SEIBERG-WITTEN INVARIANTS–AN EXPOSITORY ACCOUNT

KAPIL H. PARANJAPE AND V. PATI

We recall some constructions of spin groups in low dimensions.

1. Spin Groups

1.1. **Dimension 3.** Let W be a vector space of dimension 2. Consider the representation of $\mathrm{GL}(W)$ on $\mathrm{End}^0(W)$, the space of all traceless endomorphisms of W. There is a natural non-degenerate form $\{,\}$ on $\mathrm{End}^0(W)$ given by

$$\{f, g\} = \mathrm{Trace}_W\, f \circ g$$

Moreover, we have a sequence of isomorphisms of representations of $\mathrm{GL}(W)$,

$$\overset{3}{\wedge}\mathrm{End}^0(W) = \overset{4}{\wedge}\mathrm{End}(W) = \overset{4}{\wedge}(W^* \otimes W) = (\overset{2}{\wedge}W^*)^{\otimes 2} \otimes (\overset{2}{\wedge}W)^{\otimes 2} = \mathbf{1}$$

where $\mathbf{1}$ denotes the trivial representation. Thus we obtain a natural homomorphism $\mathrm{GL}(W) \to \mathrm{SO}(\mathrm{End}^0(W))$. Over the complex numbers this identifies $\mathrm{GL}(2)$ with the "Cspin" group of $\mathrm{SO}(3)$. The subgroup $\mathrm{SL}(2)$ is identified with the spin group.

1.2. **Dimension 4.** Let W_+ and W_- be two vector spaces of dimension 2 and let $\phi : \overset{2}{\wedge}W_- \to \overset{2}{\wedge}W_+$ be an isomorphism. Then the vector space $U = \mathrm{Hom}(W_+, W_-)$ is isomorphic to its dual via a map $B : U \to U^* = \mathrm{Hom}(W_-, W_+)$ defined by the identity

$$\phi(f(w_+) \wedge w_-) = w_+ \wedge B(f)(w_-)$$

Thus we have a non-degenerate pairing

$$(f, g) = \mathrm{Trace}_{W_+}\, B(f) \circ g = \mathrm{Trace}_{W_-}\, g \circ B(f)$$

which can be seen to be a symmetric form. The group of automorphisms of the triple (W_+, W_-, ϕ) is

$$S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-)) = \{(g, h) \mid \det(g) = \det(h)\}$$

We have a sequence of isomorphisms of representations of this group

$$\overset{4}{\wedge}(U) = \overset{4}{\wedge}(W_+^* \otimes W_-) = (\overset{2}{\wedge}W_+^*)^{\otimes 2} \otimes (\overset{2}{\wedge}W_-)^{\otimes 2} \xrightarrow{\phi\otimes\phi} \mathbf{1}$$

Thus we obtain a morphism

$$S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-)) \to \mathrm{SO}(\mathrm{Hom}(W_+, W_-))$$

Over the complex numbers this identifies the group $S(\mathrm{GL}(2) \times \mathrm{GL}(2))$ with the "Cspin" group of $\mathrm{SO}(4)$ and the subgroup $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is identified with the spin group of $\mathrm{SO}(4)$.

1.3. **Dimension 6.** Let U be a four dimensional vector space and let $\psi : \overset{4}{\wedge} U \to \mathbf{1}$ be a chosen isomorphism so that the group of automorphisms of the pair (U, ψ) is $\mathrm{SL}(U)$. Consider the pairing $<,>$ on $\overset{2}{\wedge} U$ given by the composite

$$\overset{2}{\wedge} U \otimes \overset{2}{\wedge} U \xrightarrow{\wedge} \overset{4}{\wedge} U \xrightarrow{\psi} \mathbf{1}$$

This is symmetric and non-degenerate. Moreover, we have a natural sequence of isomorphisms of representations of $\mathrm{SL}(U)$

$$\overset{6}{\wedge}(\overset{2}{\wedge} U) = (\overset{4}{\wedge} U)^{\otimes 6} \xrightarrow{\psi^{\otimes 6}} \mathbf{1}$$

Thus we have a representation of $\mathrm{SL}(U)$ in $\mathrm{SO}(\overset{2}{\wedge} U)$. Over the complex numbers this identifies $\mathrm{SL}(4)$ with the spin group of $\mathrm{SO}(6)$.

1.4. **Combination of the above.** Now consider the situation of (1.3) where $U = \mathrm{Hom}(W_+, W_-)$. In this situation U carries a non-degenerate pairing $(,)$ as described above and hence there is an induced pairing on $\overset{2}{\wedge} U$ which we also denote by $(,)$. We then have an automorphism $*$ on $\overset{2}{\wedge} U$ defined by the identity $(\alpha, \beta) = < \alpha, *\beta >$. Now the fact that $\psi = \phi \otimes \phi$ satisfies $(\psi, \psi) = 1$ implies that $*^2 = 1$. Moreover, one can see that the positive (resp. negative) eigenspace Λ^+ (resp. Λ^-) of $*$ is of dimension 3. Thus the combined representation

$$S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-)) \to \mathrm{SO}(U) \hookrightarrow \mathrm{SL}(U) \to \mathrm{SO}(\overset{2}{\wedge} U)$$

gives a morphism into $\mathrm{SO}(\Lambda^+) \times \mathrm{SO}(\Lambda^-)$. Now we have natural maps $S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-)) \to \mathrm{GL}(W_\pm)$. And hence we have representations of $S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-))$ into $\mathrm{SO}(\mathrm{End}^0(W_\pm))$. Consider the homomorphisms of representations of $S(\mathrm{GL}(W_+) \times \mathrm{GL}(W_-))$

$$\overset{2}{\wedge} U \to \mathrm{End}^0(W_+) \text{ where } f \wedge g \mapsto B(f) \circ g - B(g) \circ f$$

and similarly

$$\overset{2}{\wedge} U \to \mathrm{End}^0(W_-) \text{ where } f \wedge g \mapsto f \circ B(g) - g \circ B(f)$$

These induce isomorphisms of $\mathrm{End}^0(W_\pm)$ with $\Lambda^\pm$.

