THE INDIVIDUAL SAMPLING DISTRIBUTION OF THE MAXIMUM, THE MINIMUM AND ANY INTERMEDIATE OF THE *p*-STATISTICS ON THE NULL-HYPOTHESIS

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INTRODUCTION

Given two random samples S' and S'' of sizes n' and n'', and variance and covariance matrices $\|a'_{ij}\|$ and $\|a''_{ij}\|$ drawn from two p-variate normal populations Σ' and Σ'' with variance and covariance matrices $\|\alpha'_{ij}\|$ and $\|\alpha''_{ij}\|$, it was shown earlier by the author (1939) and also by others (1939) that the null hypothesis (for the populations) $\|\alpha'_{ij}\| = \|\alpha''_{ij}\|$ could be appropriately tested by a set of p-statistics given by the p roots $k^2_1, k^2_2, \ldots, k^2_p$ (all positive in this particular case) of the determinantal equation in $k^2 \to |a'_{ij} - k^2 a''_{ij}| = 0$. The joint sampling distribution of these p-statistics on the null hypothesis $\|\alpha'_{ij}\| = \|\alpha''_{ij}\|$ was at the same time obtained by the author (1939) and by others (1939). The joint distribution of these k_i's$(i = 1, 2, \ldots, p)$ on the non-null hypothesis (that is, when $\|\alpha'_{ij}\| \neq \|\alpha''_{ij}\|$) was also given by the author a little later (1942) and it was found that this involved as population parameters p quantities $\kappa^2_1, \kappa^2_2, \ldots, \kappa^2_p$ which are all unity when and only when $\|\alpha'_{ij}\| = \|\alpha''_{ij}\|$ and which come out as the p roots (all positive in this case) of the determinantal equation in $\kappa^2 \to |\alpha'_{ij} - \kappa^2 \alpha''_{ij}| = 0$. Closely associated with this is the second problem of multivariate analysis of variance which can be stated as follows. Given l samples $S_1, S_2, \ldots, S_l$ of sizes $n_1, n_2, \ldots, n_l$ and mean values $\bar{x}_i(r)$ and variance and covariance matrices $\|a_{ij}(r)\|$ (where i, j refer to the different variates, and r to the sample, and $i, j, = 1, 2, \ldots, p$ and $r = 1, 2, \ldots, l$) supposed to have been drawn from l p-variate normal populations $\Sigma_1, \Sigma_2, \ldots, \Sigma_l$ with mean values $m_i(r)$ and variance and covariance matrix $\|\beta''_{ij}\|$ (supposed to be the same for all populations), to test the null hypothesis $m_i(1) = m_i(2) = \ldots = m_i(l)$, $(i = 1, 2, \ldots, p)$. This is, of course, in terms of one-way classification. The same technique could by a slight twist be used for analysis of variance in terms of multiway and other types of classifications. It was shown by the author (1939) that the null hypothesis in the case just mentioned could be appropriately tested (when $l > p$) by a set of p-statistics $t^2_1, t^2_2, \ldots, t^2_p$ which come out as the p roots (all positive in this case) of the determinantal equation in $t^2 \to |b'_{ij} - t^2 b''_{ij}| = 0$, where

$$\left.\begin{aligned} b'_{ij} &= \sum_{r=1}^{l} n_r \{\bar{x}_i(r) - \bar{x}_i\} \{\bar{x}_j(r) - \bar{x}_j\}/(l-1); \quad b''_{ij} = \sum_{r=1}^{l} (n-1) a_{ij}(r) \Big/ \sum_{r=1}^{l} (n_r - 1) \\ \bar{x}_i &= \sum_{r=1}^{l} n_r \bar{x}_i(r) \Big/ \sum_{r=1}^{l} n_r, \quad (i, j = 1, 2, \ldots, p) \end{aligned}\right\} \quad \ldots (1.1)$$

The joint distribution of these p-statistics $t_1, t_2, \ldots, t_p$ on the null-hypothesis $m_i(1) = m_i(2) = \ldots =, m_i(l)$ $(i = 1, 2, \ldots, p)$ comes out to be of the same form as that of $k_1, k_2, \ldots, k_p$ of the first problem and was indicated as such by the author(1939) and by others (1939). The distribution on the non-null hypothesis $m_i(1) \neq m_i(2) \neq \ldots \neq m_i(l)$ $(i = 1, 2, \ldots, p)$ was given by the author (1942) a little later and involved as parameter a certain function of p quantities $r_1, r_2, \ldots, r_p$ where $r^2_1, r^2_2, \ldots, r^2_p$ are the p roots (all positive in this case) of the determinantal equation in $r^2 \rightarrow |\beta'_{ij} - r^2 \beta''_{ij}| = 0$ where β''_{ij} is already defined and

$$\beta'_{ij} = \sum_{r=1}^{l} n_r \{m_i(r) - \bar{m}_i\} \{m_j(r) - \bar{m}_j\}/(l-1) \; ; \; \bar{m}_i = \sum_{r=1}^{l} n_r \, m_i(r) / \sum_{r=1}^{l} n_r \qquad \ldots (1.2)$$

It was noted that r_i's $(i = 1, 2, \ldots, p)$ are zero when and only when $|\beta'_{ij}| = 0$ which happens when and only when $m_i(1) = m_i(2) = \ldots m_i(l)$, $(i = 1, 2, \ldots, p)$. It is also worth noting here that while on the respective null hypotheses the joint distribution of k_i's of the first problem has the same form as that of t_i's of the second problem, the corresponding distributions on the respective non-null hypotheses are entirely different. This feature of the problem is well-known in the univariate case and is found to be carried over into the multivariate in a manner one would be naturally led to expect. The case for $l \leqslant p$ has also been since investigated and the solution will be out shortly.

In the first case on the null hypothesis the joint sampling distribution of the p statistics $k^2_1, k^2_2, \ldots, k^2_p$ (supposed to be arranged in an ascending order of magnitude) comes out as

$$\text{Const.} \prod_{i=1}^{p} k_i^{n'-p-1} \, dk_i / (1+\lambda k^2_i)^{\frac{n'+n''-2}{2}}$$

$$\times \begin{vmatrix} (k^2_p)^{p-1} & (k^2_{p-1})^{p-1} & \ldots & (k^2_2)^{p-1} & (k^2_1)^{p-1} \\ (k^2_p)^{p-2} & (k^2_{p-1})^{p-2} & \ldots & (k^2_2)^{p-2} & (k^2_1)^{p-2} \\ \cdot & & \cdot\cdot & & \cdot \\ k^2_p & k^2_{p-1} & \ldots & k^2_2 & k^2_1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} \qquad \ldots (1.3)$$

where
$$\lambda = (n'-1)/(n''-1) \qquad \ldots (1.4)$$

while in the second case on the null hypothesis the distribution of $t_1, t_2, \ldots, t_p$ (supposed to be arranged in an ascending order of magnitude) comes out as

$$\text{Const} \prod_{i=1}^{p} t_i^{l-p-1} \, dt_i / (1+\mu \, t^2_i)^{\frac{N-1}{2}}$$

$$\times \begin{vmatrix} (t^2_p)^{p-1} & (t^2_{p-1})^{p-1} & \ldots & (t^2_2)^{p-1} & (t^2_1)^{p-1} \\ (t^2_p)^{p-2} & (t^2_{p-1})^{p-2} & \ldots & (t^2_2)^{p-2} & (t^2_1)^{p-2} \\ \cdot & & \cdot\cdot & & \cdot \\ t^2_p & t^2_{p-1} & \ldots & t^2_2 & t^2_1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} \qquad \ldots (1.5)$$

where
$$\mu = (l-1)/(N-l) \text{ and } N = \sum_{r=1}^{l} n_r \qquad \ldots (1.6)$$

It will be seen that (1.3) and (1.5) are, as noted earlier, exactly of the same form.

The object of the present paper is to find out the sampling distribution of (*i*) the maximum statistic k_p or t_p of (1.3) or (1.5), (*ii*) the minimum statistic k_1 or t_1 of (1.3) or (1.5), and (*iii*) in general any intermediate statistic k_j or t_j ($j = 2, 3, \ldots, p-1$). It is evident that one line of investigation would serve for both k's and t's. The next paper will be concerned with the sampling distribution of (*i*) k_1, k_p or $k_j (j = 2, 3, \ldots, p-1)$ on the non-null hypothesis $\|\alpha'_{ij}\| \neq \|\alpha''_{ij}\|$ and (*ii*) t_1, t_p and $t_j (j = 2, 3, \ldots, p-1)$ on the non-null hypothesis $\|\beta'_{ij}\| \neq 0$.

2. NOTATION AND MATHEMATICAL PRELIMINARIES

If in (1.3) we put $\lambda k^2_i = x'_i$ ($i = 1, 2, \ldots, p$), then (1.3) reduces to

$$\text{Const.} \prod_{i=1}^{p} x'_i{}^{\frac{n'-p-2}{2}} \, dx'_i / (1+x'_i)^{\frac{n'+n''-2}{2}}$$

$$\times \begin{vmatrix} (x'_p)^{p-1} & (x'_{p-1})^{p-1} & \ldots & (x'_2)^{p-1} & (x'_1)^{p-1} \\ (x'_p)^{p-2} & (x'_{p-1})^{p-2} & \ldots & (x'_2)^{p-2} & (x'_1)^{p-2} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ x'_p & x'_{p-1} & \ldots & x'_2 & x'_1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} \quad \ldots \quad (2.1)$$

Similarly if we put $\mu t^2_i = x''_i (i = 1, 2, \ldots p)$ (1.5) goes over into

$$\text{Const.} \prod_{i=1}^{p} x''_i{}^{\frac{l-p-2}{2}} \, dx''_i / (1+x''_i)^{\frac{N-1}{2}}$$

$$\times \begin{vmatrix} (x''_p)^{p-1} & (x''_{p-1})^{p-1} & \ldots & (x''_2)^{p-1} & (x''_1)^{p-1} \\ (x''_p)^{p-2} & (x''_{p-1})^{p-2} & \ldots & (x''_2)^{p-2} & (x''_1)^{p-2} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ x''_p & x''_{p-1} & \ldots & x''_2 & x''_1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} \quad \ldots \quad (2.2)$$

Now (2.1) and (2.2) being of the same form we can conveniently put x_i for either x'_i or x''_i, m for either $(n'-p-2)/2$ or $(l-p-2)/2$, and n for either $(n'+n''-2)/2$ or $(N-1)/2$. It is to be noticed that $(n'-p-2)/2 < (n'+n''-2)/2$ and $(l-p-2)/2 < (N-1)/2$, which means that m in any case is less than n. We can now write (2.1) or (2.2) in the form

$$\text{Const.} \prod_{i=1}^{p} x^m_i \, dx_i / (1+x_i)^n$$

$$\times \begin{vmatrix} x_p^{p-1} & x_{p-1}^{p-1} & \ldots & x_2^{p-1} & x_1^{p-1} \\ x_p^{p-2} & x_{p-1}^{p-2} & \ldots & x_2^{p-2} & x_1^{p-2} \\ \cdot & \cdot & \ldots & \cdot & \\ x_p & x_{p-1} & \ldots & x_2 & x_1 \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} \quad \ldots \quad (2.3)$$

It was shown in earlier papers that, unarranged, each of the k^2_is or t^2_is varies from 0 to ∞ and arranged in an ascending order we have

$$0 \leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_{p-1} \leqslant k_p \leqslant \infty, \text{ and } 0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{p-1} \leqslant t_p \leqslant \infty \quad \ldots \quad (2.4)$$

It is to be remembered here that (1.3) and (1.5) or (2.1) and (2.2) give the relevant distributions only when the statistics have been arranged in an ascending order. Otherwise, as pointed out in earlier papers, instead of the determinants occurring here we would have had absolute values of these determinants coming in.

