Injectivity Sets for Spherical Means on $\mathbb{R}^n$ and on Symmetric Spaces

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ABSTRACT. *A complex Radon measure μ on $\mathbb{R}^n$ is said to be of at most exponential-quadratic growth if there exist positive constants C and a such that $|\mu|(B(0,r)) \leq C\, e^{ar^2}$, $r > 0$. Let $X_{\exp}$ denote the space of all complex Radon measure on $\mathbb{R}^n$ of at most exponential-quadratic growth. Using elementary methods, we obtain injectivity sets for spherical means for $X_{\exp}$. We also discuss similar results for symmetric spaces.*

1. Introduction and Some Results for $\mathbb{R}^n$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $n \geq 2$. For $x \in \mathbb{R}^n$ and $r \geq 0$, the spherical means of f are defined by

$$Mf(x,r) = \int_{S(x,r)} f\, d\sigma$$

where $S(x,r)$ is the $(n-1)$-dimensional sphere of radius r centred at x and σ is the normalized surface measure on $S(x,r)$. It is easy to see using a Fubini argument that, for each $x \in \mathbb{R}^n$, $Mf(x,r)$ is defined $a.e.(r)$.

Definition. Let $X \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$. A set Γ of $\mathbb{R}^n$ is said to be *a set of injectivity for spherical means for X* if $f \in X$ and $Mf(x,r) = 0$ $a.e.(r)$ for each $x \in \Gamma$ implies $f = 0$ $a.e.$

The following problem has received some attention recently:

Given $X \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$, determine subsets Γ of $\mathbb{R}^n$ which are sets of injectivity for spherical means for X.

Some of the techniques used in dealing with this question are quite sophisticated, see [1], [2] etc. In [1] it is shown that if D is a bounded region in $\mathbb{R}^n$, $n \geq 2$, and Γ is the boundary of D, then Γ is a set of injectivity for the spherical means for $L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n}{n-1}$. It is also shown in [1] that if Γ is a sphere, then Γ is *not* a set of injectivity for any $L^q(\mathbb{R}^n)$, $q > \frac{2n}{n-1}$. It would therefore be worthwhile to find interesting examples of *thin* sets Γ which are sets of injectivity for

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all L^q, $1 \leq q \leq \infty$. Using fairly elementary methods we are able to produce such sets of injectivity not only for all L^qs but also a slightly wider class of functions and measures.

If one wants to talk about measures instead of functions, it is better to replace $S(x, r) = \{y \in \mathbb{R}^n : \|x - y\| = r\}$ by $B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. (Clearly for locally integrable functions, the two formulations are equivalent.) To be consistent with the vocabulary of measure theory, we switch jargon!

Let X be a subspace of complex Radon measures on $\mathbb{R}^n$, $n \geq 1$. For the definition of complex Radon measures see [3], p. 216. (Note that in the discussion that is to follow, we also allow $n = 1$.) We say a collection of Borel subsets $\mathcal{C}$ of $\mathbb{R}^n$ is *a determining class for* X if $\mu \in X$ and $\mu(C) = 0, \forall C \in \mathcal{C}$ necessarily implies $\mu = 0$. Thus, if X_F is the set of finite complex measures, the following classes are well known to be determining classes for X_F: $\mathcal{C}_1 = \{B(x, r_0) : x \in \mathbb{R}^n, r_0$ is a fixed positive number $\}$, $\mathcal{C}_2 =$ the collection of all half spaces in $\mathbb{R}^n$. See [10] and [9]. In the language just introduced, the problem can be restated as:

Given a subspace X of complex Radon measures on $\mathbb{R}^n$, determine subsets Γ of $\mathbb{R}^n$, such that $\mathcal{C} = \{B(x, r) : x \in \Gamma, r \geq 0\}$ is a determining class for X.

Before stating and proving our main result, we start with some definitions, notation, and an elementary lemma.

Definition. A complex Radon measure μ on $\mathbb{R}^n$ is said to be of *at most exponential-quadratic growth* if there exist positive constants C and α such that $|\mu|(B(0, r)) \leq C\, e^{\alpha r^2}$, $r \geq 0$. (For any compact set K of $\mathbb{R}^n$, μ restricted to K gives a finite complex measure. $|\mu|(K)$ is the total variation norm of μ restricted to K.)

