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The theorem of Kronheimer–Mrowka

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Abstract. A proof of the conjecture of Thom (algebraic curves in the complex projective plane minimize genus within their homology class) due to Kronheimer–Mrowka is presented. The proof uses the mod-2 Seiberg–Witten invariants.

Keywords. Thom conjecture; Seiberg–Witten invariants; 4-manifolds.

1. Some transversality results

1.1 *Introduction*

This note is an exposition of the proof of Thom's conjecture, namely that algebraic curves minimize genus within their homology class in $\mathbb{C}P^2$, due to Kronheimer–Mrowka (see [KM]). New invariants for 4-manifolds, called Seiberg–Witten invariants, are used in the proof.

For the sake of completeness we have included an appendix at the end on Fredholm theory. As background material, the reader may wish to consult the references [D], [PP], [W].

1.2 *Metrics*

Let X be a connected, compact, oriented 4-manifold. Fix a reference metric g_0 on X, which defines a volume form dV_{g_0} on X, and hence a trivialization of the determinant bundle $\wedge^4(T^*X)$ of the real cotangent bundle. This trivialization of $\wedge^4$ will be fixed throughout, and will be denoted dV without any ambiguity.

Thus the structure group for X is reduced to $SL(4, \mathbb{R})$, and one lets $P_X SL(4)$ denote the principal $SL(4, \mathbb{R})$ bundle on X consisting of all frames in TX on which dV yields the constant function 1 on X.

The corresponding ad-bundle, denoted $\mathrm{ad}\, \underline{sl}_4$, the vector bundle on X whose fibre over $x \in X$ is the vector space of traceless endomorphisms $\mathrm{End}^0(T_x X)$ of $T_x X$ splits into the direct sum of $\mathrm{ad}\, \underline{so}_4$ and $\mathrm{ad}\mathcal{P}$ (abuse of notation since $\mathcal{P}$ is not a Lie algebra) corresponding to the Cartan decomposition: $\underline{sl}_4 = \underline{so}_4 \oplus \mathcal{P}$. Here, the bundle $\mathrm{ad}\, \underline{so}_4$ is the adjoint bundle corresponding to the principal $SO(4, \mathbb{R})$-bundle $P_X SO(4)$ consisting of g_0-orthonormal oriented frames. It has fibre consisting of traceless g_0-skew-symmetric endos of $T_x X$ over x. $\mathrm{ad}\, \mathcal{P}$ has fibre consisting of traceless g_0-symmetric endos of $T_x X$ over x. Both are real rank-3 bundles.

Clearly, if g is *another* Riemannian metric on X with $dV_g = dV$, pointwise, then $g(u, v) = g_0((\exp h)u, (\exp h)v)$ for some section h of $\mathrm{ad}\, \mathcal{P}$, and thus the space of all C^r Riemannian metrics g whose volume form coincides with the prescribed one dV is precisely the space of C^r-sections $\Gamma^r(\mathrm{ad}\, \mathcal{P})$. This space clearly contains exactly one representative from each