Now from (2.4) it easily follows that

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{p-2} \leqslant x_{p-1} \leqslant x_p \leqslant \infty \qquad \ldots (2.41)$$

The distribution (2.3) can be conveniently rewritten in the form

$$\text{Const.} \prod_{i=1}^{p} dx_i$$

$$\begin{vmatrix} \frac{x_p^{m+p-1}}{(1+x_p)^n} & \frac{x_{p-1}^{m+p-1}}{(1+x_{p-1})^n} & \ldots & \frac{x_2^{m+p-1}}{(1+x_2)^n} & \frac{x_1^{m+p-1}}{(1+x_1)^n} \\ \frac{x_p^{m+p-2}}{(1+x_p)^n} & \frac{x_{p-1}^{m+p-2}}{(1+x_{p-1})^n} & \ldots & \frac{x_2^{m+p-2}}{(1+x_2)^n} & \frac{x_1^{m+p-2}}{(1+x_1)^n} \\ \cdot & \cdot & \ldots & \cdot & \\ \frac{x_p^{m+1}}{(1+x_p)^n} & \frac{x_{p-1}^{m+1}}{(1+x_{p-1})^n} & \ldots & \frac{x_2^{m+1}}{(1+x_2)^n} & \frac{x_1^{m+1}}{(1+x_1)^n} \\ \frac{x_p^{m}}{(1+x_p)^n} & \frac{x^{m}_{p-1}}{(1+x_{p-1})^n} & \ldots & \frac{x_2^{m}}{(1+x_2)^n} & \frac{x_1^{m}}{(1+x_1)^n} \end{vmatrix} \qquad \ldots (2.42)$$

or in a more compressed notation

$$\text{Const.} \quad F\{m_p, n ; x_p : m_{p-1}, n ; x_{p-1} : \ldots . m_2, n ; x_2 : m_1, n ; x_1\} \times \prod_{i=1}^{p} dx_i \qquad \ldots (2.43)$$

where the function F{ } stands for the determinant in (2.42) and where $m_i = m - i - 1$ $(i = p, p-1 \ldots 1)$

The incomplete probability integrals of the distributions of $(i) x_p$, $(ii) x_1$, and (iii) x_s for $(s = p-1, \ldots, 2)$ are given respectively by the p-fold integrals

$$(i) \int_0^x dx_p \int_0^{x_p} dx_{p-1} \int_0^{x_{p-1}} dx_{p-2} \ldots \int_0^{x_2} dx_1 F\{m_p, n ; x_p ; m_{p-1}, n ; x_{p-1} ; \ldots m_1 n ; x_1\} \qquad \ldots (2.5)$$

$$(ii) \int_{x_{p-1}}^{\infty} dx_p \int_{x_{p-2}}^{\infty} dx_{p-1} \ldots \int_{x_1}^{\infty} dx_2 \int_{x}^{\infty} dx_1 F\{m_p, n ; x_p ; m_{p-1}, n ; x_{p-1} ; \ldots m_1, n ; x_1\} \qquad \ldots (2.51)$$

$$(iii) \int_{x_{p-1}}^{\infty} dx_p \int_{x_{p-2}}^{\infty} dx_{p-1} \ldots \int_{x_j}^{\infty} dx_{j+1} \int_{x}^{\infty} dx_j \int_0^{x_j} dx_{j-1} \ldots \int_0^{x_2} dx_1 F\{m_p n ; x_p \ldots m_1, n ; x_1\} \qquad (2.52)$$

In (i) the order of integration is from $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots x_p$; in (ii) the order is from $x_p \rightarrow x_{p-1} \rightarrow x_{p-2} \rightarrow \ldots \rightarrow x_1$; in (iii) it is from $x_p \rightarrow x_{p-1} \rightarrow \ldots \rightarrow x_j$ on one side and from $x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_j$ on the other side. It will be also seen that (i) gives the incomplete probability integral of x_p from 0 to x whence we could, if we liked, easily obtain the incomplete integral from x to ∞, (ii) gives the incomplete probability integral of x_1 from x to ∞ whence that for the range 0 to x could be easily obtained, while (iii) gives the incomplete integral for x_j from x to ∞ whence again that from 0 to x is immediately derivable.

In the function F{ } defined by (2.43) we have the restriction (i) $\infty \geqslant x_p \geqslant x_{p-1} \geqslant x_{p-2} \geqslant \ldots \geqslant x_2 \geqslant x_1 \geqslant 0$, (ii) $m_s = m+s-1 (s=p, p-1, \ldots, 1)$ and (iii) all the n's are equal and greater than m_p. It will be necessary and hence desirable to relax (ii) and (iii) and consider m_p, n_p ; m_{p-1}, n_{p-1} ; $\ldots m_1, n_1$ subject, however, to the provisos that (α) $m_p > m_{p-1} > \ldots > m_1$, ($\beta$) $n_p > n_{p-1} > \ldots > n_1$ and (γ) $n_1 > m_p$. The more general type of function F{ } will then be

$$F\{m_p, n_p ; x_p : m_{p-1}, n_{p-1} ; x_{p-1} : \ldots m_2, n_2 ; x_2 : m_1, n_1 ; x_1\} \quad .. \quad (2.6)$$

where the x_s's are subject to (i) and m_s's and n_s's are subject to (α), (β) and (γ), and $s = p, p-1, \ldots, 2, 1$. For such a function we shall have, of course, incomplete probability integrals corresponding to (2.5), (2.51) and (2.52). Such integrals would not directly arise out of the statistical situations considered here, but it will be seen later on that in course of reduction of the main integrals, such integrals would occur as mathematical auxiliaries not open to direct statistical interpretation.

We shall denote (2.5), (2.51) and (2.52) respectively by

$$\left.\begin{aligned} &F\{x ; m_p, n ; m_{p-1}, n ; \ldots m_1, n\}, \\ &F\{m_p, n ; m_{p-1}, n ; \ldots m_1, n ; x\} \text{ and} \\ &F\{m_p, n ; \ldots m_j, n ; x ; m_{j-1}, n ; \ldots m_1, n\} \end{aligned}\right\} \quad .. \quad (2.7)$$

Or, since all the n's are equal, by

$$\left.\begin{aligned} &F\{x ; m_p, m_{p-1}, \ldots m_1 ; n\}, \\ &F\{m_p, m_{p-1}, \ldots m_1 ; n ; x\} \text{ and} \\ &F\{m_p, m_{p-1}, \ldots m_j ; x ; m_{j-1}, \ldots m_1 ; n\} \end{aligned}\right\} \quad .. \quad (2.71)$$

while the corresponding incomplete integrals arising out of (2.6) can be, of course, conveniently denoted by

$$F\{x ; m_p, n_p ; m_{p-1}, n_{p-1} ; \ldots m_1, n_1\}, \; F\{m_p, n_p ; m_{p-1}, n_{p-1} ; \ldots m_1, n_1 ; x\}$$

and

$$F\{m_p, n_p ; \ldots m_j, n_j ; x ; m_{j-1}, n_{j-1} ; \ldots m_1, n_1\} \quad .. \quad (2.72)$$

It will be seen from (2.42), (2.43), (2.5), (2.52) and (2.6) that the evaluation of the functions defined by (2.7)—(2.72) involves a consideration of the following types of integrals

$$\int_0^1 \frac{x_p^{m'_p} dx_p}{(1+x_p)^{n'_p}} \int_0^{x_p} \frac{x_{p-1}^{m'_{p-1}} dx_{p-1}}{(1+x_{p-1})^{n'_{p-1}}} \ldots \int_0^{x_2} \frac{x_1^{m'_1} dx_1}{(1+x_1)^{n'_1}} \; ; \; \int_{x_{p-1}}^{\infty} \frac{x_p^{m'_p} dx_p}{(1+x_p)^{n'_p}} \ldots \int_x^{\infty} \frac{x_1^{m'_1} dx_1}{(1+x_1)^{n'_1}};$$

and

$$\int_{x_{p-1}}^{\infty} \frac{x_p^{m'_p} dx_p}{(1+x_p)^{n'_p}} \ldots \int_{x_j}^{\infty} \frac{x_{j+1}^{m'_{j+1}} dx_{j+1}}{(1+x_{j+1})^{n'_{j+1}}} \int_x^{\infty} \frac{x_j^{m'_j} dx_j}{(1+x_j)^{n'_j}} \int_0^{x_j} \frac{x_{j-1}^{m'_{j-1}} dx_{j-1}}{(1+x_{j-1})^{n'_{j-1}}} \ldots \int_0^{x_2} \frac{x_1^{m'_1} dx_1}{(1+x_1)^{n'_1}} \quad .. \quad (2.73)$$

where $(m'_p, m'_{p-1}, \ldots m'_1)$ is any permutation of $m_p, m_{p-1}, \ldots m_1$ and $(n'_p, \ldots, n'_1)$ is the corresponding permutation of $(n_p, n_{p-1}, \ldots n_1)$.

The integrals given by (2.73) we can conveniently denote by

$$F(x ; m'_p, n'_p ; m'_{p-1}, n'_{p-1} ; \ldots m'_1, n'_1) \; ; \; F(m'_p, n'_p ; m'_{p-1}, n'_{p-1} ; m'_1, n'_1 ; x)$$

and

$$F(m'_p, n'_p ; \ldots ; m'_j, n'_j ; x ; m'_{j-1}, n'_{j-1} ; \ldots ; m'_1, n'_1) \quad .. \quad (2.74)$$

The functions F(   ) defined earlier and F(   ) defined now should be carefully distinguished. For purposes of reduction and demonstration another alternative notation to (2.7), (2.71) and (2.72) appears to be more helpful namely

$$F\left\{x;\begin{pmatrix} m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \end{pmatrix}\right\},\quad F\left\{\begin{pmatrix} m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \end{pmatrix}; x\right\}$$

$$\text{and } F\left\{\begin{matrix} m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ \cdot & \cdot & & \cdot \\ m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1 \\ \cdot & \cdot & \cdots & \cdot \\ x; (m_p, n_p & m_{p-1}, n_{p-1} & \cdots & m_1, n_1) \end{matrix}\right\} \quad \cdots \quad (2.75)$$

The notation can be illustrated for the case of three variates for which

$$F\left\{x;\begin{pmatrix} m_3, n_3 & m_2, n_2 & m_1, n_1 \\ m_3, n_3 & m_2, n_2 & m_1, n_1 \\ m_3, n_3 & m_2, n_2 & m_1, n_1 \end{pmatrix}\right\}$$

$$\equiv +\int_0^x \frac{x_3^{m_3}\,dx_3}{(1+x_3)^{n_3}}\left(\int_0^{x_3}\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}}-\int_0^{x_3}\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_0^{x_2}\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}\right)$$

$$-\int_0^x \frac{x_3^{m_2}\,dx_3}{(1+x_3)^{n_2}}\left(\int_0^{x_3}\frac{x_2^{m_3}\,dx_2}{(1+x_2)^{n_3}}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}}-\int_0^{x_3}\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_0^{x_2}\frac{x_1^{m_3}\,dx_1}{(1+x_1)^{n_3}}\right)$$

$$+\int_0^x \frac{x_3^{m_1}\,dx_3}{(1+x_3)^{n_1}}\left(\int_0^{x_3}\frac{x_2^{m_3}\,dx_2}{(1+x_2)^{n_3}}\int_0^{x_2}\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}-\int_0^{x_3}\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_0^{x_2}\frac{x_1^{m_3}\,dx_1}{(1+x_1)^{n_3}}\right)$$

$$\equiv F(x; m_3, n_3; m_2, n_2; m_1, n_1) - F(x; m_3, n_3; m_1, n_1; m_2, n_2) - F(x; m_2, n_2; m_3, n_3; m_1, n_1)$$
$$+ F(x; m_2, n_2; m_1, n_1; m_3, n_3) + F(x; m_1, n_1; m_3, n_3; m_2, n_2)$$
$$- F(x; m_1, n_1; m_2, n_2; m_3, n_3) \quad \cdots \quad (2.76)$$

Similarly,

$$F\left\{\begin{pmatrix} m_3, n_3 & m_2, n_2 & m_1, n_1 \\ m_3, n_3 & m_2, n_2 & m_1, n_1 \\ m_3, n_3 & m_2, n_2 & m_1, n_1 \end{pmatrix}; x\right\}$$

$$=+\int_{x_2}^\infty \frac{x_3^{m_3}\,dx_3}{(1+x_3)^{n_3}}\left(\int_{x_1}^\infty\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_x^\infty\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}}-\int_{x_1}^\infty\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_x^\infty\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}\right)$$

$$-\int_{x_2}^\infty \frac{x_3^{m_2}\,dx_3}{(1+x_3)^{n_2}}\left(\int_{x_1}^\infty\frac{x_2^{m_3}\,dx_2}{(1+x_2)^{n_3}}\int_x^\infty\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}}-\int_{x_1}^\infty\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_x^\infty\frac{x_1^{m_3}\,dx_1}{(1+x_1)^{n_3}}\right)$$

$$+\int_{x_2}^\infty \frac{x_3^{m_1}\,dx_3}{(1+x_3)^{n_1}}\left(\int_{x_1}^\infty\frac{x_2^{m_3}\,dx_2}{(1+x_2)^{n_3}}\int_x^\infty\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}-\int_{x_1}^\infty\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_x^\infty\frac{x_1^{m_3}\,dx_1}{(1+x_1)^{n_3}}\right)$$

$$= F(m_3,n_3; m_2,n_2; m_1,n_1; x) - F(m_3,n_3; m_1,n_1; m_2,n_2; x) - F(m_2,n_2; m_3,n_3; m_1,n_1; x)$$
$$+ F(m_2,n_2; m_1,n_1; m_3,n_3; x) + F(m_1,n_1; m_3,n_3; m_2,n_2; x) - F(m_1,n_1; m_2,n_2; m_3,n_3; x) \quad \cdots \quad (2.77)$$

and

$$F\left\{\begin{matrix} & m_3, n_3 & m_2, n_2 & m_1, n_1 \\ & m_3, n_3 & m_2, n_2 & m_1, n_1 \\ x\,; & (m_3, n_3 & m_2, n_2 & m_1, n_1) \end{matrix}\right\}$$

$$= + \int_{x_2}^{\infty} \frac{x_3^{m_3}\, dx_3}{(1+x_3)^{n_3}} \left(\int_{x}^{\infty} \frac{x_2^{m_2}\, dx_2}{(1+x_2)^{n_2}} \int_0^{x_2} \frac{x_1^{m_1}\, dx_1}{(1+x_1)^{n_1}} - \int_{x}^{\infty} \frac{x_2^{m_1}\, dx_2}{(1+x_2)^{n_1}} \int_0^{x_2} \frac{x_1^{m_2}\, dx_1}{(1+x_1)^{n_2}} \right)$$

$$- \int_{x_1}^{\infty} \frac{x_3^{m_2}\, dx_3}{(1+x_3)^{n_2}} \left(\int_{x}^{\infty} \frac{x_2^{m_3}\, dx_2}{(1+x_2)^{n_3}} \int_0^{x_2} \frac{x_1^{m_1}\, dx_1}{(1+x_1)^{n_1}} - \int_{x}^{\infty} \frac{x_2^{m_1}\, dx_2}{(1+x_2)^{n_1}} \int_0^{x_2} \frac{x_1^{m_3}\, dx_1}{(1+x_1)^{n_3}} \right)$$

$$+ \int_{x_2}^{\infty} \frac{x_3^{m_1}\, dx_3}{(1+x_3)^{n_1}} \left(\int_{x}^{\infty} \frac{x_2^{m_3}\, dx_2}{(1+x_2)^{n_3}} \int_0^{x_2} \frac{x_1^{m_2}\, dx_1}{(1+x_1)^{n_2}} - \int_{x}^{\infty} \frac{x_2^{m_2}\, dx_2}{(1+x_2)^{n_2}} \int_0^{x_2} \frac{x_1^{m_3}\, dx_1}{(1+x_1)^{n_3}} \right)$$

$$= F(m_3,n_3;\, m_2,n_2;\, x;\, m_1,n_1) - F(m_3,n_3;\, m_1,n_1;\, x;\, m_2,n_2) - F(m_2,n_2;\, m_3,n_3;\, x;\, m_1,n_1)$$
$$+ F(m_2,n_2;\, m_1,n_1;\, x;\, m_3,n_3) + F(m_1,n_1;\, m_3,n_3;\, x;\, m_2,n_2) - F(m_1,n_1;\, m_2,n_2;\, x;\, m_3,n_3) \quad .. \quad (2.78)$$