1.5. **Compact forms.** Let us fix hermitian structures $h_\pm$ on $W_\pm$ so that ϕ is an isometry. The group of automorphisms then becomes $S(\mathrm{U}(W_+) \times \mathrm{U}(W_-))$. We define a $\mathbb{C}$-anti-linear automorphism $f \mapsto f^\dagger$ defined by the identity

$$h_+(f^\dagger(w), w') = h_+(w, B(f)(w'))$$

One sees that $f^{\dagger\dagger} = f$. Thus we obtain a real vector space T so that $U = T + \iota T$. Moreover, one sees that the form $(,)$ restricts to a positive definite form on T; hence we obtain a representation $S(\mathrm{U}(W_+) \times \mathrm{U}(W_-)) \to \mathrm{SO}(T)$. The above discussion then gives us a decomposition of $\overset{2}{\wedge} T$ into $\Lambda^\pm_{\mathbb{R}}$.

We have a $\mathbb{C}$-anti-linear endomorphism $f \mapsto f^\dagger$ of $\mathrm{End}^0(W_\pm)$ given by

$$h_\pm(f^\dagger w, w') = h_\pm(w, f(w'))$$

One shows that under the isomorphism between $\mathrm{End}^0(W_\pm)$ and $\Lambda^\pm = \Lambda^\pm_{\mathbb{R}} + \iota\Lambda^\pm_{\mathbb{R}}$, we obtain identifications of $\Lambda^\pm_{\mathbb{R}}$ with the spaces $\mathrm{End}^0(W_\pm)^{ah}$ consisting of $f = -f^\dagger$.

We note that for any pair of elements Φ,Ψ of W_+ we have an element $\sigma(\Phi,\Psi)$ of $\mathrm{End}^0(W_+)$ given by

$$w \mapsto i(h_+(w,\Psi)\cdot\Phi - \frac{1}{2}h_+(\Phi,\Psi)\cdot w)$$

When $\Phi = \Psi$ this is an element of $\mathrm{End}^0(W_+)^{ah}$. We identify this with an element of $\Lambda^+_{\mathbb{R}}$.

1.6. **Unitary group case.** We now further specialise to the case when $W_+ = \mathbf{1}\oplus\det W_-$. For ease of notation we use W for W_-. In this case, we have a natural sequence of identifications

$$\mathrm{Hom}_{\mathbb{C}}(W_+,W_-) = W\oplus W^* = W\oplus\overline{W} = W\otimes_{\mathbb{R}}\mathbb{C}$$

Thus we can identify the special orthogonal representation T with the underlying real vector space of W. Now let $\overset{(2,0)}{\wedge}T$ denote the underlying real vector space to $\overset{2}{\wedge}_{\mathbb{C}}W$ and let $\overset{(1,1)}{\wedge}T$ the real vector space such that $W\otimes\overline{W} = \overset{(1,1)}{\wedge}T\otimes\mathbb{R}\mathbb{C}$. We have a natural decomposition

$$\overset{2}{\wedge}T = \overset{(2,0)}{\wedge}T\oplus\overset{(1,1)}{\wedge}T$$

The imaginary part of the hermitian metric on W gives a natural element ω of the latter space. One then computes that

$$\lambda^+_{\mathbb{R}} = \overset{(2,0)}{\wedge}T\oplus\mathbb{R}\cdot\omega \text{ and } \lambda^-_{\mathbb{R}} = \omega^\perp\cap\overset{(1,1)}{\wedge}T$$

Moreover, under the identification between $\Lambda^+_{\mathbb{R}}$ and $\mathrm{End}(W_+)^{ah}$ we obtain identifications

$$\overset{(2,0)}{\wedge}T = \mathrm{Hom}_{\mathbb{C}}(\mathbf{1},\det W) = \det W \text{ and } \mathbb{R} = \mathbb{R}\cdot\omega = \mathbb{R}i\cdot\mathbf{1}_{\det W}$$

2. Spin structures on four manifolds

Let X be a compact oriented four manifold. For any metric g on X we have the principal $\mathrm{SO}(4)$ bundle P on X which consists of oriented orthonormal frames. This corresponds to a class $[P]$ in $\mathrm{H}^1(X,\mathrm{SO}(4))$. Using the exact sequence

$$1\to\mathrm{U}(1)\to\mathrm{Spin}_c(4)\to\mathrm{SO}(4)\to 1$$

we see that we have an exact sequence

$$\mathrm{H}^1(X,\mathrm{U}(1))\to\mathrm{H}^1(X,\mathrm{Spin}_c(4))\to\mathrm{H}^1(X,\mathrm{SO}(4))\to\mathrm{H}^2(X,\mathrm{U}(1))$$

we see that the obstruction to giving a reduction of structure group from $\mathrm{SO}(4)$ to $\mathrm{Spin}_c(4)$ is given by a class in $\mathrm{H}^2(X,\mathrm{U}(1))$. Moreover, from the exact sequence

$$1\to\mathbb{Z}/2\mathbb{Z}\to\mathrm{Spin}(4)\to\mathrm{SO}(4)\to 1$$

we see that the obstruction to giving a spin structure lies in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$. Under the natural inclusion of $\mathbb{Z}/2\mathbb{Z}$ in $\mathrm{U}(1)$, the obstruction for spin maps to the obstruction for Cspin. In fact consider the diagram

