denote PAP^t). By the earlier lemma, $A_{LL} \leq 0, A_{L\bar{L}} \leq 0$ and $A_{\bar{L}L} \leq 0$. Also note that by definition, $A_{\bar{L}\bar{L}}$ is a skew symmetric matrix. For any $y \in R^n$, let $y = \begin{pmatrix} y_L \\ y_{\bar{L}} \end{pmatrix}$ denote the corresponding partition of y. Note that

$$y^t Ay = y_L^t A_{LL} y_L + y_{\bar{L}}^t A_{\bar{L}\bar{L}} y_{\bar{L}} + y_{\bar{L}}^t (A_{\bar{L}L} + A_{L\bar{L}}^t) y_L.$$

Since $A_{\bar{L}\bar{L}}$ is skew-symmetric, it follows that for all $y \in R^n$, $\;y_{\bar{L}}^t A_{\bar{L}\bar{L}} y_{\bar{L}} = 0$. It follows that for all vectors y such that y_L is positive, $y^t Ay$ is negative and hence both Ay and $A^t y$ are unisigned. To complete the proof, we need to show that none of the entries of $A_{\bar{L}\bar{L}}$ is positive. Suppose to the contrary that for some $s \in \bar{L}, r \in \bar{L}, \; a_{sr} > 0$. Choose ϵ such that

$$\epsilon \sum_{i \in L} a_{ir} + a_{sr} > 0.$$

Define the vector $\bar{y}$ by taking $y_i = \epsilon \quad \forall \; i \in L$ and $y_i = 0 \; \forall \; i \neq r \in \bar{L}$ and $y_r = 1$. Note that since each row and column of A_{LL} contains at least one negative entry and all the entries of A_{LL}, and $A_{\bar{L}L}$ are nonpositive, it follows that $(A^t y)_i < 0 \;\; \forall \; i \in L$. Also by construction $(A^t y)_r > 0$. This is a contradiction. Hence $A_{\bar{L}\bar{L}} \leq 0$ and the lemma follows. ■

THEOREM 2.3.5 *Suppose $A \in R^{n \times n}$ is a PSBD matrix with rank$(A) \geq 2$. Then A is a Q_0 matrix.*

Proof. By Theorem 2.2.2, A^t is a PSBD matrix. Also by the same theorem, either $A \in$ PSD or $(A+A^t) \leq 0$ or $A \in C_0^*$. If $A \in C_0^*$ then $A \in Q_0$ (see [7]). Now if $(A+A^t) \leq 0$, and A is not skew-symmetric then by Lemma 2.3.2 it follows that $A \leq 0$. In this case, $A \in Q_0$ [7]. However, if A is skew-symmetric then $A \in PSD$. Therefore, $A \in Q_0$. ■

Corollary 2.3.1 *Suppose A is a PSBD matrix with rank$(A) \geq 2$. Then LCP(q, A) is processable by Lemke's algorithm. If rank$(A) = 1$, (i.e., $A = ab^t,\ a, b \neq 0$) and $A \in C_0$ then LCP(q, A) is processable by Lemke's algorithm whenever $b_i = 0 \Rightarrow a_i = 0$.*

Proof. Suppose rank$(A) \geq 2$. From Theorem 2.2.2 and the proof of Theorem 2.3.5, it follows that A is either a PSD matrix or $A \leq 0$ or $A \in C_0^*$. Hence LCP(q, A) is processable by Lemke's algorithm (see [7]). For PSBD $\cap\, C_0$ matrices of rank$(A) = 1$, i.e., for $A = ab^t,\ a, b \neq 0,$ such that $b_i = 0 \Rightarrow a_i = 0$. Note that $A \in C_0^*$ by Theorem 2.2.1. Hence LCP(q, A) with such matrices are processable by Lemke's algorithm. ■

Theorem 2.3.6 *Suppose A is a PSBD $\cap\, C_0$ matrix with rank$(A) \geq 2$. Then $A \in R^{n \times n}$ is a sufficient matrix.*

Proof. Note that by Theorem 2.2.2, A^t is a PSBD $\cap\, C_0$ matrix with rank$(A^t) \geq 2$. Now by Theorem 2.2.4, A and A^t are pseudomonotone. Hence A and A^t are row sufficient by Theorem 2.2.5. Therefore, A is sufficient. ■

Corollary 2.3.2 *Suppose $A \in C_0 \cap PSBD$. Then $A \in P_0$.*

Proof. If rank$(A) = 1$, then $A \in P_0$ since $A \in C_0$. If rank$(A) \geq 2$, then the inclusion $A \in P_0$ follows from Theorem 2.3.6. ■

The following example shows that in general PSBD matrices need not belong to P_0.

Example 2.3.3 Let $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x^tAx = -2x_1x_2 < 0$ implies x_1 and x_2 are of same sign. $A \in$ PSBD, since $A^tx = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix}$ implies either $A^tx \leq 0$ or $A^tx \geq 0$ but $A \notin P_0$.

The following example shows that a PSBD matrix need not be a Q_0-matrix in general.

Example 2.3.4 Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A^tx = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$, implies either $A^tx \leq 0$ or $A^tx \geq 0$. Hence $A \in$ PSBD. Taking $q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ we note that LCP(q, A) is feasible but has no solution. Therefore, A is not a Q_0-matrix.

The following theorem provides a new sufficient condition to solve LCP(q, A) by Lemke's algorithm (See Section 1.4.2 for a discussion on Lemke's algorithm).

Theorem 2.3.7 *Suppose $A \in R^{n\times n}$ can be written as $M + N$ where $M \in$ MPSBD $\cap\, C_0^+$, rank$(M) \geq 2$ and $N \in C_0$. If the system $q + Mx - N^ty \geq 0,\ y \geq 0$ is feasible, then Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ terminates with a solution.*

Proof. Assume that the feasibility condition of the theorem holds so that there exist an $x^0 \in R^n$ and a $y^0 \in R^n_+$ such that $q + Mx^0 - N^ty^0 \geq 0$. First we shall show that for any $x \in R^n_+$, if $Ax \geq 0$ and $x^tAx = 0$, then $x^tq \geq 0$. Note that for given $x \geq 0$, $x^tAx = 0 \Rightarrow x^t(M + N)x = 0$ and since $M, N \in C_0$, this implies that $x^tMx = 0$. As M is a MPSBD matrix $x^tMx = 0 \Leftrightarrow x^t(M + M^t)x = 0 \Leftrightarrow (M + M^t)x = 0 \Leftrightarrow M^tx = 0 \Leftrightarrow Mx = 0$. See Theorem 2.2.3. Also since $Ax \geq 0$, it follows that $Nx \geq 0$ and hence $x^tN^ty^0 \geq 0$. Further since $q + Mx^0 - N^ty^0 \geq 0$ and $x \geq 0$, it follows that $x^t(q + Mx^0 - N^ty^0) \geq 0$. This implies that $x^tq \geq x^tN^ty^0 \geq 0$.

Now from Corollary 4.4.12 and Theorem 4.4.13 of [7, p.277] it follows that Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ terminates with a solution. ■

The following example shows that the class MPSBD $\cap$ C_0^+ is non-empty.

EXAMPLE 2.3.5 Let $M = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that $x^tMx = 2(x_1+x_2)(x_1+2x_2)$. Using this expression it is easy to verify that $x^tMx < 0 \Rightarrow$ either $M^tx \leq 0$ or $M^tx \geq 0$. Also it is easy to see that $M \in C_0^+$.

REMARK 2.3.2 *The above theorem cannot be extended to a PSBD matrix. Note that the class PSBD matrices includes PSD matrices. In the example below, we consider a matrix A which may be written as $M + N$, where $M \in$ non-symmetric PSD and $N \in C_0$ and show that Theorem 2.3.7 does not hold.*

EXAMPLE 2.3.6 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Taking $q = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ we note that LCP(q, A) is feasible but the problem has no solution. Therefore, A is not a Q_0 matrix.

Let $M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Note that M is a non-symmetric PSD matrix of rank 2 and $N \in C_0$ and it is easy to check that the system $q + Mx - N^ty \geq 0,\ y \geq 0$ is feasible. Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ (for example $d = e$ where e is a n dimensional column vector of all 1's) terminates with a secondary ray for this q, as LCP(q, A) has no solution. Thus if M is a non-symmetric PSD matrix, Theorem 2.3.7 does not hold.

Chapter 3

Generalized Positive Subdefinite Matrices and their Properties

3.1 Introduction

The class of generalized positive subdefinite (GPSBD) matrices is an interesting matrix class introduced by Crouzeix and Komlósi [11]. This class is a generalization of the class of symmetric positive subdefinite (PSBD) matrices introduced by Martos [46] and nonsymmetric PSBD matrices studied by Crouzeix et al. [10]. We recall that A is called a *PSBD matrix* if for all $x \in R^n,\ x^tAx < 0$ implies A^tx is unisigned. A is said to be *merely positive subdefinite* (MPSBD) if A is a PSBD matrix but not a PSD matrix. The solution set of a linear complementarity problem ($S(q, A)$) can be linked with the set of KKT-stationary points ($S^{''}(q, A)$) of the corresponding quadratic programming problem. The row sufficient matrices have been characterized by Cottle et al. [8] as the class for which the solution set of LCP(q, A) is the same as the solution set of KKT points of the corresponding quadratic program. In [11], Crouzeix and Komlósi showed that the property ($S^{''}(q, A) \subseteq S(q, A)$) holds for generalized positive subdefinite (GPSBD) matrices. However, we show that $(S''(q, A) \subseteq S(q, A))$ holds for GPSBD matrices under some additional assumptions. We produce a counterexample to show that the assumption cannot be relaxed further (see Section 3.3).

In this chapter, we study generalized positive subdefinite (GPSBD) matrices and obtain some properties of this matrix class. In Section 3.2, we introduce the notations and provide the relevant definitions used in this chapter. In Section 3.3, we show that every principal submatrix of a GPSBD matrix is also GPSBD and prove that a copositive GPSBD matrix is a P_0-matrix. We show that a copositive merely generalized positive subdefinite (MGPSBD) matrix with some additional conditions on it belongs to the class of row sufficient matrices. Further, it is shown that for a subclass of GPSBD matrices, the solution set of a linear complementarity problem is the same as the set of KKT-stationary points of the corresponding quadratic programming problem. We provide a counterexample to show that a copositive GPSBD matrix need not be sufficient. Finally, we show that if a matrix A which can be written as a sum of a copositive-plus MGPSBD matrix with an additional condition and a copositive matrix and if it satisfies a feasibility condition then Lemke's algorithm can solve LCP(q, A). This further extends the result obtained in Chapter 2 and a result of Jones [38] and Evers [18].

A large subclass of GPSBD matrices is identified as row sufficient matrices in this chapter. This has practical relevance to the study of quadratic programming and interior point algorithms. In addition, our result mentioned above on the applicability of Lemke's algorithm extends the class of LCP solvable by Lemke's algorithm.

3.2 Preliminaries

A matrix $A \in R^{n\times n}$ is called *generalized positive subdefinite matrix* (GPSBD) [11] if there exist nonnegative multipliers s_i, t_i with $s_i + t_i = 1,\ i = 1, 2, \ldots, n$ such that

$$\forall\, z \in R^n,\quad z^tAz < 0 \Rightarrow \begin{cases} \text{either} & -s_iz_i + t_i(A^tz)_i \geq 0 \text{ for all } i, \\ \text{or} & -s_iz_i + t_i(A^tz)_i \leq 0 \text{ for all } i. \end{cases} \tag{3.2.1}$$

Let S and T be two nonnegative diagonal matrices with diagonal elements s_i, t_i where $s_i + t_i = 1$ for $i = 1, \ldots, n$. Note that S and T are independent of z. A matrix $A \in R^{n\times n}$ is said to be GPSBD if there exist two nonnegative diagonal matrices S and T with $S + T = I$ such that

$$\forall\, z \in R^n,\quad z^tAz < 0 \Rightarrow \begin{cases} \text{either} & -Sz + TA^tz \geq 0 \\ \text{or} & -Sz + TA^tz \leq 0. \end{cases} \tag{3.2.2}$$

Note that GPSBD reduces to PSBD if $S = 0$. A is called *nondegenerate* GPSBD if for all $z \in R^n$, $z^tAz < 0$, implies $-Sz + TA^tz \neq 0$ and unisigned, i.e., at least one of the inequalities in (3.2.2) should hold as a strict inequality. A is said to be a *merely generalized positive subdefinite* (MGPSBD) *matrix* if A is a GPSBD matrix but not a PSBD matrix.

The following is a nontrivial example of a GPSBD matrix.

Example 3.2.1 Let $A = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. Note that $\nu_-(A + A^t) = 1$. Then for any $z = [z_1\ \ z_2\ \ z_3]^t$, $z^tAz = z_1z_2 < 0$ implies z_1 and z_2 are of opposite sign. Clearly, $A^tz = [-z_2\ \ 2z_1 + z_3\ \ \ -z_2]^t$ and for $z = [-1\ \ 1\ \ 5]^t$, $z^tAz < 0$ but A^tz is not unisigned. Therefore, A is not a PSBD matrix. However, with the choice $s_1 = 0, s_2 = 1$ and $s_3 = 0$, it is easy to check that A is a GPSBD matrix.

A careful examination of the definition of GPSBD matrix leads to the following result.

PROPOSITION 3.2.1 *Let S and T be two nonnegative diagonal matrices such that $S + T = I$. Then A is a GPSBD matrix if any one of the following conditions holds:*

(i). $Z_1 \subseteq Z_2$, where $Z_1 = \{z \mid z^tAz < 0\}$ and $Z_2 = \{z \mid z^t(-S + TA^t)z < 0\}$ and $(-S + AT)$ is PSBD.

(ii). $t_i = k,\ 0 \leq k \leq 1,\ \forall\, i$ and $(-S + AT)$ is PSBD.

(iii). $\bar{Z} = \{z \mid z^tAz < 0\} \subseteq R^n_+ \cup -R^n_+$.

(iv). $a_{ii} \geq 0,\ \forall\, i$ and there exists an $i = i_0$ such that $A_{i_0\cdot} \geq 0$ and $a_{ij} = 0,\ \forall\, i, j,\ i \neq i_0,\ j \neq i$.

(v). $a_{ii} \geq 0,\ \forall\, i$ and there exists a $j = j_0$ such that $A_{\cdot j_0} \neq 0$ and $a_{ij} = 0,\ \forall\, i, j,\ i \neq j,\ j \neq j_0$.

Proof. (i). This part follows from the fact that $z^tAz < 0 \Rightarrow z^t(-S+TA^t)z < 0$ and $(-S + AT)$ is PSBD.
(ii). When $t_i = k,\ \forall\, i$ then $Z_1 \subseteq Z_2$.
(iii). The conclusion follows if we take $s_i = 1,\ \forall\, i$.
(iv) and (v). Choose $t_i = \frac{1}{1+a_{ii}}$. ■

3.3 Main Results on GPSBD Matrices

THEOREM 3.3.1 *Suppose A is a GPSBD matrix. Then $A_{\alpha\alpha} \in$ GPSBD where $\alpha \subseteq \{1, \ldots, n\}$.*

Proof. Let $x_\alpha \in R^{|\alpha|}$ and

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $\alpha \subseteq \{1, \dots, n\}$.

Suppose that $x_\alpha^t A_{\alpha\alpha} x_\alpha < 0$. Now define $z \in R^n$ by taking $z_\alpha = x_\alpha$ and $z_{\bar{\alpha}} = 0$. Then $z^t A z = x_\alpha^t A_{\alpha\alpha} x_\alpha$. Since A is a GPSBD matrix, $z^t A z = x_\alpha^t A_{\alpha\alpha} x_\alpha < 0 \Rightarrow$ either $(-S + TA^t)z \geq 0$ which implies that $(-S_{\alpha\alpha} + T_{\alpha\alpha} A_{\alpha\alpha}^t)x_\alpha \geq 0$ or $(-S + TA^t)z \leq 0$ which implies $(-S_{\alpha\alpha} + T_{\alpha\alpha} A_{\alpha\alpha}^t)x_\alpha \leq 0$. Hence $A_{\alpha\alpha} \in$ GPSBD. ■

This theorem says that GPSBD is a complete class in the sense of [7, 3.9.5]. Crouzeix et al. [10, Theorem 2.1] observed that if A is a PSBD matrix with rank$(A) \geq 2$, then A^t is also PSBD. However, this is not true for GPSBD matrices in general. This is shown in the following example.

Example 3.3.1 Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}$. Take $z = [1 \ \frac{5}{2} \ -1]^t$. Clearly, A and A^t are not PSBD. Now with the choice $s_1 = 1, s_2 = \frac{1}{2}$, $s_3 = \frac{1}{2}$, it is easy to see that A is a GPSBD matrix. Now consider A^t. Note that for $z = [1 \ \frac{5}{2} \ -1]^t$,

$$(-S + TA)z = \begin{bmatrix} (1-2s_1)z_1 - (1-s_1)z_2 \\ -(1-s_2)z_1 + (1-2s_2)z_2 \\ 10(1-s_3)z_1 + (1-2s_3)z_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s_1 - \frac{3}{2} \\ -4s_2 + \frac{3}{2} \\ -8s_3 + 9 \end{bmatrix}.$$

It is easy to see that no s_1, s_2, s_3 exists for which A^t satisfies the definition of a GPSBD matrix.

The following theorem is observed in [11]. For the sake of completeness, we include the proof here.

Theorem 3.3.2 *Suppose A is a nondegenerate MGPSBD matrix. Then $\nu_-(A + A^t) = 1$.*

Proof. Let $B = A + A^t$. Note that B has at least one negative eigenvalue since A is not PSD. Suppose for contradiction there exist $\lambda_1, \lambda_2, z_1, z_2$ so that

$$Bz_1 = 2\lambda_1 z_1,\ Bz_2 = 2\lambda_2 z_2,\ ||z_1||^2 = ||z_2||^2 = 1,$$

$$\lambda_1 \leq \lambda_2 < 0 \text{ and } z_1^t z_2 = 0.$$

It is easy to see that $z_1^t A z_1 < 0$ and $z_2^t A z_2 < 0$. Without loss of generality, assume that $\tilde{A}^t z_1 \leq 0$ and $\tilde{A}^t z_2 \geq 0$ where $\tilde{A}^t = -S + TA^t$.

Now for $\mu \in [0,1]$, we define $z(\mu) = \mu z_1 + (1-\mu) z_2$. Then $z(\mu)^t B z(\mu) = 2\mu^2 \lambda_1 + 2(1-\mu)^2 \lambda_2 < 0$. Since A is a nondegenerate GPSBD matrix, it follows that $0 \neq \tilde{A}^t z(\mu) = \mu \tilde{A}^t z_1 + (1-\mu)\tilde{A}^t z_2 \in R^n_+ \cup -R^n_+$, since $\tilde{A}^t z(\mu)$ is unisigned. Now, since $\tilde{A}^t z(0) \geq 0$ and $\tilde{A}^t z(1) \leq 0$, it follows that there exists a $\bar{\mu} \in (0,1)$ such that $\tilde{A}^t z(\bar{\mu}) = 0$, which contradicts the nondegeneracy assumption. ■

Theorem 3.3.3 *Suppose A is a nondegenerate MGPSBD matrix where all the diagonal entries of T are positive. Then $(A + A^t)z = 0 \Rightarrow Az = A^t z = 0$.*

Proof. Let $B = A + A^t$. Suppose there exist λ_1, z_1 with $\lambda_1 < 0$ and $||z_1||^2 = 1$ such that $Bz_1 = 2\lambda_1 z_1$. Let z_0 satisfy $Bz_0 = 0$. For $\mu \in R$, define $z(\mu) = z_1 + \mu z_0$. Then it is easy to see that $z^t(\mu) B z(\mu) = 2\lambda < 0$. Without loss of generality, assume that $\tilde{A}^t z_1 \leq 0$ where $\tilde{A}^t = -S + TA^t$. Since A is a nondegenerate MGPSBD matrix, $\tilde{A}^t z(\mu) \neq 0$. Now for all $\mu \in R$, $0 \neq \tilde{A}^t z(\mu) = \tilde{A}^t z_1 - \mu S z_0 + \mu T A^t z_0 \in R^n_+ \cup -R^n_+$, since $\tilde{A}^t z(\mu)$ is unisigned. Now, since $0 \neq \tilde{A}^t z(\mu)$ is unisigned, the terms containing the coefficient μ should vanish. Therefore, $A^t z_0 = 0$. Since $Bz_0 = 0$, it follows that $Az_0 = 0$. ■

Theorem 3.3.4 *Suppose $A \in GPSBD \cap C_0$. Then $A \in P_0$.*

Proof. To show that $A \in P_0$, it is enough to show that $A^t \in P_0$. In view of a result of Fiedler and Pták [21, Theorem 1.3] it is enough to show that for any nonzero z, $\max\limits_{z_i \neq 0}[z_i (A^t z)_i] \geq 0$. Given a nonzero $z \in R^n$, let $I_1 = \{i : z_i > 0\}$ and $I_2 = \{i : z_i < 0\}$. We need to consider three cases.

