

## MARKOFF'S THEOREM WITH LINEAR RESTRICTIONS ON PARAMETERS

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### INTRODUCTION

In an earlier paper by the author (1945) the problem of linear estimation has been dealt with under very general conditions. The problem is, given the vector  $Y = (y_1, y_2, \dots, y_n)$  of  $n$  stochastic variates such that  $E(Y) = TA'$  where  $T = (\tau_1, \tau_2, \dots, \tau_m)$  is a row matrix of  $m$  unknown parameters and  $A$  is a known matrix with  $n$  rows and  $m$  columns and also the dispersion matrix  $\Delta$  of the stochastic variates, to find a linear function  $BY'$  such that

$$(a) \quad E(BY') = LT' \quad \text{independently of } T \quad \dots \quad (1-1)$$

$$\text{and} \quad (b) \quad V(BY') \quad \text{is minimum} \quad \dots \quad (1-11)$$

where  $LT'$  is a given parametric function.

The object of the present paper is to extend the above problem to cases where the parameters  $\tau_1, \tau_2, \dots, \tau_m$  are subject to  $k$  linear restrictions  $G = TR'$  where  $G$  and  $R$  are known matrices the former being a row matrix with  $k$  elements and the latter being a matrix with  $k$  rows and  $m$  columns. The problem is to find a linear function  $b_0 + BY'$  (if one exists, not necessarily homogeneous in  $y$ 's) such that

$$(a) \quad E(b_0 + BY') = LT' \quad \text{subject to } G = TR' \quad \dots \quad (1-2)$$

$$(b) \quad V(b_0 + BY') \quad \text{is minimum} \quad \dots \quad (1-21)$$

The discussion on tests of linear hypotheses given by Rao (1945) holds good in these situations also.

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## 2. SOLUTION TO THE GENERAL PROBLEM

If  $b_0$  and  $B$  exist such that (1.2) is satisfied then

$$LT' = E(b_0 + BY') = b_0 + BAT' \quad \dots (2.1)$$

is true whenever  $G' = RT'$  which gives that for a suitable vector  $D$  we get

$$b_0 + (BA - L)T' = D(G' - RT') \quad \dots (2.11)$$

$$\text{or } BA - DR = L \text{ and } b_0 = DG' \quad \dots (2.12)$$

Also if  $b_0$  and  $B$  satisfy (2.12) then  $E(b_0 + BY') = LT'$  which gives that the necessary and sufficient condition that  $LT'$  is estimable is that there exist  $b_0$  and  $B$  such that for a suitable  $D$  the relations  $BA + DR = L$  and  $b_0 = DG'$  are identically true.

From the set of  $b_0$ ,  $B$  and  $D$  satisfying (2.12) we have to select those for which  $V(BY') = BA B'$  is the least. Introducing Lagrangian multipliers  $2d$  and  $2C = 2(c_1, c_2, \dots, c_m)$  we have to minimise  $B \Lambda B' - 2C(A'B' + R'D' - L') + 2d(b_0 - DG')$  with respect to  $d$ ,  $b_0$  and the elements of  $B$ ,  $D$  and  $C$ . This leads to

$$B\Lambda - CA' = 0, \quad BA + DR = L \quad \dots (2.2)$$

$$CR' = dG, \quad d = 0, \quad b_0 = DG' \quad \dots (2.21)$$

Defining  $\Lambda^{-1}$  as the inverse of  $\Lambda$  we get

$$B = CA'\Lambda^{-1}, \quad CA'\Lambda^{-1}A + DR = L \quad \dots (2.22)$$

$$CR' = 0, \quad b_0 = DG' \quad \dots (2.23)$$

So we get that  $b_0$  and  $B$  satisfying (1.2) and (1.21) are given by  $B = CA'\Lambda^{-1}$  and  $b_0 = DG'$  where  $C$  and  $D$  satisfy the equations

$$CA'\Lambda^{-1}A + DR = L, \quad CR' = 0 \quad \dots (2.24)$$

As shown earlier (Rao : 1945) we can easily prove that  $b_0$  and  $B$  are unique for all  $C$  and  $D$  satisfying (2.24) and the variance associated with  $b_0 + BY'$  derived from above is the least.

## 3. THE ANALOGUE OF MARKOFF'S RULE

It has been shown above that the best estimate of  $LT'$  is given by  $DG' + CA'\Lambda^{-1}Y'$  where  $C$  and  $D$  satisfy (2.24). If we construct the quantities  $Q = (q_1, q_2, \dots, q_m)$  by the transformation  $Q = Y'\Lambda^{-1}A$  with  $E(Q') = A'\Lambda^{-1}AT' = H'$  we can restate the result of section (2) as if there exist  $C$  and  $D$  such that  $LT' = CH' + DRT'$  and  $CR' = 0$  then the best unbiased estimate of  $LT'$  is  $DG' + CQ'$  and this is unique for all  $C$  and  $D$  satisfying the above relation.

We may translate the above result into the following rule. Let us construct the linear equations

$$Q = H + SR \text{ and } CR' = 0 \quad \dots (3.2)$$

in  $r_1, r_2, \dots, r_m$  and  $k$  pseudo-variables  $s_1, s_2, \dots, s_k$  of the vector  $S = (s_1, s_2, \dots, s_k)$  and get  $\hat{T} = (t_1, t_2, \dots, t_m)$  and  $\hat{S} = (s_1, s_2, \dots, s_k)$  as solutions. Because of the identity

$$LT' = DRT' + CH' \quad \text{where } CR' = 0 \quad \dots (3.21)$$

$$\hat{L}\hat{T}' = D\hat{R}\hat{T}' + C\hat{H}' = DG' + C(Q' - R'\hat{S}) = DG' + CQ' \quad \dots (3.22)$$

we get the result that the substitution of any solution of (3.2) in an estimable parametric function  $LT'$  leads to its best estimate.