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Spin}(4) & \to & \mathrm{SO}(4) & \to 1 \\ & \downarrow & & \downarrow & & \| & \\ 1 \to & \mathrm{U}(1) & \to & \mathrm{Spin}_c(4) & \to & \mathrm{SO}(4) & \to 1 \\ & \downarrow & & \downarrow & & & \\ & \mathrm{U}(1) & = & \mathrm{U}(1) & & & \\ & \downarrow & & \downarrow & & & \\ & 1 & & 1 & & & \end{array}$$

By the associated diagram of cohomologies

$$\begin{array}{ccccccccc} & \downarrow & & \downarrow & & & & \downarrow \\ \to & \mathrm{H}^1(X,\mathbb{Z}/2\mathbb{Z}) & \to & \mathrm{H}^1(X,\mathrm{Spin}(4)) & \to & \mathrm{H}^1(X,\mathrm{SO}(4)) & \to & \mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z}) \\ & \downarrow & & \downarrow & & \| & & \downarrow \\ \to & \mathrm{H}^1(X,\mathrm{U}(1)) & \to & \mathrm{H}^1(X,\mathrm{Spin}_c(4)) & \to & \mathrm{H}^1(X,\mathrm{SO}(4)) & \to & \mathrm{H}^2(X,\mathrm{U}(1)) \\ & \downarrow & & \downarrow & & & & \downarrow \\ & \mathrm{H}^1(X,\mathrm{U}(1)) & = & \mathrm{H}^1(X,\mathrm{U}(1)) & & & & \mathrm{H}^2(X,\mathrm{U}(1)) \\ & \downarrow & & \downarrow & & & & \end{array}$$

we see that the distinct lifts of a given $\mathrm{SO}(4)$ bundle to a $\mathrm{Spin}_c(4)$ bundle correspond exactly to the different lifts of the Spin(4) obstruction class in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$ to a class in $\mathrm{H}^1(X,\mathrm{U}(1))$. We note that the latter is the group of metrised complex line bundles.

Now we have a natural exact sequence (the exponential sequence) of sheaves

$$0 \to \mathbb{Z} \to \mathcal{C}^\infty \to \mathrm{U}(1) \to 1$$

which gives the natural isomorphisms $\mathrm{H}^{i+1}(X,\mathbb{Z}) = \mathrm{H}^i(X,\mathrm{U}(1))$. Moreover, under these isomorphisms the exact sequence

$$\to \mathrm{H}^1(X,\mathrm{U}(1)) \to \mathrm{H}^1(X,\mathrm{U}(1)) \to \mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(X,\mathrm{U}(1)) \to \mathrm{H}^2(X,\mathrm{U}(1))$$

is the same as the exact sequence

$$\to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^3(X,\mathbb{Z}) \to \mathrm{H}^3(X,\mathbb{Z})$$

To summarise, the obstruction to giving a $\mathrm{Spin}_c(4)$ structure is the image in $\mathrm{H}^3(X,\mathbb{Z})$ of the obstruction to a Spin(4) which lies in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$. If the former is zero then the different $\mathrm{Spin}_c(4)$ structures correspond to the different lifts of the Spin(4) obstruction class to $\mathrm{H}^2(X,\mathbb{Z})$.

In the case when the principal bundle is the one associated with the metrised tangent bundle as above we have the result that the obstruction to having a spin structure is given by $w_2(X)$ in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$; the second Stiefel-Whitney class of X. Then we have Wu's formula which implies that for any y in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$ we have $w_2(X) \cap y = y \cap y$. Now consider the image w of $w_2(X)$ in $\mathrm{H}^3(X,\mathbb{Z})$; this is a 2-torsion class. Let $\mathrm{H}^2(X,\mathbb{Z})_\tau$ denote the group of torsion elements in $\mathrm{H}^2(X,\mathbb{Z})$. There is a natural duality between the 2-torsion in $\mathrm{H}^3(X,\mathbb{Z})$ and the group $\mathrm{H}^2(X,\mathbb{Z})_\tau \otimes \mathbb{Z}/2\mathbb{Z}$; this duality is given as follows. Let $a \in \mathrm{H}^2(X,\mathbb{Z})_\tau$ be a torsion class and let $b \in \mathrm{H}^3(X,\mathbb{Z})$ be a 2-torsion class. Let b' be a class in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$ whose image is b. Let a' be the image of a in $\mathrm{H}^2(X,\mathbb{Z}/2\mathbb{Z})$, then

$< a, b >= (a', b')$. By this identification we have

$$< a, w >= (a', w_2(X)) = \mathrm{Trace}(a' \cap w_2(X)) = \mathrm{Trace}(a' \cap a') = 0$$

for all a in $\mathrm{H}^2(X, \mathbb{Z})_\tau$. But then by the duality we see that w is 0. Hence, in this case we obtain that $w_2(X)$ is the reduction modulo 2 of an integral cohomology class; in other words *an oriented compact Riemannian four manifold always has a* $\mathrm{Spin}_c(4)$ *structure*.

3. Monopole equations and their moduli space

In this section we describe the monopole moduli spaces and compute the expected dimension.

3.1. **Connections for Spin structures.** Let (X, g, c) be a compact oriented Riemannian four manifold with a Spin_c structure (denoted by c). Let Q denote the corresponding principal $\mathrm{Spin}_c(4)$ bundle over X. Then the principal bundle of oriented orthonormal frames on X is given by $P = Q/\mathrm{U}(1)$. We have a natural torsion free connection on this bundle called the Riemannian connection. The pull-back of this to Q gives us a 1-form on Q with values in Lie(SO(4) which is invariant for the action of $\mathrm{Spin}_c(4)$. Now consider the principal U(1) bundle $Q/\mathrm{Spin}(4)$ associated with Q which is just the space of all unit vectors in the line bundle $L = \det(W_\pm)$. Let A be connection on this line bundle. We can pull this back to a form on Q. Adding the above two forms together we obtain a connection on Q which we shall denote by ∇_A since the Riemannian connection is unique whereas A can be varied.