Case-I: $I_2 = \emptyset$. Then $z^t A^t z = z^t A z \geq 0$ as $A \in C_0$. Hence $\max\limits_i [z_i (A^t z)_i] \geq 0$.

Case-II: $I_1 = \emptyset$. Then $(-z)^t A^t (-z) = z^t A z \geq 0$ as $A \in C_0$. Hence $\max\limits_i [z_i (A^t z)_i] \geq 0$.

Case-III: $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. Suppose that $\max_i [z_i(A^tz)_i] < 0$. Now $z^tAz = z^tA^tz = \sum_i [z_i(A^tz)_i] < 0$. This implies $-s_iz_i + t_i(A^tz)_i \geq 0, \ \forall \ i$ or $-s_iz_i + t_i(A^tz)_i \leq 0, \ \forall \ i$. Suppose $-s_iz_i + t_i(A^tz)_i \geq 0, \ \forall \ i$. Then for all $i \in I_1$, $-s_iz_i^2 + t_i[z_i(A^tz)_i] \geq 0$. But since $\max_i [z_i(A^tz)_i] < 0$, we get $-s_iz_i^2 + t_i[z_i(A^tz)_i] < 0, \ \forall \ i \in I_1$. This leads to a contradiction. Therefore, $\max_i [z_i(A^tz)_i] \geq 0$. Similarly, when $-s_iz_i + t_i(A^tz)_i \leq 0, \ \forall \ i \in I_2$, we consider $-s_iz_i^2 + t_i[z_i(A^tz)_i] \geq 0$. But since $\max_i [z_i(A^tz)_i] < 0$, we get $-s_iz_i^2 + t_i[z_i(A^tz)_i] < 0, \ \forall \ i \in I_2$. This leads to a contradiction. Therefore, $\max_i [z_i(A^tz)_i] \geq 0$. This completes the proof. ■

In Chapter 2, it is shown that if A is a copositive PSBD matrix of rank ≥ 2, then A is sufficient. However, the following example shows that a GPSBD matrix need not be sufficient.

EXAMPLE 3.3.2 Consider the copositive GPSBD matrix A in Example 3.3.1. Note that A is not PSBD. It is easy to check that A is a row sufficient matrix. However, A is not a column sufficient matrix.

We prove the following result on row sufficiency.

THEOREM 3.3.5 *Suppose $A \in MGPSBD \cap C_0$ with $0 < t_i < 1$ for all i. Then A is a row sufficient matrix.*

Proof. Suppose $z_i(A^tz)_i \leq 0$ for $i = 1, \ldots, n$. Let $I_1 = \{i \ : \ z_i > 0\}$ and $I_2 = \{i \ : \ z_i < 0\}$. We need to consider three cases.

Case-I: $I_2 = \emptyset$. Then $z^tAz = z^tA^tz = \sum_i z_i(A^tz)_i \leq 0$. Since $A \in C_0$, $[z_i(A^tz)_i] = 0, \ \forall \ i$.

Case-II: $I_1 = \emptyset$. Then $(-z)^tA^t(-z) = z^tA^tz = \sum_i z_i(A^tz)_i \leq 0$. Since $A \in C_0$, $[z_i(A^tz)_i] = 0, \ \forall \ i$.

Case-III: Suppose there exists a vector z such that $z_i(A^tz)_i \leq 0$ for $i = 1, 2, \ldots, n$ and $z_k(A^tz)_k < 0$ for at least one $k \in \{1, 2, \ldots, n\}$. Let $I_1 \neq \emptyset$ and

$I_2 \neq \emptyset$. $z^tAz = z^tA^tz = \sum_i [z_i(A^tz)_i] < 0$. This implies $-s_iz_i + t_i(A^tz)_i \geq 0, \ \forall\, i$ or $-s_iz_i + t_i(A^tz)_i \leq 0, \ \forall\, i$.

Without loss of generality, assume $-s_iz_i + t_i(A^tz)_i \geq 0, \ \forall\, i$. Then for all $i \in I_1$, $-s_iz_i^2 + t_iz_i(A^tz)_i \geq 0$. This implies $[z_i(A^tz)_i] \geq \frac{s_i}{t_i}z_i^2 > 0, \ \forall\, i \in I_1$. Therefore, $\sum_{i \in I_1} [z_i(A^tz)_i] > 0$. Since $z_i(A^tz)_i \leq 0$ for $i = 1, \ldots, n$, this leads to a contradiction. Therefore, $[z_i(A^tz)_i] = 0, \ \forall\, i$. So A is row sufficient. ∎

Note that the assumption in the above theorem $0 < t_i < 1, \ \forall\, i$, can not be relaxed.

Example 3.3.3 Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 10 & 0 & 0 \end{bmatrix}$. Note that A is copositive but it is not row sufficient.

For $z = [-1 \ \ -1 \ \ 1]^t$, $z^tAz < 0$. Clearly A is not PSBD. Now for $z = [-1 \ \ -1 \ \ 1]^t$,

$$(-S + TA^t)z = \begin{bmatrix} (1-2s_1)z_1 + (1-s_1)2z_2 + (1-s_1)10z_3 \\ (1-2s_2)z_2 \\ -s_3z_3 \end{bmatrix} = \begin{bmatrix} 7 - 6s_1 \\ 2s_2 - 1 \\ -s_3 \end{bmatrix}.$$

It is easy to see that no s_1, s_2, s_3 exists where $0 < s_i < 1, \forall\, i$, i.e., no t_1, t_2, t_3 exists where $0 < t_i < 1, \ \forall\, i$ for which the definition of GPSBD matrix is satisfied. However, with the choice $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ s_3 = 0$, A is a MGPSBD matrix.

The following result is a consequence of the characterization of row sufficient matrices observed by Cottle et al. [8].

Lemma 3.3.1 *Suppose $A \in MGPSBD \cap C_0$ with $0 < t_i < 1$ for all i. For each vector $q \in R^n$, if (z^*, u^*) is a Karush-Kuhn-Tucker pair of the quadratic program $QP(q, A)$:* $[\min z^t(Az+q); z \geq 0, \ Az+q \geq 0]$, *then z^* solves $LCP(q, A)$:* $[z \geq 0, \ Az + q \geq 0, \ z^t(Az + q) = 0]$.

Proof. From Theorem 3.3.5 and [8, Theorem 4, p.238], the result follows. ∎

REMARK 3.3.1 *The Example 3.3.3 and Lemma 3.3.1 show that the statement* $S''(q, A) \subseteq S(q, A)$ *in [11] holds for GPSBD matrices only with some additional assumptions as stated in Theorem 3.3.5.*

THEOREM 3.3.6 *Assume A is a nonnegative merely generalized positive subdefinite matrix with* $0 < t_i < 1,\ \forall\ i$. *Then* $A \in C_0^*$.

Proof. By Theorem 3.3.5, A is row sufficient. Therefore, $A \in Q_0$ [8, p.239]. Now by Theorem 1.4.4, for any nonnegative Q_0-matrix $A_{i\cdot} \neq 0 \Rightarrow a_{ii} > 0$. Let $\alpha = \{i \mid a_{ii} > 0\}$. Then $A_{\bar{\alpha}\cdot} = 0$. For any $x \geq 0$ such that $x^t Ax = 0$, we have $x_\alpha = 0$. Hence $x^t A = 0$, so $A \in C_0^*$. ∎

From Theorem 3.3.5, it follows that LCP(q, A) where A is a copositive MGPSBD matrix with $0 < t_i < 1$ is processable by Lemke's algorithm.

The following theorem extends the result of Evers [18] and the result obtained in [49] for solving LCP(q, A) by Lemke's algorithm when A satisfies certain conditions stated in the following theorem. The proof follows along the similar lines of the proof given in Theorem 2.3.7.

THEOREM 3.3.7 *Suppose* $A \in R^{n\times n}$ *can be written as* $M + N$ *where* $M \in$ *MGPSBD* $\cap\, C_0^+$, *is nondegenerate with* $0 < t_i < 1,\ \forall\ i$ *and* $N \in C_0$. *If the system* $q + Mx - N^t y \geq 0,\ y \geq 0$ *is feasible, then Lemke's algorithm for LCP*(q, A) *with covering vector* $d > 0$ *terminates with a solution.*

Proof. Suppose there exist an $x^* \in R^n$ and a $y^* \in R^n_+$ such that $q + Mx^* - N^t y^* \geq 0$. First we shall prove that for any $x \geq 0$, if $Ax \geq 0$ and $x^t Ax = 0$, then $x^t q \geq 0$. Note that for given $x \geq 0$, $x^t Ax = 0 \Rightarrow x^t(M + N)x = 0$ and since $M, N \in C_0$, this implies that $x^t Mx = 0$. As M is a nondegenerate MGPSBD matrix, by Theorem 3.3.3 we get $x^t Mx = 0 \Rightarrow x^t(M+M^t)x = 0 \Rightarrow (M+M^t)x =$

$0 \Rightarrow M^t x = 0 \Rightarrow Mx = 0$. Also since $Ax \geq 0$, it follows that $Nx \geq 0$ and hence $x^t N^t y^* \geq 0$. Further since $q + Mx^* - N^t y^* \geq 0$ and $x \geq 0$, it follows that $x^t(q + Mx^* - N^t y^*) \geq 0$. This implies that $x^t q \geq x^t N^t y^* \geq 0$.

Now from Corollary 4.4.12 and Theorem 4.4.13 of [7, p.277], it follows that Lemke's algorithm for LCP(q, A) with covering vector $d > 0$ terminates with a solution. ∎

The following example demonstrates that the class MGPSBD $\cap$ C_0^+ is nonempty.

Example 3.3.4 Consider the copositive-plus matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}$.

Take $z = [-1 \ \ -1 \ \ 1]^t$. It is easy to check that A is not MPSBD. However, with choice $s_i = \frac{1}{2} \, \forall \, i$, A is a MGPSBD matrix.

Chapter 4

Fully Copositive and Fully Semimonotone Matrices

4.1 Introduction

In linear complementarity theory, the issue of uniqueness of solution has received much attention. A result of Samelson, Thrall and Wesler [89] which was later discovered independently by Ingleton [36] and Murty [66] states that LCP(q, A) has a unique solution for all $q \in R^n$ if and only if A has positive principal minors. However, the literature on the uniqueness of solution is rather small. Cottle and Stone [9] introduced a matrix class $\mathcal{U}$ for which LCP(q, A) has a unique solution for all $q \in \text{int } K(A)$ where $K(A)$ is the union of all complementary cones corresponding to A. Further, in the same paper Cottle and Stone [9] enlarged the class $\mathcal{U}$ by demanding uniqueness of solution for LCP(q, A) only for those q's which lie in the interior of a full complementary cone. This is a geometric characterization of the class E_0^f of so-called fully semimonotone matrices.

Stone [97] studied various properties of E_0^f and conjectured that $E_0^f \cap Q_0 \subseteq P_0$. In [60], the conjecture was resolved for E_0^f-matrices of order 4 and for some subcases under various assumptions on A. In the same paper, E_0^f was replaced by its subclass (defined in the next section) called fully copositive matrices (C_0^f); the conjecture was shown to hold for C_0^f-matrices with positive diagonal entries. Murthy and Parthasarathy [61] proved that $C_0^f \cap Q_0 \subseteq P_0$, that $C_0^f \cap Q_0$ matrices of order 2 are PSD and that a bisymmetric Q_0-matrix is PSD if and only if it is fully copositive. It is known that PSD matrices are sufficient. They conjectured that if $A \in C_0^f \cap Q_0$ then it is PSD.

In this chapter, we study C_0^f and E_0^f-matrices. In Section 4.2, we present the required definitions, results and introduce the notations used in this chapter. In Section 4.3, we present a different proof of the result that $C_0^f \cap Q_0$ matrices are sufficient. We also consider C_0^f-matrices with positive diagonal entries [60] and show that they are column sufficient. We provide an example to show that $C_0^f \cap Q_0 \not\subseteq$ PSD and thus settle the conjecture made by Murthy and Parthasarathy [61]. Finally, in Section 4.4 we consider E_0^f-matrices introduced by Cottle and Stone [9] and partially address Stone's conjecture [97] that $E_0^f \cap Q_0 \subseteq P_0$ by showing that $E_0^f \cap D^c$ matrices are contained in P_0 where D^c is Doverspike's class of matrices for which all the strongly degenerate complementary cones of $(I, -A)$ are contained in the boundary of *pos* $(I, -A)$. See Section 1.4 for details. This generalizes the result of Sridhar [94] to the effect that $E_0^f \cap R_0 \subseteq P_0$.

4.2 Preliminaries

We require the following definitions, theorems and lemma in the next section.

Definition 4.2.1 We say that A is *fully semimonotone* (E_0^f) if every legitimate PPT of A is in E_0. By a *legitimate principal pivot transform* we mean the PPT obtained from A by performing a principal pivot on its nonsingular principal submatrices.

DEFINITION 4.2.2 We say that A is *fully copositive* (C_0^f) if every legitimate PPT of A is C_0.

Note that $P \subseteq P_0 \subseteq E_0^f \subseteq E_0$ and $C_0^f \subseteq E_0^f$. If A belongs to any one of the classes $E_0, C_0, E, C, E_0^f, C_0^f$ or the class of sufficient matrices then so is (i) any principal submatrix of A and (ii) any principal permutation of A. For details on the class of fully copositive matrices, see [60], [61], [63] and [64].

In [63], Murthy et al. proved that $C_0^f \cap Q_0$ matrices are sufficient. The elements of $C_0^f \cap Q_0$ are completely Q_0-matrices ([61]) and share many properties of positive semidefinite (PSD) matrices. Symmetric $C_0^f \cap Q_0$ matrices are PSD. The principal pivoting algorithm of Graves ([33]) for solving LCPs with PSD matrices also processes matrices in the class $C_0^f \cap Q_0$.

THEOREM 4.2.1 *([61, Theorem 4.5]) Suppose $A \in C_0^f \cap Q_0$. Then $A \in P_0$.*

THEOREM 4.2.2 *([61, Theorem 3.3]) Let $A \in C_0^f$. The following statements are equivalent:*

(a) *A is a Q_0-matrix.*

(b) *for every PPT M of A, $m_{ii} = 0 \Rightarrow m_{ij} + m_{ji} = 0, \ \forall\, i, j \in \{1, 2, \ldots, n\}$.*

(c) *A is a completely Q_0-matrix.*

THEOREM 4.2.3 *([61, Theorem 4.9]) If $A \in R^{2\times 2} \cap C_0^f \cap Q_0$, then A is a PSD matrix.*

LEMMA 4.2.1 *([63, Lemma 14]) Let $A \in P_0$ and $q \in R^n$. If (w, z) and (y, x) are two distinct solutions of LCP(q, A) then there exists an index i, $1 \le i \le n$, such that either $z_i = x_i = 0$ or $w_i = y_i = 0$.*

4.3 Fully Copositive Matrices and the Conjecture of Murthy and Parthasarathy

It is known that positive semidefinite matrices are sufficient. Murthy and Parthasarathy [63] proved that $C_0^f \cap Q_0$ matrices are sufficient. Here, we show that this result is a consequence of the following result proved by Cottle and Guu [5].

THEOREM 4.3.1 *$A \in R^{n\times n}$ is sufficient if and only if every matrix obtained from it by means of a PPT operation is sufficient of order 2.*

As a consequence we have the following theorem.

THEOREM 4.3.2 *Let $A \in C_0^f \cap Q_0$. Then A is sufficient.*

Proof. Note that all 2×2 submatrices of A or its PPTs are $C_0^f \cap Q_0$ matrices since A and all its PPTs are completely Q_0-matrices. Now by Theorem 4.2.3, all 2×2 submatrices of A or its PPTs are positive semidefinite, and hence sufficient. Therefore A or every matrix obtained by means of a PPT operation is sufficient of order 2. Now by Theorem 4.3.1, A is sufficient. $\blacksquare$

REMARK 4.3.1 *In [61], it is shown that Graves's principal pivoting algorithm [33] for solving LCP(q, A) where A is positive semidefinite also processes LCP(q, A) with $A \in C_0^f \cap Q_0$. By Theorem 4.3.2, it follows that Cottle's principal pivoting method also processes LCP(q, A) when $A \in C_0^f \cap Q_0$. See [3] and other references cited therein.*

Murthy and Parthasarathy [60] proved the following theorem.

THEOREM 4.3.3 *Suppose $A \in R^{n\times n} \cap C_0^f$. Assume that $a_{ii} > 0 \ \forall \ i \in \{1, 2, \ldots, n\}$. Then $A \in P_0$.*

In contrast to the above we observe that with the assumption of positive diagonal entries, a C_0^f-matrix is a column sufficient matrix and that if a matrix A with positive diagonal entries, and its transpose are in C_0^f, then such a matrix is in Q_0 and hence it is a completely Q_0-matrix.

THEOREM 4.3.4 *Let $A \in R^{n\times n} \cap C_0^f$. Assume that $a_{ii} > 0 \; \forall \; i \in \; \{1, 2, \ldots, n\}$. Then*
(i) A is column sufficient.
(ii) In addition, if $A^t \in C_0^f$, then A is a completely Q_0-matrix.

Proof. We shall first show that A is column sufficient.

Let $q \in R^n$ and consider the solution set $S(q, A)$ of the LCP(q, A). From Theorem 4.3.3, it follows that $A \in P_0$. From Theorem 4.3 in [104], it follows that A is sufficient if $n = 1$ or 2. Let us make the induction hypothesis that if $B \in R^{(n-1)\times(n-1)} \cap C_0^f$ with the assumption $b_{ii} > 0, \; \forall \; i = 1, \ldots, n-1$, then B is sufficient of order $(n-1)$. Let $A \in C_0^f$ be of order n with $a_{ii} > 0, \; \forall \; i$. To show that A is column sufficient, it is enough to show that $S(q, A)$ is convex $\forall \; q \in \; R^n$ by Theorem 6 in [8]. Let (w, z), (y, x) be two solutions to LCP(q, A) and let $0 < \lambda < 1$ be given. Now since $A \in P_0$, from Lemma 4.2.1 it follows that there is an index i, $1 \le i \le n$, such that either $x_i = z_i = 0$ or $w_i = y_i = 0$.

Case (i): $x_i = z_i = 0$.

In this case $x_\alpha \ne z_\alpha \in \; S(q_\alpha, A_{\alpha\alpha})$ where $\alpha = \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. From the induction hypothesis $\lambda \, x_\alpha + (1-\lambda) z_\alpha \in \; S(q_\alpha, A_{\alpha\alpha})$. Hence it follows that $\lambda \, x + (1-\lambda) z \in \; S(q, A)$.

Case (ii): $y_i = w_i = 0$.

Without loss of generality, we assume that $i = 1$. We have $a_{11} > 0$ by the hypothesis of the theorem. Let LCP$(\bar{q}, M)$ be the PPT of LCP(q, A) with respect to $\alpha = \{1\}$. Let $(\bar{y}, \bar{x}), (\bar{w}, \bar{z})$ be the solutions to LCP$(\bar{q}, M)$ corresponding to the solutions (y, x), (w, z) of LCP(q, A), respectively. It follows that $\bar{x}_1 = 0$ and

$\bar{z}_1 = 0$. From here it follows that $\lambda\,(\bar{y}, \bar{x}) + (1-\lambda)(\bar{w}, \bar{z}) \in\ S(\bar{q}, M)$ and hence $\lambda\,(y, x) + (1-\lambda)(w, z) \in\ S(q, A)$. Thus it follows that $S(q, A)$ is convex.

By the principle of induction, it follows that A is column sufficient for all n.

Now to conclude (ii) under the additional assumption that A^t is a C_0^f-matrix. we proceed as follows. As $A^t \in C_0^f$ and has positive diagonal entries, from the part(i) it follows that A^t is also column sufficient. Thus A is sufficient and hence $A \in Q_0$. Since the above arguments apply to every principal submatrix of A, it follows that A is a completely Q_0-matrix. ■

The following example shows that in the above theorem for the stronger conclusion in (ii), it is necessary to assume that A^t is also a C_0^f-matrix.