Let $X_{\exp}(\mathrm{C}, \alpha)$ denote the collection of all Radon measures satisfying the above estimate. Let $X_{\exp}$ denote the space of all complex Radon measures on $\mathbb{R}^n$ of at most exponential-quadratic growth.

Clearly $X_{\exp}$ contains all finite measures as well as infinite measures of at most polynomial growth.

Definition. A subset Γ of $\mathbb{R}^n$ is called *an NA-set* if the only real analytic function (defined on an open set containing Γ) vanishing on Γ is the zero function.

Examples of NA-sets are subsets Γ whose closure $\bar{\Gamma}$ is a set of positive Lebesgue measure. As has been pointed out to us by Pati, examples of *thin* NA-sets in $\mathbb{R}^2$ are continuous curves Γ such that for *any* choice D of discrete points, $\Gamma \backslash D$ is *not* an analytic one-dimensional submanifold of $\mathbb{R}^2$. The proof that such a curve Γ is an NA-set relies on the non-trivial fact that the zero set of a non-zero real valued real analytic function on $\mathbb{R}^2$ can always be made into a one-dimensional analytic submanifold by removing a discrete set of points. This fact, in turn, is a consequence of much more general results about real analytic varieties [11], p. 133). More generally, there are plenty of $(n-1)$-dimensional non-analytic submanifolds of $\mathbb{R}^n$ which cannot be contained in the zero set of a non-trivial real analytic function.

We now state a lemma which will be needed in the sequel.

Lemma 1.

Let $0 \neq \mu \in X_{\exp}(\mathrm{C}, \alpha)$ and let $\phi_\alpha(x) = e^{-2\alpha\|x\|^2}$, $x \in \mathbb{R}^n$. Then $\mu * \phi_\alpha(x)$ is a non-trivial function on $\mathbb{R}^n$ and is in fact the restriction to $\mathbb{R}^n$ of an entire function on $\mathbb{C}^n$. Consequently $\mu * \phi_\alpha(x)$ is a non-trivial real analytic function on $\mathbb{R}^n$.

Proof. Define a measure ν by $d\nu(y) = e^{-2\alpha\|y\|^2} d\mu(y)$. Since $\mu \in X_{\exp}(\mathrm{C}, \alpha)$, it is easy to see that ν is a finite complex measure and that its Fourier transform $\hat{\nu}$ extends to an entire function on $\mathbb{C}^n$. The lemma now follows from the observation that $\mu * \phi_\alpha(x) = e^{-2\alpha\|x\|^2} \hat{\nu}(4\alpha i x)$. $\square$

We are now in a position to state and prove the main proposition.

Proposition 1.

Let Γ be an NA-subset of $\mathbb{R}^n$ and $C = \{B(x, r) : x \in \Gamma, r \geq 0\}$. Then C is a determining class for $X_{\exp}$.

Proof. If $\mu \in X_{\exp}$, then there exists positive numbers C and α such that $\mu \in X_{\exp}(C, \alpha)$. Suppose $\mu(B(x, r)) = 0$ for all $r \geq 0$. Then an easy approximation argument shows, since $e^{-2\alpha\|x-y\|^2}$ is a function which is radial with respect to the origin at x and decreasing sufficiently rapidly at infinity, that $\int e^{-2\alpha\|x-y\|^2} d\mu(y) = 0, \forall x \in \Gamma$. Thus, $\mu * \phi_\alpha$ vanishes on Γ. If μ is non-trivial then, by Lemma 1, $\mu * \phi_\alpha$ is a non-trivial real analytic function vanishing on Γ. Since Γ is an NA-set, we have a contradiction. The proof of the proposition is now complete. □

A corollary to the proposition is:

Corollary 1.

If Γ is an NA-set, then Γ is a set of injectivity for the spherical means for any of the spaces $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty, n \geq 2$.

Proof. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty, n \geq 2$. Then $\int_{B(0,r)} |f(x)| dx \leq C\|f\|_p r^{\frac{n}{q}}$. Hence, identifying the function f with the complex radon measure $f(x)\, dx$, we have $f \in X_{\exp}$. The result now follows from Proposition 1. □

Remarks.

(1) In the result of Agranovsky et al. [1] quoted earlier, if we take the bounded region D to be such that the boundary ∂D is *badly behaved*, then ∂D will be a set of injectivity for M for all L^qs. In fact, if ∂D is *sufficiently badly behaved*, *any* $(n-1)$ submanifold of ∂D will be a set of injectivity.