A detailed expansion for the case $p = 3$ makes evident the meaning of the notation for any general p. Going back to (2.42) we notice that in an actual p-variate problem $n_p = n_{p-1} = \ldots = n_1 = n$(suppose) and $m_s = m+s-1$ $(s = p,\ p-1, \ldots 1)$ although as we shall see later on, even there, for purposes of reduction the auxiliary functions F{ } have to be considered in which $n_p \neq n_{p-1} \neq \ldots n_1$ and $m_s \neq m+s-1$. If, however, $n_p = n_{p-1} = \ldots = n_1 = n$, then we can also more appropriately replace (2.75) by

$$F\left\{ x\,; \begin{pmatrix} m_p & m_{p-1} & .. & m_1 \\ \cdot & \cdot & .. & \cdot \\ m_p & m_{p-1} & .. & m_1 \\ m_p & m_{p-1} & .. & m_1 \end{pmatrix}; n \right\};\quad F\left\{ \begin{pmatrix} m_p & m_{p-1} & .. & m_1 \\ \cdot & \cdot & .. & \cdot \\ m_p & m_{p-1} & .. & m_1 \\ m_p & m_{p-1} & .. & m_1 \end{pmatrix}; n\,; x \right\}$$

and
$$F\left\{ \begin{matrix} & m_p & m_{p-1} & .. & m_1 & \\ & \cdot & \cdot & .. & \cdot & \\ x; & (m_p & m_{p-1} & .. & m_1)\,; & n \\ & \cdot & \cdot & .. & \cdot & \\ & m_p & m_{p-1} & .. & m_1 & \end{matrix} \right\} \quad .. \quad (2.79)$$

The following is a group of fundamental theorems needed for reduction of the present problem

$$\int_0^x \frac{x^m dx}{(1+x)^n} Q(x) = -\frac{1}{n-m-1} \left[\frac{x^m}{(1+x)^{n-1}} Q(x) \right]_0^x + \frac{1}{n-m-1} \int_0^x \frac{x^m\, dx}{(1+x)^{n-1}} \cdot \frac{\partial Q}{\partial x}$$
$$+ \frac{m}{n-m-1} \int_0^x \frac{x^{m-1}\, dx}{(1+x)^n} Q(x) = -\frac{1}{n-m-1} \frac{x^m}{(1+x)^{n-1}} Q(x) + \frac{1}{n-m-1} \int_0^x \frac{x^m\, dx}{(1+x)^{n-1}} \frac{\partial Q}{\partial x}$$
$$+ \frac{m}{n-m-1} \int_0^x \frac{x^{m-1}\, dx}{(1+x)^n} Q(x) \quad .. \quad (2{\cdot}8)$$

Here $Q(x)$ is any function of x which conforms to the usual conditions of differentiability and integrability from 0 to x and the result (2.8) is obtained by integrating $1/(1+x)^{n-m}$ and differentiating $\{x^m/(1+x)^m\}\, Q(x)$. m is, of course, less than n as in all situations to be considered here. We can express (2.8) in a more convenient notation.

$$F(x;\, m,n;\, Q) = -(1/n-m-1)\, F_0(m,n-1;\, x)Q(x) + (1/n-m-1)\, F(x;\, m,n-1;\, Q')$$
$$+ (m/n-m-1) F(x;\, m-1,\, n;\, Q) \quad .. \quad (2{\cdot}81)$$

where $F(x;\, m,n;\, Q) = \int_0^x \frac{x^m\, dx}{(1+x)^n} Q(x),\ F_0(m,n-1\,;\, x) = \frac{x^m}{(1+x)^{n-1}}$

and $Q' = \frac{\partial Q}{\partial x}$ $\quad .. \quad (2.82)$

It follows from (2.81) by transposition that

$$F(x;m,n;Q)= (1/m+1)\ F_0(m+1,n-1;x)Q(x)-(1/m+1)\ F(x;\ m+1,n-1;\ Q')$$
$$+(m+1/n-m-2)\ F(x;m+1,n;Q) \quad .. \quad (2.83)$$

Similarly

$$\int_x^\infty \frac{x^m\,dx}{(1+x)^n}Q(x)= \frac{1}{n-m-1}\ \frac{x^m}{(1+x)^{n-1}}\ Q(x)+\frac{1}{n-m-1}\int_x^\infty \frac{x^m dx}{(1+)^{n-1}}\ \frac{\partial Q}{\partial x}$$
$$+\frac{m}{n-m-1}\int_x^\infty \frac{x^{m-1}\,dx}{(1+x)^n}\,Q(x)$$

or $\quad F(m,n;\ Q;\ x)= (1/n-m-1)\ F_0(m,\ n-1;x)Q(x)+(1/n-m-1)\ F(m,\ n-1;\ Q';\ x)$
$$+(m/n-m-1)\ F(m-1,\ n;\ Q;\ x) \quad .. \quad (2.84)$$

and by transposition

$$F(m,n\ ;\ Q;\ x)=-(1/m+1)\ F_0(m+1,n-1;\ x)Q(x)-(1/m+1)\ F(m+1,n-1\ ;\ Q';\ x)$$
$$+(n-m-2/m+1)\ F(m+1,n\ ;\ Q\ ;\ x) \quad .. \quad (2.85)$$

This is one group of theorems mostly needed here. There is another gro p which we should not directly use here but which might be useful in other types of reduction

$$\int_0^x \frac{x^m\,dx}{(1+x)^n}Q(x)=-\frac{1}{n-1}\left[\frac{x^m}{(1+x)^{n-1}}\ Q(x)\right]_0^x+\frac{1}{n-1}\int_0^x \frac{x^m\,dx}{(1+x)^{n-1}}\ \frac{\partial Q}{\partial x}+\frac{m}{n-1}\int_0^x \frac{x^{m-1}\,dx}{(1+x)^{n-1}}\ Q(x)$$

or $\quad F(x;\ m,n;\ Q)=-(1/n-1)F_0(m,n-1;\ x)Q(x)+(1/n-1)\ F(x;\ m,n-1;Q')$
$$+(n/m+1)\ F(x;\ m+1,n-1;Q)$$

and the associated theorem

$$F(x;\ m,n;\ Q)= (1/m-1)\ F_0(m+1,n\ ;x)Q(x)-(1/m+1)\ F(x\ ;\ m+1,n;Q')$$
$$+(n/m-1)\ F(x;\ m+1,n-1;\ Q) \quad .. \quad (2.86)$$

(2.86) is evidently obtained by integrating $1./(1+x)^n$ and differentiating $x^mQ(x)$. We have similarly

$$F(m,n;\ Q;\ x)= (1/n-1)\ F_0(m,n-1\ ;\ x)Q(x)+(1/n-1)\ F(m,n-1\ ;Q'\ ;x)$$
$$+(m/n-1)\ F(m-1,n-1\ ;Q\ ;x)$$

and alternatively

$$F(m,n\ ;\ Q\ ;\ x)=-(1/m+1)\ F_0(m+1,n\ ;x)\ Q(x)-(1/m+1)\ F(m+1,n;Q';x)$$
$$+(n/m+1)\ F(m+1,n+1\ ;Q\ ;x) \quad .. \quad (2.87)$$

The way in which the theorems (2.8—2.85) are proposed to be used for purposes of reduction here may be illustrated as follows for the case of two variates ($p=2$). With $n_1=n_2=n$ we have

$$F(x;\ m_2,n\ ;\ m_1,n) \equiv \int_0^x \frac{x_2^{m_2}\,dx_2}{(1+x_2)^n}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x)^n}$$
$$=F(x\ ;\ m_2,\ n\ ;\ Q),\ \text{where}\ Q(x_2)=F(x_2;m_1,n)\ . \quad .. \quad (2.88)$$

Remembering that $Q' = x_2^{m_1}/(1 + x_2)^{n_1}$ we see that the right side of (2.88) reduces by using (2.81) to

$$-(1/n-m_1-1)\, F_0(m_1,n_1-1\,;\,x)\, F(x\,;\,m_1,n)+(1/n-m_1-1)\, F(x\,;\,m_1+m_2;\,2n-1)$$
$$+(m_2/n-m_1-1)\;\; F(x\,;\,m_2-1\,;\,n\,;\,m_1,n). \qquad .. \quad (2.89)$$

This is if we integrate out by parts over x_2, trying to reduce m_2 in successive stages.

On the other hand we might integrate out by parts over x_1 trying to decrease m_1 in successive stages. In this case

$$F(x\,;\,m_2,n\,;\,m_1,n) = \int_0^x \frac{x_2^{m_2}\,dx_2}{(1+x_2)^n} \int_0^{x_2} \frac{x_1^{m_1}\,dx_1}{(1+x_1)^n}$$

$$= \int_0^x \frac{x_2^{m_2}\,dx_2}{(1+x_2)^n}\left[-(1/n-m_1-1)\,\frac{x_2^{m_1}}{(1+x_2)^{n-1}}+(m_1/n-m_1-1)\int_0^{x_2}\frac{x_1^{m_1-1}\,dx_1}{(1+x_1)^n}\right]$$

$$=-(1/n-m_1-1)\, F(x\,;\,m_2+m_1,\,2n-1)+(m_1/n-m_1-1)\;\; F(x\,;\,m_2,n\,;\,m_1-1,n) \qquad .. \quad (2.9)$$

This is obtained by using (2.81) and remembering that here $Q=1$. (2.89) is to be used when $m_2 > m_1$, and (2.9) when $m_2 < m_1$, our object in any case being to get to $F(x;\, m',\, n;\, m',\, n)$ where m' is m_1 or m_2 whichever is less.

As an alternative procedure we might also have had by using (2.83)

$$F(x\,;\,m_2,n\,;\,m_1,n)= (1/m_2+1)\, F_0(m_2+1,n-1;\,x)\, F(x;\,m_1,n)-(1/m_2+1)\, F(x;\,m_2+1+m_1,2n-1)$$
$$+(n-m_2-2/m_2+2)\; F(x\,;\,m_2+1,\,n\,;\,m_1,n) \qquad .. \quad (2.91)$$

or $$F(x\,;\,m_2,n\,;\,m_1,n)= (1/m_1+1)\; F(x;\,m_1+1+m_2,\,2n-1)$$
$$+(n-m_1-2/m_1+1)\;\; F(x\,;\,m_2,n\,;\,m_1+1,n) \qquad .. \quad (2.92)$$

We shall use (2.91) or (2.92) according as $m_1 > m_2$ or $m_2 > m_1$ our object there being in any case to get to $F(x\,;\; m'',\, n\,;\; m'',\, n)$ where m'' is m_1 or m_2 whichever is greater.

For $p = 2$, using (2.89) or (2.9), we have (for the distribution of the maximum statistic).

$$F(x\,;\,m+1,\,n\,;\,m,n)\equiv F\left\{x\,;\begin{pmatrix} m+1 & m \\ m+1 & m \end{pmatrix};n\right\}$$

$$=+F(x\,;\,m+1,n\,;\,m,n)-F(x\,;\,m,n\,;\,m+1,n)$$
$$=-(1/n-m-2)\; F_0(m+1,n-1\,;\,x)\; F(x\,;\,m,n)+(1/n-m-2)\; F(x\,;\,2m+1,2n-1)$$
$$+(m+1/n-m-2)\; F(x;\, m,n;\, m,n)+(1/n-m-2)\; F(x\,;\,2m+1,2n-1)$$
$$-(m+1/n-m-2)\; F(x\,;\,m,n\,;\,m,n)=(2/n-m-2)\; F(x;\,2m+1,2n-1)$$
$$-(1/n-m-2)\; F_0(m+1,n-1\,;\,x)\; F(x;\,m,n) \qquad .. \quad (2.93)$$

Alternatively by using (2.91) and (2.92) we could have

$$F\left\{x\,;\begin{pmatrix} m+1 & m \\ m+1 & m \end{pmatrix};n\right\}$$

$$=(2/m+1)F(x;\,2m+1,\,2n-1)-(1/m+1)F_0(m+1,n-1;\,x)\;\; F(x;\,m+1,n) \qquad .. \quad (2.94)$$

With a general m_2 and m_1 (but $m_2 > m_1$) we should have had in place of (2.93) and (2.94)

$$F(x; m_2, n; m_1, n) \equiv F\left\{x; \begin{pmatrix} m_2 & m_1 \\ m_2 & m_1 \end{pmatrix}; n\right\}$$

$$= (2/n-m_2+1)\ F(x\,; m_2+m_1, 2n-1)-(1/n-m_2-1)\ F_0(m_2, n-1; x)\ F(x\,; m_1, n)$$

$$+(m_2/n-m_2-1)\quad F\left\{x; \begin{pmatrix} m_2-1 & m_1 \\ m_2-1 & m_1 \end{pmatrix}; n\right\} \quad .. \quad (2.95)$$

or alternatively

$$= (2/m_1+1)\ F(x\,; m_1+m_2, 2n-1)-(1/m_1+1)\ F(m_1+1, n-1\,; x)\ F_0(x\,; m_2, n)$$

$$+(n-m_1-2/m_1+1)\ F\left\{x\,; \begin{pmatrix} m_2 & m_1+1 \\ m_2 & m_1+1 \end{pmatrix}; n\right\} \quad .. \quad (2.96)$$

(2.95) or (2.96) gives us a recursion formula by which we get after a few steps to

$$F\left\{x\,; \begin{pmatrix} m_1 & m_1 \\ m_1 & m_1 \end{pmatrix}; n\right\} \text{ or } F\left\{x\,; \begin{pmatrix} m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}; n\right\} \quad .. \quad (2.961)$$

either of which is zero by definition of the functions F(   ).