Example 4.3.1

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that the above matrix is a C_0^f-matrix but it is not a Q_0-matrix. For example, the vector

$$q = \begin{bmatrix} -8 \\ -5 \\ 2 \end{bmatrix}$$

is feasible but LCP(q, A) has no solution. It is also easy to verify that A^t is not a C_0^f-matrix.

Murthy and Parthasarathy [61] proved that if $A \in R^{2\times 2} \cap C_0^f \cap Q_0$, then A is positive semidefinite and conjectured that this will be true for all $n \times n$ matrices. However, we present below a counterexample to this conjecture.

EXAMPLE 4.3.2

$$A = \begin{bmatrix} 1 & \frac{7}{4} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that principal submatrices of order 2 of $A + A^t$ are positive definite but $\det(A+A^t) < 0$. Therefore A is not positive semidefinite. However, $A \in Q_0$ since A is strictly copositive.

We now show that $A \in C_0^f$.

Note that there are four distinct PPTs of A, each of which happens to correspond to four choices of the index set α. The first of these PPTs is the strictly copositive matrix itself. It is the PPT of A corresponding to

$$\alpha = \emptyset \text{ or } \alpha = \{3\}.$$

The other PPTs are

$$M_1 = \begin{bmatrix} 1 & -\frac{7}{4} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & \frac{7}{4} & -\frac{7}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & -\frac{7}{4} & \frac{7}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The index set α to which these PPTs correspond are, respectively,

(1) $\{1\}$ and $\{1,3\}$;

(2) $\{2\}$ and $\{2,3\}$;

(3) $\{1,2\}$ and $\{1,2,3\}$.

The copositivity of the matrices M_1, M_2 and M_3 can be demonstrated by determinantal criteria such as those given in [6] or by an analysis of the corresponding quadratic forms which can be rewritten as follows:

(1) $x^t M_1 x = (x_1 - \frac{7}{8}x_2)^2 + \frac{15}{64}x_2^2 + x_2x_3 + x_3^2$

(2) $x^t M_2 x = (x_2 - \frac{1}{2}x_3)^2 + x_1^2 + \frac{7}{4}x_1x_2 - \frac{7}{4}x_1x_3 + \frac{3}{4}x_3^2$

(3) $x^t M_3 x = (x_3 - \frac{1}{2}x_2)^2 + x_1^2 + \frac{7}{4}x_1x_3 - \frac{7}{4}x_1x_2 + \frac{3}{4}x_2^2$

Hence $A \in C_0^f \cap Q_0$. But A is not positive semidefinite.

4.4 Fully semimonotone Matrices and Stone's Conjecture

Stone ([97]) conjectured that within the class of Q_0-matrices, fully semimonotone matrices are P_0. Note that $E_0^f \cap D^c \subseteq Q_0$. We now prove a special case of Stone's conjecture [97] by showing that $E_0^f \cap D^c \subseteq P_0$. This generalizes the result $E_0^f \cap R_0 \subseteq P_0$, due to Sridhar ([94]).

THEOREM 4.4.1 *Let A be an $n \times n$ real matrix. Let $\mathcal{K}(A)$ denote the union of all the facets of the complementary cones of $(I, -A)$. Consider $q \in R^n \backslash \mathcal{K}(A)$ where q is nondegenerate with respect to A. Let $\beta \subseteq \{1, 2, \ldots, n\}$ be such that* $\det(A_{\beta\beta}) \neq 0$ *and let $\bar{M}$ be a PPT of A with respect to β. Then*

$$\deg_{\bar{M}}(\bar{q}) \ = \ \frac{\det(A_{\beta\beta})}{|\det(A_{\beta\beta})|}. \deg_A(q).$$

Proof. Note that this is a generalization of Theorem 6.6.23 in [7]. This theorem asserts the conditions of Theorem 6.6.23 without assuming that A is R_0, for the local degree when it is defined. The proof of this theorem is similar to that of Theorem 6.6.23 in [7, p. 595]. ■

Let $\zeta = pos\,(I, -A)$ and let $\mathcal{C}(A)$ denote the union of all strongly degenerate cones of $(I, -A)$. Further suppose that $\mathcal{C}(A)$ is contained in the boundary of ζ. Then ζ, being convex, is a connected component of $R^n \backslash \mathcal{C}(A)$. Hence by Theorem 6.1.17, [7, p.515] it follows that if q and q' are two nondegenerate vectors in ζ, then

$$\deg_A(q) = \deg_A(q') \tag{4.4.1}$$

We denote this common degree of A, restricted to ζ, by $\deg_\zeta(A)$. Let $\bar{M}$ be any PPT of A with respect to a given index set $\beta \subseteq \{1, 2, \ldots, n\}$ such that $\det(A_{\beta\beta}) \neq 0$. Let $pos(I, -\bar{M}) = \bar{\zeta}$. We now have the following theorem.

THEOREM 4.4.2 *Let* $A \in R^{n\times n} \cap\ E_0^f \cap D^c$. *Then* $\deg_\zeta(A) = 1$ *where* $\zeta = pos\,(I, -A)$.

Proof. It is well-known that if $A \in R^{n\times n} \cap E_0^f \cap D^c$ then the strongly degenerate complementary cones of $(I, -A)$ are contained in the boundary of $pos\,(I, -A)$. See [14]. Further since A is a Q_0-matrix, $pos(I, -A) = \{q \mid S(q, A) \neq \emptyset\} = \zeta$, is a convex set. Hence the interior of ζ is a connected component of $R^n \setminus \mathcal{C}(A)$. Thus $\deg_\zeta(A)$ is well defined. Further if $q^* > 0$ then LCP(q^*, A) has a unique solution, namely $w = q^*,\ z = 0$. Hence $\deg_A(q^*) = 1$. It follows from (4.4.1) that $\deg_\zeta(A) = 1$. ∎

We now prove our main result.

THEOREM 4.4.3 *Let* $A \in R^{n\times n} \cap E_0^f \cap D^c$. *Then* A *is a* P_0*-matrix.*

Proof. Suppose not. Then there is a set $\beta \subseteq \{1, \ldots, n\}$ such that $\det(A_{\beta\beta}) < 0$. Let $\bar{M}$ be the PPT of A with respect to β. Note that $\bar{M}$ is again a $E_0^f \cap D^c$-matrix and hence $\deg_{\bar{\zeta}}(\bar{M}) = 1$ by Theorem 4.4.2 where $pos(I, -\bar{M}) = \bar{\zeta}$. Now however, from Theorem 4.4.1, it follows that for any $q \in \zeta$ which is nondegenerate with respect to A,

$$\deg_{\bar{M}}(\bar{q}) \;=\; \frac{\det(A_{\beta\beta})}{|\det(A_{\beta\beta})|} \cdot \deg_A(q) = -1.\deg_A(q) = -1.$$

Therefore $\deg_{\bar{\zeta}}(\bar{M}) = -1$ which is a contradiction. ∎

We conclude this chapter after stating a corollary whose proof is obvious since $E_0^f \cap L \subseteq E_0^f \cap D^c$.

COROLLARY 4.4.1 *Suppose* $A \in E_0^f \cap L$. *Then* A *is a* P_0*-matrix.*

Chapter 5

Almost Type Classes of Matrices with Q-property

5.1 Introduction

The notion of an *almost type* class was introduced by Väliaho [103], [102]. There he defined and thoroughly investigated the class of almost copositive matrices and show that such matrices are of crucial importance in deriving criteria for copositivity. Olech et al. [75] introduced the class of almost N-matrices, namely the class of matrices whose determinant is positive and all proper principal minors are negative. Pye [82] studied the class of *almost* P_0*-matrices* of order n whose determinant is negative and all proper principal minors are nonnegative.

Let $\mathcal{Y}$ be the class of all square matrices of all orders that satisfy a particular property. Then a square matrix A is *almost-*$\mathcal{Y}$ if $\mathcal{Y}$ contains all principal submatrices of A except for A itself. For example, for an *almost* N_0 (N)*-matrix,* $\det A_{\alpha\alpha} \leq 0$ (< 0) $\forall\, \alpha \subset \{1, 2, \ldots, n\}$ and $\det A > 0$. We say that A is an *almost* P_0 (P)*-matrix* if $\det A_{\alpha\alpha} \geq 0$ (> 0) $\forall\ \alpha \subset \{1, 2, \ldots, n\}$ and $\det A < 0$. The almost type classes are referred as matrices of exact order 1 in Mohan et al. [58].

In linear complementarity theory, much of the research is devoted to finding constructive characterizations of matrices having the Q-property. The class of matrices due to Saigal [88] for which LCP$(0, A)$ has a unique solution and LCP(q, A) has an odd number of solutions for some nondegenerate q with respect to A is a large class satisfying the Q-property. The almost P_0 and almost C_0 classes satisfying the Q-property are in R_0.

In Section 5.2, some notations, definitions and some well-known results in linear complementarity and matrix games are presented which will be used in the sequel. In Section 5.3, we introduce almost $\bar{N}$-matrix (a new subclass of almost N_0-matrices which are obtained as a limit of a sequence of almost N-matrices) and obtain a sufficient condition for the almost $\bar{N}$ class with positive value to possess the Q-property. We give a counterexample to show that an almost $\bar{N} \cap Q$ matrix need not be an R_0-matrix. In Section 5.4, we consider a generalization of almost $\bar{N}$-matrix, namely, $\bar{N}$-matrix of exact order 2 and extend the results proved for almost $\bar{N}$ class to this class. In Section 5.5, we introduce another new class called $\bar{E}(d)$ and show that $\bar{E}(d) \cap R_0 \subset Q$. Finally, in Section 5.6, we show that Pang's conjecture is true if E_0 is replaced by almost C_0. We also consider almost P_0-matrices and give a game theoretic proof of necessary and sufficient condition for this class to possess the Q-property.

5.2 Preliminaries

In this section, we define some well-known matrix classes and state results which will be used in Chapter 5 and Chapter 6.

Definition 5.2.1 A matrix $A \in R^{n\times n}$ is said to be an N_0 (N)-*matrix of exact order* k, $(1 \leq k \leq n)$ if every principal submatrix of order $(n-k)$ is an N_0 (N)-matrix and every principal minor of order r, $(n-k) < r \leq n$ is positive.

N-matrices of exact order 1 and 2 are studied in detail by Sridhar [96] and Mohan, Parthasarathy and Sridhar [58].

DEFINITION 5.2.2 $A \in R^{n\times n}$ is said to be an *almost copositive* if it is copositive of order $(n-1)$ but not of order n. A copositive matrix $A \in R^{n\times n}$ is said to be an *almost strictly copositive (almost copositive plus)* if it is strictly copositive (copositive plus) of order $(n-1)$ but not of order n.

DEFINITION 5.2.3 $A \in R^{n\times n}$ is said to be *copositive of exact order* 2 if it is copositive of order $(n-2)$ but not of order n and $(n-1)$. Similarly, a copositive matrix $A \in R^{n\times n}$ is said to be *strictly copositive (copositive-plus) of exact order* 2 if it is strictly copositive (copositive-plus) of order $(n-2)$ but not of order n and $(n-1)$.

REMARK 5.2.1 *Almost copositive matrices are also called exact order matrices of order* $(n-1)$ *in Väliaho [102]. However, in this dissertation we mean almost copositive matrices as copositive matrices of exact order* 1. *For details see [102] and [103].*

We make use of the following result on the class R_0 due to Murty [66] and Saigal [88].

THEOREM 5.2.1 *If* $A \in R_0$ *and LCP*(q, A) *has an odd number of solutions for a nondegenerate* q, *then* $A \in Q$.

The following results were proved by Väliaho [102] for symmetric almost copositive matrices. However, it is easy to see that these results hold for nonsymmetric almost copositive matrices as well.

THEOREM 5.2.2 *Let* $A \in R^{n\times n}$ *be almost copositive. Then* A *is PSD of order* $(n-1)$ *and* A *is PD of order* $(n-2)$.

THEOREM 5.2.3 *Suppose A is almost strictly copositive. Then A is PSD and PD of order $(n-1)$.*

THEOREM 5.2.4 *Suppose that A is almost copositive-plus (or copositive-plus of exact order 1). Then it is strictly copositive of exact order 2.*

The following result on semimonotone matrices is due to Pang [77].

THEOREM 5.2.5 ([77]) *Suppose $A \in E_0 \cap Q$. Then the system $Ax = 0,\ x > 0$ has no solution.*

The inconsistency of the above system is equivalent to the fact that any nonzero solution to LCP$(0, A)$ must have some zero components. Further, every nontrivial solution of LCP$(0, A)$ has at least two nonzero coordinates.

The following results will be used in the sequel.

THEOREM 5.2.6 ([60]) *Suppose $A \in R^{n\times n}$ is an almost P_0-matrix. Let $B = A^{-1}$. Then there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ such that $B_{\alpha\alpha} \leq 0$, $B_{\bar{\alpha}\bar{\alpha}} \leq 0, B_{\bar{\alpha}\alpha} \geq 0$ and $B_{\alpha\bar{\alpha}} \geq 0$.*

THEOREM 5.2.7 ([60, p.1271]) *Suppose $A \in Q$ (Q_0). Assume that $A_{i\cdot} \geq 0$ for some $i \in \{1, 2, \ldots, n\}$. Then $A_{\alpha\alpha} \in Q$ (Q_0), where $\alpha = \{1, 2, \ldots, n\}\backslash\{i\}$.*

THEOREM 5.2.8 ([88, p.45]) *A sufficient condition for LCP(q, A) to have even number of solutions for all q for which each solution is nondegenerate is that there exists a vector $z > 0$ such that $z^t A < 0$.*

THEOREM 5.2.9 ([63, p.195]) *Let $A \in R^{n\times n}$ be an E_0-matrix with $n \geq 3$. Suppose any one of the following conditions holds:*

(i) Every principal submatrix of order $(n-1)$ is an R_0-matrix.

(ii) Every principal submatrix of order less than or equal to $(n-2)$ is an R_0-matrix.

Then A is a Q-matrix if and only if A is an R_0-matrix.

5.3 Almost $\bar{N}$-matrices

The class of $\bar{N}$-matrices was introduced by Mohan and Sridhar in [57]. The class of almost N-matrices is studied in [58]. We introduce here a new matrix class almost $\bar{N}$ which is a subclass of the almost N_0-matrices. See also [75, p.119].

DEFINITION 5.3.1 A matrix $A \in R^{n\times n}$ is said to be an *almost $\bar{N}$-matrix* if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are almost N-matrices such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

EXAMPLE 5.3.1 Let $A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Note that A is an almost N_0-matrix. It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence of almost N-matrices

$$A^{(k)} = \begin{bmatrix} -1 & 2 & 2 \\ \frac{1}{k} & -\frac{1}{k} & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

REMARK 5.3.1 *It is well-known that for P_0 (almost P_0)-matrices, by perturbing the diagonal entries alone one can get a sequence of P (almost P)-matrices that converges to an element of P_0 (almost P_0). However, this is not true for N_0 (almost N_0)-matrices. One of the reasons is that an N (almost N)-matrix needs to have all its entries nonzero. In the above example, we can see that even though the matrix $A \in$ almost N_0, it cannot be obtained as a limit point of almost N-matrices by perturbing the diagonal. However, we show in the above example that $A \in$ almost $\bar{N}$.*

The following example shows that an almost N_0-matrix need not be an almost $\bar{N}$-matrix.

EXAMPLE 5.3.2 Let $A = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$. Here A is an almost N_0-matrix. However, it is easy to verify that A is not an almost $\bar{N}$-matrix since we cannot get A as a limit point of a sequence of almost N-matrices.

Now we consider almost N_0-matrices and ask the following question. Suppose $A \in$ almost N_0. Then is it true that (i) $A \in Q$ implies $A \in R_0$ (ii) $A \in R_0$ implies $A \in Q$?

In the sequel, we partially settle the above questions. The following example demonstrates that $A \in$ almost $N_0 \cap Q$ but $A \notin R_0$.

EXAMPLE 5.3.3 Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$. It is easy to check that $A \in$ almost N_0. Now taking a PPT with respect to $\alpha = \{1, 3\}$ we get $M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$. Now $A \in Q$ since M (a PPT of A) $\in Q$ *(see [63, p. 193])*. However, $(0, 1, 0, 0)$ solves LCP$(0, A)$. Hence $A \notin R_0$.

The following example due to Olech et al. *[75, p.120]* shows that an almost N_0-matrix, even with value positive, need not be a Q-matrix or an R_0-matrix.

EXAMPLE 5.3.4 Let $A = \begin{bmatrix} -2 & -2 & -2 & 2 \\ -2 & -1 & -3 & 3 \\ -2 & -3 & -1 & 3 \\ 2 & 3 & 3 & 0 \end{bmatrix}$ $q = \begin{bmatrix} -1001 \\ -500 \\ -500 \\ 500 \end{bmatrix}$. It is easy to check that $A \in$ almost N_0 but $A \notin Q$ even though $v(A)$ is positive. Furthermore,

$A \notin R_0$.

However, if $A \in$ almost $\bar{N} \cap R_0$ and $v(A) > 0$, then we show that $A \in Q$.

In the statement of some theorems that follow, we assume that $n \geq 4$, to make use of the sign pattern stated in the following lemma.

LEMMA 5.3.1 *Suppose $A \in R^{n\times n}$ is an almost $\bar{N}$-matrix of order $n \geq 4$. Then there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ such that A can be written in the partitioned form as (if necessary, after a principal rearrangement of its rows and columns)*

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Proof. This follows from Remark 3.1 in [58, p. 623] and from the definition of almost $\bar{N}$-matrices. ■

REMARK 5.3.2 *In the proof of the sign pattern in Lemma 5.3.1, we assume $n \geq 4$ since lemma requires that all the principal minors of order 3 or less are negative.*

THEOREM 5.3.1 *Suppose $A \in E_0 \cap$ almost $\bar{N}$ $(n \geq 4)$. Then there exists a principal rearrangement*

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A where $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices and $B_{\bar{\alpha}\alpha}$, $B_{\alpha\bar{\alpha}}$ are nonnegative matrices.

Proof. Note that A is an almost $\bar{N}$-matrix of order $n \geq 4$. By Lemma 5.3.1 there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ satisfying

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

$A \in E_0$ implies $A_{\alpha\alpha} \in E_0$. It is easy to see that there exist permutation matrices $\mathcal{L} \in R^{|\alpha|\times|\alpha|}$ and $\mathcal{M} \in R^{|\bar{\alpha}|\times|\bar{\alpha}|}$ such that $\mathcal{L}A_{\alpha\alpha}\mathcal{L}^t$ and $\mathcal{M}A_{\bar{\alpha}\bar{\alpha}}\mathcal{M}^t$ are strict upper triangular matrices. Let

$$P = \begin{bmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{M} \end{bmatrix}$$

be a permutation matrix. Then

$$B = PAP^t = \begin{bmatrix} \mathcal{L}A_{\alpha\alpha}\mathcal{L}^t & \mathcal{L}A_{\alpha\bar{\alpha}}\mathcal{M}^t \\ \mathcal{M}A_{\bar{\alpha}\alpha}\mathcal{L}^t & \mathcal{M}A_{\bar{\alpha}\bar{\alpha}}\mathcal{M}^t \end{bmatrix}$$

where $\mathcal{L}A_{\alpha\alpha}\mathcal{L}^t$ and $\mathcal{M}A_{\bar{\alpha}\bar{\alpha}}\mathcal{M}^t$ are nonpositive strict upper triangular matrices and $\mathcal{L}A_{\alpha\bar{\alpha}}\mathcal{M}^t$, $\mathcal{M}A_{\bar{\alpha}\alpha}\mathcal{L}^t$ are nonnegative matrices. Hence the result. ■

The following example shows that almost $\bar{N} \cap E_0$ is nonempty.

EXAMPLE 5.3.5 Let $A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Here A is an $E_0 \cap N_0$-matrix. It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence $A^{(k)} = \begin{bmatrix} -\frac{1}{k} & -1 & \frac{2}{k} & 2 \\ -\frac{1}{k} & -\frac{1}{k} & 1 & \frac{2}{k} \\ \frac{4}{k} & 1 & -\frac{1}{k} & -1 \\ 1 & \frac{2}{k} & -\frac{1}{k} & -\frac{1}{k} \end{bmatrix}$ of almost N-matrices which converges to A as $k \to \infty$.

