Comment.Math.Univ.Carolin. 41,1 (2000)107–110

Boundedness of linear maps

T.S.S.R.K. Rao

Abstract. In this short note we consider necessary and sufficient conditions on normed linear spaces, that ensure the boundedness of any linear map whose adjoint maps extreme points of the unit ball of the domain space to continuous linear functionals.

Keywords: bounded linear maps, extreme points, barrelled spaces

Classification: 46B20

Introduction

Let X, Y be normed linear spaces and $T : X \to Y$ be a linear map. In this note we are interested in studying some "weak" continuity conditions on T, that will imply continuity. Motivation for this work comes from a recent work of Labuschagne and Mascioni [6], where they characterize linear maps between C^* algebras whose adjoints preserve extreme points. A small step in their work consists of showing, using C^* algebra methods, the continuity of such a map. In this note we first show that if X and Y are normed linear spaces such that for each extreme point y^* of the dual unit ball Y_1^*, $y^* \circ T$ is an extreme point of X_1^*, then T is bounded.

Let X_1 denote the closed unit ball of X and let $\partial_e X_1$ denote the set of extreme points. Since boundedness of the set $T(X_1)$ in the weak topology implies the boundedness of T, a natural question that can be asked now is: Is T bounded if one merely assumes that for all $y^* \in \partial_e Y_1^*$, $y^* \circ T \in X^*$? Here we give necessary and sufficient conditions on X and Y so that any such linear map T is bounded.

Main results

We first show the continuity of T when the "adjoint" preserves extreme points of the dual ball. Let $\mathcal{L}(X,Y)$ denote the space of bounded operators from X to Y.

Proposition 1. *Let X, Y be normed linear spaces. Let $T : X \to Y$ be a linear map such that for each $y^* \in \partial_e Y_1^*$, $y^* \circ T \in \partial_e X_1^*$; then $T \in \partial_e \mathcal{L}(X,Y)_1$.*

PROOF: Let $x \in X$. Choose a $y^* \in \partial_e Y_1^*$ such that $\|T(x)\| = y^*(T(x))$. By hypothesis $y^* \circ T$ is a functional of norm one. Thus $\|T\| \leq 1$. That T is an extreme point can be proved using the hypothesis and the Krein-Milman theorem (see [2, p. 148]). □

Remark 1. Operators whose adjoints preserve extreme points are known as "nice" operators (see [7] and the references listed therein). The analogous question, "when are elements of $\partial_e \mathcal{L}(X,Y)_1$ nice operators ?" received considerable attention, we again refer the reader to [7] and the references listed there for more information.

From now on we study conditions on X or Y that will result on the boundedness of T under the assumption $y^* \circ T \in X^*$ for every $y^* \in \partial_e Y_1^*$. We may assume w.l.o.g. that Y is a Banach space. To show the weak boundedness of $T(X_1)$, it is enough to show that $y^* \circ T \in X^*$ for every $y^* \in Y_1^*$. If Y_1^* is the convex hull of its extreme points then T is bounded without any further assumptions. This for example is the case when Y is a finite dimensional space or the space of trace class operators on a complex Hilbert space.

We recall that any infinite dimensional C^* algebra contains an isometric copy of c_0.

Theorem 1. *Let Y be a Banach space containing no isomorphic copy of c_0. For every normed linear space X, every linear operator $T : X \to Y$ such that $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$, is bounded.*

Proof: Let X be a normed linear space and $T : X \to Y$ be a linear map such that $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$. To show that T is bounded it is enough to show that for every sequence $\{x_n\}_{n\geq 1} \subset X_1$, $\{T(x_n)\}_{n\geq 1}$ is a bounded sequence in Y. For any $y^* \in \partial_e Y_1^*$, since $y^* \circ T \in X^*$, we have that $\{y^*(T(x_n))\}_{n\geq 1}$ is a bounded sequence of scalars. Therefore it follows from [3] that $\{T(x_n)\}_{n\geq 1}$ is a bounded sequence in Y. □

Example. Let $Y = c_0$ and $X =$ span$\{e_n\}_{n\geq 1}$, where $\{e_n\}_{n\geq 1}$ is the canonical Schauder basis of c_0. Then by defining $T : X \to Y$ by $T(\sum_{n=1}^k \alpha_n e_n) = \sum_{n=1}^k \alpha_n n e_n$, we see that T is a linear map and $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$ and T is not bounded.

In the next proposition we show that the Example described above works as a counterexample whenever the range space contains an isomorphic copy of c_0. Our proof involves the notion of an M-ideal whose definition we now recall from [5].