This enables us to evaluate $F(x\,; m_2, n\,; m_1, n)$ in terms of functions lik $F(x\,; m', n')$ which are readily evaluated from the incomplete B-function tables and functions like $F_0(x\,; m', n')$ which are of course directly calculated.

Similarly we have by using (2.84)

$$F(m_2, n; m_1, n; x) \equiv \int_{x_1}^{\infty} \frac{x_2^{m_2}\, dx_2}{(1+x_2)^n} \int_{x}^{\infty} \frac{x_1^{m_1}\, dx_1}{(1+x_1)^n}$$

$$= (1/n-m_2-1)\ F(m_2+m_1, 2n-1; x)+(m_2/n-m_2-1)\ F(m_2-1, n: m_1\ n: x) \quad .. \quad (2.97)$$

or on the other hand

$$= (1/n-m_1-1)\ F_0(m_1, n-1\,; x)\ F(m_2, n\,; x)-(1/n-m_1-1)\ F(m_1+m_2, 2n-1\,; x)$$

$$+(m_1/n-m_1-1)\ F(m_2, n\,; m_1-1, n\,; x) \quad .. \quad (2.98)$$

The form (2.97) (obtained by integration by parts over x_2) or (2.98) (obtained by integration by parts over x_1) is to be used according as $m_2 > m_1$ or $m_1 > m_2$ our object being in any case to get to $F(m'\,; n\,; m', n\,; x)$ where m' is the lesser of m_2 and m_1
Alternatively by using (2.85)

$$F(m_2, n\,; m_1, n\,; x) = -(1/m_1+1)\ F(m_2+m_1+1, 2n-1\,; x)$$
$$+(n-m_1-2/m_1+1)\ F(m_2+1, n\,; m_1, n\,; x)$$

$$= -(1/m_1+1)\ F_0(m_1+1, n-1\,; x) F(m_2, n\,; x)+(1/m_1+1)\ F(m_1+m_2+1, 2n-1\,; x)$$

$$+(n-m_1-2/m_1+1)\ F(m_2, n\,; m_1+1, n\,; x) \quad .. \quad (2.99)$$

Of the two forms given by (2.99) the first or the second is to be used according as $m_2 > m_1$ or $m_1 > m_2$ our object in any case being to get to $F(m^*, n\,;\, m^*, n\,;\, x)$ where m^* is the greater of m_2 and m_1. Using (2.97) and (2.98) we easily see (with $m_2 > m_1$) that

$$F(m_2, n\,;\, m_1, n\,;\, x) \equiv F\left\{\begin{pmatrix} m_2 & m_1 \\ m_2 & m_1 \end{pmatrix}; n\,;\, x\right\}$$

$$= (2/n-m_2-1)\ F(m_2+m_1, 2n-1\,;\, x) - (1/n-m_2-1)\ F_0(m_2, n-1\,;\, x)F(m_1, n\,;\, x)$$

$$+(m_2/n-m_2-1)\ \ F\left\{\begin{pmatrix} m_2-1 & m_1 \\ m_2-1 & m_1 \end{pmatrix}; n\,;\, x\right\} \qquad \ldots \quad (2.991)$$

and using (2.99) we have

$$F\left\{\begin{pmatrix} m_2 & m_1 \\ m_2 & m_1 \end{pmatrix}; n\,;\, x\right\} = +(2/m_1+1)\ F(m_2+m_1+1, 2n-1\,;\, x)$$

$$-(1/m_1+1)\ F_0(m_1+1, n-1\,;\, x)\ F(m_2, n\,;\, x)$$

$$+(n-m_1-2/m_1+1)\ F\left\{\begin{pmatrix} m_2 & m_1+1 \\ m_2 & m_1+1 \end{pmatrix}; n\,;\, x\right\} \quad \ldots \quad (2.992)$$

By using (2.991) or (2.992) we can by successive stages get to

$$F\left\{\begin{pmatrix} m_1 & m_1 \\ m_1 & m_1 \end{pmatrix}; n\,;\, x\right\} \quad \text{or} \quad F\left\{\begin{pmatrix} m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}; n\,;\, x\right\}$$

either of which is zero by definition. This means that we can express $F(m_2, n\,;\, m_1, n\,;\, x)$ in terms of functions of the form $F(m', n'\,;\, x)$ which are calculable in terms of the incomplete B-functions tables. For the distribution of the minimum statistic (for $p = 2$)

$$F(m+1,\ n\ ;\, m, n\,;\, x) \equiv F\left\{\begin{pmatrix} m+1 & m \\ m+1 & m \end{pmatrix}; n\,;\, x\right\}$$

$$\left.\begin{aligned} &= (2/n-m-2)\ F(2m+1, 2n-1\,;\, x) - (1/n-m-2)\ F_0(m+1, n-1\,;\, x)F(m, n\,;\, x) \\ &= (2/m+1)\ F(2m+2, 2n-1\,;\, x) - (1/m+1)\ F_0(m+1, n-1;\, x)\ F(m+1, n;\, x) \end{aligned}\right\} \quad (2{\cdot}993)$$

Except for a constant factor (2.93) and (2.94) give us two alternative forms for the distribution of the maximum statistic for $p = 2$, and (2.993) gives us two alternative forms for the distribution of the minimum statistic for the case $p = 2$.

3. THE ACTUAL DERIVATION OF THE DISTRIBUTION OF THE MAXIMUM AND THE MINIMUM STATISTIC FOR THE GENERAL CASE.

Before proceeding with the actual derivation for the general case we shall prove a few more auxiliary theorems or lemmas.

It follows from the notation and definition discussed in §2 that whether in
$F\{x\,;(m_p, m_{p-1}, \ldots m_1)\,;\, n\}$ or $F\{(m_p, m_{p-1}, \ldots m_1)\,;\, n\,;\, x\}$
that is, in

$$F\left\{x\,;\begin{pmatrix} m_p & m_{p-1} & \cdots & m_1 \\ m_p & m_{p-1} & \cdots & m_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p & m_{p-1} & \cdots & m_1 \end{pmatrix};n\right\} \quad \text{or} \quad F\left\{\begin{pmatrix} m_p & m_{p-1} & \cdots & m_1 \\ m_p & m_{p-1} & \cdots & m_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p & m_{p-1} & \cdots & m_1 \end{pmatrix};n\,;x\right\}$$

if we put $m_s = m_{s'} (s \neq s'$, and $s, s' = p, p-1 \ldots 1)$ then F( ) becomes zero. Stated otherwise, if in the pseudo-determinantal forms we put any two columns equal then F( ) becomes zero. This is evident if we look at the expansions (2.76) or (2.77).

Turning now to functions F( ) we notice first that

$$F(x\,;m_2,n\,;m_1,n)+F(x\,;m_1,n\,;m_2,n)$$
$$=\int_0^x \frac{x_2^{m_2}\,dx_2}{(1+x_2)^n}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x_1)^n}+\int_0^x\frac{x_2^{m_1}\,dx_2}{(1+x_2)^n}\int_0^{x_2}\frac{x_1^{m_2}\,dx_1}{(1+x_1)^n}$$
$$=F(x\,;m_2,n)F(x\,;m_1,n) \qquad \ldots (3.1)$$

Likewise, if in $F(x\,;m_p,n\,;m_{p-1},n\,;\ldots m_1,n)$ we permute the parameters $m_p, m_{p-1}, \ldots m_1$ over the different places then

$$\Sigma F(x\,;m'_p,n\,;m'_{p-1},n;\ldots m'_1,n)=\prod_{s=1}^{p} F(x\,;m_s,n) \qquad \ldots (3.11)$$

where $(m'_p, m'_{p-1}, \ldots, m'_1)$ is any permutation of $(m_p, m_{p-1}, \ldots, m_1)$ and where the summation Σ extends over all such possible permutations.

Similarly we have also

$$\Sigma F(m'_p,n\,;m'_{p-1},n\,;\ldots m'_1,n\,;x)=\prod_{s=1}^{p} F(m_s,n\,;x) \qquad \ldots (3.12)$$

Going back to the expansions (2.74) and (2.75) we notice that we could have generalised

$$F\left\{x;\begin{pmatrix} m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \\ m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \end{pmatrix}\right\} \text{and } F\left\{\begin{pmatrix} m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \\ m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \\ \cdot & \cdot & \cdots & \cdot \\ m_p,n_p & m_{p-1},n_{p-1} & \cdots & m_1,n_1 \end{pmatrix};x\right\}$$

to

$$F\left\{x\,;\begin{pmatrix} m_{pp},n_{pp} & m_{p,p-1},n_{p,p-1} & \cdots & m_{p1},n_{p1} \\ \cdot & \cdot & \cdots & \cdot \\ m_{1p},n_{1p} & m_{1,p-1},n_{1,p-1} & \cdots & m_{11},n_{11} \end{pmatrix}\right\}\text{and } F\left\{\begin{pmatrix} m_{pp},n_{pp} & m_{p,p-1},n_{p,p-1} & \cdots & m_{p1},n_{p1} \\ \cdot & \cdot & \cdots & \cdot \\ m_{1p},n_{1p} & m_{1,p-1},n_{1,p-1} & \cdots & m_{11},n_{11} \end{pmatrix};x\right\}$$

or to $\quad F\{x\,;(m_{ij}, n_{ij})\}$ and $F\{(m_{ij}, n_{ij})\,;x\} \ldots (i, j=p, p-1, \ldots 1) \qquad \ldots (3.13)$

The special case we have considered earlier is obtained if we put $m_{ij} = m_{oj}$, $n_{ij} = n_{oj}$ $(j=p, \ldots, 1)$ and the more special case directly arising out of the distribution problem to be considered here is obtained if further we put $m_{oj} = m_o + j$, $n_{oj} = n$ $(j = p, p-1, \ldots, 1)$. The law of expansion of (3.13) is, of course, evident.

Of great use for later investigations will be an appropriate notation which covers the case of all other rows of the pseudo-determinants being like $(m_p, n_p\,;\, m_{p-1}, n_{p-1} \ldots m_1, n_1)$

and only one row being like $(m'_p, n'_p\,;\ m'_{p-1}, n'_{p-1};\ \ldots m'_1, n'_1)$. Where $(m'_p, n'_p\,;\ m'_{p-1}, n'_{p-1}\,;\ \ldots m'_1, n'_1)$ occurs in the s-th row we shall change our symbols a little and write for (3.13)

$$F\{x\,;(m^s_{ij}, n^s_{ij})\} \text{ and } F\{(m^s_{ij}, n^s_{ij});x\} \quad .. \quad (3.14)$$

where $m^s_{ij}=m_{sj},\ n^s_{ij}=n_{sj}\ (i\neq s \text{ and } j=p,p-1,\ ..\ 1)$

and $m^s_{ij}=m'_j,\ n^s_{ij}=n'_j(i=s, \text{ and } j=p,\ p-1,\ ..\ 1)$ .. (3.15)

For the more special use arising directly out of the distribution problem we have, of course,

$$\left.\begin{array}{l} m^s_{ij}=m_o+j\,;\ n^s_{ij}=n(i\neq s, \text{ and } j=p-1,\ ..\ 0) \\ m^s_{ij}=m'_j\,;\ n^s_{ij}=n'_j(i=s, \text{ and } j=p-1,\ ..\ 0) \end{array}\right\} \quad .. \quad (3.16)$$