THEOREM 5.3.2 *Suppose $A \in R^{n\times n}$ is an almost $\bar{N} \cap Q_0 \cap E_0$-matrix with $n \geq 4$. Then there exists a principal rearrangement B of A such that all the leading principal submatrices of B are Q_0-matrices.*

Proof. Note that A is an almost $\bar{N} \cap Q_0 \cap E_0$-matrix with $n \geq 4$. Then by Theorem 5.3.1 there exists a principal rearrangement

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices and $B_{\bar{\alpha}\alpha}$, $B_{\alpha\bar{\alpha}}$ are nonnegative matrices. It is easy to conclude from the structure of B that $B_{n\cdot} \geq 0$. Note that $B \in Q_0$, since B is a principal rearrangement of A. Therefore, by Theorem 5.2.7, $B_{\beta\beta} \in Q_0$ where $\beta = \{1, 2, \ldots, n\} \setminus \{n\}$. Repeating the same argument, it follows that all leading principal submatrices of B are Q_0.
■

THEOREM 5.3.3 *Let $A \in$ almost $\bar{N} \cap R^{n\times n}, n \geq 4$ with $v(A) > 0$. Then $A \in Q$ if $A \in R_0$.*

Proof. Let $A \in$ almost $\bar{N} \cap R_0$. Then by Lemma 5.3.1, there exists $\emptyset \neq \alpha \subseteq \{1, 2, \ldots, n\}$, $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Now consider $A_{\alpha\alpha}$. Suppose $A_{\alpha\alpha}$ contains a nonnegative column vector. Then clearly LCP$(0, A)$ has a nontrivial solution which contradicts our hypothesis that $A \in R_0$. Hence every column of $A_{\alpha\alpha}$ should have at least one negative entry. Hence there exists an $x \in R^{|\alpha|}$, $x > 0$, such that $x^t A_{\alpha\alpha} < 0$. It now follows from Theorem 5.2.8 that for any $q_\alpha > 0$, where q_α is nondegenerate with respect to $A_{\alpha\alpha}$, LCP$(q_\alpha, A_{\alpha\alpha})$ has r solutions ($r \geq 2$ and even). Similarly, LCP$(q_{\bar{\alpha}}, A_{\bar{\alpha}\bar{\alpha}})$ has s solutions ($s \geq 2$ and even) for any $q_{\bar{\alpha}} > 0$, where $q_{\bar{\alpha}}$ is nondegenerate with respect to $A_{\bar{\alpha}\bar{\alpha}}$. Now suppose (w^i_α, z^i_α) is a solution for LCP$(q_\alpha, A_{\alpha\alpha})$. Note that $w = \begin{bmatrix} w^i_\alpha \\ q_{\bar{\alpha}} \end{bmatrix}$ and $z = \begin{bmatrix} z^i_\alpha \\ 0 \end{bmatrix}$ solves LCP(q, A). Similarly, associated with every solution $(w^i_{\bar{\alpha}}, z^i_{\bar{\alpha}})$ we can construct a solution of LCP(q, A). Thus LCP(q, A) has

$(r+s-1)$ solutions accounting for only once the solution $w = q, z = 0$. Thus there are an odd number $(r + s - 1 \geq 3)$ of solutions to LCP(q, A) with all solutions nondegenerate. We shall show that $(r + s - 1) \leq 3$ and hence there are only 3 solutions to LCP(q, A). Since q is nondegenerate with respect to A, this is a finite set [66, p.85]. Suppose $(\bar{w}, \bar{z})$ is a nondegenerate solution to LCP(q, A). Then $(\bar{w}, \bar{z}) \in S(q, A)$. Now since A is a limit point of almost N-matrices $\{A^{(k)}\}$, we note that the complementary basis corresponding $(\bar{w}, \bar{z})$ will also yield a solution to LCP$(q, A^{(k)})$ for all k sufficiently large. By Theorem 3.2 [58, p. 625], which asserts that there are exactly 3 solutions for LCP$(q, A^{(k)})$ for any nondegenerate $q(> 0)$ with respect to $A^{(k)}$, we obtain $(r + s - 1) \leq |S(q, A)| \leq |S(q, A^{(k)})| = 3$. But $(r+s-1) \geq 3$. Hence LCP(q, A) has exactly 3 solutions for any nondegenerate $q(> 0)$ with respect to A. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 5.2.1 that $A \in Q$. ■

COROLLARY 5.3.1 *Suppose $A \in$ almost $\bar{N} \cap R_0$ with $v(A) > 0$. Then $|deg(A)| =$ odd.*

Proof. This follows from the fact that LCP(q, A) has 3 solutions for any nondegenerate $q(> 0)$ with respect to A and $A \in R_0$. ■

However, the converse of Theorem 5.3.3 is not true.

Consider the matrix $A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$ in example 5.3.3 which is also a Q-matrix. Note that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence of almost N-matrices $A^{(k)} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -\frac{1}{k^2} & -\frac{1}{k} & -\frac{1}{k} \\ 1 & -\frac{1}{k} & -\frac{1}{k} & -1 \\ 1 & -\frac{1}{k} & -1 & -\frac{1}{k} \end{bmatrix}$. However, $A \notin R_0$.

The converse of the Theorem 5.3.3 is not true for $n < 4$ is illustrated in the following example.

EXAMPLE 5.3.6 Consider the matrix $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. It is easy to see that $A \in$ almost $\bar{N}$ since we can get A as a limit point of the sequence $A^{(k)} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -\frac{1}{k} \end{bmatrix}$ of almost N-matrices which converges to A as $k \to \infty$.

We show that $A \in Q$, by showing that its $A^{-1} \in Q$. Now look at $A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Suppose that $q_1 \geq q_2$ in LCP(q, A) where $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$.

It is easy to see that $A_{23} \in Q$ and $A_{13} \in Q$. Since $A_{23} \in Q$ there exists a solution $\left(\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}, \begin{bmatrix} z_2 \\ z_3 \end{bmatrix} \right)$ to LCP$(q_\alpha, A_{\alpha\alpha})$ where $\alpha = \{2, 3\}$. Now define $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $z = \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix}$ where $w_1 = w_2 + q_1 - q_2 + \frac{1}{3} z_2$. It is easy to check that (w, z) is a solution to LCP(q, A).

If $q_1 < q_2$, then we can get a solution to LCP(q, A) using a solution to LCP$(q_\alpha, A_{\alpha\alpha})$ where $\alpha = \{1, 3\}$. Now define $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ 0 \\ z_3 \end{bmatrix}$ where $w_2 = w_1 + q_2 - q_1 + \frac{1}{3} z_1$. It is easy to check that (w, z) is a solution to LCP(q, A). Since q is arbitrary, it follows that $A \in Q$. However, $A \notin R_0$.

5.4 A generalization of Almost $\bar{N}$-matrix

Mohan, Parthasarathy and Sridhar [58] introduced N-matrix of exact order 2 as a generalization of the almost N-matrix studied by Olech, Parthasarathy and Ravindran [75]. In this section we introduce a new class of matrices as a generalization of the almost $\bar{N}$-matrix introduced in the earlier section. This class originates from the limit of a sequence of N-matrices of exact order 2.

DEFINITION 5.4.1 A matrix $A \in R^{n\times n}$ is said to be an *$\bar{N}$-matrix of exact order 2* if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are N-matrices of exact order 2 such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

EXAMPLE 5.4.1 Let $A = \begin{bmatrix} 0 & -90 & -80 & -70 & 0 \\ -90 & -2 & -2 & -2 & 2 \\ -70 & -2 & -1 & -3 & 3 \\ -50 & -2 & -3 & -0.8 & 3 \\ 0 & 2 & 3 & 3 & 0 \end{bmatrix}$. Here A is an N_0-matrix of exact order 2.

Also A is an $\bar{N}$-matrix of exact order 2 since we can get A as a limit point of the sequence of N-matrices of exact order 2

$$A^{(k)} = \begin{bmatrix} -\frac{1}{k^2} & -90 & -80 & -70 & \frac{1}{k} \\ -90 & -2 & -2 & -2 & 2 \\ -70 & -2 & -1 & -3 & 3 \\ -50 & -2 & -3 & -0.8 & 3 \\ \frac{1}{k} & 2 & 3 & 3 & -\frac{1}{k^2} \end{bmatrix}.$$

The following example shows that the class of $\bar{N}$-matrices of exact order 2 is a proper subclass of N_0-matrices of exact order 2.

EXAMPLE 5.4.2 Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$. It is easy to check that A is an N_0-matrix of exact order 2. Note that all 3×3 principal submatrices of A are the limit of a sequence of 3×3, N matrices. However, A is not an $\bar{N}$-matrix of exact order 2, since we cannot get A as a limit of a sequence of N-matrices of exact order 2 such that the principal submatrices $A_{\alpha\alpha}$ for $\alpha = \{1, 2, 4\}$ and $\alpha = \{1, 2, 5\}$ belongs to the class N.

REMARK 5.4.1 *It is easy to see that Lemma 5.3.1 holds for $\bar{N}$-matrices of exact order* 2 *for* $n \geq 5$.

THEOREM 5.4.1 *For $n \geq 5$, let $A \in \bar{N} \cap R^{n \times n}$ be of exact order 2 with $v(A) > 0$. Suppose there exists at most one nonpositive principal submatrix of order $(n-1)$ and the values of the proper principal submatrices of order ≥ 2 of A which contains at least one positive entry are positive. Then $A \in Q$ if $A \in R_0$.*

Proof. There are two cases.

Case-I: Suppose there is a nonpositive principal submatrix of order $(n-1)$. We may assume, without loss of generality that $A_{\alpha\alpha} \leq 0$ where $\alpha = \{2, \ldots, n\}$. Since $A \in \bar{N} \cap R^{n \times n}, n \geq 5$ of exact order 2 with $v(A) > 0$ and $A \in R_0$, the sign pattern of A can be written as

$$A = \begin{bmatrix} - & \oplus & \oplus & \ldots & \oplus \\ + & & & & \\ + & & & A_{\alpha\alpha} & \\ \vdots & & & & \\ + & & & & \end{bmatrix}$$

where the sign symbol $\oplus$ denotes a nonnegative real number. Choose a $q > 0$ which is nondegenerate with respect to A and the partitioned form of q is $q = [q_1, \quad q_\alpha]^t$ where $|\alpha| = (n-1)$. By repeating a similar argument as in Theorem 5.3.3, we can show that LCP$(q_\alpha, A_{\alpha\alpha})$ has r solutions ($r \geq 2$ and even). Similarly, LCP(q_1, a_{11}) has 2 solutions. Thus there are an odd number ($r + 1 \geq 3$) of solutions to LCP(q, A) with all solutions nondegenerate.

Now we show that for this q ($q > 0$), LCP(q, A) has no other solution. Suppose $(\hat{w}, \hat{z})$ is another solution distinct from the odd number of solutions listed above.

Let $\beta = \{i \ : \ \hat{z}_i > 0\}$. Since $(\hat{w}, \hat{z})$ is different from the solutions listed above, it follows that the index $1 \in \beta$ and $\beta \cap \{2, \ldots, n\} \neq \phi$. Note that all $A_{\beta\beta}$ contains at least one positive entry. So, by assumption $v(A_{\beta\beta}) > 0$.

Now $\hat{w} - A\hat{z} = q$, leads to $A_{\beta\beta}\hat{z} < 0$ which contradicts our assumption $v(A_{\beta\beta}) > 0$.

Thus LCP(q, A) has an odd number of solutions. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 5.2.1 that $A \in Q$.

Case-II: Suppose there is no nonpositive principal submatrix of order $(n-1)$.

Then by Remark 5.4.1, there exists a $\emptyset \neq \alpha \subseteq \{1, 2, \ldots, n\}$ so that A can be written in the partitioned form as

$$A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ and $A_{\alpha\bar{\alpha}} \geq 0$.

Now consider $A_{\alpha\alpha}$. We proceed as in Theorem 5.3.3. Thus there are an odd number (≥ 3) of solutions to LCP(q, A) with all solutions nondegenerate. As before (Case-I) we can show that there are no other solutions. Since $A \in R_0$ and LCP(q, A) has an odd number of solutions, it follows from Theorem 5.2.1 that $A \in Q$. $\blacksquare$

REMARK 5.4.2 *Now since A is a limit of a sequence $\{A^{(k)}\}$ of N-matrices of exact order 2, we note that the complementary basis corresponding to a solution will also yield a solution to LCP$(q, A^{(k)})$ for all k sufficiently large. Hence there are exactly 5 solutions [58, p. 634] for LCP$(q, A^{(k)})$ for any nondegenerate $q(> 0)$ with respect to $A^{(k)}$. Therefore,* $3 \leq |S(q, A)| \leq |S(q, A^{(k)})| = 5$.

5.5 $\bar{E}(d)$-matrices

Garcia [27] introduced the class of matrices $E(d)$ which is dependent only on d as a generalization of E_0. For a given $d > 0$, $E(d)$ is the class of matrices for which LCP(d, A) has a unique solution $w = d,\ z = 0$. Now we ask the following question.

Is $E(d)$ closed for a given $d > 0$?

The answer is no and it is illustrated in the following example.

EXAMPLE 5.5.1 Consider the following matrix
$A = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$ and $A^{(k)} = \begin{bmatrix} -2 & 3 \\ -3 - \frac{1}{k} & 4 \end{bmatrix}$.
It is easy to see that for $d = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, LCP$(d, A^{(k)})$ has a unique solution $w = d,\ z = 0$ but LCP(d, A) has 2 solutions. Thus we have a sequence $\{A^{(k)}\}$ of matrices where $A^{(k)} \in E(d)$ and as $k \to \infty$, $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. However, $A \notin E(d)$. Thus the class $E(d)$ is not closed.

We now introduce a new matrix class.

DEFINITION 5.5.1 For a given positive vector $d \in R^n$, a matrix $A \in R^{n \times n}$ is said to be an *$\bar{E}(d)$-matrix* if there exists a sequence $\{A^{(k)}\}$ where $A^{(k)} = [a_{ij}^{(k)}]$ are in $E(d)$ such that $a_{ij}^{(k)} \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Note that the matrix A in the above example belongs to $\bar{E}(d)$. Although $E(d)$ is not closed, so that LCP(d, A) may have more than one solution, we have the following theorem.

THEOREM 5.5.1 *Suppose $A \in \bar{E}(d) \cap R_0$ for a given positive vector $d \in R^n$. Then $A \in Q$.*

Proof. Since $A \in \bar{E}(d)$ there exists a sequence $\{A^{(k)}\}$ of matrices such that $A^{(k)} \in E(d)$ and $A^{(k)} \to A$. Note that d is nondegenerate with respect to $A^{(k)}$ for all k and $d > 0$. Suppose d is degenerate with respect to A. Since the set $\{q \mid q \text{ is degenerate with respect to } A\}$ has dimension $\leq (n-1)$, it follows that we can find an $\epsilon > 0$ and $d^* \in N_\epsilon(d)$ where $N_\epsilon(d)$ is the ϵ-neighborhood of d such that d^* is nondegenerate with respect to A and also $A^{(k)}$ for all k. Now let $S(d^*, A) = \{(w, z) \,|\, (w, z) \text{ is a solution to LCP}(d^*, A)\}$. Note that $S(d^*, A) \neq \emptyset$, since $(d^*, 0) \in S(d^*, A)$ and also $S(d^*, A)$ is finite since d^* is nondegenerate with respect to A [66, p.85].

Let $\epsilon > 0$ be given. Suppose $(w^*, z^*) \in S(d^*, A)$. Thus for k large enough $S(d^*, A^{(k)}) \cap N_\epsilon(w^*, z^*) \neq \emptyset$ where $N_\epsilon(w^*, z^*)$ is the ϵ-neighborhood of (w^*, z^*). To see this, let B be the complementary basis submatrix of $(I, -A)$ induced by (w^*, z^*) and let $B^{(k)}$ be the corresponding complementary basis submatrix of $(I, -A^{(k)})$. Note that $B^{(k)}$ is arbitrarily close to B for large k and hence $(B^{(k)})^{-1}d^*$ can be made arbitrarily close to $B^{-1}d^*$ and in particular $(B^{(k)})^{-1}d^* > 0$. Therefore the corresponding solution (w^k, z^k) of LCP$(d^*, A^{(k)}) \in N_\epsilon(w^*, z^*)$.

Thus every solution of LCP(d^*, A) corresponds to a distinct solution of LCP$(d^*, A^{(k)})$ for k sufficiently large. Hence $\emptyset \neq |S(d^*, A)| \leq |S(d^*, A^{(k)})| = 1$, since LCP$(d^*, A^{(k)})$ has a unique solution by our choice of d^*. Therefore LCP(d^*, A) has a unique nondegenerate solution. Using Theorem 5.2.1, it follows that $A \in Q$. ∎

COROLLARY 5.5.1 *Suppose $A \in \bar{E}(d) \cap R_0$. Then $deg(A) = 1$.*

Proof. This follows from the uniqueness of the solution of LCP(q, A) for a nondegenerate $q = d^* > 0$ and $\det A_{\emptyset\emptyset} = 1$. ■

For implementation of Lemke's algorithm (see Section 1.4.2 for details), one needs a positive vector d. The above proof uses the fact that if $A^{(k)} \to A$ with $A^{(k)} \in E(d)$ one can find a d^* by perturbing d slightly, to be used as covering vector for processing LCP(q, A) by Lemke's algorithm. In the above example one may take $d^* = \begin{bmatrix} 2 \\ 2.98 \end{bmatrix}$. It is easy to check that LCP(d^*, A) has a unique solution $w = d^*,\ z = 0$.

5.6 Almost C_0 and almost P_0-matrices

Väliaho [103], [102] introduced symmetric almost C_0-matrices. The following example shows that an almost C_0-matrix need not be an E_0-matrix.

Example 5.6.1 Consider the following matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix}$. It is easy to see that $A \in$ almost C_0 but $A \notin E_0$.

Pang [77] proved the following theorem.

Theorem 5.6.1 *Suppose $A \in E_0$. If $A \in R_0$ then $A \in Q$.*

Pang conjectured that the converse must be true, i.e., $E_0 \cap Q \subset R_0$. However, this was disproved by Jeter and Pye [37]. Murthy et al. [62] showed that the conjecture is not true even if E_0-matrix is replaced by C_0-matrix. Here we show that if the class of E_0-matrices is replaced by the class of almost C_0-matrices then Pang's conjecture is true. We present a game theoretic proof.

Theorem 5.6.2 *Suppose $A \in$ almost C_0 with $n \geq 3$. If $A \in Q$ then $A \in R_0$.*

Proof. $A \in Q$ implies $v(A) > 0$. Suppose $v(A_{\alpha\alpha}) < 0$ for $\alpha \subset \{1, 2, \ldots, n\}$. Then there exists a mixed strategy y such that $y_\alpha^t A_{\alpha\alpha} < 0$. Define $x \in R_+^n$ such that $y_\alpha = x_\alpha$ and $x_{\bar{\alpha}} = 0$. Hence $x^t Ax = y_\alpha^t A_{\alpha\alpha} y_\alpha < 0$ which contradicts the fact that submatrices of order $(n-1)$ are copositive. Therefore, $v(A_{\alpha\alpha}) \geq 0 \; \forall \; \alpha \subseteq \{1, 2, \ldots, n\}$. It follows from Remark 1.4.1 that $A \in E_0$. From Theorem 5.2.2, it follows that A is PD of order $(n-2)$. Hence every principal submatrix of order less than or equal to $(n-2)$ is an R_0-matrix. Since $A \in Q$, by Theorem 5.2.9 it follows that $A \in R_0$. ■

To prove the converse we need the additional assumption $v(A) > 0$.

THEOREM 5.6.3 *Suppose $A \in$ almost C_0 with $v(A) > 0$. If $A \in R_0$ then $A \in Q$.*

Proof. Using a similar argument we can see that $A \in E_0$. Since $E_0 \cap R_0$-matrix is a Q_0-matrix with $v(A) > 0$, it follows from the Remark 1.4.1 that $A \in Q$. ■

THEOREM 5.6.4 *Suppose a copositive matrix A is almost copositive-plus with $n \geq 3$. Then $A \in Q$ if and only if $A \in R_0$.*

Proof. By Theorem 5.2.4, A is strictly copositive of exact order 2. So, by definition every principal submatrix of order less than or equal to $(n-2)$ is a strictly copositive matrix. It follows that every principal submatrix of order less than or equal to $(n-2)$ is an R_0-matrix. Since every C_0-matrix is an E_0-matrix, by Theorem 5.2.9, it follows that $A \in Q$ if and only if $A \in R_0$. ■

The following result was proved by W. C. Pye [82]. We present a game theoretic proof.