3.2. **Dirac equation.** Fixing A for the time being we have the differential operator $\nabla_A : W_+ \to W_+ \otimes T^*X$ induced by the connection as above. One the one hand the Riemannian structure gives us a natural (flat) identification between T^*X and TX and on the other we have seen that TX can be thought of as a subspace of $\mathrm{Hom}_\mathbb{C}(W_+, W_-)$; moreover, this identification is also invariant under the connection (flat). Thus by contraction we obtain the composite differential operator of order 1

$$D_A = D_{A,+} : W_+ \to W_+ \otimes T^*X \to W_+ \otimes TX \to W_-$$

This is called the Dirac operator. As seen earlier we have a natural identification $\mathrm{Hom}_\mathbb{C}(W_+, W_-) = \mathrm{Hom}_\mathbb{C}(W_-, W_+)$. Thus we also obtain an operator $D_A^* = -D_{A,-} : W_- \to W_+$. We have the identity (see Section 4)

$$\int_X h_-(-D_{A,+}\Phi, \Psi) = \int_X h_+(\Phi, D_{A,-}\Psi)$$

so that we see that D_A^* is the adjoint of D_A. This justifies the notation.

The first monopole equation is the Dirac equation $D_A(\Phi) = 0$.

3.3. **Second monopole equation.** Consider the curvature F_A of the connection A on L. This gives a two form with values in the Lie algebra of U(1) which is just $\mathbb{R}$. Let F_A^+ denote the projection into Λ^+. We have also defined the map $\sigma : W_+ \otimes \overline{W_+} \to \mathrm{End}(W_+)^{ah}$. Moreover, we have obtained an identification of Λ^+ with the space of skew-Hermitian endomorphisms of W_+. The second monopole equation is

$$F_A^+ = \sigma(\Phi, \Phi)$$

3.4. **Gauge group.** Let $\mathcal{G} = \mathrm{Map}(X, \mathrm{U}(1))$ and consider the action of this group on the space $\mathcal{N}$ of pairs (A, Φ) where A is a connection on the line bundle L and Φ a section of W_+ given by

$$g \cdot (A, \Phi) = (g \cdot A, g \cdot \Phi) = (A - (1/2\pi i)g^{-1}dg, g\Phi)$$

We see easily that if (A, Φ) satisfies the monopole equation then so does $g \cdot (A, \Phi)$. In fact we have

$$D_{g \cdot A}(g \cdot \Phi) = g \cdot D_A\Phi \text{ and } F_{g \cdot A} = F_A \text{ and } \sigma(g\Phi, g\Psi) = \sigma(\Phi, \Psi)$$

Thus we may consider the 'moduli space' of monopoles

$$M = M_c = \{(A, \Phi) \mid D_A\Phi = 0 \text{ and } F_A^+ = \sigma(\Phi, \Phi)\}/\mathcal{G}$$

We will show that for 'good' metrics this is a compact orientable manifold. We shall also find out how it depends on this choice of metric.

Now let $\mathcal{W}_\pm$ be the spaces of sections of $W_\pm$. We have a map

$$\nu : \mathcal{N} \to \mathcal{W}_- \text{ given by } (A, \Phi) \mapsto (D_A\Phi)$$

Let $\mathcal{M}$ denote the inverse image $\nu^{-1}(0)$ and let $\mathcal{M}^*$ denote the open subset consisting of pairs (A, Φ) where $\Phi \neq 0$. The differential of the map $(A, \Phi) \to D_A\Phi$ is given by

$$(a, \phi) \mapsto D_A\phi + 2\pi i a \circ \Phi$$

Suppose Ψ is orthogonal to the image. Then we obtain the equations

$$D_A^*\Psi = 0 \text{ and } \Phi \otimes \Psi = 0$$

by orthogonality with the image of vectors of the form $(a, 0)$ and $(0, \phi)$ respectively. Now a solution of an elliptic operator vanishes on an open set only if it is identically 0. Thus we see that $\psi = 0$; in other words ν is a submersion when restricted to the space $\mathcal{N}^*$ consisting of pairs (A, Φ) where $\Phi \neq 0$. Thus $\mathcal{M}^*$ is a manifold (albeit of infinite dimension).

The group $\mathcal{G}$ acts freely on $\mathcal{M}^*$ since a solution of an elliptic operator cannot vanish on an open set unless it is 0. Consider the space Ω^{2+} consisting of 2-forms invariant under $*$ with the trivial action of $\mathcal{G}$. The map $\mathcal{M} \to \Omega^{2+}$ given by $F_A^+ - \sigma(\Phi, \Phi)$ factors thorugh the quotient $\mathcal{M}/\mathcal{G}$. We thus obtain a 'complex' $\mathcal{G} \to \mathcal{M}^* \to \Omega^{2+}$. The moduli space can be thought of as being its 'cohomology'.

3.5. **Virtual dimension of the moduli space.** To compute the dimension of the moduli space we need to compute the cohomology of the complex of differentials of the complex $\mathcal{G} \to \mathcal{M}^* \to \Omega^{2+}$. The tangent space to $\mathcal{G}$ at identity can be identified with Ω^0 the space of fucntions and the tangent space to Ω^{2+} can be identified with itself since it is a vector space. We have an exact sequence

$$0 \to T\mathcal{M}^* \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \to 0$$