It will be seen that instead of the single suffix notation $(m_p, n_p\,;\ m_{p-1}, n_{p-1}\,;\ \ldots m_1, n_1)$ we use for convenience the two-way and double suffix notation

$$(m_{op},\ n_{op}\,;\ m_{o,p-1}\,;\ n_{o,p-1},\ \ldots\ m_{o1},\ n_{o1})$$

Pursuing the line suggested by (3.1), (3.11) and (3.12) we have now

$$F\left\{x\,;\begin{pmatrix} m'_p,n'_p & m'_{p-1},n'_{p-1} & .. & m'_1,n'_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ . & . & .. & . \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \end{pmatrix}\right\} - F\left\{x\,;\begin{pmatrix} m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ m'_p,n'_p & m'_{p-1},n'_{p-1} & .. & m'_1,n'_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ . & . & .. & . \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \end{pmatrix}\right\}$$

$$+F\left\{x\,;\begin{pmatrix} m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \\ m'_p,n'_p & m'_{p-1},n'_{p-1} & .. & m'_1,n'_1 \\ . & . & .. & . \\ m_p, n_p & m_{p-1}, n_{p-1} & .. & m_1, n_1 \end{pmatrix}\right\} - F\left\{\cdot \qquad \right\} + F\left\{ \qquad \right\} - \ldots$$

$$+(-1)^{p-1}\ F\left\{x\,;\begin{pmatrix} m_p, n_p & m_{p-1},n_{p-1} & .. & m_1,n_1 \\ m_p, n_p & m_{p-1},n_{p-1} & .. & m_1,n_1 \\ . & . & .. & . \\ m'_p,n'_p & m'_{p-1},n'_{p-1} & .. & m'_1,n'_1 \end{pmatrix}\right\}$$

$$=F(x;m'_p,n'_p)F\left\{x\,;\begin{pmatrix} m_{p-1},n_{p-1} & .. & m_1,n_1 \\ m_{p-1},n_{p-1} & .. & m_1,n_1 \\ . & .. & . \\ m_{p-1},n_{p-1} & .. & m_1,n_1 \end{pmatrix}\right\} - F(x;m'_{p-1},n'_{p-1})F\left\{x\,;\begin{pmatrix} m_p,n_p & m_{p-1},n_{p-1} & .. & m_1,n_1 \\ m_p,n_p & m_{p-1},n_{p-1} & .. & m_1,n_1 \\ . & . & .. & . \\ m_p,n_p & m_{p-1},n_{p-1} & .. & m_1,n_1 \end{pmatrix}\right\}$$

$$+\ldots+(-1)^{p-1}F(x\,;m'_1,n'_1)\ F\ \left\{x\,;\begin{pmatrix} m_p,n_p & .. & m_2,n_2 \\ m_p,n_p & .. & m_2,n_2 \\ . & .. & . \\ m_p,n_p & .. & m_2,n_2 \end{pmatrix}\right\} \quad .. \quad (3\cdot17)$$

or in the more compact notation of (3.14) and (3.15)

$$\sum_{s=1}^{p-1} (-1)^{p-s}\quad F\{x\,;\ (m^s_{ij},n^s_{ij})\}$$

$$=\sum_{j=p}^{1} (-1)^{p-j}F(x\,;\ m'_j,n'_j)\ F\left\{x\,;\begin{pmatrix} m_{op},n_{op} & .. & m_{o,j+1},n_{o,j+1} & m_{o,j-1},n_{o,j-1} & .. & m_{o1},n_{o1} \\ m_{op},n_{op} & .. & m_{o,j+1},n_{o,j+1} & m_{o,j-1},n_{o,j-1} & .. & m_{o1},n_{o1} \\ . & .. & . & . & .. & . \\ m_{op},n_{op} & .. & m_{o,j+1},n_{o,j+1} & m_{o,j-1},n_{o,j-1} & .. & m_{o1},n_{o1} \end{pmatrix}\right\} \quad (3.18)$$

where m'_{ij} and n'_{ij} are given by (3.15) and $m_{ij} = m_{oj}, n_{ij} = n_{oj}$ $(j = p, p-1, \ldots, 1)$; we could if we like drop the 'o' of the double suffix throughout; this will be done in some of the investigations to follow, specially in cases where no confusion is likely to be caused thereby.

This summation $\sum\limits_{j=1}^{p}$ carries obvious limitations at the extremities. We should have likewise

$$\sum_{s=0}^{p-1} (-1)^{p-s} F\{(m'_{ij}, n'_{ij})\,;\, x\} = \sum_{j=0}^{p-1} (-1)^{p-j}\, F(m'_j, n'_j;\, x)\; F\{Co(m_{pj}, n_{pj});\, x\} \quad .. \quad (3.19)$$

The mechanism by which (3.17), that is, (3.18) or (3.19) is obtained may be illustrated by considering in particular the case of $p = 3$ and working it out in detail. Here we have

$$F\left\{x\,;\begin{pmatrix} m'_3,n'_3 & m'_2,n'_2 & m'_1,n'_1 \\ m_3,n_3 & m_2,n_2 & m_1,n_1 \\ m_3,n_3 & m_2,n_2 & m_1,n_1 \end{pmatrix}\right\} - F\left\{x\,;\begin{pmatrix} m_3,n_3 & m_2,n_2 & m_1,n_1 \\ m'_3,n'_3 & m'_2,n'_2 & m'_1,n'_1 \\ m_3,n_3 & m_2,n_2 & m_1,n_1 \end{pmatrix}\right\}$$

$$+F\left\{x\,;\begin{pmatrix} m_3,n_3 & m_2,n_2 & m_1,n_1 \\ m_3,n_3 & m_2,n_2 & m_1,n_1 \\ m'_3,n'_3 & m'_2,n'_2 & m'_1,n'_1 \end{pmatrix}\right\}$$

$$= \int_0^x \frac{x_3^{m'_3}dx_3}{(1+x_3)^{n'_3}}\left(\int_0^{x_3}\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}} - \int_0^{x_3}\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_0^{x_2}\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}\right)$$

$$+\left(\int_0^x\frac{x_3^{m_2}\,dx_3}{(1+x_3)^{n_2}}\int_0^{x_3}\frac{x_2^{m'_2}\,dx_2}{(1+x_2)^{n'_2}}\int_0^{x_2}\frac{x_1^{m_1}\,dx_1}{(1+x_1)^{n_1}} - \int_0^x\frac{x_3^{m_1}\,dx_3}{(1+x_3)^{n_1}}\int_0^{x_3}\frac{x_2^{m'_2}\,dx_2}{(1+x_2)^{n'_2}}\int_0^{x_2}\frac{x_1^{m_2}\,dx_1}{(1+x_1)^{n_2}}\right)$$

$$+\left(\int_0^x\frac{x_3^{m_2}\,dx_3}{(1+x_3)^{n_2}}\int_0^{x_3}\frac{x_2^{m_1}\,dx_2}{(1+x_2)^{n_1}}\int_0^{x_2}\frac{x_1^{m'_1}\,dx_1}{(1+x_1)^{n'_1}} - \int_0^x\frac{x_3^{m_1}\,dx_3}{(1+x_3)^{n_1}}\int_0^{x_3}\frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n_2}}\int_0^{x_2}\frac{x_1^{m'_1}\,dx_1}{(1+x_1)^{n'_1}}\right)$$

$$+ \;(\;)\; + \;(\;)\; + \;(\;) \;\ldots\ldots\ldots\ldots$$

$$= F(x\,;m'_3,n'_3)\;F\left\{x\,;\begin{pmatrix} m_2,n_2 & m_1,n_1 \\ m_2,n_2 & m_1,n_1\end{pmatrix}\right\} - F(x\,;m'_2,n'_2)F\left\{x\,;\begin{pmatrix} m_3,n_3 & m_1,n_1 \\ m_3,n_3 & m_1,n_1\end{pmatrix}\right\}$$

$$+F(x; m'_1,n'_1)\;F\left\{x\,;\begin{pmatrix} m_3,n_3 & m_2,n_2 \\ m_3,n_3 & m_2,n_2\end{pmatrix}\right\} \quad .. \quad (3.2)$$

We can at this stage fruitfully take up the main distribution problem.

$$F\left\{x\,;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_1,n \\ m_p,n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\} = \Sigma \pm F(x\,;m'_p,n\,;m'_{p-1},n\,;\ldots m'_1,n)$$

where $(m'_p, m'_{p-1}, \ldots, m'_1)$ is any permutation of the symbols $(m_p, m_{p-1}, \ldots, m_1)$ and the sign + or − is to be taken according as it is an even permutation or an odd permutation. The number of terms in the summation is, of course, $p\,!$. Suppose that $m_p > m_{p-1} > \ldots > m_1$ and suppose further we try to decrease m_p in successive stages till we get to m_{p-1}. Suppose that in any term $F(x\,;m'_p, n\,;m'_{p-1}, n;\ldots m'_1, n)$, m'_s is the largest of $(m'_p, m'_{p-1}, \ldots, m'_1)$, that is, $m'_s = m_p$. Then using the fundamental relations (2.81) and (2.82) it is easily seen that

$$F(x;m'_p,n;m'_{p-1},n;\ldots m'_1,n) = (1/n-m_p-1)\;F(x;m'_p,n;\ldots m'_{s+1}+m_p, 2n-1;m'_{s-1},n;\ldots m'_1,n)$$

$$+(1/n-m_p-1)\;F(x\,;m'_p,n\,;\ldots m'_{s+1},n\,;m'_{s-1}+m_p,\,2n-1\,;\,\ldots\,m'_1,n)$$

$$+(m_p/n-m_p-1)\;F(x\,;m'_p,n\,;m'_{p-1},n\,;\,\ldots\,m'_{s+1},n;\;m_p-1,n\,;\ldots m'_1,n) \quad .. \quad (3.21)$$

he above relations will hold for $s = p-1, p-2, \ldots, 2$

If $s = p$ we shall have instead

$$F(x;\, m_p,n;\, m'_{p-1},n;\, \ldots\, m'_1,n) = -(1/n-m_p-1)\; F_0(x;\, m_p,n-1)\; F(x;\, m'_{p-1},n;\, m'_{p-2},n;\, \ldots\, m'_1,n)$$
$$+(1/n-m_p-1)\; F(x;\, m_p+m_{p-1},\, 2n-1\,;\, m'_{p-2},n;\, \ldots m'_1,n)$$
$$+(m_p/n-m_p-1)\;\; F(x;\, m_p-1,\, n;\, m'_{p-1},n;\, \ldots\, m'_1,n) \qquad \ldots \quad (3.22)$$

and finally if $s = 1$ we shall have

$$F(x;\, m'_p,n;\, m'_{p-1},n;\, \ldots\, m'_2,n;\, m_1,n) = -(n/n-m_p-1)\; F(x\,;\, m'_p,n;\, m'_{p-1},n;\, \ldots m'_2+m_p,\, 2n-1)$$
$$+(m_p/n-m_p-1)\; F(x\,;\, m'_p,n\,;\, m'_{p-1},n\,;\, \ldots\, m'_2,n\,;\, m_p-1,\, n) \qquad \ldots \quad (3.23)$$

The left hand sides of (3.21), (3.22) and (3.23) are all p-fold incomplete integrals, but the first two terms on the right hand side of (3.21), the second term and the second factor of the first term on the right hand side of (3.22) and the first term on the right hand side of (3.23) are all $(p-1)$-fold incomplete integrals. The last terms on the right hand side of (3.21) and (3.23) are all p-fold incomplete integrals again but with one parameter m_p decreased (by one) to m_p-1. As a more convenient notation we can replace

(i) $F(x\,;\, m'_p,n\,;\, \ldots\, m'_{s+1}+m'_p,\, 2n-1\,;\, m'_{s-1},n\,;\, \ldots\, m'_1,n)$ by

$F(x\,;\, m'_p,n\,;\, \ldots\, m'_{s+1},n\,;\, \overleftarrow{m_p},\overleftarrow{n-1}\,;\, m'_{s-1},n\,;\, \ldots\, m'_1,n)$

where $\overleftarrow{m_p}$ is always added to the 'm' on the left and $\overleftarrow{n-1}$ to 'n' on the left.

(ii) $F_0(x\,;\, m_p,n-1)\; F(x\,;\, m'_{p-1},n\,;\, m'_{p-2},n;\, \ldots\, m'_1,n)$ by

$F(x\,;\, \overleftarrow{m_p},\overleftarrow{n-1};\, m'_{p-1},n;\, \ldots m'_1,n)$ $\qquad \ldots \quad (3.24)$

(iii) $F(x\,;\, m'_p,n\,;\ldots m'_{s+1},n\,;\, m_p+m_{s-1},\, 2n-1\,;\, m_{s-2},n\,;\, \ldots\, m'_1,n)$ by

$F(x\,;\, m'_p,n\,;\, \ldots\, m'_{s+1},n\,;\, \overrightarrow{m_p},\overrightarrow{n-1}\,;\, m'_{s-1},n\,;\, \ldots\, m'_1,n)$

where $\overrightarrow{m_p}$ is always added to the 'm' on the right and $\overrightarrow{n-1}$ to the 'n' on the right

Using the results (2.81)—(3.23) and the notation (3.24) we find that

$$F\left\{x\,;\begin{pmatrix} m_p,n & m_{p-1},n & \ldots & m_1,n \\ m_p,n & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ m_p,n & m_{p-1},n & \ldots & m_1,n \end{pmatrix}\right\}$$

$$= -(1/n-m_p-1)\; F\left\{x;\begin{pmatrix} \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & \ldots & m_1,n \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & \ldots & m_1,n \end{pmatrix}\right\}$$

$$+(1/n-m_p-1)\; F\left\{x\,;\begin{pmatrix} \overrightarrow{m_p},\overrightarrow{n-1} & m_{p-1},n & \ldots & m_1,n \\ \overrightarrow{m_p},\overrightarrow{n-1} & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ \Box & m_{p-1},n & \ldots & m_1,n \end{pmatrix}\right\}$$

$$+(m_p/n-m_p-1)F\left\{x\,;\begin{pmatrix} m_p-1,n & m_{p-1},n & \ldots & m_1,n \\ m_p-1,n & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ m_p-1,n & m_{p-1},n & \ldots & m_1,n \end{pmatrix}\right\} \qquad \ldots \quad (3.25)$$

where in the second term on the right hand sides of (3.25), $\Box$ in the pseudo-determinant merely means that the corresponding terms in the formal expansion are not to be considered at all ; $\Box$ is introduced merely to write the pseudo-determinant in a complete form.