THEOREM 5.6.5 *Let A be a nonsingular almost $P_0 \cap R^{n\times n}$ matrix with $v(A) > 0$. Then the following statements hold.*

(i) If $A \in R_0$, then $A \in Q$.

(ii) If $A \in Q$, then $A \in R$.

Proof. Note that $A \in E_0$ since $v(A_{\alpha\alpha}) \geq 0 \ \forall \ \alpha \subseteq \{1, 2, \ldots, n\}$. Assume $A \in R_0$. Then $A \in Q_0$ with $v(A) > 0$. Hence $A \in Q$. Conversely assume that $A \in Q$. We show that $A \in R$. Suppose $A \notin R$. Let z be a nontrivial solution of $\mathrm{LCP}(te, A)$ where $t \geq 0$. There are two cases.

Case(a): $t > 0$. Let $\beta = \{i \mid \ z_i = 0\}$. Let $\alpha = \{1, 2, \ldots, n\} \setminus \beta$. Note that $(w_\alpha, z_\alpha), \ z_\alpha \neq 0$ is a solution of $\mathrm{LCP}(te_\alpha, A_{\alpha\alpha})$. Hence $A_{\alpha\alpha} z_\alpha < 0, \ z_\alpha > 0$. Therefore, $z_\alpha^t A_{\alpha\alpha}^t < 0, \ z_\alpha > 0$. But this implies $v(A_{\alpha\alpha}^t) < 0$. This contradicts the fact that $A_{\alpha\alpha}^t$ is a P_0-matrix. Therefore, $\mathrm{LCP}(te, A)$ where $t > 0$ has no nontrivial solution.

Case(b): $t = 0$. By Theorem 5 [82, p.441], it follows that if $\mathrm{LCP}(0, A)$ has a nontrivial solution then $A \notin Q$. Hence $A \in R$. ∎

Chapter 6

Principal Pivot Transforms of Some More Classes of Matrices

6.1 Introduction

Introduced by Tucker [101], the concept of principal pivot transforms (PPTs) plays an important role in the study of linear complementarity theory. In principal pivoting algorithm for linear complementarity problem, PPTs are used to exchange the role of basic and nonbasic variables. The PPT, under the name *sweep operator* also plays an important role in statistics mainly because of conceptual and computational advantages it enjoys in solving least squares regression problems. The PPT also appears under the term *gyration* and is mentioned in a survey of Schur complements by Cottle [2]. See also [100].

Tucker [101] proved that if the diagonal entries for every PPT of A are positive, then A is a P-matrix. However if the diagonal entries for every PPT of A is nonnegative, then A need not be a P_0-matrix. Cottle and Stone [9] introduced the notion of a fully semimonotone matrix (E_0^f) by requiring that every PPT of such a matrix be a semimonotone matrix. We recall that A is called *fully* *copositive* (C_0^f) if every legitimate PPT of A is C_0.

Motivated by the class of almost C_0 matrices introduced by Väliaho, we introduce two new classes of matrices based on principal pivot transforms in this chapter. One of the new classes has the property that its PPTs are either C_0 or almost C_0 with at least one PPT almost C_0, and the other class has the property that its PPTs are either E_0 or almost C_0 with at least one PPT almost C_0.

In Section 6.2, some notations, definitions and a few well-known results in linear complementarity and matrix games are presented that will be used in the next section. In Section 6.3, we present some results on the class for which PPTs are either in C_0 (E_0) or almost C_0 with at least one PPT almost C_0. The almost classes studied in this chapter have algorithmic significance. If A belongs to the intersection of this class and Q_0, then LCP(q, A) can be processed by Lemke's algorithm. For a description of Lemke's algorithm see Section 1.4.2. For many results we present proofs which use some terminology from matrix games. Finally in Section 6.4, we consider the problem of characterizing a class of matrices whose member possess at least one PPT that is a Z-matrix.

6.2 Preliminaries

The following definitions and notations are needed in Section 6.4.

Definition 6.2.1 We say that a matrix A is called an $N(P)$-*matrix of exact order* k, $1 \leq k \leq n$, if every principal submatrix of order $(n-k)$ is an N-matrix (P-matrix) and if every principal minor of order r, $n-k < r \leq n$ is positive (negative).

Definition 6.2.2 A is called a *matrix of exact order* k if it is a P-matrix or a N-matrix of exact order k.

Note that an $N(P)$-matrix is an $N(P)$-matrix of exact order 0, and an almost $N(P)$-matrix is an $N(P)$-matrix of exact order 1.

DEFINITION 6.2.3 An N-matrix of *exact order* 1 is of *first category* if both A and A^{-1} have at least one positive entry, otherwise it is N-matrix of *exact order* 1 of *second category.*

DEFINITION 6.2.4 A P-matrix of *exact order* 1 is of *first category* if A^{-1} has a positive entry otherwise it is said to be of *second category.*

Given a matrix $A \in R^{n\times n}$, let $B_i \in R^{(n-1)\times(n-1)}$, $i = 1, 2, \ldots, n$ denote the principal submatrices of A, obtained by deleting the i^{th} row and i^{th} column of A. Note that if A is of exact order k, then $B_i,\ 1 \leq i \leq n$, are matrices of exact order $(k-1)$.

DEFINITION 6.2.5 We say that a matrix $A(A \not< 0)$ of *exact order* 2 is of the *first category* if there exists at most one index k $(1 \leq k \leq n)$ such that the $(n-1)\times(n-1)$ exact order 1 principal submatrix B_k is nonpositive and every $(n-1)\times(n-1)$ principal submatrix B_i which is $\not< 0$, $1 \leq i \leq n$ is exact order 1 of the first category. We say that it is of the *second category*, if all B_i are of the second category.

For further details, see [58].

We make use of the following result in the sequel.

LEMMA 6.2.1 *Let $M \in R^{n\times n}$ be a PPT of a given matrix $A \in R^{n\times n}$. Then $v(A) > 0$ if and only if $v(M) > 0$.*

Proof. It is enough to show that $v(A) > 0 \Rightarrow v(M) > 0$. Let $v(A) > 0$. Then there exists a $z > 0$ such that $Az > 0$.

$$\text{Let } \begin{bmatrix} w_\alpha \\ w_{\bar\alpha} \end{bmatrix} = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar\alpha} \\ A_{\bar\alpha\alpha} & A_{\bar\alpha\bar\alpha} \end{bmatrix} \begin{bmatrix} z_\alpha \\ z_{\bar\alpha} \end{bmatrix}.$$

Premultiplying by $\begin{bmatrix} -A_{\alpha\alpha} & 0 \\ -A_{\bar{\alpha}\alpha} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}^{-1} = \begin{bmatrix} -A_{\alpha\alpha}^{-1} & 0 \\ -A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1} & I_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$ and rewriting we get

$$\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix}$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}}{=}{-}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Since $\begin{bmatrix} z_\alpha \\ w_{\bar{\alpha}} \end{bmatrix} > 0$ and $\begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} > 0$, it follows that $v(M) > 0$. ■

If A is a Q-matrix then $v(A) > 0$ [63]. Since any PPT M of a Q-matrix is again a Q-matrix, it follows that for any Q-matrix $v(M) > 0$ in all of its PPTs M. It is easy to prove that for any matrix A with $v(A) > 0$, $A \in Q$ if and only if $A \in Q_0$.

The following results are needed in sequel.

THEOREM 6.2.1 ([60]) *Suppose $A \in R^{n\times n}$ ($n \geq 3$) is a nonsingular N_0-matrix. Then there exists a nonempty subset α of $\{1, 2, \ldots, n\}$ such that* $A = \begin{bmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$, $A_{\alpha\alpha} \leq 0$, $A_{\bar{\alpha}\bar{\alpha}} \leq 0, A_{\bar{\alpha}\alpha} \geq 0$ *and* $A_{\alpha\bar{\alpha}} \geq 0$.

THEOREM 6.2.2 ([60]) *Suppose $A \in R^{n\times n}$ ($n \geq 3$) is a nonsingular $E_0 \cap N_0$ matrix. Then there exists a principal rearrangement*

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \emptyset$, $\alpha \neq \{1, 2, \ldots, n\}$, $B_{\bar{\alpha}\alpha} \geq 0$, $B_{\alpha\bar{\alpha}} \geq 0$ and $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are strict upper triangular nonpositive matrices.

It is easy to observe the following.

THEOREM 6.2.3 *Assume $A \in R^{n\times n}$ ($n \geq 3$) is an $E_0 \cap N_0 \cap Q_0$ matrix. Then there exists a principal rearrangement B of A such that all the leading principal submatrices of B are Q_0-matrices.*

Proof. By Theorem 6.2.2, there exists a principal rearrangement

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of A such that $\alpha \neq \emptyset$, $\alpha \neq \{1, 2, \ldots, n\}$, $B_{\bar{\alpha}\alpha} \geq 0$, $B_{\alpha\bar{\alpha}} \geq 0$ and $B_{\alpha\alpha}$, $B_{\bar{\alpha}\bar{\alpha}}$ are strict upper triangular nonpositive matrices. It is easy to conclude from the structure of B that $B_{n\cdot} \geq 0$. Note that $B \in Q_0$, since B is a principal rearrangement of A. Therefore by Theorem 5.2.7, $B_{\beta\beta} \in Q_0$ where $\beta = \{1, 2, \ldots, n\} \backslash \{n\}$. Similarly, we can show that the other leading principal submatrices of B are Q_0. ∎

6.3 Some PPT Based Matrix Classes and their Subclasses

Stone [97] conjectured that a fully semimonotone Q_0-matrix has nonnegative principal minors. Various subclasses of E_0^f, C_0^f were studied earlier in [9], [60], [61], [48]. In this section, we consider some more classes, defined using principal pivot transforms. One of these classes has the property that its PPTs are either C_0 or almost C_0 with at least one PPT almost C_0. The other class considered in this chapter has the property that its PPTs are either E_0 or almost C_0 with at least one PPT almost C_0. Note that an almost C_0-matrix is not necessarily E_0. We show that the intersection of this class and Q_0 is in E_0^f by showing that this class is in P_0.

Definition 6.3.1 A is said to be an *almost fully copositive (almost C_0^f)* matrix if its PPTs are either C_0 or almost C_0 and there exists at least one PPT M of A for some $\alpha \subset \{1, 2, \ldots, n\}$ that is almost C_0.

EXAMPLE 6.3.1 The following matrix A is almost fully copositive.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

THEOREM 6.3.1 *If $A \in R^{n\times n} \cap$ almost $C_0^f \cap Q_0$ $(n \geq 3)$, then A is a P_0-matrix.*

Proof. Suppose M is a PPT of A so that $M \in$ almost C_0. By Theorem 5.2.2, all the principal submatrices of order $(n-1)$ of M are PSD. Now to show $M \in P_0$ it is enough to show that $\det M \geq 0$. Suppose $\det M < 0$. Then M is an almost P_0-matrix. Therefore $M^{-1} \in N_0$ follows from Theorem 1.4.6. Now, by Theorem 6.2.1 there exists a nonempty subset $\alpha \subseteq \{1, 2, \ldots, n\}$ satisfying

$$M^{-1}_{\alpha\alpha} \leq 0,\ M^{-1}_{\bar{\alpha}\bar{\alpha}} \leq 0,\ M^{-1}_{\alpha\bar{\alpha}} \geq 0 \text{ and } M^{-1}_{\bar{\alpha}\alpha} \geq 0. \tag{6.3.1}$$

But M^{-1} is a PPT of M and by definition of almost C_0^f, $M^{-1} \in$ almost C_0 or $M^{-1} \in C_0$. We consider the following cases:

Case (i) : $M^{-1} \in$ almost C_0. Note that by Theorem 5.2.2, the principal submatrices of order $(n-2)$ are PD. Therefore the diagonal entries of M^{-1} are positive. But $M^{-1} \in N_0$ and hence contradicts (6.3.1). Therefore $\det(M) \geq 0$ and $M \in P_0$. Since M is a PPT of A it follows that of $A \in P_0$.

Case (ii). $M^{-1} \in C_0 \cap Q_0$. Since $M^{-1} \in N_0$ we must have $M^{-1}_{\alpha\alpha} = 0,\ M^{-1}_{\bar{\alpha}\bar{\alpha}} = 0$. Therefore

$$M^{-1} = \begin{bmatrix} 0 & M^{-1}_{\alpha\bar{\alpha}} \\ M^{-1}_{\bar{\alpha}\alpha} & 0 \end{bmatrix}$$

But this contradicts that M^{-1} is a Q_0-matrix. See Theorem 1.4.4. Therefore $M \in P_0$. ■

Now we consider the matrix class whose members have PPTs that are either E_0 or almost C_0 with at least one PPT that is almost C_0. The following example shows that this class is nonempty.

EXAMPLE 6.3.2 Consider the following matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

It is easy to verify that $A \in Q_0$ and all its PPTs are either E_0 or almost C_0.

THEOREM 6.3.2 *Suppose $A \in R^{n\times n} \cap Q_0$ $(n \geq 3)$ and the PPTs of A are either E_0 or almost C_0 with at least one PPT almost C_0. Then $A \in P_0$.*

Proof. Suppose M be a PPT of A so that $M \in$ almost C_0. By Theorem 5.2.2, all the submatrices of order $(n-1)$ of M are PSD. Now to complete the proof, we need to show that $\det M \geq 0$. Suppose $\det M < 0$. Then M is an almost P_0-matrix. Therefore $M^{-1} \in N_0$ and by Theorem 6.2.1 there exists a nonempty subset $\alpha \subseteq \{1, 2, \ldots, n\}$ satisfying

$$M^{-1}_{\alpha\alpha} \leq 0,\ M^{-1}_{\bar{\alpha}\bar{\alpha}} \leq 0,\ M^{-1}_{\alpha\bar{\alpha}} \geq 0 \text{ and } M^{-1}_{\bar{\alpha}\alpha} \geq 0. \tag{6.3.2}$$

But M^{-1} is a PPT of M and by definition $M^{-1} \in$ almost C_0 or $M^{-1} \in E_0$. We consider the following cases:

Case (i) : $M^{-1} \in$ almost C_0. Then the diagonal entries of M^{-1} are positive. But $M^{-1} \in N_0$ and contradicts (6.3.2). Therefore $\det(M) \geq 0$ and $M \in P_0$. Since M is a PPT of A it follows that of $A \in P_0$.

Case (ii) : $M^{-1} \in E_0 \cap Q_0$. Since $M^{-1} \in E_0 \cap N_0$ then by Theorem 6.2.2 there exists a principal rearrangement

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha\bar{\alpha}} \\ B_{\bar{\alpha}\alpha} & B_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

of M^{-1} such that $B_{\alpha\alpha},\ B_{\bar{\alpha}\bar{\alpha}}$ are nonpositive strict upper triangular matrices and $B_{\alpha\bar{\alpha}}, B_{\bar{\alpha}\alpha}$ are nonnegative matrices.

Take $\alpha = \{1, 2, \ldots, p\}$ and $\gamma = \{1, 2, \ldots, (p+1)\}$. Note that by Theorem 6.2.3, $B_{\gamma\gamma} \in Q_0$. Consider

$$B = \begin{bmatrix} B_{\alpha\alpha} & B_{\alpha(p+1)} & B_{\alpha\bar{\gamma}} \\ B_{(p+1)\alpha} & B_{(p+1)(p+1)} & B_{(p+1)\bar{\gamma}} \\ B_{\bar{\gamma}\alpha} & B_{\bar{\gamma}(p+1)} & B_{\bar{\gamma}\bar{\gamma}} \end{bmatrix}$$

Note that

$$B_{\bar{\alpha}\bar{\alpha}} = \begin{bmatrix} B_{(p+1)(p+1)} & B_{(p+1)\bar{\gamma}} \\ B_{\bar{\gamma}(p+1)} & B_{\bar{\gamma}\bar{\gamma}} \end{bmatrix}$$

is a strict upper triangular matrix nonpositive matrix. Therefore $B_{(p+1)(p+1)} = 0$ and $B_{\bar{\gamma}(p+1)} = 0$.

Now look at the principal submatrix $B_{\gamma\gamma}$ of order $(p+1)$. We shall show that $B_{\alpha(p+1)} = 0$. Suppose $b_{i_0(p+1)} > 0$ for some $i_0 \in \alpha$. Since $b_{i_0(p+1)} > 0$ there exists a q_γ such that $q_{i_0} < 0$ and $q_i > 0$ for all $i \in \gamma,\ i \neq i_0$, and the set of feasible solution $F(q_\gamma, B_{\gamma\gamma})$ of LCP$(q_\gamma, B_{\gamma\gamma})$ is nonempty. Let (w_γ, z_γ) be a solution of LCP$(q_\gamma, B_{\gamma\gamma})$. Then $z_{p+1} > 0$. Now $B_{(p+1)\alpha} \geq 0$ implies $w_{p+1} > 0$ contradicts $B_{\gamma\gamma} \in Q_0$. Therefore $B_{\alpha(p+1)} = 0$. Hence B is singular. But this leads to a contradiction. Therefore $A \in P_0$. ■

REMARK 6.3.1 *Note that Theorem 6.3.1 also follows from Theorem 6.3.2. However, in the proof of Theorem 6.3.1, we use different arguments that use the structure of a C_0-matrix.*

THEOREM 6.3.3 *Let $A \in E_0^f$ have exactly one zero principal minor. Assume that $A \in Q_0 \backslash Q$. Then there exists a PPT M of A such that the following holds: (i) rank$(M) = n - 1$, (ii) $Mz = 0$ and $\pi^t M = 0$ for some vectors $z, \pi > 0$.*

Proof. Assume $\det(A_{\alpha\alpha}) = 0$ for some $\alpha \subseteq \{1, 2, \ldots, n\}$. Let M be a PPT of A with respect to a nonsingular principal submatrix, say $A_{\beta\beta}$ of A such that $\det(M) = 0$. This follows from Theorem 1.4.6. Hence rank$(M) = n - 1$. Since

$M \in E_0^f$, LCP(d, M) has a unique solution for $d > 0$. Note that $M \in Q_0 \backslash Q$ since M is a PPT of A. Thus there exists a $q \in R^n$ such that LCP(q, M) does not have a solution. Therefore Lemke's algorithm when applied LCP(q, M) terminates in a secondary ray. Since no proper principal minor of M is zero and $M \in E_0^f$, it follows that, we get a positive vector z such that $Mz = 0$. Now we show that there is a positive vector $\pi > 0$ such that $\pi^t M = 0$. Without loss of generality, assume that z and π are probability vectors. Note that $M^t \in E_0$. Therefore $v(M^t) \geq 0$ by Theorem 1.4.3. Let $0 \neq \pi \geq 0$ be the optimal strategy for M^t. Therefore $M^t \pi \geq 0$. Now since $z^t M^t = 0$, therefore $v(M^t) = 0$ which implies $M^t \pi = 0$. Since $\det(M^t) = 0$ and the principal minors are nonzero, it follows that there is a positive vector $\pi > 0$ such that $\pi^t M = 0$. ■

The class of Q_0-matrices identified in the above theorem is contained in the class of Q_0-matrices of order n and rank $(n - 1)$ with positive vectors z and π satisfying $Mz = 0$ and $\pi^t M = 0$ mentioned in [15]. Note that the class is not contained in any well-known classes of Q_0-matrices such as those studied in Garcia [27]. Lemke's algorithm is not applicable for this class. However, Algorithm-I of Eagambaram and Mohan[15] can be applied to solve LCP(q, A) with A in the class identified above. Finally, we conclude the chapter by mentioning an open problem associated with PPTs in the next section.

6.4 Characterization of Matrices for which at least one PPT is a Z-matrix: An Open Problem

The principal pivot transform of a Z-matrix need not be a Z-matrix. However Väliaho [103] observed that the inverse of a symmetric almost copositive matrix is a Z-matrix. Mohan et al. [58] considered a class of matrices of exact order 2 whose inverses belong to class Z and observed the following result.

THEOREM 6.4.1 *Let $A \in R^{n\times n}(n \geq 5)$ be a matrix of exact order 2. $A^{-1} \in Z$ if and only if $v(A) < 0$ and A is of second category with each $B_i \not< 0$.*

For the class stated in the theorem the following result on algorithmic significance was also proved by Mohan et al. [58].

THEOREM 6.4.2 *Let $A \in R^{n\times n}(n \geq 5)$ be a matrix of exact order 2 of the second category with $B_i \not< 0$ for $1 \leq i \leq n$. Then a solution to LCP(q, A), if one exists, can be computed by obtaining a solution to LCP$(-A^{-1}q, A^{-1})$, in at most n steps.*

However, the complete characterization of the class of matrices for which at least one PPT is a Z-matrix remains an interesting open problem.