Where we have identified the tangent space of $\mathcal{A}$ with Ω^1 the space of 1-forms. Thus the complex of differentials

$$T\mathcal{G} \to T\mathcal{M}^* \to \Omega^{2+}$$

is quasi-isomorphic to the complex

$$\Omega^0 \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \oplus \mathcal{W}_-$$

where the maps are

$$h \mapsto (-dh, 2\pi i h\Phi) \text{ and } (a, \psi) \mapsto (d^+a - \Im\sigma(\Phi, \phi), D_A\phi + 2\pi i a \circ \Phi)$$

for $h \in \Omega^0$, $a \in \Omega^1$ and $\phi \in \mathcal{W}_+$. Here d^+ denotes the exterior derivative combined with the projection to Ω^{2+} and $\Im\sigma(\Phi, \phi)$ denotes the skew-Hermitian part of $\sigma(\Phi, \phi)$. This complex is homotopic to the complex where the first map is $h \mapsto (-dh, 0)$ and the second is $(a, \phi) \mapsto (d^+a, D_A\phi)$ since the difference between these two complexes is given by compact operators. Thus the index of our complex of differentials is the index of the complex

$$\Omega^0 \to \Omega^1 \oplus \mathcal{W}_+ \to \Omega^{2+} \oplus \mathcal{W}_-$$

where the maps are

$$h \mapsto (-dh, 0) \text{ and } (a, \psi) \mapsto (d^+a, D_A\phi)$$

This is a topological invariant for the pair (X, c) by the Atiyah-Singer Index theorem; we call this the virtual dimension of the moduli space. In case we can find a point $\delta \in \Omega^{2+}$ which is a regular value for the map $\mathcal{M}^* \to \Omega^{2+}$ we see that this index will be the dimension of

$$M_{c,\delta} = \{(A, \Phi) \mid D_A\Phi = 0 \text{ and } F_A^+ = \sigma(\Phi, \Phi) + \delta\}/cG$$

We call this the perturbed moduli space. We will show that such a value of δ exists that $M_{c,\delta}$ is a compact orientable manifold whose dimension is the virtual dimension.

4. Differential Calculus

We derive various identities among differential operators in the context of Spin_c connections.

4.1. The Adjoint of the Dirac operator.

We have defined the Dirac operator as the composite

$$D_{A,+} = D_A : W_+ \xrightarrow{\nabla_A} TX^* \otimes W_+ \to W_-$$

where the latter map is the contraction under the identification of TX^* with $TX \subset \text{Hom}_{\mathbb{C}}(W_+, W_-)$. We have similarly the Dirac operator $D_{A,-} : W_- \to W_+$ since we have an identification of $\text{Hom}_{\mathbb{C}}(W_+, W_-)$ with its dual space $\text{Hom}_{\mathbb{C}}(W_-, W_+)$. In terms of an orthonormal frame of tangent vectors $\{e_i\}$ we obtain a sequence of identities:

$$(D_A\Phi, \Psi) = \sum_i (e_i \circ \nabla_{e_i}\Phi, \Psi)$$

and since $(f \circ \Phi, \Psi) = (f^\dagger \circ \Phi, \Psi) = (\Phi, f \circ \Psi)$ for all f in TX,

$$(D_A\Phi, \Psi) = \sum_i (\nabla_{e_i}\Phi, e_i \circ \Psi)$$

Now the fact that ∇ is a metric connection means that

$$(\nabla_{e_i}\Phi, e_i \circ \Psi) = e_i(\Phi, e_i \circ \Psi) - (\Phi, \nabla_{e_i}(e_i \circ \Psi))$$

Let $d\tau$ denote the volume form then for any function f and any vector field v we have,

$$v(f)d\tau = d(f(v \lrcorner d\tau)) - fd(v \lrcorner d\tau)$$

Thus we obtain small

$$(D_A\Phi, \Psi)d\tau = \sum_i d((\Phi, e_i \circ \Psi)e_i \lrcorner d\tau) - \sum_i (\Phi, e_i \circ \Psi)d(e_i \lrcorner d\tau) - \sum_i (\Phi, \nabla_{e_i}(e_i \circ \Psi))$$

For any vector field v we have the identity

$$d(v \lrcorner d\tau) = \sum_j (e_j, \nabla_{e_j} v) d\tau$$

Moreover, since $(e_j, e_j) = \delta_{i,j}$ is a constant we have

$$\sum_i (\Phi, e_i \circ \Psi) d(e_i \lrcorner d\tau) = \sum_{i,j} (\Phi, e_i \circ \Psi)(e_j, \nabla_{e_j} e_i) d\tau = -\sum_{i,j} (\Phi, e_i \circ \Psi)(\nabla_{e_j} e_j, e_i) d\tau$$

The other term can be written as follows

$$\nabla_{e_i}(e_i \circ \Psi) = (\nabla_{e_i} e_i) \circ \Psi + e_i \circ \nabla_{e_i} \Psi$$

and

$$(\nabla_{e_i} e_i) \circ \Psi = \sum_j (\nabla_{e_i} e_i, e_j) e_j \circ \Psi$$

Combining the above identities we obtain

$$(D_A \Phi, \Psi) = \sum_i d((\Phi, e_i \circ \Psi) e_i \lrcorner d\tau) - \sum_i (\Phi, e_i \circ (\nabla_{e_i} \Psi)) d\tau$$

Hence

$$\int_X (-D_{A,+}\Phi, \Psi) = \int_X (\Phi, D_{A,-}\Psi)$$

and $-D_{A,-}$ is the adjoint operator of $D_{A,+}$.