If we use now the relations (3.17), (3.18), (3.21)—(3.23) and the notation (3.24), the right hand side of (3.25) would simplify thus :

$$F\left\{x;\begin{pmatrix} \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\}$$

$$=F_0(x\ ;m,n)\quad F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-1},n & .. & m_1,n \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \\ . & & .. & . \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\}$$

$$-F\left\{x,\begin{pmatrix} m_{p-1}+m_p,2n-1 & m_{p-1}+m_p,2n-1 & .. & m_1+m_p,2n-1 \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_{p-1},n & m_{p-1},n & & m_1,n \end{pmatrix}\right\}$$

$$+\ (\ )\ -\ (\ )\ ..$$

$$+F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-1},n & .. & m_1,n \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \\ & . & .. & . \\ m_{p-1}+m_p,2n-1 & m_{p-1}+m_p,2n-1 & .. & m_1+m_p,2n-1 \end{pmatrix}\right\}$$

$$=\quad F_0(x;\ m_p,n)\quad F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-1},n & .. & m_1,n \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_{p-1},n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\}$$

$$-F(x\ ;m_{p-1}+m_p,2n-1)\ F\left\{x\ ;\begin{pmatrix} m_{p-2},n & m_{p-3},n & .. & m_1,n \\ m_{p-2},n & m_{p-3},n & .. & m_1,n \\ . & . & .. & . \end{pmatrix}\right\}$$

$$+F(x\ ;m_{p-2}+m_p,\ 2n-1)\ F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-3},n & .. & m_1,n \\ m_{p-1},n & m_{p-3},n & .. & m_1,n \\ . & . & .. & . \end{pmatrix}\right\}$$

$$-\ (\ )\ +\ (\ )\ ..$$

$$\pm F_0(x\ ;m_1+m_p,\ 2n-1)F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-2},n & .. & m_2,n \\ m_{p-1},n & m_{p-2},n & .. & m_2,n \\ . & . & .. & . \end{pmatrix}\right\}$$

$$=F_0(x\ ;m_p,n)\ F\left\{x\ ;\begin{pmatrix} m_{p-1},n & m_{p-2},n & .. & m_1,n \\ m_{p-1},n & m_{p-2},n & .. & m_1,n \\ . & . & .. & . \end{pmatrix}\right\}$$

$$+\sum_{s=p-1}^{1}(-1)^{p-s-1}\ F(x;\ m_s+m_p,\ 2n-1)\ F\left\{x;\begin{pmatrix} m_{p-1},n & .. & m_{s+1},n & m_{s-1},n & .. & m_1,n \\ m_{p-1},n & .. & m_{s+1},n & m_{s-1},n & .. & m_1,n \\ . & .. & . & . & .. & . \end{pmatrix}\right\}\ ..\quad (3.26)$$

Likewise with obvious limitations at the extremities, which the $\sum_{s=p-1}$ summations will always be supposed to carry

$$F\left\{x\,;\begin{pmatrix} \vec{m}_p,\vec{n}-1 & m_{p-1},n & \cdot\cdot & m_1,n \\ \vec{m}_p,\vec{n}-1 & m_{p-1},n & \cdot\cdot & m_1,n \\ & & \cdot\cdot & \cdot \\ \square & m_{p-1},n & \cdot\cdot & m_1,n \end{pmatrix}\right\}$$

$$= F\left\{x\,;\begin{pmatrix} m_{p-1}+m_p,2n-1 & m_{p-2}+m_p,2n-1 & \cdot\cdot & m_1+m_p,2n-1 \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \cdot \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \end{pmatrix}\right\}$$

$$-F\left\{x;\begin{pmatrix} m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ m_{p-1}+m_p,2n-1 & m_{p-2}+m_p,2n-1 & \cdot\cdot & m_1+m_p,2n-1 \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \end{pmatrix}\right\}$$

$$+\ (\ )\ -\ (\ )\ \ldots\ldots$$

$$\pm F\left\{x;\begin{pmatrix} m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \cdot \\ m_{p-1}+m_p,2n-1 & m_{p-2}+m_p,2n-1 & \cdot\cdot & m_1+m_p,2n-1 \end{pmatrix}\right\}$$

(the last term being + or − according as p is odd or even)

$$= \sum_{s=p-1}^{1}(-1)^{p-s-1}\,F(x;\ m_s+m_p,2n-1)F\left\{x\,;\begin{pmatrix} m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \\ m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \end{pmatrix}\right\} \quad (3\cdot27)$$

Combining (3.26) and (3.27) we have now (3.26) reducing to

$$F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & \cdot\cdot & m_1,n \\ m_p,n & m_{p-1},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \cdot \\ m_p,n & m_{p-1},n & \cdot\cdot & m_1,n \end{pmatrix}\right\}$$

$$= -(1/n-m_p-1)\quad F_0(x\,;\ m_p,n-1)F\left\{x;\begin{pmatrix} m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \cdot \\ m_{p-1},n & m_{p-2},n & \cdot\cdot & m_1,n \end{pmatrix}\right\}$$

$$+(2/n-m_p-1)\sum_{s=p-1}^{1}(-1)^{p-s-1}F(x;\ m_s+m_p,2n-1)F\left\{x;\begin{pmatrix} m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \\ m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ m_{p-1},n & \cdot\cdot & m_{s+1},n & m_{s-1},n & \cdot\cdot & m_1,n \end{pmatrix}\right\}$$

$$+(m_p/n-m_p-1)\,F\left\{x;\begin{pmatrix} m_p-1,n & m_{p-1},n & \cdot\cdot & m_1,n \\ \cdot & \cdot & \cdot\cdot & \cdot \\ m_p-1,n & m_{p-1},n & \cdot\cdot & m_1,n \end{pmatrix}\right\} \qquad \cdot\cdot \quad (3\cdot28)$$

The left hand side is a p-th order pseudo-determinant. On the right hand side the first term involves a $(p-1)$th order pseudo-determinant, the second group of terms each involves a $(p-2)$-th order pseudo-determinant, the third term is a pseudo-determinant of the p-th order no doubt, but the parameter m_p is replaced by m_p-1. (3.28) provides us with a recursion chain by which m_p is reduced in successive stages to m_{p-1} and as soon as this happens the whole pseudo-determinant becomes zero as we have already observed.

Proceeding along the recursion chain suggested by (3.28) we have

$$F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_1,n \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\}$$

$$=-F\left\{x;\begin{pmatrix} m_{p-1},n & m_{p-2},n & .. & m_1,n \\ . & . & .. & . \\ m_{p-1},n & m_{p-2},n & .. & m_1,n \end{pmatrix}\right\}\sum_{r=1}^{m_p-m_{p-1}} ({}_{m}P_{r-1}/{}_{n-m_p-2+r}P_r)\ F_0(x;m_p-r+1,n-1)$$

$$+\sum_{s=p-1}^{1}(-1)^{p-s-1}F\left\{x;\begin{pmatrix} m_{p-1},n & .. & m_{s+1},n & m_{s-1},n & .. & m_1,n \\ . & .. & . & . & .. & . \\ m_{p-1},n & .. & m_{s+1},n & m_{s-1},n & .. & m_1,n \end{pmatrix}\right\}\sum_{r=1}^{m_p-m_{p-1}} 2Q_s$$

where $\qquad Q_s=({}_{m_p}P_{r-1}/{}_{n-m_p-2+r}P_r)\ F(x\,;m_s+m_p-r+1,2n-1) \qquad .. \quad (3\cdot29)$

or in a different notation already considered earlier

$$F(x\,;(m_p,n\,;\,m_{p-1},n\,;\,..\,m_1,n))$$

$$=-F(x\,;(m_{p-1},n\,;\,m_{p-2},n;\,..\,m_1,n))\sum_{r=1}^{m_p-m_{p-1}} {}_{m_p}P_{r-1}/{}_{n-m_p-2+r}P_r\ F(x\,;m_s+m_p-r+1,2n-1)$$

$$+\sum_{s=p-1}^{1}(-1)^{p-s-1}F(x;\,(m_{p-1},n;\,..\,m_{s+1},n;\,m_{s-1},n;\,..\,m_1,n))\sum_{r=1}^{m_p-m_{p-1}} 2Q_s \qquad .. \quad (3\cdot3)$$

The p-fold pseudo-determinant is thus thrown back on $(p-1)$ and $(p-2)$-fold ones, and these again on $(p-2)$ and $(p-3)$ and so on till we get to 1-fold cases, that is, functions of the type $F(x\,;\,m,n)$ which are easily calculated from the incomplete B—function tables. Now (3.3) is the reduction formula if one attempts to reduce m_p to m_{p-1} ; on the other hand if one tried to raise m_1 to m_2 then using again the same body of theorems one would have had

$$F\left\{x;\begin{pmatrix} m_p,n\,, & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\}$$

$$=\ (1/m_1+1)\ F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_2,n & \overleftarrow{m_1+1},\overleftarrow{n-1} \\ m_p,n & m_{p-1},n & .. & m_2,n & \overleftarrow{m_1+1},\overleftarrow{n-1} \\ . & . & .. & . & \end{pmatrix}\right\}$$

$$+(1/m_1+1)\ F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_2,n & \overrightarrow{m_1+1},\overrightarrow{n-1} \\ . & . & .. & .. & \overrightarrow{\cdot}\ \ \overrightarrow{\cdot} \\ m_p,n & m_{p-1},n & .. & m_2,n & m_1+1,n-1 \\ m_p,n & m_{p-1},n & .. & m_2,n & \square \end{pmatrix}\right\}$$

$$+(n-m_1-2/m_1+1)F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_2,n & m_1+1,n \\ m_p,n & m_{p-1},n & .. & m_2,n & m_1+1,n \\ . & . & .. & . & . \end{pmatrix}\right\}$$

$$=\ (-1)^{p-1}\,(1/m_1+1)\ F_0(x;\,m_1+1,n-1)\ F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_2,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_2,n \end{pmatrix}\right\}$$

$$+(-1)^p\,(2/m_1+1)\sum_{s=p}^{2}(-1)^{s-s}\,F(x;\,m_s+m_1+1,\ 2n-1)\ F\left\{x;\begin{pmatrix} m_p,n & .. & m_{s+1},n & m_{s-1},n & .. & m_2,n \\ . & .. & . & . & .. & . \\ m_p,n & .. & m_{s+1},n & m_{s-1},n & .. & m_2,n \end{pmatrix}\right\}$$

$$+(n-m_1-2/m_1+1)\ F\left\{x;\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_2,n & m_1+1,n \\ . & . & .. & . & . \\ m_p,n & m_{p-1},n & .. & m_2,n & m_1+1,n \end{pmatrix}\right\} \qquad .. \quad (3\cdot31)$$

$$=(-1)^{p-1} F\left\{x; \begin{pmatrix} m_p,n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix}\right\} \sum_{r=1}^{m_2-m_1} ({}^{n-m_1-r}P_{r-1}/{}^{m_1+r}P_r)\ F_0(x;\ m_1+r, n-1)$$

$$+(-1)^p \sum_{s=p}^{2} (1)^{p-s} F\left\{x; \begin{pmatrix} m_p,n & m_{s+1},n & m_{s-1},n & .. & m_2,n \\ . & . & . & .. & . \\ m_p,n & m_{s+1},n & m_{s-1},n & .. & m_2,n \end{pmatrix}\right\} \sum_{r=1}^{m_2-m_1} 2Q_s \qquad .. \quad (3.32)$$

where $\qquad Q_s=({}^{n-m_1-r}P_{r-1}/{}^{m_1+r}P_r)\ F(x\ ;\ m_1+r+m_s,\ 2n-1)$

and where, as in (3.29), the left hand side is a p-th order pseudo-determinant while on the right hand side the first term involves a $(p-1)$-th and the second involves up to $(p-2)$-th order pseudo-determinants. In a different notation already used (3.32) can be replaced by

$$F\{x\ ;\ (m_p,n\ ;\ m_{p-1},n\ ;\ ..\ m_1,n)\}$$

$$=(-1)^{p-1} F\{x\ ;\ (m_p,n\ ;..m_2,n)\} \quad \times \sum_{r=1}^{m_2-m_1} ({}^{n-m_1-r}P_r/{}^{m_1+r}P_r)\ F_0(x\ ;\ (m_1+r,\ n-1)$$

$$+(-1)^p \sum_{s=p}^{2} (-1)^{p-s} F\{x\ ;\ (m_p,n\ ;\ ..\ m_{s+1},n\ ;\ m_{s-1},n\ ;\ ..\ m_1,n)\} \quad \sum_{r=1}^{m_2-m_1} 2Q_s$$

where $\qquad Q_s=({}^{n-m_1-r}P_{r-1}/{}^{m_1+r}P_{r-1})\ F\{x\ ;\ (m_s+r+m_1,\ 2n-1)\} \qquad .. \quad (3{\cdot}33)$

In the actual distribution problem of maximum statistic for p variates we have

$$m_j=m+j+1 \quad (j=p,\ p-1,\ \ldots,\ 1)$$

Hence (3.3) is to be replaced by

$$F\{x;\ (m+p-1,\ n\ ;\ m+p-2,\ n\ ;\ ..\ m,n)\}$$

$$=-F\{x\ ;\ (m+p-2,\ n;\ m+p-3,\ ;\ ..\ m,n)\} \quad F_0(x\ ;\ m+p-1,\ n-1)/n-m-p$$

$$+(2/n-m-1) \sum_{s=p-2}^{1} (-1)\ F\{x\ ;\ (m+p-2,n\ ..\ m+s,n\ ;\ m+s-2,\ n;\ ..\ m,n)\}\ Q_s$$

where $\qquad Q_s=F(x\ ;\ 2m+p+s-2, 2n-1) \qquad ..\quad 3.34)$

and (3·33) is replaced by

$$F\{x\ ;\ (m+p-1,\ n\ ;\ m+p-2,\ n\ :\ ..\ m,n)\}$$

$$=\ (-1)^{p-1} F\{x\ ;\ (m+p-1,\ n\ ;\ m+p-2,\ n\ ;\ ..\ m+1,n)\ (1/m+1)\ F_0(x\ ;\ m+1,\ n-1)$$

$$+(-1)^p \sum_{s=p}^{2} (-1)^{p-s}\ F\{x\ ;\ (m+p-1,\ n\ ;\ ..\ m+s,\ n\ ;\ m+s-2,\ n\ ;\ ..\ m+1,\ n)\}\ Q_s$$

where $\qquad Q_s=(2,m+1)\ \ F(x\ ;\ 2m+s,\ 2n-1) \qquad ..\quad (3.35)$

This is so far as the distribution of the maximum of p-statistics is concerned. We shall now turn to the distribution of the minimum statistic.