If  $NT'$  is any given parametric function then  $E(NT')=NT'$  when and only when  $NT'$  is estimable. This may be used as the criterion of estimability. A non-estimable parametric function will reveal itself either in violation of this criterion or in giving different expressions for different solutions.

The equations (3.2) may be called normal equations appropriate for this case and are readily obtained by minimising the expression.

$$\sum \sum \lambda^{ij} (y_i - \theta_j) (y_j - \theta_i) \quad \dots (3.4)$$

where  $\lambda^{ij}$  ( $i, j = 1, 2, \dots, n$ ) are the elements of  $\Lambda^{-1}$ , with respect to  $\tau_1, \tau_2, \dots, \tau_n$ , subject to the restrictions  $G=TR'$  thus leading to the usual rule of least squares.

The equations (3.2) are always solvable. For this it is enough to show that if there exist  $U$  and  $W$  such that

$$U(H' + R'S') + WRT' \equiv 0 \quad \dots (3.5)$$

then  $UQ' + DG' \equiv 0$  which is true for we can show that both the expectation and the variance of  $UQ' + DG'$  vanish if (3.5) holds.

The above discussion shows that we can add a consistent and is convenient and sometimes a conventionally chosen set of equations to (3.2) and get solutions for substitution.

From the necessary and sufficient condition for estimability (2.12) we derive that the number of estimable parametric functions is equal to the rank of the matrix obtained by adjoining  $R$  to  $A$ .

#### 4. VARIANCES AND COVARIANCES OF ESTIMATES

The best estimate of  $LT'$  is given by  $DG' + CQ'$  where  $C$  and  $D$  satisfy (2.24). It follows that

$$V(DG' + CQ') = CA'\Lambda^{-1}AC' = (I - DR)C' = LC' \quad \dots (4.1)$$

If  $LT'$  and  $MT'$  are estimated by  $DG' + CQ'$  and  $\Delta G' + \chi Q'$ , then

$$\text{cov.} (DG' + CQ') (\Delta G' + \chi Q') = \text{cov.} (CQ') (\chi Q') = L\chi' = MC' \quad \dots (4.11)$$

which are analogous to those derived by Rao in (1945).

We can also show that the intrinsic properties of normal equations, discussed in the previous paper, hold good in this case also. These results, in particular cases, have been already discussed by the author elsewhere (Rao : 1943).

#### 5. TESTS OF LINEAR HYPOTHESES

The nature of linear hypothesis is the assignment of the value of one or more linear parametric functions. The case of a single parametric function does not present any difficulty. If the  $y$ 's form a multivariate normal system, we construct the normal variate with zero mean and unit variance by taking the ratio of the deviation of the estimate from the given value to its standard deviation and test for its significance. Thus if  $LT'$  is an estimable parametric function with a specified value  $\xi$  and its best unbiased estimate is  $DG' + CQ'$  then to test the hypothesis  $LT' = \xi$  we construct the statistic

$$v = (DG' + CQ' - \xi) / \sqrt{C'L} \quad \dots (5.0)$$

which can be referred to a probability integral table of normal deviates.

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We can extend this result to test the composite hypothesis that a number of parametric functions  $L_1T', L_2T', \dots, L_pT'$  have assigned values  $\xi_1, \xi_2, \dots, \xi_p$ . If  $E$  and  $\xi$  be the vectors defined by  $(L_1T' \ L_2T' \ \dots \ L_pT')$  and  $(\xi_1 \xi_2 \dots \xi_p)$  then for the hypothesis to be consistent it follows that if there exists two vectors  $M=(m_1 \ m_2 \ \dots \ m_p)$  and  $N=(n_1 \ n_2 \ \dots \ n_p)$  such that  $ME' = NRT'$  or 0, then  $M\xi' = NG'$  or 0 respectively. In this case we can replace the composite hypothesis by finding an independent set of parametric functions  $M_1T', M_2T', \dots, M_mT'$  and corresponding values  $\eta_1, \eta_2, \dots, \eta_m$  derived by linear combinations from  $L_1T', L_2T', \dots, L_pT'$  such that no linear combination of the new set can be derived from the parametric functions  $TR'$  occurring in the linear restrictions  $G=TR'$ . If the best estimates of  $M_1T' - \eta_1, M_2T' - \eta_2, \dots, M_mT' - \eta_m$  are given by the elements of the vector  $P=(P_1 \ P_2 \ \dots \ P_m)$  with their dispersion matrix given by  $D$ , then following the argument, as in the previous paper, we get the statistic  $V$ , suitable for testing the composite hypothesis as the root of the determinantal equation

$$|P'P - VD| = 0 \quad \dots (5.1)$$

The statistic  $V$  is distributed as  $\chi^2$  with  $m$  degrees of freedom on the null hypothesis and as the Bessel function variate defined in the previous paper, on the non null hypothesis.

Since  $V$  is invariant when the matrices  $P'P$  and  $D$  are pre or post multiplied by non singular square matrices, it follows that  $V$  is the same for any set of independent functions  $M_1T', M_2T', \dots, M_mT'$  chosen above.

The necessary statistics when the variances and covariances are not known are obtained by studentising the above statistics with the help of suitable quadratic estimates of the variances and covariances. The general problem of quadratic estimation is being considered for this purpose and it will be shown elsewhere, that many problems concerning linear hypothesis can be answered with the help of two distributions viz the  $t$  and  $F$  distributions.

### REFERENCES

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