Definition. *Let Z be a Banach space. A closed subspace $Y \subset Z$ is said to be an M-ideal if Z^* is the ℓ^1 direct sum of $Y^\perp$ and another closed subspace $N \subset Z^*$.*

It is easy to see (Lemma I.1.5, [5]) that $\partial_e Z_1^* = \partial_e (Y^\perp)_1 \cup \partial_e N_1$. We also note that N is canonically isometric to Y^*.

Proposition 2. *Let Y be a Banach space containing an isomorphic copy of c_0. Then Y can be renormed such that for the new norm on Y there is a normed linear space X and a linear map $T : X \to Y$ such that for every $y^* \in \partial_e Y_1$, $y^* \circ T \in X^*$, but T is not bounded.*

Proof: Since we are interested in renorming Y, by applying Lemma 8.1 in Chapter 2 of [1], we may assume that Y contains an isometric copy of c_0. It now follows from Proposition II.2.10 in [5] that we can renorm Y so that c_0 becomes an M-ideal in Y. We also note that c_0 still has the supremum norm. Now let X and T be as in the above Example. For any $y^* \in \partial_e Y_1^*$(w.r. to the new norm) either $y^* \in \partial_e {\ell^1}_1$ or $y^* \in {c_0}^\perp$. Thus $y^* \circ T \in X^*$. Also T is not bounded. $\square$

Our next theorem gives a necessary and sufficient condition on the domain space for the validity of a similar result.

Theorem 2. *Let X be a normed linear space. For every Banach space Y, every linear operator $T : X \to Y$ such that $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$ is bounded, iff X is barrelled.*

Proof: Let X be barrelled and let $T : X \to Y$ be a linear map such that $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$. It is easy to see that $\{y^* \circ T : y^* \in \partial_e Y_1^*\}$ is a pointwise bounded family of functionals on X. Now by invoking the uniform boundedness theorem for barrelled spaces (Theorem 9-3-4 in [8]) we conclude that $T(X_1)$ is a bounded set.

Conversely suppose that X is a normed linear space such that for all Banach spaces Y every linear operator $T : X \to Y$ such that $y^* \circ T \in X^*$ for all $y^* \in \partial_e Y_1^*$ is bounded. We shall show that every weak* compact set $K \subset X^*$ is a norm bounded set. It would then follow from Theorem 9-3-4 of [8] again that X is barrelled.

Let $K \subset X^*$ be a weak* compact set. Take $Y = C(K)$, the Banach space of continuous functions on K. If we now define $T : X \to Y$ by $T(x)(k) = k(x)$ for $x \in X$ and $k \in K$, then clearly T is a linear map. Since elements of $\partial_e C(K)_1^*$ are given by evaluation functionals $\delta(k)$, $k \in K$ it is easy to see that T satisfies the hypothesis and hence it is a bounded operator. Since $T^*(\delta(k)) = k$, we conclude that K is a norm bounded set. $\square$

Remark 2. When the range space is a separable, real Banach space and contains no copy of c_0, it follows from Theorem 4 in [4] that one can weaken the hypothesis in Theorem 1 to $y^* \circ T \in X^*$ for all weak* exposed points $y^* \in Y_1^*$. Similarly when X is separable, since any weak* compact set $K \subset X^*$ is metrizable we see that for any $k \in K$, $\delta(k)$ is a weak* exposed point.

Acknowledgment. Thanks are due to Professor Y. Abramovich for suggesting that I consider barrelled spaces.

References

[1] Deville R., Godefroy G., Zizler V., *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64**, New York, 1993.
[2] Diestel J., *Sequences and series in Banach spaces*, GTM **92**, Springer, Berlin, 1984.
[3] Fonf V.P., *Weakly extremal properties of Banach spaces*, Math. Notes **45** (1989), 488–494.
[4] Fonf V.P., *On exposed and smooth points of convex bodies in Banach spaces*, Bull. London Math. Soc. **28** (1996), 51–58.
[5] Harmand P., Werner D., Werner W., *M-ideals in Banach spaces and Banach algebras*, Springer LNM **1547**, Berlin, 1993.

[6] Labuschagne L.E., Mascioni V., *Linear maps between C^* algebras whose adjoints preserve extreme points of the dual unit ball*, Advances in Math. **138** (1998), 15–45.
[7] Rao T.S.S.R.K., *On the extreme point intersection property*, "Function spaces, the second conference", Ed. K. Jarosz, Lecture Notes in Pure and Appl. Math. **172**, Marcel Dekker, 1995, pp. 339–346.
[8] Wilansky A., *Modern methods in topological vector spaces*, McGraw Hill, New York, 1978.

Indian Statistical Institute, R.V. College Post, Bangalore- 560 059, India

Current address:
220 Math. Sci. Bldg., University of Missouri-Columbia, Columbia MO 65211

E-mail: tss@isibang.ac.in

(*Received* January 8, 1999)