Chapter 7

Linear Complementarity and Two Classes of Structured Stochastic Games

7.1 Introduction

Stochastic games were first formulated by Shapley [92] in 1953. The games considered by Shapley are now called two-person zero-sum discounted games with finite state and action space. In this fundamental paper, Shapley [92] proved the existence of a value and optimal stationary strategies for discounted case which gave a method for iterative computation of the value of a stochastic game with discounted payoff. Gillette [28] studied the undiscounted case or limiting average payoff case. Since then there have been a number of papers on stochastic game dealing with the problem of finding sufficient conditions for the existence of their value and their optimal or ϵ-optimal strategies. As a generalization of Shapley's stochastic games, nonzero-sum stochastic games have been considered by many researchers. See [25], [98], [93] and the references therein. The theory of stochastic games has been applied to study many practical problems like search problems, military applications, advertising problems, the traveling inspector model, and various economic applications. For details see [24].

Another major area of research in this field is to identify those classes of zero-sum stochastic games for which there is a possibility of obtaining a finite step algorithm to compute a solution. Many of the results in this area are for zero-sum games with special structures. We will refer to these zero-sum stochastic games with special structure collectively as the class of *structured stochastic games.* The class of structured stochastic games contains single controller games, switching controller games, games with state independent transitions and separable rewards and additive reward and additive transitions (ARAT) games. For the above class of structured stochastic games, it is known that optimal stationary strategies exist and the game satisfies the orderfield property (i.e., the solution to the game lie in the same ordered field as the data of the game (e.g., rational)). Many of the researchers have attempted either to give a finite step method of computing a value and optimal strategies or at least to give a constructive proof for their existence.

Filar and Schultz [23] observed that an undiscounted zero-sum stochastic game possesses optimal stationary strategies if and only if a global minimum with optimum value zero can be found to an appropriate linearly constrained nonlinear program. Perhaps, a more interesting problem is the reduction of these nonlinear programs to linear complementarity problems or linear programs.

We look at the problem of formulating zero-sum structured stochastic game as linear complementarity problem. The linear complementarity problem arises in some classes of stochastic game problems, for example, see [51], [52], [53] and [90].

In Section 7.2, we present the necessary definitions and theorems to be used in subsequent sections. In Section 7.3 and Section 7.4, we formulate the problem of computing the value vector and optimal stationary strategies for zero-sum undiscounted switching controller and zero-sum ARAT stochastic games as linear complementary problems. We conclude this chapter by indicating some areas of further research in Section 7.5.

7.2 Preliminaries

A stochastic game with a finite state space and action space is defined below.

A two-player finite state/action space zero-sum stochastic game is defined by the following objects.

1. A state space $S = \{1, 2, \ldots, N\}$.
2. For each $s \in S$, finite action sets $A(s) = \{1, 2, \ldots, m_s\}$ for Player I and $B(s) = \{1, 2, \ldots, n_s\}$ for Player II.
3. A reward law $R(s)$ for $s \in S$ where $R(s) = [r(s, i, j)]$ is an $m_s \times n_s$ matrix whose $(i, j)^{th}$ entry denotes the payoff from Player II to Player I corresponding to the choices of action $i \in A(s),\ j \in B(s)$ by Player I and Player II respectively.
4. A transition law $q = (q_{ij}(s, s') :\ (s, s') \in S \times S, i \in A(s),\ j \in B(s))$, where $q_{ij}(s, s')$ denotes the probability of a transition from state s to state s' given that Player I and Player II choose actions $i \in A(s), j \in B(s)$ respectively.

The game is played in stages $t = 0, 1, 2, \ldots$ At some stage t, the players find themselves in a state $s \in S$ and independently choose actions $i \in A(s), j \in B(s)$. Player II pays Player I an amount $r(s, i, j)$ and at stage $(t + 1)$, the new state is $s^{'}$ with probability $q_{ij}(s, s^{'})$. Play continues at this new state.

The players guide the game via strategies and in general, strategies can depend on complete histories of the game until the current stage. We are however concerned with the simpler class of *stationary strategies* which depend only on the current state s and not on stages. So for Player I, a stationary strategy

$$f \in F_S = \{f_i(s) \mid s \in S, i \in A(s), f_i(s) \geq 0, \sum_{i \in A(s)} f_i(s) = 1\}$$

indicates that the action $i \in A(s)$ should be chosen by Player I with probability $f_i(s)$ when the game is in state s.

Similarly for Player II, a stationary strategy

$$g \in G_S = \{g_j(s) \mid s \in S, j \in B(s), g_j(s) \geq 0, \sum_{j \in B(s)} g_j(s) = 1\}$$

indicates that the action $j \in B(s)$ should be chosen with probability $g_j(s)$ when the game is in state s.

Here F_S and G_S will denote the set of all stationary strategies for Player I and Player II, respectively. Let $f(s)$ and $g(s)$ be the corresponding m_s- and n_s-dimensional vectors, respectively.

Fixed stationary strategies f and g induce a Markov chain on S with transition matrix $P(f, g)$ whose $(s, s^{'})^{th}$ entry is given by

$$P_{ss'}(f, g) = \sum_{i \in A(s)} \sum_{j \in B(s)} q_{ij}(s, s^{'}) f_i(s) g_j(s)$$

and the expected current reward vector $r(f, g)$ has entries defined by

$$r_s(f, g) = \sum_{i \in A(s)} \sum_{j \in B(s)} r(s, i, j) f_i(s) g_j(s) = f^t(s) R(s) g(s)$$

With fixed general strategies f, g and an initial state s, the stream of expected payoff to Player I at stage t, denoted by $v_s^t(f, g)$, $t = 0, 1, 2, \ldots$ is well defined and the resulting discounted and undiscounted payoffs are

$$\phi_s^\beta(f, g) = \sum_{t=0}^{\infty} \beta^t v_s^t(f, g) \text{ for a } \beta \in (0, 1)$$

and

$$\phi_s(f,g) = \lim_{T\uparrow\infty} \inf \frac{1}{T+1} \sum_{t=0}^{T} v_s^t(f,g).$$

A pair of strategies (f^*, g^*) is optimal for Player I and Player II in the undiscounted game if for all $s \in S$

$$\phi_s(f, g^*) \le \phi_s(f^*, g^*) = v_s^* \le \phi_s(f^*, g),$$

for any strategies f and g of Player I and Player II. The number v_s^* is called the *value of the game* starting in state s and $v^* = (v_1^*, v_2^*, \ldots, v_N^*)$ is called the *value vector.* The definition for discounted case is similar.

We require the following definition and the results established by Filar and Schultz [23, Theorem 2.1, 2.2].

DEFINITION 7.2.1 A pair of optimal stationary strategies (f^*, g^*) for an undiscounted stochastic game is *asymptotically stable* if there exist a $\beta_0 \in (0,1)$ and stationary strategy pairs (f^β, g^β) optimal in the β discounted stochastic game for each $\beta \in (\beta_0, 1)$ such that

(i) $\lim_{\beta\uparrow 1} f^\beta = f^*,\ \lim_{\beta\uparrow 1} g^\beta = g^*$

(ii) for all $\beta \in (\beta_0, 1)$, $r(f^\beta, g^\beta) = r(f^*, g^*)$, $P(f, g^\beta) = P(f, g^*)$ for $f \in F_S$ and $P(f^\beta, g) = P(f^*, g)$ for $g \in G_S$ where $P(f,g)$ is the transition matrix and $r(f,g)$ is the current expected reward vector which are defined earlier.

THEOREM 7.2.1 *([23, Theorem 2.1]) An undiscounted stochastic game possesses value vector v^* and optimal stationary strategies f^* for Player I and g^* for Player II if and only if there exists a solution $(v^*, t^*, u^*, f^*, g^*)$ with t^*, $u^* \in R^{|S|}$ to the following nonlinear system SYS1a.*

SYS1a: *Find (v, t, u, f, g) where $v, t, u \in R^{|S|}$, $f \in F_S$ and $g \in G_S$ such that*

$$v_s - \sum_{s' \in S} v_{s'} \sum_{j=1}^{n_s} q_{ij}(s, s') g_j(s) \ge 0,\ i \in A(s), s \in S \tag{7.2.1}$$

$$v_s + t_s - \sum_{s' \in S} t_{s'} \sum_{j=1}^{n_s} q_{ij}(s,s')g_j(s) - [R(s)g(s)]_i \geq 0,\ i \in A(s), s \in S \quad (7.2.2)$$

$$-v_s + \sum_{s' \in S} v_{s'} \sum_{i=1}^{m_s} q_{ij}(s,s')f_i(s) \geq 0,\ j \in B(s), s \in S \quad (7.2.3)$$

$$-v_s - u_s + \sum_{s' \in S} u_{s'} \sum_{i=1}^{m_s} q_{ij}(s,s')f_i(s) + [f(s)R(s)]_j \geq 0,\ j \in B(s), s \in S \quad (7.2.4)$$

THEOREM 7.2.2 *([23, Theorem 2.2]) If a stochastic game possesses asymptotically stable stationary optimal strategies then feasibility of the nonlinear system (SYS1b) is both necessary and sufficient for existence of a stationary optimal solution.*

SYS1b: *Find (v, t, f, g) where $v, t \in R^{|S|}$, $f \in F_S$ and $g \in G_S$ such that (7.2.1), (7.2.2), (7.2.3) are satisfied and*

$$-v_s - t_s + \sum_{s' \in S} t_{s'} \sum_{i=1}^{m_s} q_{ij}(s,s')f_i(s) + [f(s)R(s)]_j \geq 0,\ j \in B(s), s \in S \quad (7.2.5)$$

7.3 Switching Controller Stochastic Games

The class of switching controller stochastic games was introduced by Filar [22]. In a switching controller stochastic game the law of motion is controlled by Player I alone when the game is played in a certain subset of states and Player II alone when the game is played in other states. In other words, a switching controller game is a stochastic game in which the set of states is partitioned into sets S_1 and S_2 where the transition function is given by

$$q_{i,j}(s,s^{'}) = \begin{cases} q_i(s,s^{'}), & \text{for } s' \in S, s \in S_1, i \in A(s) \text{ and } \forall j \in B(s) \\ q_j(s,s^{'}), & \text{for } s' \in S, s \in S_2, j \in B(s) \text{ and } \forall i \in A(s) \end{cases} \quad (7.3.1)$$

While the above transition structure is a natural generalization of the single controller game from the algorithmic point of view, this class of games appear to be more difficult.

The game structure was used to develop a finite algorithm in Vrieze et al. [106] but that algorithm requires solving a large number of single controller stochastic games. In [51] and [52], Mohan, Neogy and Parthasarathy formulated a single controller game as solving a single linear complementarity problem and proved that Lemke's algorithm can solve such an LCP. Mohan and Raghavan [56] proposed an algorithm for discounted switching controller games which is based on two linear programs. Schultz [90] formulated the discounted switching controller game as a linear complementarity problem.

For an undiscounted switching controller game, Filar and Schultz [23] formulated the problem of computing a value vector and an optimal pair of stationary strategies as a bilinear programming problem. In this section, we consider the problem of formulating zero-sum undiscounted switching controller games as a linear complementarity problem.

THEOREM 7.3.1 *For an undiscounted, zero-sum, switching controller game, the value vector and an optimal pair of stationary strategies can be derived from any solution to the following system of linear and nonlinear inequalities (SYS2). Conversely, for such a game, a solution of the SYS2 can be derived from any pair of asymptotically stable stationary strategies.*

SYS2: *Find $(v, t, \theta, \eta, f, g)$ where $v, t, \theta, \eta, \in R^{|S|}$, $f \in F_S$ and $g \in G_S$ such that*

$$v_s - \sum_{s' \in S} v_{s'} q_i(s, s') \geq 0, \ i \in A(s), s \in S_1 \tag{7.3.2}$$

$$-v_s + \theta_s \geq 0, \ s \in S_1 \tag{7.3.3}$$

$$v_s + t_s - \sum_{s' \in S} t_{s'} q_i(s, s') - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_1 \tag{7.3.4}$$

$$-v_s - t_s + \eta_s + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_1 \tag{7.3.5}$$

$$-v_s + \sum_{s' \in S} v_{s'} q_j(s, s') \geq 0, \ j \in B(s), s \in S_2 \tag{7.3.6}$$

$$v_s - \theta_s \geq 0, \ s \in S_2 \tag{7.3.7}$$

$$v_s + t_s - \eta_s - [R(s)g(s)]_i \geq 0,\ i \in A(s), s \in S_2 \tag{7.3.8}$$

$$-v_s - t_s + \sum_{s' \in S} t_{s'} q_j(s,s') + [f(s)R(s)]_j \geq 0,\ j \in B(s), s \in S_2 \tag{7.3.9}$$

$$f \in F_S,\ g \in G_S \tag{7.3.10}$$

$$f_i(s)[v_s - \sum_{s' \in S} v_{s'} q_i(s,s')] = 0,\ i \in A(s), s \in S_1 \tag{7.3.11}$$

$$f_i(s)[-v_s + \theta_s] = 0,\ s \in S_1,\ i \in A(s) \tag{7.3.12}$$

$$f_i(s)[v_s + t_s - \sum_{s' \in S} t_{s'} q_i(s,s') - [R(s)g(s)]_i] = 0,\ i \in A(s), s \in S_1 \tag{7.3.13}$$

$$g_j(s)[-v_s - t_s + \eta_s + [f(s)R(s)]_j] = 0,\ j \in B(s), s \in S_1 \tag{7.3.14}$$

$$g_j(s)[v_s - \theta_s] = 0,\ s \in S_2, j \in B(s) \tag{7.3.15}$$

$$g_j(s)[-v_s + \sum_{s' \in S} v_{s'} q_j(s,s')] = 0,\ j \in B(s), s \in S_2 \tag{7.3.16}$$

$$f_i(s)[v_s + t_s - \eta_s - [R(s)g(s)]_i] = 0,\ i \in A(s), s \in S_2 \tag{7.3.17}$$

$$g_j(s)[-v_s - t_s + \sum_{s' \in S} t_{s'} q_j(s,s') + [f(s)R(s)]_j] = 0,\ j \in B(s), s \in S_2 \tag{7.3.18}$$

Proof. We prove this theorem by showing that a feasible solution to SYS2 can be used to derive a solution of SYS1b and, by Theorem 7.2.2, this solution solves the switching controller game. Conversely, we show that any solution of SYS1b can be used to construct a solution of SYS2. Note that the existence of asymptotic stable stationary strategies for a switching controller game has been proved by Filar [22]. Let $z^* = (v^*, t^*, \theta^*, \eta^*, f^*, g^*)$ be a feasible solution of the SYS2. From (7.3.11) through (7.3.18) we get

$$\theta_s^* = \begin{cases} \sum_{s' \in S} \sum_{i=1}^{m_s} v_{s'}^* q_i(s,s') f_i^*(s), & s \in S_1 \\ \sum_{s' \in S} \sum_{j=1}^{n_s} v_{s'}^* q_j(s,s') g_j^*(s), & s \in S_2 \end{cases} \tag{7.3.19}$$

$$\eta_s^* = \begin{cases} \sum_{s' \in S} \sum_{i=1}^{m_s} t_{s'}^* q_i(s,s') f_i^*(s), & s \in S_1 \\ \sum_{s' \in S} \sum_{j=1}^{n_s} t_{s'}^* q_j(s,s') g_j^*(s), & s \in S_2 \end{cases} \tag{7.3.20}$$

Now substituting the value of θ_s^* and η_s^* in the system of inequalities (7.3.2) through (7.3.9) we get the system of inequalities in SYS1b. Note that the inequalities (7.3.2) and (7.3.7) yield after substitution

$$v_s^* - \sum_{s' \in S} v_{s'}^* q_i(s,s')[\sum_{j=1}^{n_s} g_j^*(s)] \geq 0, \ i \in A(s), s \in S_1$$

i.e.,

$$v_s^* - \sum_{s' \in S} v_{s'}^* \sum_{j=1}^{n_s} q_i(s,s') g_j^*(s) \geq 0, \ i \in A(s), s \in S_1$$

since $\sum_{j=1}^{n_s} g_j^*(s) = 1$. Substituting θ_s^* in (7.3.7) and combining with the above using the definition of a switching controller game we get

$$v_s^* - \sum_{s' \in S} v_{s'}^* \sum_{j=1}^{n_s} q_{i,j}(s,s') g_j^*(s) \geq 0, \ i \in A(s), s \in S$$

which is same as (7.2.1). Inequalities (7.2.2), (7.2.3) and (7.2.5) can be obtained similarly.

Conversely, from any solution (v^*, t^*, f^*, g^*) of SYS1b we define θ_s^*, η_s^* as in (7.3.19), (7.3.20). Rewriting SYS1b using the switching controller assumption (7.3.1) and using (7.3.19), (7.3.20), we get (7.3.2) through (7.3.9). Finally, using (7.3.19), (7.3.20) and (7.3.2) through (7.3.9) we get (7.3.11) through (7.3.18). So, (v^*, t^*, f^*, g^*) is a solution of SYS1b and the simplified SYS1b system is exactly the same system of linear and nonlinear inequalities in SYS2 after substitution of θ_s^*, η_s^*. Therefore, $(v^*, t^*, \theta_s^*, \eta_s^*, f^*, g^*)$ is a solution of SYS2. ■

Corollary 7.3.1 *For an undiscounted, zero-sum, switching controller game, the values v_s for $s \in S$ and optimal stationary strategies $f(s)$ and $g(s)$ for $s \in S$ can be computed by solving an LCP.*

Proof. It is enough to show that SYS2 in Theorem 7.3.1 can be written as an LCP.

First we consider the inequalities (7.3.2) through (7.3.4) and (7.3.8) and let

$$w_1^1(s,i) = v_s - \sum_{s' \in S} v_{s'} q_i(s,s') \geq 0, \ i \in A(s), s \in S_1 \tag{7.3.21}$$

$$w_2^1(s,i) = -v_s + \theta_s \geq 0, \ i \in A(s), \ s \in S_1 \tag{7.3.22}$$

$$w_3^1(s,i) = v_s + t_s - \sum_{s' \in S} t_{s'} q_i(s,s') - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_1 \tag{7.3.23}$$

Also let $w_c^1(s,i) = w_1^1(s,i) + w_2^1(s,i) + w_3^1(s,i)$. We impose the complementarity condition as

$$f_i(s) w_c^1(s,i) = 0, \ i \in A(s), s \in S_1$$

which will imply $f_i(s)w_1^1(s,i) = 0$, $f_i(s)w_2^1(s,i) = 0$ and $f_i(s)w_3^1(s,i) = 0$, for $s \in S_1$. Therefore, we get (7.3.11) through (7.3.13). Let

$$w_d^1(s,i) = v_s + t_s - \eta_s - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_2$$

and from $f_i(s)w_d^1(s,i) = 0, \ i \in A(s), \ s \in S_2$ we get (7.3.17). Similarly, consider inequalities (7.3.6), (7.3.7) and (7.3.9) and let

$$w_1^2(s,j) = -v_s + \sum_{s' \in S} v_{s'} q_j(s,s') \geq 0, \ j \in B(s), s \in S_2,$$

$$w_2^2(s,j) = v_s - \theta_s \geq 0, \ j \in B(s), \ s \in S_2,$$

$$w_3^2(s,j) = -v_s - t_s + \sum_{s' \in S} t_{s'} q_j(s,s') + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_2.$$

Write $w_c^2(s,j) = w_1^2(s,j) + w_2^2(s,j) + w_3^2(s,j)$. Now the complementarity condition

$$f_i(s) w_c^2(s,j) = 0, \ j \in B(s), s \in S_2$$

implies $g_j(s)w_1^2(s,j) = 0$, $g_j(s)w_2^2(s,j) = 0$, $g_j(s)w_3^2(s,j) = 0$. Thus we get (7.3.15), (7.3.16) and (7.3.18) of SYS2. Now consider (7.3.5) and write

$$w_d^2(s,j) = -v_s - t_s + \eta_s + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_1$$

and from complementarity condition $g_j(s)w_d^2(s,j) = 0, \ s \in S_1$, we get (7.3.14).