By an entirely similar chain of reasoning we show that the adjoint $\nabla^* : TX \otimes W_- \to W_+$ of ∇ on W_+ is given by

$$\nabla(v \otimes \Phi) = -(\sum_i (e_i, \nabla_{e_i} v)\Phi + \nabla_v \Phi)$$

In invariant terms, we can describe this as the composite

$$TX \otimes W_+ \xrightarrow{-\nabla} TX^* \otimes TX \otimes W_+ \xrightarrow{\text{Trace} \otimes \mathbf{1}} W_+$$

4.2. **The Weitzenbock formula.** We now compute the composite $D_A^* D_A \Phi$. As before we choose a local orthonormal frame $\{e_i\}$ for X. We then have

$$D_A^* D_A \Phi = \sum_i -D_A(e_i \circ \nabla_{e_i} \Phi) = -\sum_{i,j} e_j \circ \nabla_{e_j}(e_i \circ \nabla_{e_i} \Phi))$$

We expand the summand to obtain

$$e_j \circ \nabla_{e_j} e_i \circ \nabla_{e_i} \Phi + e_j \circ e_i \nabla_{e_j} \nabla_{e_i} \Phi$$

As above the first term above can be expanded again as

$$\sum_k (e_k, \nabla_{e_j} e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi = -\sum_k (\nabla_{e_j} e_k, e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi$$

We obtain the formula

$$D_A^* D_A \Phi = \sum_{i,j,k} (\nabla_{e_j} e_k, e_i) e_j \circ e_k \circ \nabla_{e_i} \Phi - \sum_{i,j} e_j \circ e_i \circ \nabla_{e_j} \nabla_{e_i} \Phi$$

Now defining $\nabla^2_{V,W} = \nabla_V \nabla_W - \nabla_{\nabla_V W}$,

$$D_A^* D_A \Phi = -\sum_{i,j} e_j \circ e_i \circ \nabla^2_{e_j, e_i} \Phi$$

Similar calculations yield the formula

$$\nabla_A^*\nabla_A\Phi = -\sum_i \nabla^2_{e_i,e_i}\Phi$$

Now the difference gives us

$$D_A^*D_A\Phi - \nabla_A^*\nabla_A\Phi = -\sum_{i\neq j} e_j \circ e_i \circ \nabla^2_{e_j,e_i}\Phi$$

From the definition of $\nabla^2_{\cdot,\cdot}$ we have

$$\nabla^2_{V,W} - \nabla^2_{V,W} = \nabla_V\nabla_W - \nabla_{\nabla_V W} - \nabla_W\nabla_V - \nabla_{\nabla_W V} = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]}$$

using the fact that the connection is torsion free. Since we have an orthonormal basis we have $e_i \circ e_j = -e_j \circ e_i$ so that we obtain

$$D_A^*D_A\Phi - \nabla_A^*\nabla_A\Phi == -\sum_{i<j} e_j \circ e_i \circ R(e_j, e_i)\Phi$$

where $R(V,W) = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]}$ is the curvature tensor.

4.3. **The Curvature tensors.** The Spin_c connection has been expressed as a sum of the Riemannian connection and the U(1) connection A on L. Thus the curvature tensor R is also the sum of the Riemann curvature tensor S and the curvature of A. The former can be expressed as

$$S(V,W) = \sum_{k,l}(S(V,W)e_l, e_k)e_k \circ e_l$$

Thus we obtain

$$\sum_{i,j} e_j \circ e_i \circ S(e_j, e_i) = \sum_{i,j,k,l}(S(e_j,e_i)e_l, e_k)e_j \circ e_i \circ e_k \circ e_l$$

By the orthonormality of e_i's we easily resolve the latter to obtain $\sum_{i,j}(S(e_j,e_i)e_i, e_j)$ which is the negative of the scalar curvature s. The curvature of A considered as an operator on W_+ acts as $2\pi i F_A$. Thus the final (Weitzenbock) formula reads

$$D_A^*D_A - \nabla_A^*\nabla_A = s - 2\pi i F_A$$

4.4. **Extrema.** Let x be a point of our manifold where (Φ,Φ) attains a maximum. Then for any vector v at x we have $v((\Phi,\Phi))(x) = 0$. Thus consider the following identity (where $\Re$ denotes the real part)

$$\Re(\nabla_A^*\nabla_A\Phi,\Phi) = -\sum_i \frac{1}{2}(e_ie_i(\Phi,\Phi) - (\nabla_{e_i}\Phi, \nabla_{e_i}\Phi)) + \sum_{i,j}(e_j, \nabla_{e_i}e_i)\Re(\nabla_{e_i}\Phi,\Phi)$$

Since $e_i(\Phi,\Phi)(x) = 0$ the last term vanishes at x. Moreover, since x is a local maximum for (Φ,Φ) the term $e_ie_i(\Phi,\Phi)(x)$ is negative. Thus we see that $\Re(\nabla_A^*\nabla_A\Phi,\Phi)$ is positive at x.

5. The Seiberg-Witten invariants

In this section we construct the Seiberg-Witten invariants. First of all we fix a four manifold X, a Riemannian metric g and a Spin_c structure c. At the end of the section we will discuss the independence of the invariants on the metric considered.

5.1. **Statement of the basic construction.** Let M_δ denote the fibre of $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ over the point δ. We wish to show that there is a δ such that this is a compact manifold. To show this we need to show

1. There are regular values for $\mathcal{M}^*/\mathcal{G} \to \Omega^{2+}$.
2. There are regular values as above such that the fibre of $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ is contained in $\mathcal{M}^*$.
3. The map $\mathcal{M}/\mathcal{G} \to \Omega^{2+}$ is proper.

5.2. **Properness.** Let δ_i be a convergent sequence of elements in Ω^{2+}. This means that the sequence converges in the L^2_k Sobolev norm for every k. Let (A_i, Φ_i) be such that $D_{A_i}\Phi_i = 0$ and $F^+_{A_i} - \sigma(\Phi_i, \Phi_i) = \delta_i$. To show properness we need to find a convergent subsequence of (A_i, Φ_i); for which it is enough to show that this sequence is bounded in the L^2_k Sobolev norm for every k.