Remembering (2.84) and (2.85) suppose we try to expand

$$F\{(m_p,n\ ;\ m_{p-1},n\ ;\ ..\ m_1,n)\ ;\ x\}=F\left\{\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix};x\right\}$$

with the proviso that $m_p>m_{p-1}> \ldots >m_1$

As indicated earlier we can write this as $\Sigma\ F(m'_p,\ n\ ;\ m'_{p-1},\ n\ ;\ \ldots.\ m'_1,\ n\ ;\ x)$ where $(m'_p, m'_{p-1}, \ldots, m'_1)$ is any permutation of $(m_p,\ m_{p-1},\ \ldots,\ m_1)$ and the summation is over all such permutations.

Consider the typical term $F(m'_p, n\,;\, m'_{p-1}, n\,;\, \ldots m'_s, n\,;\, m'_{s-1}, n\,;\, \ldots\,;\, m'_1, n\,;\, x)$ and suppose that $m'_s = m_p$. Suppose further that we try to reduce m_p in successive stages to m_{p-1}. Then using (2.84) we have

$$F(m'_p,n\,;\, ..\, m'_{s+1},n\,;\, m_p,n\,;\, m'_{s-1},n\,;\, ..\, m'_1,n\,;\, x)$$

$$\begin{aligned}
&=(1/n-m_p-1)\ F(m'_p,n\,;\, ..\, m'_{s+1},n\,;\, m_p+m'_{s-1}, 2n-1\,;\, m'_{s-2}, n\,;\, ..\, m'_1,n\,;\, x) \\
&-(1/n-m_p-1)\ F(m'_p,n\ ..\ m'_{s+1}+m_p, 2n-1\,;\, m'_{s-1},n\,;\, ..\, m'_1,n\,;\, x) \\
&+(m_p/n-m_p-1)\ F(m'_p,n\,;\, ..\, m'_{s+1},n\,;\, m_p-1, n\,;\, m'_{s-1}, n\,;\, ..\, m'_1, n;\, x) \qquad .. \quad (3.36)
\end{aligned}$$

When $s = p$, the middle term would not be there, and when $s=1$, the first term is to be replaced by $F_0(m_p, n-1\,;\, x)\ F(m'_p, n\,;\, m'_{p-1}, n\,;\, \ldots m'_2, n\,;\, x)/(n-m_p-1)$. As earlier introduce now the notation

$$F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, \overrightarrow{m_p},\overrightarrow{n-1}\,;\, m'_{s-1},n\,;\ ..\ m'_1,n\,;\, x)$$

and $F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, \overleftarrow{m_p},\overleftarrow{n-1}\,;\, m'_{s-1},n\,;\ ..\ m'_1,n\,;\, x)$ for

$F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, m'_{s-1}+m_p, 2n-1\,;\, m'_{s-2},n\,;\ ..\ m'_1,n\,;\, x)$ and

$F(m'_p,n;\ ..\ m'_{s+1}+m_p, 2n-1;\, m'_{s-1},n;\ ..\ m'_1,n;\, x)$ respectively, with the proviso

that $F(\overleftarrow{m_p}, \overleftarrow{n-1}; m'_{p-1},n;\, ..\, m'_1,n';\, x)$ would denote $F_0(m_p,n-1;\, x)\ F(m'_{p-1},n;\, ..\, m'_1,n;\, x)$. Then we can replace (3,36) by

$$\begin{aligned}
&F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, m_p,n\,;\, m'_{s-1}, n;\ ..\ m'_1,n\,;\, x) \\
&=(1/n-m_p-1)\ F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, \overrightarrow{m_p},\overrightarrow{n-1}\,;\, m'_{s-1},n\,;\, ..\, m'_1,n\,;\, x) \\
&-(1/n-m_p-1)\ F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, \overleftarrow{m_p},\overleftarrow{n-1}\,;\, m'_{s-1},n\,;\, ..\, m'_1,n\,;\, x) \\
&+(m_p/n-m_p-1)\ F(m'_p,n\,;\ ..\ m'_{s+1},n\,;\, m_p-1,n\,;\, m'_{s-1},n\ \ ..\ m'_1,n\,;\, x)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&F\left\{\begin{pmatrix} m_p,n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p,n & m_{p-1},n & .. & m_1,n \end{pmatrix}; x\right\} \\
&= (1/n-m_p-1)\ F\left\{\begin{pmatrix} \square & m_{p-1},n & .. & m_1,n \\ \overrightarrow{m_p},\overrightarrow{n-1} & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ \overrightarrow{m_p},\overrightarrow{n-1} & m_{p-1},n & .. & m_1,n \end{pmatrix}; x\right\} \\
&-(1/n-m_p-1)\ F\left\{\begin{pmatrix} \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ \overleftarrow{m_p},\overleftarrow{n-1} & m_{p-1},n & .. & m_1,n \end{pmatrix}; x\right\} \\
&+(m_p/n-m_p-1)\ F\left\{\begin{pmatrix} m_p-1,n & m_{p-1},n & .. & m_1,n \\ m_p-1,n & m_{p-1},n & .. & m_1,n \\ . & . & .. & . \\ m_p-1,n & m_{p-1},n & .. & m_1,n \end{pmatrix}; x\right\} \qquad .. \quad (3.37)
\end{aligned}$$

Expanding and proceeding exactly as for the case of the maximum statistic we have

$$F\{(m_p,n\,;\,m_{p-1},n\,;\,.\,.\,m_1,n)\,;\,x\}$$

$$=-(1/n-m_p-1)F_0(m_p,\,n-1\,;\,x)\times F\{(m_{p-1},n\,;\,m_{p-2},\,n\,;\,.\,.\,m_1,n)\,;\,x\}$$

$$+(2/n-m_p-1)\sum_{s=p-1}^{1} F(m_s+m_p,\,2n-1\,;\,x)\,Q_3$$

$$+(m_p/n-m_p-1)\,F\left\{\begin{pmatrix} m_p-1,n & m_{p-1},n & .\,. & m_1,n \\ m_p-1,n & m_{p-1},n & .\,. & m_1,n \\ m_p-1,n & m_{p-1},n & .\,. & m_1,n \end{pmatrix} x\right\} \qquad ..\quad (3.38)$$

where $$Q_3=F\left\{\begin{pmatrix} m_{p-1},n & .\,. & m_{s+1},n & m_{s-1},n & .\,. & m_1,n \\ . & & & . & & . \\ m_{p-1},n & .\,. & m_{s+1},n & m_{s-1},n & .\,. & m_1,n \end{pmatrix};x\right\}$$

We have thus by successive stages as before

$$F\{(m_p,n\,;\,.\,.\,m_1,n)\,;\,x\}$$

$$=-F\{(m_{p-1},n\,;\,m_{p-2},n\,;\,.\,.\,m_1,n)\,;\,x\}\ \times\sum_{r=1}^{m_p-m_{p-1}} ({}_{m_p}P_{r-1}/{}_{n-m_p-1}P_r)\,F_0(m_p-r+1,\,n-1\,;\,x)$$

$$+2\sum_{s=p-1}^{1}(-1)^{p-s-1}\,F\{(m_{p-1},n\,;\,.\,.\,m_{s+1},n\,;\,m_{s-1},n\,;\,.\,.\,m_1,n)\,;\,x\}\ \sum_{r=1}^{m_p-m_{p-1}} Q_4$$

where $$Q_4=({}_{m_p}P_{r-1}/{}_{n-m_p-1}P_r)\ F(m_s+m_p-r+1,\,2n-1\,;\,x) \qquad ..\quad (3.39)$$

This is if we reduce m_p by successive stages to m_{p-1} ; on the other hand if we try to increase m_1 by successive stages to m_2 then proceeding exactly as for the case of the maximum statistic we have, by using (2.85),

$$F\{(m_p,n\,;\,.\,.\,m_1,n)\,;\,x\}=-(1/m_1+1)\ F\left\{\begin{pmatrix} m_p,n & m_{p-1},n & .\,. & m_2,n & \square \\ m_p,n & m_{p-1},n & .\,. & m_2,n & \overrightarrow{m_1+1},\overrightarrow{n-1} \\ . & . & .\,. & . & \\ m_p,n & m_{p-1},n & .\,. & m_2,n & \overrightarrow{m_1+1},\overrightarrow{n-1} \end{pmatrix}\right\}$$

$$+(1/m_1+1)\ F\left\{\begin{pmatrix} m_p,n & m_{p-1},n & .\,. & m_2,n & \overleftarrow{m_1+1}, & \overleftarrow{n-1} \\ m_p,n & m_{p-1},n & .\,. & m_2,n & \overleftarrow{m_1+1}, & \overleftarrow{n-1} \\ . & . & .\,. & . & & \\ m_p,n & m_{p-1},n & .\,. & m_2,n & \overleftarrow{m_1+1}, & \overleftarrow{n-1} \end{pmatrix};x\right\}$$

$$+(n-m_1-2/m_1+1)\ F\left\{\begin{pmatrix} m_p,n & .\,. & m_2,n & m_1+1,n \\ . & .\,. & . & . \\ m_p,n & .\,. & m_2,n & m_1+1,n \end{pmatrix};x\right\} \qquad ..\quad (3.40)$$

whence we have

$$F\{(m_p,n\ .\,.\ m_1,n);\,x\}$$

$$=\ (-1)^{p-1}\ F\{m_p,n;\,m_{p-1},n;\ .\,.\ m_2,n);\,x\}\ \times\sum_{r=1}^{m_2-m_1} ({}_{n-m_1-2}P_r/{}_{m_1+r}P_r)\ F_0(m_p+r,n-1)$$

$$-\sum_{s=p}^{2}(-1)^{p-s}\ F\{m_p,n;\ .\,.\ m_{s+1},n;\,m_{s-1},n;\ .\,.\ m_2,n);\ x\}\ \sum_{r=1}^{m_2-m_1} 2Q_5$$

where $$Q_5=({}_{n-m_1-2}P_{r-1}/{}_{m_1+r}P_r)\ F(m_1+r+m_s,\,2n-1;\,x) \qquad ..\quad (3.41)$$

In the actual distribution problem we have $m_j = m + j - 1$ $(j = p, \ldots, 1)$ and hence (3.39) and (3.41) reduce respectively to

$$F\{(m+p-1, n\ ;\ .\ .\ m,n)\ ;\ x\}$$
$$= -F\{(m+p-2, n\ ;\ .\ .m,n);\ x\} \times (1/n-m-p)\ F_0(m+p-1,\ n-1\ ;\ x)$$
$$+2 \sum_{s=1}^{p} (-1)^{p-s-1}\ F\{(m+p-2,\ n\ ;\ .\ .\ m+s,n\ ;\ m+s-2,\ n\ ;\ .\ .\ m,n)\ ;\ x\}Q_s$$

where $$Q_s = (1/n-m-p)F(2m+p+s-2,\ 2n-1\ ;\ x) \qquad \ldots\ (3.42)$$

and

$$F\{(m+p-1,\ n\ ;\ .\ .\ m,n);\ x\}$$
$$= \ (-1)^{p-1}\ F\{(m+p-1,\ n\ ;\ .\ .\ m+1,n)\ ;\ x\}\ \ (1/m+1)\ \ F_0(m+1,\ n-1\ ;\ x)$$
$$+(-1)^p \sum_{s=p}^{2} (-1)^{p-s}\ F\{(m+p-1,\ n;\ .\ .\ m+s,\ n\ ;\ m+s-2,\ n\ ;\ .\ .\ m+1,n)\ ;\ x\}2Q_s$$

where $$Q_s = (1/m+1)F(2m+s,\ 2n-1\ ;\ x) \qquad \ldots\ (3.43)$$

We shall now apply (3.34) for the maximum statistic and (3.42) for the minimum statistic to the particular case of three or four variates, that is, for $p = 3$ and $p = 4$.