Now we express the variables v_s, θ_s, η_s and t_s as difference of nonnegative variables as a standard method of representing free variables, i.e.,

$$v_s = \bar{v}_s - \hat{v}_s,\ \theta_s = \bar{\theta}_s - \hat{\theta}_s,\ \eta_s = \bar{\eta}_s - \hat{\eta}_s,\ \ t_s = \bar{t}_s - \hat{t}_s.$$

Now we write down the constraints pertaining to probability vector $f(s)$ and $g(s)$ as follows.

$$\bar{w}_1(s) = -1 + \sum_{i \in A(s)} f_i(s) \geq 0, s \in S$$

$$\hat{w}_1(s) = 1 - \sum_{i \in A(s)} f_i(s) \geq 0, s \in S$$

$$\bar{w}_2(s) = -1 + \sum_{j \in B(s)} g_j(s) \geq 0, s \in S$$

$$\hat{w}_2(s) = 1 - \sum_{j \in B(s)} g_j(s) \geq 0, s \in S$$

The complementarity conditions involving the inequalities related to the probability vector constraints are $\bar{v}_s \bar{w}_1(s) = 0$, $\hat{v}_s \hat{w}_1(s) = 0$, $\bar{\theta}_s \bar{w}_2(s) = 0$, $\hat{\theta}_s \hat{w}_2(s) = 0$ for $s \in S$.

We introduce a few dummy variables and inequalities in order to arrive at the standard LCP formulation.

$$\bar{w}_3(s) = -\bar{k}^3(s) + \sum_{i \in A(s)} \xi_1^1(s,i) \geq 0, s \in S_1,$$

$$\bar{w}_3(s) = -\bar{k}^3(s) + \sum_{j \in B(s)} \xi_1^2(s,j) \geq 0, s \in S_2,$$

$$\hat{w}_3(s) = -\hat{k}^3(s) + \sum_{i \in A(s)} \xi_2^1(s,i) \geq 0, s \in S_1,$$

$$\hat{w}_3(s) = -\hat{k}^3(s) + \sum_{j \in B(s)} \xi_2^2(s,j) \geq 0, s \in S_2,$$

$$\bar{w}_4(s) = -\bar{k}^4(s) + \sum_{i \in A(s)} \xi_3^1(s,i) \geq 0, s \in S_1,$$

$$\bar{w}_4(s) = -\bar{k}^4(s) + \sum_{j \in B(s)} \xi_3^2(s,j) \geq 0, s \in S_2,$$

$$\hat{w}_4(s) = -2 + \sum_{i \in A(s)} f_i(s) + \sum_{j \in B(s)} g_j(s) \geq 0, \ s \in S$$

In the above inequalities $\bar{k}^3(s), \hat{k}^3(s), \bar{k}^4(s)$ and $\hat{k}^4(s)$ for $s \in S$ are appropriate constants.

The complementarity conditions pertaining to the above inequalities are given below.

$$\bar{t}_s \bar{w}_3(s) = 0, \ \hat{t}_s \hat{w}_3(s) = 0,$$

$$\bar{\eta}_s \bar{w}_4(s) = 0, \ \hat{\eta}_s \hat{w}_4(s) = 0, \ \text{where} \ \ s \in S$$

The complementarity relationships of the dummy variables which appears in the above inequalities are

$$\xi_1^1(s,i) w_1^1(s,i) = 0, \ i \in A(s), s \in S_1, \ \xi_2^1(s,i) w_2^1(s,i) = 0, \ i \in A(s), \ s \in S_1$$

$$\xi_3^1(s,i) w_3^1(s,i) = 0, \ i \in A(s), s \in S_1, \ \xi_1^2(s,j) w_1^2(s,j) = 0, \ j \in B(s), s \in S_2,$$

$$\xi_2^2(s,j) w_2^2(s,j) = 0, \ j \in B(s), \ s \in S_2, \ \xi_3^2(s,j) w_3^2(s,j) = 0, \ j \in B(s), s \in S_2$$

Finally the LCP formulation is as follows:

LCP1:

$$w_c^1(s,i) \ = v_s + \theta_s + t_s - \sum_{s' \in S} v_{s'} q_i(s,s') - \sum_{s' \in S} t_{s'} q_i(s,s') - [R(s)g(s)]_i \geq 0, \quad i \in A(s), s \in S_1 \tag{7.3.24}$$

$$w_d^1(s,i) = v_s + t_s - \eta_s - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_2 \tag{7.3.25}$$

$$w_c^2(s,j) \ = -v_s - t_s - \theta_s + \sum_{s' \in S} v_{s'} q_j(s,s') + \sum_{s' \in S} t_{s'} q_j(s,s') + [f(s)R(s)]_j \geq 0, \quad j \in B(s), s \in S_2 \tag{7.3.26}$$

$$w_d^2(s,j) = -v_s - t_s + \eta_s + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_1 \tag{7.3.27}$$

$$w_1^1(s,i) = v_s - \sum_{s' \in S} v_{s'} q_i(s,s') \geq 0, \ i \in A(s), s \in S_1 \tag{7.3.28}$$

$$w_2^1(s,i) = -v_s + \theta_s \geq 0,\ i \in A(s),\ s \in S_1 \tag{7.3.29}$$

$$w_3^1(s,i) = v_s + t_s - \sum_{s' \in S} t_{s'} q_i(s,s') - [R(s)g(s)]_i \geq 0,\ i \in A(s), s \in S_1 \tag{7.3.30}$$

$$w_1^2(s,j) = -v_s + \sum_{s' \in S} v_{s'} q_j(s,s') \geq 0,\ j \in B(s), s \in S_2 \tag{7.3.31}$$

$$w_2^2(s,j) = v_s - \theta_s \geq 0,\ j \in B(s),\ s \in S_2 \tag{7.3.32}$$

$$w_3^2(s,j) = -v_s - t_s + \sum_{s' \in S} t_{s'} q_j(s,s') + [f(s)R(s)]_j \geq 0,\ j \in B(s), s \in S_2 \tag{7.3.33}$$

$$\bar{w}_1(s) = -1 + \sum_{i \in A(s)} f_i(s) \geq 0, s \in S \tag{7.3.34}$$

$$\hat{w}_1(s) = 1 - \sum_{i \in A(s)} f_i(s) \geq 0, s \in S \tag{7.3.35}$$

$$\bar{w}_2(s) = -1 + \sum_{j \in B(s)} g_j(s) \geq 0, s \in S \tag{7.3.36}$$

$$\hat{w}_2(s) = 1 - \sum_{j \in B(s)} g_j(s) \geq 0, s \in S \tag{7.3.37}$$

$$\bar{w}_3(s) = -\bar{k}^3(s) + \sum_{i \in A(s)} \xi_1^1(s,i) \geq 0, s \in S_1 \tag{7.3.38}$$

$$\bar{w}_3(s) = -\bar{k}^3(s) + \sum_{j \in B(s)} \xi_1^2(s,j) \geq 0, s \in S_2 \tag{7.3.39}$$

$$\hat{w}_3(s) = -\hat{k}^3(s) + \sum_{i \in A(s)} \xi_2^1(s,i) \geq 0, s \in S_1 \tag{7.3.40}$$

$$\hat{w}_3(s) = -\hat{k}^3(s) + \sum_{j \in B(s)} \xi_2^2(s,j) \geq 0, s \in S_2 \tag{7.3.41}$$

$$\bar{w}_4(s) = -\bar{k}^4(s) + \sum_{i \in A(s)} \xi_3^1(s,i) \geq 0, s \in S_1 \tag{7.3.42}$$

$$\bar{w}_4(s) = -\bar{k}^4(s) + \sum_{j \in B(s)} \xi_3^2(s,j) \geq 0, s \in S_2 \tag{7.3.43}$$

$$\hat{w}_4(s) = -2 + \sum_{i \in A(s)} f_i(s) + \sum_{j \in B(s)} g_j(s) \geq 0,\ s \in S \tag{7.3.44}$$

$$f_i(s) w_c^1(s,i) = 0,\ i \in A(s), s \in S_1 \tag{7.3.45}$$

$$f_i(s) w_d^1(s,i) = 0,\ i \in A(s), s \in S_2 \tag{7.3.46}$$

$$g_j(s)w_c^2(s,j) = 0,\ j \in B(s), s \in S_2 \tag{7.3.47}$$

$$g_j(s)w_d^2(s,j) = 0,\ j \in B(s), s \in S_1 \tag{7.3.48}$$

$$\xi_1^1(s,i)w_1^1(s,i) = 0,\ i \in A(s), s \in S_1 \tag{7.3.49}$$

$$\xi_2^1(s,i)w_2^1(s,i) = 0,\ i \in A(s),\ s \in S_1 \tag{7.3.50}$$

$$\xi_3^1(s,i)w_3^1(s,i) = 0,\ i \in A(s), s \in S_1 \tag{7.3.51}$$

$$\xi_1^2(s,j)w_1^2(s,j) = 0,\ j \in B(s), s \in S_2 \tag{7.3.52}$$

$$\xi_2^2(s,j)w_2^2(s,j) = 0,\ j \in B(s),\ s \in S_2 \tag{7.3.53}$$

$$\xi_3^2(s,j)w_3^2(s,j) = 0,\ j \in B(s), s \in S_2 \tag{7.3.54}$$

$$\bar{v}_s\bar{w}_1(s) = 0,\ s \in S \tag{7.3.55}$$

$$\hat{v}_s\hat{w}_1(s) = 0, s \in S \tag{7.3.56}$$

$$\bar{\theta}_s\bar{w}_2(s) = 0, s \in S \tag{7.3.57}$$

$$\hat{\theta}_s\hat{w}_2(s) = 0,\ s \in S \tag{7.3.58}$$

$$\bar{t}_s\bar{w}_3(s) = 0, s \in S \tag{7.3.59}$$

$$\hat{t}_s\hat{w}_3(s) = 0,\ s \in S \tag{7.3.60}$$

$$\bar{\eta}_s\bar{w}_4(s) = 0,\ s \in S \tag{7.3.61}$$

$$\hat{\eta}_s\hat{w}_4(s) = 0,\ s \in S \tag{7.3.62}$$

$$\begin{gathered}\bar{v}_s, \hat{v}_s, \bar{t}_s, \hat{t}_s, \bar{\theta}_s, \hat{\theta}_s, \bar{\eta}_s, \hat{\eta}_s, \bar{w}_i(s), \hat{w}_i(s) \geq 0,\ s \in S,\ i = 1,2,3,4, \\ \xi_1^1(s,i), \xi_2^1(s,i), \xi_3^1(s,i) \geq 0,\ i \in A(s), s \in S_1, \\ \xi_1^2(s,j), \xi_2^2(s,j), \xi_3^2(s,j) \geq 0,\ j \in B(s), s \in S_2, \\ f(s), g(s) \geq 0,\ s \in S \end{gathered} \tag{7.3.63}$$

where as usual $f(s) = (f_i(s))_{i \in A(s)}$ and $g(s) = (g_i(s))_{j \in B(s)}$.

It is easy to check that a solution of LCP1 also solves SYS2. ■

EXAMPLE 7.3.1 Consider the undiscounted switching controller stochastic game with 2 states and $m_1 = m_2 = n_1 = n_2 = 2$. The rewards and transition probabilities are as follows:

$$R(1) = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, R(2) = \begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} q_1(1,1) \\ q_2(1,1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} q_1(1,2) \\ q_2(1,2) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} q_1(2,1) \\ q_2(2,1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \begin{bmatrix} q_1(2,2) \\ q_2(2,2) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

The values v_1 and v_2 for the undiscounted, zero-sum, switching controller game, and optimal stationary strategies $f(1), f(2)$ and $g(1), g(2)$ can be computed by solving the following LCP(q, M).

Here the decision variable is $z = [z_1 \ \ z_2 \ \ z_3]^t$ where
$z_1 = [f_1(1), f_2(1), f_1(2), f_2(2), g_1(1), g_2(1), g_1(2), g_2(2)]^t$,
$z_2 = [\xi_1^1(1,1), \xi_1^1(1,2), \xi_2^1(1,1), \xi_2^1(1,2), \xi_3^1(1,1), \xi_3^1(1,2), \xi_1^2(2,1), \xi_1^2(2,2), \xi_2^2(2,1), \xi_2^2(2,2), \xi_3^2(2,1), \xi_3^2(2,2)]^t$, $z_3 = [\bar{v}_1, \hat{v}_1, \bar{v}_2, \hat{v}_2, \bar{\theta}_1, \hat{\theta}_1, \bar{\theta}_2, \hat{\theta}_2, \bar{t}_1, \hat{t}_1, \bar{t}_2, \hat{t}_2, \bar{\eta}_1, \hat{\eta}_1, \bar{\eta}_2, \hat{\eta}_2]^t$ and $v_s = \bar{v}_s - \hat{v}_s,\ \theta_s = \bar{\theta}_s - \hat{\theta}_s,\ \eta_s = \bar{\eta}_s - \hat{\eta}_s,\ \ t_s = \bar{t}_s - \hat{t}_s,$ for $s = 1, 2$.

The matrix M and q are shown in the following partitioned form.

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, \ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where

$$M_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & -1 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{13} = \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 & 1 & -1 & 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & -1 & 1 & 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix}
0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$M_{31} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix},$$

$$M_{32} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$M_{12} = [0]_{8\times 12},\ M_{22} = [0]_{12\times 12},\ M_{33} = [0]_{16\times 16},\ q_1 = [0]_{8\times 1},\ q_2 = [0]_{12\times 1},$
$q_3 = [-1\ 1\ -1\ 1\ -1\ 1\ -1\ 1\ -10\ -20\ -30\ -40\ -50\ -60\ -2\ -2]^t.$

7.4 Additive Reward Additive Transition Games (ARAT Games)

ARAT games have been studied in the literature earlier by Raghavan et al. [85]. Both the discounted and the limiting average criterion of evaluation of strategies have been considered. It is known for example, that for a β-discounted zero-sum ARAT game, the value exists and both players have stationary optimal strategies, which may also be taken as pure strategies. Raghavan et al. [85], have shown that undiscounted ARAT game possesses uniformly discounted optimal stationary strategies and therefore asymptotically stable optimal stationary strategies. In [85], a finite step method to compute a pair of pure stationary optimal strategies and the value of the game has been suggested. However this approach involves solving a series (finite number) of Markov decision problems. See also [24] and [84].

A stochastic game is said to be an *Additive-Reward- Additive Transition game (ARAT game)* if the reward

(i) $r(s,i,j) = r_i^1(s) + r_j^2(s)$ for $i \in A(s),\ j \in B(s),\ s \in S$

and the transition probabilities

(ii) $q_{i,j}(s,s^{'}) = q_i^1(s,s^{'}) + q_j^2(s,s^{'})$ for $i \in A(s), j \in B(s),\ (s,s') \in S \times S.$

The following lemma was proved by Filar and Schultz [23].

Lemma 7.4.1 *([23, Lemma 2.4]) (i) If $(v^*, t^*, u^*, f^*, g^*)$ satisfy SYS1a, then for all $s \in S$*

$$v_s^* = [P(f^*, g^*)v^*]_s.$$

(ii) If $(v^*, t^*, u^*, f^*, g^*)$ solves SYS1b, then for all $s \in S$

$$v_s^* + t_s^* = [P(f^*, g^*)t^* + r(f^*, g^*)]_s$$

Mohan, Neogy, Parthasarathy [53] formulated the undiscounted ARAT game as a vertical linear complementarity problem. For the LCP formulation of discounted version of ARAT game, see [54]. In this section, we present a new LCP formulation for the undiscounted ARAT game.

THEOREM 7.4.1 *For an undiscounted zero-sum ARAT game, the value vector and an optimal pair of stationary strategies can be derived from any solution to the following system of linear and nonlinear inequalities (SYS3). Conversely, for such a game, a solution of SYS3 can be derived from any pair of asymptotically stable stationary strategies.*

SYS3: *Find* $(\theta, \eta, \phi, \gamma, f, g)$ *where* $\theta, \eta, \phi, \gamma \in R^{|S|}$, $f \in F_S$ *and* $g \in G_S$ *such that*

$$\phi_s - \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_i^1(s, s') \geq 0, \ i \in A(s), s \in S \tag{7.4.1}$$

$$\gamma_s - \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_i^1(s, s') - r_i^1(s) \geq 0, \ i \in A(s), s \in S \tag{7.4.2}$$

$$-\theta_s + \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_j^2(s, s') \geq 0, \ j \in B(s), s \in S \tag{7.4.3}$$

$$-\eta_s + \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_j^2(s, s') + r_j^2(s) \geq 0, \ j \in B(s), s \in S \tag{7.4.4}$$

$$f_i(s)[\phi_s - \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_i^1(s, s')] = 0, \ i \in A(s), s \in S \tag{7.4.5}$$

$$f_i(s)[\gamma_s - \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_i^1(s, s') - r_i^1(s)] = 0, \ i \in A(s), s \in S \tag{7.4.6}$$

$$g_j(s)[-\theta_s + \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_j^2(s, s')] = 0, \ j \in B(s), s \in S \tag{7.4.7}$$

$$g_j(s)[-\eta_s + \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_j^2(s, s') + r_j^2(s)] = 0, \ j \in B(s), s \in S \tag{7.4.8}$$

$$f \in F_S, \ g \in G_S \tag{7.4.9}$$

Proof. This theorem is proved by following a similar approach as in Theorem 7.3.1. We show that a feasible solution of SYS3 also solves SYS1b and by Theorem 7.2.2, this solution solves the game problem. Let $(\theta^*, \eta^*, \phi^*, \gamma^*, f^*, g^*)$ be a feasible solution of the SYS3.

We define

$$v_s^* = \theta_s^* + \phi_s^* \ \text{ for } s \in S \tag{7.4.10}$$

$$t_s^* = \eta_s^* + \gamma_s^* - \theta_s^* - \phi_s^* \ \text{ for } s \in S \tag{7.4.11}$$

From (7.4.10) and (7.4.11) we get

$$\eta_s^* + \gamma_s^* = v_s^* + t_s^* \ \text{ for } s \in S$$

Substituting v_s^* for $(\theta_s^* + \phi_s^*)$ and $(v_s^* + t_s^*)$ for $(\eta_s^* + \gamma_s^*)$ in (7.4.1) through (7.4.4) and (7.4.5) through (7.4.8) we get

$$\phi_s^* - \sum_{s'=1}^{N} v_{s'}^* q_i^1(s,s') \geq 0, \ i \in A(s), s \in S \tag{7.4.12}$$

$$\gamma_s^* - \sum_{s'=1}^{N} t_{s'}^* q_i^1(s,s') - r_i^1(s) \geq 0, \ i \in A(s), s \in S \tag{7.4.13}$$

$$-\theta_s^* + \sum_{s'=1}^{N} v_{s'}^* q_j^2(s,s') \geq 0, \ j \in B(s), s \in S \tag{7.4.14}$$

$$-\eta_s^* + \sum_{s'=1}^{N} t_{s'}^* q_j^2(s,s') + r_j^2(s) \geq 0, \ j \in B(s), s \in S \tag{7.4.15}$$

$$\phi_s^* = \sum_{s'=1}^{N} \sum_{i=1}^{m_s} v_{s'}^* q_i^1(s,s') f_i^*(s), \ s \in S \tag{7.4.16}$$

$$\gamma_s^* = \sum_{s'=1}^{N} \sum_{j=1}^{n_s} t_{s'}^* q_i^1(s,s') f_i^*(s) + \sum_{i=1}^{m_s} r_i^1(s) f_i^*(s), \ s \in S \tag{7.4.17}$$

$$\theta_s^* = \sum_{s'=1}^{N} \sum_{j=1}^{n_s} v_{s'}^* q_j^2(s,s') g_j^*(s), \ s \in S \tag{7.4.18}$$

$$\eta_s^* = \sum_{s'=1}^{N} \sum_{j=1}^{n_s} t_{s'}^* q_j^2(s,s') g_j^*(s) + \sum_{j=1}^{n_s} r_j^2(s) g_j^*(s), \ s \in S \tag{7.4.19}$$

Adding (7.4.12) and (7.4.18) we get

$$\theta_s^* + \phi_s^* - \sum_{s'=1}^{N} \sum_{j=1}^{n_s} v_{s'}^* q_j^2(s,s') g_j^*(s) - \sum_{s'=1}^{N} v_{s'}^* q_i^1(s,s') \geq 0, \ i \in A(s), \ s \in S \tag{7.4.20}$$

Therefore

$$\theta_s^* + \phi_s^* - \sum_{s'=1}^{N} v_{s'}^* \sum_{j=1}^{n_s} [q_j^2(s,s')g_j^*(s) + q_i^1(s,s')g_j^*(s)] \geq 0, \ i \in A(s), \ s \in S \tag{7.4.21}$$

Substituting v_s^* for $(\theta_s^* + \phi_s^*)$ we get (7.2.1)

$$v_s^* - \sum_{s'=1}^{N} \sum_{j=1}^{n_s} v_{s'}^* q_{ij}(s,s') g_j^*(s) \geq 0, \ i \in A(s), \ s \in S \tag{7.4.22}$$

Adding (7.4.13) and (7.4.19) we get (7.2.2)

$$\eta_s^* + \gamma_s^* - \sum_{s'=1}^{N} t_{s'}^* [\sum_{j=1}^{n_s} q_j^2(s,s') + q_i^1(s,s')] g_j^*(s) - \sum_{j=1}^{n_s} [r_j^2(s) + r_i^1(s)] g_j^*(s) \geq 0,$$

$$i \in A(s), s \in S \tag{7.4.23}$$

This implies

$$v_s^* + t_s^* - \sum_{s'=1}^{N} t_{s'}^* \sum_{j=1}^{n_s} q_{ij}(s,s') g_j^*(s) - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S \tag{7.4.24}$$

Subtracting (7.4.16) from (7.4.14) and subtracting (7.4.17) from (7.4.15) we get (7.2.3) and (7.2.5) respectively. Since $f \in F_S$ and $g \in G_S$ the variables satisfy SYS1b and by Theorem 7.2.2, this yields an optimal solution to undiscounted ARAT game.

