Sankhyā : The Indian Journal of Statistics
1986, Volume 48, Series B, Pt. 3, pp. 297-300

CHARACTERISATION OF THE PARENT DISTRIBUTION BY INEQUALITY MEASURES ON ITS TRUNCATIONS

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SUMMARY. It is shown that for Gini-index, H-index and Dalton's measure, the values of the inequality index for all upper (or lower) truncations of a distribution determine the distribution uniquely up to scale change. Moreover, the γ-entropy measure for all upper truncations determine the distribution unique up to translation shift.

1. Introduction

It is shown in this paper that if for some measures of inequality (in income), the upper α-truncated distributions corresponding to two income distributions F and G have the same inequality measure for every α in (0, 1), then F and G are equal except for possible change in scale. The specific inequality indices considered in this paper are Gini-index, coefficient of variation, entropy measure, measures derived from Mellin's transform, and Dalton's measure.

Bhattacharya (1963) and Schelling (1934) (see Piesch, 1975) have proved independently that a necessary and sufficient condition for an arbitrary lower truncation to leave the Lorenz curve unchanged, is that the continuous density function has the Pareto form with index greater than 1. Ord *et al.* (1983) have shown that if the Gini-index or H-index (based on Mellin's transform) is invariant for all upper truncations, then the parent distribution is Pareto. This generalizes the result of Schelling (1934) and Bhattacharya (1963). Furthermore, Ord *et al.* (1983) have shown that a distribution with continuous density having support $(0, \infty)$ is exponential if and only if, the entropy inequality measure is invariant for all upper truncations.

Our paper thus generalizes the results of Ord *et al.* (1983) as well as those of Bhattacharya (1963) and Schelling (1934).

2. The main result

Let F be the distribution function of a nonnegative random variable X, and F_α be the distribution function of X, given $X \geqslant Z_\alpha(F)$, where $Z_\alpha(F)$

AMS (1980) *subject classification :* 62P20.

Key words and phrases : Gini-index, coefficient of variation, entropy measure, measures derived from Mellin's transform, Lorenz curve.

is the upper α-quantile $(0 < \alpha < 1)$ of F. Let $\mu(F) > 0$ be the mean of the distribution F. We shall assume throughout that the inequality measure $I(F)$ for any distribution F is scale-invariant. For the following theorem we have considered $I(F)$ to be any one of the following: Coefficient of variation, Gini-index, measures derived from Mellin's transform, and Dalton's measur?. (see Nygard and Sandstrom (1981), Ord *et al.* (1983), Marshall and Olkin (1979)). Note that the above inequality indices are special cases of the following general functional form, or related to this form by one-to-one correspondence :

$$I(F) = \int_0^\infty S_F(x)\, dF(x)/T\,[\mu(F)], \qquad \dots \quad (2.1)$$

where $S_F(x)$ is either $\int_x^\infty t\, dF(t)$, or a suitably chosen strictly convex function S of x, and T is some suitable function.

Theorem 2.1 : *If for any two distribution functions F and G on $(0, \infty)$*

$$I(F_\alpha) = I(G_\alpha) \qquad \dots \quad (2.2)$$

for all $0 < \alpha < 1$, then G is a scale-transform of F. Conversely, if G is a scale-transform of F, then $I(F_\alpha) = I(G_\alpha)$ for all $0 < \alpha < 1$.

Proof: Suppose, for distributions F and G with $\mu(F) > 0$ and $\mu(G) > 0$, we have $I(F_\alpha) = I(G_\alpha)$ for all $0 < \alpha < 1$, but F is not a scale-transform of G. Without any loss of generality, we may assume that $\mu(F) = \mu(G) = 1$, since $I(F_\alpha)$ is not affected by a scale transformation of F.

Let

$$L_F(\alpha) = \int_0^{Z_\alpha(F)} t\, dF(t)/\mu(F). \qquad \dots \quad (2.3)$$

Since F is not a scale-transform of G by our assumption, the Lorenz curves corresponding to F and G will be different. Note that the set of all points $\alpha \in [0, 1]$ for which $L_F(\alpha) = L_G(\alpha)$ is closed. Since $L_F(\alpha)$ is continuous in α, we can get α_1 and α_2, $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$ such that

$$L_F(\alpha) \neq L_G(\alpha) \text{ for all } \alpha \in (\alpha_1, \alpha_2),$$

$$L_F(\alpha_i) = L_G(\alpha_i) \text{ for } i = 1, 2. \qquad \dots \quad (2.4)$$

Without loss of generality let us assume

$$L_F(\alpha) < L_G(\alpha), \text{ for oll } \alpha \in (\alpha_1, \alpha_2) \qquad \dots \quad (2.5)$$

since $L_F(\alpha) - L_G(\alpha)$ has the same sign in (α_1, α_2).