(2) One should be able to use the methods in this paper to take care of measures μ, whose growth at infinity is allowed to be much worse than exponential-quadratic.

(3) The classes of functions and measures considered in Proposition 1 are more general than those in [2], where compactly supported functions and distributions are considered. But we should emphasise that the main result in [2] is a *characterization* of sets of injectivity for $C_c(\mathbb{R}^2)$.

(4) For an excellent survey of early work on problems of *integral geometry* similar in spirit to the kind of problem considered in this paper, see [12].

2. Symmetric Spaces

We now turn our attention to symmetric spaces of the non-compact type and of real rank 1. Since such spaces are analytically diffeomorphic to $\mathbb{R}^n$, for some $n \geq 2$, the definition of NA-sets is equally meaningful for such spaces. We will only give a very brief sketch of our arguments.

Let S denote such a space. We recall a few basic facts about S. (For further details see [5].) If G is the connected component of the group of isometries of S, then G is a non-compact semi-simple Lie group and S can be identified with G/K for a suitable choice of a maximal compact subgroup K of G.

Let $B(x, r)$ denote the geodesic ball of radius r centered at $x \in S$, i.e., $B(x, r) = \{y \in S : d(x, y) \leq r\}$, where d is the metric on S given by the Riemannian structure on S. Denote by x_0, the point in S corresponding to eK under the natural identification $S \leftrightarrow G/K$. Real analytic functions on S can be identified with real analytic functions on G invariant under the right action of K (and more generally functions on S can be identified with functions on G invariant under the right action of K).

Depending on the context, dx will denote both the Haar measure on G as well as the G-invariant measure on S and dk will denote the normalized Haar measure on K. In what follows below, depending on convenience, we view a right K-invariant function on G either as a function on G or a function on S, and without any change of notation.

Fix p, $1 \leq p < \infty$. (Note that *for the moment* we are excluding $p = \infty$.) For $f \in L^p(G)$, there exists a sequence $\{\phi_n\}$ of real analytic functions on G in $L^1(G)$, which are "rapidly decreasing" at ∞ such that $f * \phi_n \to f$ in L^p and such that $f * \phi_n$ are analytic vectors for the right regular representation of G on $L^p(G)$. (This follows from the work of Harish–Chandra on analytic vectors [4]. See also [6].) Again from the work of Nelson [7] on analytic vectors it follows that $f * \phi_n$ is actually a real analytic function on G. Further if f is in $L^p(S)$ (i.e., f is right K-invariant), we can consider $\phi_n^\#$ defined by $\phi_n^\#(x) = \int_K \int_K \phi_n(k_1 x k_2) dk_1 dk_2$ and it is easy to see that $f * \phi_n^\#$ are also real analytic functions in $L^p(S)$ with $f * \phi_n^\# \to f$ in $L^p(S)$. (If h is a real analytic function on G, in view of the compactness of K, so is $h^K(x) = \int_K h(xk)dk$. In this case it is easy to see that $f * \phi_n^\# = (f * \phi_n)^K$, using the right K-invariance of f.)

We now make the following observations: Since S is of rank-1, K acts transitively on $\{x \in S : d(x, x_0) = r\}$. It therefore follows that if g is a K-invariant function on S (or equivalently a K-bi-invariant function on G), which decreases sufficiently rapidly at infinity, the u-translate of g, i.e., function $x \to g(u^{-1} \cdot x)$, $u \in G$, can be approximated by finite linear combinations of indicator functions of balls centred at $u \cdot x_0$.

Suppose now $f \in L^p(S)$ and y is a point in S such that $\int_{B(y,r)} f(x)dx = 0, \forall r \geq 0$. Hence, by the approximation alluded to earlier, $\int_S f(x) g(Y^{-1}x) dx = 0$, where $y = YK$ (i.e., $y = Y \cdot x_0$). In terms of convolution on the group, this implies $f * \check{g}(y) = 0$, where for a function h on G, $\check{h}$ denotes the function $\check{h}(z) = h(z^{-1})$. Since g is K-bi-invariant, so is $\check{g}$, and since g is a more or less arbitrary K-bi-invariant function, it follows, by taking $\check{g} = \phi_n^\#$, that $f * \phi_n^\#(y) = 0$, where the $\phi_n^\#$s are as described earlier. Now, suppose

$$\int_{B(z,r)} f(x)dx = 0, \ \ \forall z \in \Gamma, r \geq 0$$

where Γ is an NA-subset of S.