Let B be a fixed smooth connection on L; we express $A_i = B + a_i$ where a_i are 1-forms. Consider the function $h_i = G * d * a_i$ and let $g^0_i = \exp(2\pi i h_i)$. Then $g^0_i \cdot A_i = B + a_i - dh_i$ and we obtain $*d*(a_i - dh_i) = 0$. Now we can choose g_i so that the Hramonic part of a_i lies in the fundamental domain for $\mathrm{H}^1(X, \mathbb{Z})$ in $\mathrm{H}^1(X, \mathbb{R})$. Thus upto gauge invariance we can replace A_i by another so that $*d*a_i = 0$ and the harmonic part α_i of a_i is bounded. Let $b_i = a_i - \alpha_i$. The second monopole equation becomes

$$d^+ b_i = \sigma(\Phi_i, \Phi_i) - F^+_B + \delta_i$$

So that an L^2_k bound on Φ_i will give an L^2_k bound on d^+b_i. But now $b_i = G*d*d^+b_i$ by the above construction of b_i; here G is the Green's operator. Thus we obtain a bound on the L^2_{k+1} norm of b_i since G is 2-smoothing.

Let us write $\Phi_i = \Psi_i + \phi_i$ where $D_B\Psi_i = 0$ and ϕ_i is orthogonal to the space of solutions of D_B. Hence $\phi = GD^*_B D_B \Phi$ and the first monopole equation becomes

$$D_B\Phi_i = -(b_i + \alpha_i) \circ \Phi_i$$

so that an L^2_k bound on Φ_i and b_i gives us an L^{k+1}_2 bound on ϕ_i. We also need to find a way to uniformly bound Ψ_i. We do this by finding a uniform bound for Φ_i.

Let x_i be a point where (Φ_i, Φ_i) attains a supremum. Applying the Weitzenbock formula we see that at x_i we have

$$0 \geq \Re(\nabla^*_{A_i}\nabla_{A_i}\Phi_i, \Phi_i)(x_i) = -\Re(s\Phi_i, \Phi_i)(x_i) + 2\pi\Im(F_{A_i}\Phi_i, \Phi_i)(x_i)$$

Note that F_{A_i} is a skew-Hermitian endomorphism of W_+ and thus

$$\Im(F_{A_i}\Phi_i, \Phi_i) = (F_{A_i}\Phi_i, \Phi_i)$$

The second monopole equation gives us

$$F_{A_i}\Phi_i = \sigma(\Phi_i, \Phi_i)\Phi_i + \delta_i\Phi_i$$

and the expression for σ gives us

$$\sigma(\Phi_i, \Phi_i)\Phi_i = \frac{i}{2}(\Phi_i, \Phi_i)\Phi$$

Combining the above we obtain

$$(\Phi_i, \Phi_i)(x_i) \leq \max\{0, -s + \| \delta_i \|\}$$

Thus we uniformly bound Φ_i in the C^0-norm. This gives us uniform bounds for Ψ_i and ϕ_i in the C^0-norm. Now Ψ_i are solutions of the Dirac equation $D_B\Psi_i = 0$.

Thus the set of C^0-bounded solutions is a compact set; in particular, we obtain L^2_k bounds on Ψ_i for all i.

The above arguments applied inductively gives the required result. We note that the above arguments also prove that the solutions of the monopole equations are smooth since any solution which is bounded in L^2_k norm for some k is actually bounded in all $L^2_{k'}$ norms as above.

5.3. **Regular values.** Now consider the compact space M_0 of solutions of the unperturbed monopole equations. For each point (A, Φ) of M_0 we have a neighbourhood U in $\mathcal{N}$ of (A, Φ) and a finite dimensional linear space $H \subset \Omega^{2+}$ such that the composite

$$U \to \mathcal{N} \to \mathcal{W}_- \times \Omega^{2+} \to H^\perp$$

is a submersion. By compactness we can find a common H and a saturated (for $\mathcal{G}$) open set U in $\mathcal{N}$ containing the inverse image of M_0 such that the above composite is a submersion. Since the derivative is a Fredholm map, the fibre over 0 is a finite dimensional manifold N. We now consider the map of finite dimensional manifolds $N \to H$. By Sard's theorem we have a dense subset of H which consists of regular values.

Now assume that b_2^+ which is the codimension in Ω^{2+} of the δ's of the form $F_B^+ + d^+b$ is greater than zero. Then the collection of those δ for which the fibre is contained in $\mathcal{M}^*$ is a non-empty open set. If $b_2^+ > 1$ then this open set is even path-wise connected. Thus in this situation the cobordism class of the fibre is independent of the regular value chosen.

5.4. **Dependence on the metric.** Let $\mathcal{C}$ denote the space of all metrics g on X under which the fixed volume form $d\tau$ has norm one. We have a natural map $\mathcal{C} \to G$ where G denotes the Grassmannian of rank b_2^+ quotients of $\mathrm{H}^2(X, \mathbb{R})$. The corresponding tangent level map is

$$\mathrm{Hom}(\Lambda^-, \Lambda^+) = T\mathcal{C} \to TG = \mathrm{Hom}(\mathrm{H}_g^{2-}, \mathrm{H}_g^{2+})$$

Where a map $f : \Lambda^- \to \Lambda^+$ goes to its harmonic projection.

For any class $c = c_1(L)$ in $\mathrm{H}^2(X, \mathbb{R})$ let S_c denote the subvariety of G where the class c goes to zero in H^{2+}. At a point of S_c the tangent space to S_c is given by the kernel of the evaluation map $g \mapsto g(c)$. Consider the composite map

$$\mathrm{Hom}(\Lambda^-, \Lambda^+) \to \mathrm{Hom}(\mathrm{H}_g^{2-}, \mathrm{H}_g^{2+}) \to \mathrm{H}_g^{2+}$$

If we show that this map is surjective, then the space of all metrics under which the class c becomes $*$-anti-invariant will be of codimension b_2^+. The argument of the previous section will apply to show that the Seiberg-Witten invariant is independent of the metric when $b_2^+ > 1$.

To show that the above map is surjective suppose that d is perpendicular to the image. We will then obtain that $c \otimes d$ is identically zero. But now if $c \neq 0$ then it is represented by a harmonic form which cannot vanish on an open set. Thus d must vanish on an open set. But we represent d by a harmonic form too. Thus $d = 0$ as required.