Note that (2.4) implies

$$\mu(F_{\alpha_i}) = \mu(G_{\alpha_i}), \text{ for } i = 1, 2. \qquad \dots \quad (2.6)$$

Thus, we must have

$$\int_{Z_{\alpha_i}(F)}^{\infty} S_F(x)\, dF(x) = \int_{Z_{\alpha_i}(G)}^{\infty} S_G(x)\, dG(x),\ i = 1, 2. \qquad \dots \quad (2.7)$$

The relation (2.7) implies

$$\int_{Z_{\alpha_2}(F)}^{Z_{\alpha_1}(F)} S_F(x)\, dF(x) = \int_{Z_{\alpha_2}(G)}^{Z_{\alpha_1}(G)} S_G(x)\, dG(x). \qquad \dots \quad (2.8)$$

Case I : Suppose now

$$S_F(x) = \int_x^{\infty} t\, dF(t). \qquad \dots \quad (2.9)$$

Then assuming F and G to be continuous, (2.5) contradicts (2.8) ; note that the Lorenz curves corresponding to F and G cannot both become straight lines in (α_1, α_2).

Case II : Suppose $S_F(x)$ is a strictly convex function S of x.

Let X and Y denote random variables with distribution functions F and G, respectively. Now note that the conditional distribution of X, given $Z_{\alpha_2}(F) \leqslant X \leqslant Z_{\alpha_1}(F)$, is Lorenz dominated by the conditional distribution of Y, given $Z_{\alpha_2}(G) \leqslant Y \leqslant Z_{\alpha_1}(G)$. This follows from (2.4) and (2.5) ; as a matter of fact, (2.4) implies that the above conditional distributions have the same mean. Now, it follows from Ross (1983) or Atkinson (1970), or Ryff (1963, 1965) that

$$\int_{Z_{\alpha_2}(F)}^{Z_{\alpha_1}(F)} S(x)\, dF(x) > \int_{Z_{\alpha_2}(G)}^{Z_{\alpha_1}(G)} S(x)\, dG(x), \qquad \dots \quad (2.10)$$

which contradicts (2.8) with $S_F = S_G = S$.

Remark 2.1 : It follows from the above proof that an analogous result holds also for lower truncations.

Remark 2.2 : Consider the following density on $(0, c)$:

$$f(x) = \frac{\beta+1}{1+\beta}\, x^{\beta},\ -1 < \beta < \infty. \qquad \dots \quad (2.11)$$

Note that all the indices considered above for this density are invariant with respect to lower truncations. Hence, any continuous distribution for which the above inequality indices are invariant with respect to lower truncation has the density of form (2.11).

3. Truncation and entropy-measure of inequality

Definition 3.1 : The γ-entropy $(-1 < \gamma < \infty)$ of a distribution function F on $(0, \infty)$ with density f is defined as

$$e_\gamma(F) = \frac{1}{\gamma} \int_0^\infty (1 - f^\gamma(x))\, dF(x), \text{ for } \gamma \neq 0$$

and

$$e_0(F) = -\int_0^\infty \log f(x)\, dF(x).$$

The following theorem can be proved following an argument analogous to the proof of Theorem 2.1 :

Theorem 3.1 : *If f is positive on an interval in* $[0, \infty)$, *then* $\{e_\gamma(F_\alpha) : 0 < \alpha < 1\}$ *determines F uniquely except for a possible shift in translation.*

Remark 3.1 : It is an interesting problem to find a necessary and sufficient condition for $\{C_\alpha,\ 0 < \alpha < 1\}$ to correspond to $\{I(F_\alpha),\ 0 < \alpha < 1\}$ for some given inequality measure I.

Acknowledgement. The author is thankful to Professor Somesh Das Gupta and Dr. Satya Ranjan Chakrabarti for their help.

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Paper received : *May, 1985.*
Revised : *June, 1986.*