Thus, from the discussion above we find that the real analytic function $f * \phi_n^\#$ vanishes on the NA-set Γ. Hence, since each $f * \phi_n^\#$ is a real analytic function on S we conclude $f * \phi_n^\# \equiv 0$. That $f = 0$ a.e. now follows from the fact $f * \phi_n^\# \to f$ in L^p for $1 \leq p < \infty$.

Next, we take up the case $p = \infty$: Let $f \in L^\infty(S)$. Define $\psi_k = \frac{1}{m(B(x_0, \frac{1}{k}))} \chi_k$ where χ_k denotes the indicator function of the ball $B(x_0, \frac{1}{k})$ and $m(B(x_0, \frac{1}{k}))$ is the measure of the ball. ψ_k, considered as a function on G, is a K-bi-invariant L^1-function, and hence $f * \psi_k$ is a continuous function vanishing at ∞, i.e., $f * \psi_k \in C_0(S)$, the Banach space of continuous functions vanishing at ∞. Let $\phi_n^\#$ be as in the preceding paragraph. Then for $h \in C_0(G)$, $\{h * \phi_n\}$ will be analytic vectors for the right regular action of G on the Banach space $C_0(G)$ and hence, as before, are analytic functions on G. Again, as before, $\{h * \phi_n^\#\}$ are also analytic functions on G.

Now, consider $f * \psi_k * \phi_n^\#$ for a fixed k. $\psi_k * \phi_n^\#$ is a K-bi-invariant function in $C_0(S)$ and hence as before we would have $(f * (\psi_k * \phi_n^\#))(y) = 0$, where y is a point in S with the property that $\int_{B(y,r)} f(x)dx = 0, \ \forall r \geq 0$, i.e., $((f * \psi_k) * \phi_n^\#)(y) = 0$.

Suppose $\int_{B(z,r)} f(x)dx = 0, \ \forall z \in \Gamma, r \geq 0$, where Γ is an NA-subset of S. Then, from above, $f * \psi_k * \phi_n^\#$ vanishes on Γ. The real analyticity of $f * \psi_k * \phi_n^\#$ forces $f * \psi_k * \phi_n^\# \equiv 0$. Hence by allowing $n \to \infty$ and observing that $f * \psi_k * \phi_n^\# \to f * \psi_k$ in C_0, we have $f * \psi_k \equiv 0$, for each k. Since $f * \psi_k \to f$ as $k \to \infty$, at least in the sense of distributions, we conclude $f = 0$ a.e.

This completes the proof of Proposition 2 below.

Proposition 2.

Let S be a symmetric space of the non-compact type and of real rank 1, and let Γ be an NA-subset of S. If f is in $L^p(S)$, $1 \leq p \leq \infty$, and $\int_{B(y,r)} f(x)dx = 0, \forall y \in \Gamma, r \geq 0$, then $f = 0$ a.e.

Remarks.

(1) For a discussion of analytic vectors, see also [6].

(2) Just as in the case of $\mathbb{R}^n$, in Proposition 2, we could have just as well considered averages over geodesic *spheres* of radius r rather than integrals over geodesic *balls* of radius r.

(3) By considering the heat kernel, some of the results in this paper can be extended to arbitrary Riemannian manifolds [8].

Next, a brief look at compact symmetric spaces of rank 1: Let S be one such. Clearly, because of compactness, instead of considering all the L^ps, it is enough to consider $L^1(S)$. One knows that all geodesics in S are closed and are of the same length, $2L$ say. It is therefore enough to consider geodesic balls of radius $r \leq L$. Using some Peter–Weyl theory and standard facts about compact symmetric spaces of rank 1, we can prove the following:

Proposition 3.

Let Γ be an NA-subset of S. If $f \in L^1(S)$ and $\int_{B(x,r)} f = 0, \forall x \in \Gamma, 0 \leq r \leq L$, then $f = 0$ a.e.

(Since S is a real analytic manifold, the definition of an NA-set poses no problem. We can strengthen the above proposition somewhat; for instance, the above statement will be true if we just assume that Γ has the property that the only G-finite function that vanishes on Γ is the zero function. Here G is the group of isometries of S. In the case when S is a sphere, see [2], p. 405, Theorem 7.1.)

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