To prove the converse, we show that any solution to SYS1b $-$ which always exists for these games, since they possess asymptotically stable optimal stationary strategies $-$ can be used to derive a feasible solution for SYS3. Assume that (v^*, t^*, f^*, g^*) be a feasible solution of the SYS1b. From (7.2.1), (7.2.2), (7.2.3), (7.2.5) and using the definition of ARAT game we get

$$v_s^* - \sum_{s'=1}^{N} \sum_{j=1}^{n_s} v_{s'}^* q_j^2(s,s') g_j^*(s) - \sum_{s'=1}^{N} v_{s'}^* q_i^1(s,s') \geq 0, \ i \in A(s), \ s \in S \tag{7.4.25}$$

$$v_s^* + t_s^* - \sum_{s'=1}^{N} \sum_{j=1}^{n_s} t_{s'}^* q_j^2(s,s') g_j^*(s) - \sum_{s'=1}^{N} t_{s'}^* q_i^1(s,s') - \sum_{j=1}^{n_s} r_j^2(s) g_j^*(s)$$

$$-r_i^1(s) \geq 0,\ i \in A(s),\ s \in S \tag{7.4.26}$$

$$-v_s^* + \sum_{s'=1}^{N}\sum_{i=1}^{m_s} v_{s'}^* q_i^1(s,s') f_i^*(s) + \sum_{s'=1}^{N} v_{s'}^* q_j^2(s,s') \geq 0,\ j \in B(s),\ s \in S \tag{7.4.27}$$

$$-v_s^* - t_s^* + \sum_{s'=1}^{N}\sum_{i=1}^{m_s} t_{s'}^* q_i^1(s,s') f_i^*(s) + \sum_{s'=1}^{N} t_{s'}^* q_j^2(s,s') + \sum_{i=1}^{m_s} r_i^1(s) f_i^*(s)$$
$$+r_j^2(s) \geq 0,\ j \in B(s),\ s \in S \tag{7.4.28}$$

Take $\theta_s^*, \eta_s^*, \phi_s^*$ and γ_s^* for $s \in S$ as in (7.4.16) through (7.4.19). Adding (7.4.16) and (7.4.18) we get

$$\begin{aligned}\theta_s^* + \phi_s^* &= \sum_{s'=1}^{N} v_{s'}^* [\sum_{i=1}^{m_s} q_i^1(s,s') f_i^*(s) + \sum_{j=1}^{n_s} q_j^2(s,s') g_j^*(s)] \\ &= [P(f^*, g^*)v^*]_s = v_s^* \end{aligned} \tag{7.4.29}$$

by Lemma 7.4.1 (i). Similarly, using Lemma 7.4.1 (ii) and from (7.4.17) and (7.4.19) we get

$$\eta_s^* + \gamma_s^* = [P(f^*, g^*)t^* + r(f^*, g^*)]_s = v_s^* + t_s^* \tag{7.4.30}$$

From (7.4.25), (7.4.29) and using the definition of θ_s^* in (7.4.18) we get (7.4.1).

$$\theta_s^* + \phi_s^* - \theta_s^* - \sum_{s'=1}^{N} (\theta_{s'}^* + \phi_{s'}^*) q_i^1(s,s') \geq 0,\ i \in A(s),\ s \in S \tag{7.4.31}$$

From (7.4.26), (7.4.19), (7.4.29) and (7.4.30) we get (7.4.2) of SYS3. From (7.4.27), (7.4.29) and the definition of ϕ^* in (7.4.16) yields (7.4.3) of SYS3.

$$-\theta_s^* - \phi_s^* + \sum_{s'=1}^{N} (\theta_{s'}^* + \phi_{s'}^*) q_j^2(s,s') + \phi_s^* \geq 0,\ j \in B(s),\ s \in S \tag{7.4.32}$$

Similarly from (7.4.28), (7.4.29),(7.4.30) and (7.4.17) we get (7.4.4) of SYS3. From (7.4.16) through (7.4.19), (7.4.29) and (7.4.30), we get (7.4.5) through (7.4.8). Since, $f \in F_S$ and $g \in G_S$, we obtain a feasible solution $(\theta^*, \eta^*, \phi^*, \gamma^*, f^*, g^*)$ which satisfies SYS3. ∎

COROLLARY 7.4.1 *For an undiscounted, zero-sum, ARAT game, the values v_s for $s \in S$ and optimal stationary strategies $f(s)$ and $g(s)$ for $s \in S$ can be computed by solving an LCP.*

Proof. We shall show that SYS3 in Theorem 7.4.1 can be written as a linear complementarity problem.

First we consider the inequalities (7.4.1) through (7.4.4). Let

$$w^1(s,i) = \phi_s - \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_i^1(s,s') \geq 0,\ i \in A(s), s \in S$$

$$w^2(s,i) = \gamma_s - \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_i^1(s,s') - r_i^1(s) \geq 0,\ i \in A(s), s \in S$$

$$w^3(s,j) = -\theta_s + \sum_{s'=1}^{N} (\theta_{s'} + \phi_{s'}) q_j^2(s,s') \geq 0,\ j \in B(s), s \in S$$

$$w^4(s,j) = -\eta_s + \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'}) q_j^2(s,s') + r_j^2(s) \geq 0,\ j \in B(s), s \in S$$

Let $w_c^1(s,i) = w^1(s,i) + w^2(s,i)$ and $w_c^2(s,j) = w^3(s,j) + w^4(s,j)$. Now by imposing the complementarity condition $f_i(s) w_c^1(s,i) = 0,\ i \in A(s), s \in S$ implies $f_i(s) w^1(s,i) = 0,\ \ f_i(s) w^2(s,i) = 0$ for $i \in A(s), s \in S$. Thus we obtain (7.4.5) and (7.4.6) of SYS3. Similarly, by imposing the complementarity condition $g_j(s) w_c^2(s,j) = 0, j \in B(s), s \in S$ implies $g_j(s) w^3(s,j) = 0,\ \ g_j(s) w^4(s,j) = 0$ for $j \in B(s), s \in S$. Thus we get (7.4.7) and (7.4.8) of SYS3.

Now we express each of the variables $\theta_s, \eta_s, \phi_s, \gamma_s$ as difference of two non-negative variables using the standard method of representing free variables, i.e., $\theta_s = \bar{\theta}_s - \hat{\theta}_s,\ \eta_s = \bar{\eta}_s - \hat{\eta}_s,\ \phi_s = \bar{\phi}_s - \hat{\phi}_s,\ \gamma_s = \bar{\gamma}_s - \hat{\gamma}_s$. The inequalities pertaining to probability vectors are as follows.

$$\bar{w}_5(s) = -1 + \sum_{i \in A(s)} f_i(s) \geq 0, s \in S$$

$$\hat{w}_5(s) = 1 - \sum_{i \in A(s)} f_i(s) \geq 0, s \in S$$

$$\bar{w}_6(s) = -1 + \sum_{j \in B(s)} g_j(s) \geq 0, s \in S$$

$$\hat{w}_6(s) = 1 - \sum_{j \in B(s)} g_j(s) \geq 0, s \in S$$

The complementarity conditions involving inequality related to the probability vector are $\bar{\theta}_s \bar{w}_5(s) = 0$, $\hat{\theta}_s \hat{w}_5(s) = 0$ $\bar{\eta}_s \bar{w}_6(s) = 0$, $\hat{\eta}_s \hat{w}_6(s) = 0$ for $s \in S$.

Now we introduce few dummy variables and inequalities in order to obtain the standard LCP formulation. Consider

$$\bar{w}_7(s) = -\bar{k}^1(s) + \sum_{i \in A(s)} \xi^1(s,i) \geq 0, s \in S,$$

$$\hat{w}_7(s) = -\hat{k}^1(s) + \sum_{i \in A(s)} \xi^2(s,i) \geq 0, s \in S,$$

$$\bar{w}_8(s) = -\bar{k}^2(s) + \sum_{j \in B(s)} \xi^3(s,j) \geq 0, s \in S,$$

$$\hat{w}_8(s) = -\hat{k}^2(s) + \sum_{j \in B(s)} \xi^4(s,j) \geq 0, s \in S,$$

In the above inequalities $\bar{k}^1(s), \hat{k}^1(s), \bar{k}^2(s)$ and $\hat{k}^2(s)$ for $s \in S$ are appropriate constants.

The complementarity conditions for the above inequalities are given below.

$$\bar{\phi}_s \bar{w}_7(s) = 0,\ \hat{\phi}_s \hat{w}_7(s) = 0,$$

$$\bar{\gamma}_s \bar{w}_8(s) = 0,\ \hat{\gamma}_s \hat{w}_8(s) = 0,\ \text{where}\ \ s \in S$$

The complementarity relationship of the dummy variables which appears in the above inequalities are

$$\xi^1(s,i)w^1(s,i) = 0,\ i \in A(s), s \in S,\ \ \xi^2(s,i)w^2(s,i) = 0,\ i \in A(s),\ s \in S,$$

$$\xi^3(s,j)w^3(s,j) = 0,\ j \in B(s), s \in S,\ \ \xi^4(s,j)w^4(s,j) = 0,\ j \in B(s), s \in S.$$

Finally, the LCP formulation of undiscounted ARAT game is as follows:

$$w_c^1(s,i) = \phi_s + \gamma_s - \sum_{s'=1}^{N} (\eta_{s'} + \gamma_{s'}) q_i^1(s,s') - r_i^1(s) \geq 0,\ i \in A(s), s \in S \quad (7.4.33)$$

$$w_c^2(s,j) = -\theta_s - \eta_s + \sum_{s'=1}^{N}(\eta_{s'} + \gamma_{s'})q_j^2(s,s') + r_j^2(s) \geq 0,\ j \in B(s), s \in S \tag{7.4.34}$$

$$w^1(s,i) = \phi_s - \sum_{s'=1}^{N}(\theta_{s'} + \phi_{s'})q_i^1(s,s') \geq 0,\ i \in A(s), s \in S \tag{7.4.35}$$

$$w^2(s,i) = \gamma_s - \sum_{s'=1}^{N}(\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'})q_i^1(s,s') - r_i^1(s) \geq 0,\ i \in A(s), s \in S \tag{7.4.36}$$

$$w^3(s,j) = -\theta_s + \sum_{s'=1}^{N}(\theta_{s'} + \phi_{s'})q_j^2(s,s') \geq 0,\ j \in B(s), s \in S \tag{7.4.37}$$

$$w^4(s,j) = -\eta_s + \sum_{s'=1}^{N}(\eta_{s'} + \gamma_{s'} - \theta_{s'} - \phi_{s'})q_j^2(s,s') + r_j^2(s) \geq 0,\ j \in B(s), s \in S \tag{7.4.38}$$

$$\bar{w}_5(s) = -1 + \sum_{i \in A(s)} f_i(s) \geq 0, s \in S \tag{7.4.39}$$

$$\hat{w}_5(s) = 1 - \sum_{i \in A(s)} f_i(s) \geq 0, s \in S \tag{7.4.40}$$

$$\bar{w}_6(s) = -1 + \sum_{j \in B(s)} g_j(s) \geq 0, s \in S \tag{7.4.41}$$

$$\hat{w}_6(s) = 1 - \sum_{j \in B(s)} g_j(s) \geq 0, s \in S \tag{7.4.42}$$

$$\bar{w}_7(s) = -\bar{k}^1(s) + \sum_{i \in A(s)} \xi^1(s,i) \geq 0, s \in S, \tag{7.4.43}$$

$$\hat{w}_7(s) = -\hat{k}^1(s) + \sum_{i \in A(s)} \xi^2(s,i) \geq 0, s \in S, \tag{7.4.44}$$

$$\bar{w}_8(s) = -\bar{k}^2(s) + \sum_{j \in B(s)} \xi^3(s,j) \geq 0, s \in S, \tag{7.4.45}$$

$$\hat{w}_8(s) = -\hat{k}^2(s) + \sum_{j \in B(s)} \xi^4(s,j) \geq 0, s \in S, \tag{7.4.46}$$

$$f_i(s)w_c^1(s,i) = 0,\ i \in A(s), s \in S \tag{7.4.47}$$

$$g_j(s)w_c^2(s,j) = 0, j \in B(s), s \in S \tag{7.4.48}$$

$$\xi^1(s,i)w^1(s,i) = 0,\ i \in A(s), s \in S, \tag{7.4.49}$$

$$\xi^2(s,i)w^2(s,i)=0,\ i\in A(s),\ s\in S, \tag{7.4.50}$$

$$\xi^3(s,j)w^3(s,j)=0,\ j\in B(s), s\in S, \tag{7.4.51}$$

$$\xi^4(s,j)w^4(s,j)=0,\ j\in B(s), s\in S \tag{7.4.52}$$

$$\bar{\theta}_s\bar{w}_5(s)=0,\ s\in S \tag{7.4.53}$$

$$\hat{\theta}_s\hat{w}_5(s)=0,\ s\in S \tag{7.4.54}$$

$$\bar{\eta}_s\bar{w}_6(s)=0,\ s\in S \tag{7.4.55}$$

$$\hat{\eta}_s\hat{w}_6(s)=0,\ s\in S \tag{7.4.56}$$

$$\bar{\phi}_s\bar{w}_7(s)=0,\ s\in S \tag{7.4.57}$$

$$\hat{\phi}_s\hat{w}_7(s)=0,\ s\in S \tag{7.4.58}$$

$$\bar{\gamma}_s\bar{w}_8(s)=0,\ s\in S \tag{7.4.59}$$

$$\hat{\gamma}_s\hat{w}_8(s)=0,\ s\in S \tag{7.4.60}$$

$$f_i(s)\geq 0,\ i\in A(s), g_j(s)\geq 0,\ j\in B(s), \bar{\theta}_s, \hat{\theta}_s, \bar{\eta}_s, \hat{\eta}_s, \bar{\phi}_s, \hat{\phi}_s, \bar{\gamma}_s, \hat{\gamma}_s\geq 0,\ s\in S$$

$$\xi^1(s,i), \xi^2(s,i)\geq 0,\ i\in A(s), \xi^3(s,j), \xi^4(s,j)\geq 0,\ j\in B(s), s\in S \tag{7.4.61}$$

It is easy to deduce that by solving the above linear complementarity problem, one can compute a solution to SYS3. ■

Example 7.4.1 Consider a two player zero-sum undiscounted ARAT game with 2 states and $m_1 = m_2 = n_1 = n_2 = 2$. The rewards and transition probabilities are as follows:

$r_1^1(1)=4,\ \ q_1^1(1,1)=\frac{1}{2},\ q_1^1(1,2)=0,$
$r_2^1(1)=5,\ \ q_2^1(1,1)=\frac{1}{2},\ q_2^1(1,2)=0,$
$r_1^1(2)=3,\ \ q_1^1(2,1)=0,\ q_1^1(2,2)=\frac{1}{2},$
$r_2^1(2)=4,\ \ q_2^1(2,1)=0,\ q_2^1(2,2)=\frac{1}{2},$

$r_1^2(1) = 3,\ \ q_1^2(1,1) = \frac{1}{2},\ q_1^2(1,2) = 0,$

$r_2^2(1) = 6,\ \ q_2^2(1,1) = 0,\ q_2^2(1,2) = \frac{1}{2},$

$r_1^2(2) = 6,\ \ q_1^2(2,1) = 0,\ q_1^2(2,2) = \frac{1}{2},$

$r_2^2(2) = 2,\ \ q_2^2(2,1) = \frac{1}{2},\ q_2^2(2,2) = 0.$

The values and optimal stationary strategies $f(1), f(2)$ and $g(1), g(2)$ can be computed by solving the following LCP(q, M).

Here the decision variable is $z = [z_1\ \ z_2\ \ z_3]^t$ where

$z_1 = [f_1(1), f_2(1), f_1(2), f_2(2), g_1(1), g_2(1), g_1(2), g_2(2), \xi^1(1,1), \xi^1(1,2), \xi^1(2,1),$
$\xi^1(2,2), \xi^2(1,1), \xi^2(1,2), \xi^2(2,1), \xi^2(2,2), \xi^3(1,1), \xi^3(1,2), \xi^3(2,1), \xi^3(2,2),$
$\xi^4(1,1), \xi^4(1,2), \xi^4(2,1), \xi^1(2,2)]^t,\ z_2 = [\bar{\theta}_1, \hat{\theta}_1, \bar{\theta}_2, \hat{\theta}_2, \bar{\eta}_1, \hat{\eta}_1, \bar{\eta}_2, \hat{\eta}_2]^t,$
$z_3 = [\bar{\phi}_1, \hat{\phi}_1, \bar{\phi}_2, \hat{\phi}_2, \bar{\gamma}_1, \hat{\gamma}_1, \bar{\gamma}_2, \hat{\gamma}_2]^t,\ \theta_s = \bar{\theta}_s - \hat{\theta}_s,$
$\eta_s = \bar{\eta}_s - \hat{\eta}_s,\ \ \phi_s = \bar{\phi}_s - \hat{\phi}_s,\ \gamma_s = \bar{\gamma}_s - \hat{\gamma}_s$ and $v_s = \theta_s + \phi_s$ for $s = 1, 2.$

The matrix M and q are shown in the following partitioned form.

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix},\ q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

where $M_{11} = [0]_{24\times 8},\ M_{12} = [0]_{24\times 16},$

$$M_{13} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0.5 & -0.5 & -1 & 1 & 0.5 & -0.5 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
0 & 0 & 0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
0 & 0 & 0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
-0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.5 & -0.5 & -1 & 1 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 & 0 & 0 \\
0 & 0 & -0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & -1 & 1 & 0.5 & -0.5 & 0 & 0 & 0.5 & -0.5 \\
0 & 0 & -0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & 0.5 & -0.5 \\
-0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & 0 & 0 & -0.5 & 0.5 & -1 & 1 & 0.5 & -0.5 & 0 & 0
\end{bmatrix},$$

$$M_{21} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{bmatrix},$$

$$M_{32} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix},$$

$M_{22} = [0]_{8\times 16},\ M_{23} = [0]_{8\times 16},\ M_{31} = [0]_{8\times 8},\ M_{33} = [0]_{8\times 16},$

$q_1 = [-4\ -5\ -3\ -4\ 3\ 6\ 6\ 2\ 0\ 0\ 0\ 0\ -4\ -5\ -3\ -4\ 0\ 0\ 0\ 0\ 3\ 6\ 6\ 2]^t,$

$q_2 = [-1\ 1\ -1\ 1\ -1\ 1\ -1\ 1]^t$ and $q_3 = [-10\ -20\ -30\ -40\ -50\ -60\ -70\ -80]^t.$

7.5 Concluding Remarks and Areas of Further Research

For switching controller and ARAT games, nice theoretical properties have been observed by several researchers. In this chapter, we present a linear complementarity formulation for solving undiscounted switching controller and ARAT games. This gives an alternative proof of the orderfield property for these two classes games. Of course, more work can be done on algorithms for these two classes of structured games. The applicability of Lemke's algorithm for solving the LCP formulation presented should be explored. While implementing available pivoting algorithms on these two formulations, a special initialization scheme may be necessary and use of suitable degeneracy resolving mechanism may be needed. Investigation for other formulations using the inequalities in SYS1a, SYS1b, SYS2 and SYS3 as well as other solution methods are also areas of further research.

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