6. The case of Kähler manifolds

We now specialise to the case of Kähler surfaces.

6.1. **Spin structures.** For any four manifold with almost complex structure and (hermitian) metric we have a natural Spin_c structure given by taking $W_+^0 = \overset{2}{\wedge}_{\mathbb{C}} TX \oplus \mathbf{1}$ and $W_-^0 = TX$. The inclusion of TX in $\mathrm{Hom}_{\mathbb{C}}(W_+^0, W_-^0)$ is the natural one as discussed at the end of section 1. Thus any Spin_c structure on X is given by $W_+ = M \otimes_{\mathbb{C}} \overset{2}{\wedge}_{\mathbb{C}} TX \oplus M$ and $W_- = TX \otimes_{\mathbb{C}} M$. For ease of notation we adopt the standard convention $\overset{2}{\wedge} TX^* = K_X$.

6.2. **Spin_c connections.** Any U(2) connection on TX gives a connection on all associated bundles. In particular we obtain connections on $W_\pm^0$. However, in order that these be Spin_c connections it is necessary that the induced connection on TX be the Riemannian (torsion-free) connection. This can only happen if the (almost) complex structure is parallel with respect to the Riemannian connection; thus in this case the manifold must be Kähler.

To give a connection in the general Spin_c structure we need in addition to give a U(1) connection on M.

6.3. **The First monopole equation.** Consider a Spin_c connection as above. We then obtain a Dirac operator on $M \oplus M \otimes K_X^{-1}$. By the above discussion we note that the restriction of this to M is the composite

$$M \to M \otimes_{\mathbb{R}} TX^* = M \otimes_{\mathbb{C}} TX^* \oplus M \otimes_{\mathbb{C}} \overline{TX^*} \to M \otimes_{\mathbb{C}} TX$$

Here we have used the identification of TX^* with $\overline{TX}$ given by the hermitian structure. The first map in the above composite is the U(1) connection on M. Thus we see that the restriction of the Dirac operator to M is $\nabla^{(0,1)}$. We similarly show that the restriction of the Dirac operator to $M \otimes K_X^{-1}$ is also $\nabla^{(0,1)}$ for the induced U(1) connection on this line bundle.

6.4. **The Second Monopole equation.** Following Section 1 we compute that the $(2,0)$ part of $\sigma(\Phi,\Phi)$ for $\Phi = (\alpha,\beta)$ is $\overline{\alpha}\beta$ and the $(1,1)$ part of is $\frac{1}{2}(\|\beta\|^2 - \|\alpha\|^2)$. Thus the second monopole equation becomes

$$(F_A^+)^{(2,0)} = \overline{\alpha}\beta \text{ and } (F_A^+)^{(1,1)} = \frac{1}{2}(\|\beta\|^2 - \|\alpha\|^2)\omega$$

6.5. **The Weitzenbock formula.** We next apply the Weitzenbock formula for any pair (A,Φ)

$$D_A^* D_A \Phi = \nabla_A^* \nabla_A \phi + s\Phi - 2\pi i F_A^+ \Phi$$

to obtain an equality of global inner products

$$(D_A\Phi, D_A\Phi)_X = (\nabla_A\Phi, \nabla_A\Phi)_X + (s\Phi,\Phi)_X + 2\pi\Im(F_A^+\Phi,\Phi)_X$$

On the other hand we compute the global norm of $F_A - \sigma(\Phi,\Phi)$ as follows

$$\|F_A^+ - \sigma(\Phi,\Phi)\|_X^2 = \|F_A^+\|_X^2 + \|\sigma(\Phi,\Phi)\|_X^2 - 2\Re(F_A^+, \sigma(\Phi,\Phi))_X$$

The last term is computed by the integral of the function

$$\Re\,\mathrm{Trace}_{W_+}(F_A^+ \circ \sigma(\Phi,\Phi)) = -\Im(F_A^+\Phi,\Phi) + \frac{1}{2}\,\mathrm{Trace}(F_A^+)\,\|\Phi\|^2$$

Now $\mathrm{Trace}(F_A^+)$ is 0. Thus adding the above two identities we obtain

$$\|D_A\Phi\|_X^2 + \|F_A^+ - \sigma(\Phi,\Phi)\|_X^2 = \|\nabla_A\Phi\|_X^2 + (s\Phi,\Phi)_X + 2\pi(\|F_A^+\|_X^2 + \|\sigma(\Phi,\Phi)\|_X^2)$$

We note that the right hand side is equal to

$$\|\nabla_A\alpha\|_X^2 + \|\nabla_A\beta\|_X^2 + (s\alpha,\alpha)_X + (s\beta,\beta)_X + 2\pi\,\|F_A^+\|_X^2 + 2\pi(\|\alpha\|_X^2 + \|\beta\|_X^2)^2$$

which is invariant under a change of sign for α or β.

Now suppose that (A, Φ) solve the monopole equations and consider the pair (A, Φ_1) where $\Phi_1 = (\alpha, -\beta)$. By the above discussion we see that (A, Φ_1) is also a solution for the monopole equations. But then we must have

$$(F_A)^{(2,0)} = \overline{\alpha}\beta = -\overline{\alpha}\beta = 0$$

Thus we obtain the fact that F_A is a holomorphic connection on $M^{\otimes 2} \otimes K_X$. Moreover, by ellipticity of the Dirac operator (and its components) we must have that either α or β is zero according as $(F_A^+)^{(1,1)}$ is a positive or negative multiple of ω. By the first monopole equation it then follows that α and β are holomorphic sections of the corresponding line bundles.

Institute of Mathematical Sciences, Chennai 600 113,India
E-mail address: kapil@math.imsc.ernet.in

Stat-Math Unit, Indian Statistical Institute, Bangalore 560 059, India
E-mail address: pati@isibang.ac.in