For $p = 3$, (3.34) becomes $\quad F\{x\ ;\ (m+2,\ n\ ;\ m+1,\ n\ ;\ m,n)\}$

$$\begin{aligned}
&= -F\{x\ ;(m+1,\ n\ ;\ m,n)\}\ \ (1/n-m-3)\ \ F_0(x\ ;\ m+2,\ n-1) \\
&\quad +(2/n-m-3)F(x\ ;\ 2m+3,\ 2n-1)F(x\ ;\ m,n) \\
&\quad -(2/n-m-3)\ \ F(x\ ;\ 2m+2,\ 2n-1)F(x,\ m+1,\ n) \\
&= \ \ (1/n-m-3)[2F(x\ ;\ 2m+3,\ 2n-1)F(x\ ;\ m,n)-2F(x\ ;\ 2m+2,\ 2n-1)F(x\ ;\ m+1,n) \\
&\quad -(1/n-m-2)\ F_0(x\ ;\ m+2,\ n-1)(-F_0(x\ ;\ m+1,\ n-1)\ F(x\ ;\ m,n) \\
&\quad +2F(x\ ;\ 2m+1,\ 2n-1)]
\end{aligned} \qquad \ldots\ (3.44)$$

and (3.42) becomes $\quad F\{(m+2,\ n\ ;\ m+1,\ n\ ;\ m,n)\ ;\ x\}$

$$\begin{aligned}
&= -(1/n-m-3)\ F_0(m+2,\ n-1\ ;\ x)\ \ F\{(m+1,\ n\ ;\ m,n)\ ;\ x\} \\
&\quad +(2/n-m-3)\ [F(2m+3,\ 2n-1\ ;\ x)\ F(m,n\ ;\ x) \\
&\quad -F(2m+2,\ 2n-1\ ;\ x)F(m+1,\ n\ ;\ x)] \\
&= -(1/n-m-3)[(1/n-m-2)\ \ F_0(m+2,\ n-1\ ;\ x)(-F_0(m+1,\ n-1\ ;\ x) \\
&\qquad\qquad \times F(m+1,\ n\ ;\ x)+2F(2m+2,\ 2n-1\ ;\ x)) \\
&\quad +F(2m+3,\ 2n-1\ ;\ x)F(m,n\ ;\ x)-F(2m+2,\ 2n-1\ ;\ x)F(m+1,\ n\ ;\ x)]
\end{aligned} \qquad \ldots\ (3.45)$$

For $p = 4$, (3.34) becomes $\quad F\{x\ ;\ (m+3,\ n\ ;\ m+2,\ n\ ;\ m+1,\ n\ ;\ m,n)\}$

$$\begin{aligned}
&= \ \ (1/n-m-4)[-F_0(x\ ;\ m+3,\ n-1 \times (3.44) \\
&\quad +(2/n-n-3)\ F(x\ ;\ 2m+5,\ 2n-1;\ x)\ (-F_0(x\ ;\ m+1,\ n-1)F(x;\ m,n) \\
&\quad +2F(x\ ;\ 2m+1,\ 2n-1)) \\
&\quad +(2/n-m-3)F(x\ ;\ 2m+3,\ 2n-1)(-F_0(x\ ;\ m+2,\ n-1)F(x\ ;\ m+1,\ n) \\
&\quad +2F(x\ ;\ 2m+3, 2n-1)) \\
&\quad -(2/n-m-3)F(x\ ;\ 2m+4,\ 2n-1)(-F_0(x\ ;\ m+2,\ 2n-1)\ F(x\ ;\ m,n) \\
&\quad -(m+2/n-m-2)\ F_0(x\ ;\ m+1,\ n-1)F(\ x;\ m,n) \\
&\quad +2F(x\ ;\ 2m+3,\ 2n-1)+(2m+4/n-m-2)F(x\ ;\ \ 2m+2,\ n-1)
\end{aligned} \qquad \ldots\ (3.46)$$

For $p = 4$, (3.42) becomes $F\{(m+3, n\,;\, m+2, n\,;\, m+1, n\,;\, m,n)\,;\, x\}$

$$
\begin{aligned}
&= -(1/n-m-4)[F_o(m+3, n-1\,;\, x) \times (3.45) \\
&\quad +(2/n-m-3)F(2m+5, n-1\,;\, x)(-F_o(m+1, n-1\,;\, x)\ F(m,n\,;\, x) \\
&\quad +2F(2m+1, 2n-1\,;\, x))\ (+(2\ n-m-3)F(2m+3, 2n-1\,;\, x) \\
&\quad \times(-F_o(m+2, n-1\,;\, x)\ F(m+1, n\,;\, x)+2F(2m+3, 2n-1\,;\, x)) \\
&\quad -(2/n-m-3)F(2m+4\,;\, 2n-1\,;\, x)(-F_o(m+2, n-1\,;\, x)F(m,n\,;\, x) \\
&\quad +(m+2/n-m-2)F_o(m+1, n-1\,;\, x)\ F(m,n\,;\, x) \\
&\quad +2F(2m+2, 2n-1\,;\, x)+2(m+2/n-m-2)F(2m+1, 2n-1\,;\, x))]
\end{aligned}
\qquad \ldots \quad (3.47)
$$

We can if we like also apply (3.42) and (3.43) to find out the actual forms for the cases $p=3$ and $p = 4$ when reduction up to the final stage (that is, where we can directly use the incomplete B-fuction tables) has been effected. This, however, would appear to be hardly necessary for purposes of illustration after the detailed reduction already given by (3.46) and (3.47). It is more worth while noting from (3.34), (3.35), (3.42), (3.43) that for final reduction of the p-variate problem it is only necessary to calculate

$$
\begin{aligned}
&F(x\,;\, m+j, n-1) \equiv F(m+j\,;\, n-1\,;\, x)(j=p, p-1, \ldots 1),\ F(x\,;\, 2m+i, 2n-1),\ F(2m+i, 2n-1\,;\, x) \\
&\qquad (i=2p-2, 2p-3, \ldots 1),\ F(x\,;\, m+j'\,;\, n-1),\ F(m+j', n-1\,;\, x)(j'=p-1, \ldots 1)
\end{aligned}
$$

It is evident from a look at (3.34), (3.35), (3.42) and (3.43) as well as at the more general forms (3.3), (3.33), (3.39) and (3.42) that once the basic quantities indicated just now have been calculated from the incomplete B-function tables, all functions like

$$
\begin{aligned}
&F\{(m+p-1, n;\, m+p-2, n;\, \ldots\, m,n);\, x\},\ F\{(m+p-2, n;\, \ldots\, m+p-3, n;\, \ldots\, m,n);\, x\}, \\
&\quad \ldots\ldots\ F\{(m+1, n\,;\, m,n)\,;\, x\}\ \text{and also}\ F\{x\,;\, (m+p-1, n\,;\, \ldots\, m,n)\}, \\
&\quad \ldots\ldots\ F\{x\,;\, (m+p-2, n, \ldots\, m,n)\},\ F\{x\,;\, (m+1, n\,;\, m,n)\}
\end{aligned}
$$

can be evaluated. A slight adaptation and adjustment of the recursion formulae to the needs of a compact computational procedure would be, however, necessary. This is possible and has been actually done, but is not worth while indicating in the present paper. It will, however, be discussed in detail and the actual tabulational and computational procedure will be set forth in a later paper by the author and other collaborators.

4. THE DERIVATION OF THE DISTRIBUTION OF $x_j(j = p-1, p-2, \ldots, 2)$

We shall consider now the functions.

$$
F\{m_p,n\,;\, m_{p-1},n\,;\, \ldots\, m_{j+1},n\,;\, m_j,n\,;\, x\,;\, m_{j-1},n\,;\, \ldots\, m_1,n\}\ \ldots\ (j=p-1, p-2, \ldots, 2)
$$

$$
\equiv F\left\{\begin{array}{l|l} & \begin{array}{cccc} m_p,n & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ m_p,n & m_{p-1},n & \ldots & m_1,n\,. \end{array} \\ x; & \begin{array}{cccc} (m_p,n & m_{p-1},n & \ldots & m_1,n) \\ m_p,n & m_{p-1},n & \ldots & m_1,n \\ \cdot & \cdot & \ldots & \cdot \\ m_p,n & m_{p-1},n & \ldots & m_1,n \end{array} \end{array}\right\} \qquad \ldots \quad (4)
$$

Suppose that we tried to reduce m_p successively by 1 ; the pseudo-determinant is easily expanded into the sum of a number of terms of which the typical one is either of the following different types.

(*i*) $F(m'_p, n\,;\, \ldots\, m'_j, n\,;\, x\,;\, m'_{j-1}, n\,;\, \ldots\, m'_s, n\,;\, \ldots\, m'_1, n)$

(*ii*) $F(m'_p, n\,;\, \ldots\, m'_s, n\,;\, \ldots\, m'_j, n\,;\, x\,;\, m'_{j-1}, n\, \ldots\, m'_1, n)$ ....

(*iii*) $F(m'_p, n\,;\, \ldots\, m'_s, n\,;\, x\,;\, m'_{j-1}, n\,;\, \ldots\, m'_1,n)$

where $m'_s = m_p$ and where in the three different cases we have respectively,

$$(i)\ s < j \qquad (ii)\ s > j \qquad (iii)\ s = j$$

The whole pseudo-determinant is $\Sigma(i)$, $\Sigma(ii)$ or $\Sigma(iii)$ where the summation is taken over all permutations of $(m_p, m_{p-1}, \ldots m_1)$, a typical permutation being denoted as before by $(m'_p, m'_{p-1} \ldots m'_2, m'_1)$. It follows from (2.81), (2.82) and (2.84) that

$$\begin{aligned}
&(i)\ \ F(m'_p,n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_s,n\,;\ldots m'_1,n)\\
&= -(1/n-m_p-1)\ \ F(m_p,n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n;\ldots m'_{s+1}+m_p,2n-1\,;\,m'_{s-1},n\,;\ldots m'_1,n)\\
&\quad +(1/n-m_p-1)\ \ F(m_p,n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_{s+1},n\,;\,m'_{s-1}+m_p,2n-1\,;\ldots m'_1,n)\\
&\quad +(m_p/n-m_p-1)\ F(m_p,n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_{s+1},n\,;\,m_p-1,n\,;\ldots m'_1,n) \quad \ldots (4.1)
\end{aligned}$$

except when $s = 1$ in which case the second term would be absent. In the notation previously introduced we can write this in the alternative form.

$$\begin{aligned}
&-(1/n-m_p-1)\ F(m'_p,n;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n;\ldots m'_{s+1},n\,;\,\overleftarrow{m_p},\overleftarrow{n-1}\,;\,m'_{s-1},n;\ldots m'_1,n)\\
&+(1/n-m_p-1)\ F(m'_p,n;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_{s+1},n\,;\,\overrightarrow{m_p},\overrightarrow{n-1}\,;\,m'_{s-1},n\,;\ldots m'_1,n)\\
&+(m_p/n-m_p-1)F(m'_p,n;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_{s+1},n\,;\,m_p-1,n;\ldots m'_1,n) \quad \ldots (4.11)
\end{aligned}$$

Similarly

$$\begin{aligned}
&(ii)\ \ F(m'_p,n\,;\ldots m'_{s+1},n\,;\,m_p,n\,;\,m'_{s-1},n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&= \ (1/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{s+1},n\,;\,\overrightarrow{m_p},\overrightarrow{n-1}\,;\,m'_{s-1},n\,;\ldots m'_j,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&\quad -(1/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{s+1},n\,;\,\overleftarrow{m_p},\overleftarrow{n-1}\,;\,m'_{s-1},n\,;\ldots m'_jn\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&\quad +(m_p/n-m_p-1)\ F(m'_p,n;\ldots m'_{s+1},n;\,m_p-1,n;\,m'_{s-1},n;\ldots m'_j,n;\,x\,;\,m'_{j-1},n;\ldots m'_1,n) \quad (4.12)
\end{aligned}$$

except when $s = p$ in which case the second term would be absent, and finally

$$\begin{aligned}
&(iii)\ \ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,m_p,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&\qquad = \ (1/n-m_p-1)\ \ F_0(x\,;\,m_p,n-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,x)\ F(x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&\qquad\quad +(1/n-m_p-1)\ \ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,m'_{j-1}+m_p,2n-1\,;\,x\,;\ldots m'_1,n)\\
&\qquad\quad -(1/n-m_p-1)\ \ F(m'_p,n\,;\ldots m'_{j+1}+m_p,2n-1\,;\,x\,;\,m'_{j-1}n\,;\ldots m'_1,n)\\
&\qquad\quad +(m_p/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,m_p-1,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n)
\end{aligned}$$

or in a different notation

$$\begin{aligned}
&= \ (1/n-m_p-1)\ F_0(x\,;\,m_p,n-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,x)\ F(x\,;\,m'_{j-1},n\,;\ldots m'_1,n)\\
&\quad +(1/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,m_p,n-1\,;\,x\,;\,\overleftarrow{m'_{j-1}},\overleftarrow{n}\,;\ldots m'_1,n)\\
&\quad -(1/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,\overleftarrow{m_p},\overleftarrow{n-1}\,;\,x\,;\,m'_{j-1},n-1\,;\ldots m'_1,n)\\
&\quad +(m_p/n-m_p-1)\ F(m'_p,n\,;\ldots m'_{j+1},n\,;\,m_p-1,n\,;\,x\,;\,m'_{j-1},n\,;\ldots m'_1,n) \quad \ldots (4.13)
\end{aligned}$$

The mechanism by which (4.13) is obtained becomes evident if we consider (for purposes of illustration) the case of three variables

$$\int_{x_2}^{\infty}\frac{x_3^{m'_3}\,dx_3}{(1+x_3)^n}\int_{x}^{\infty}\frac{x_2^{m_3}\,dx_2}{(1+x_2)^n}\int_{0}^{x_2}\frac{x_1^{m'_1}\,dx_1}{(1+x_1)^n} \cdot \int_{x}^{\infty}\frac{x_2^{m_3}\,dx_2}{(1+x_2)^n}\cdot\phi(x_2)\,\psi(x_2),$$

where $$\phi(x_2)=\int_{x_2}^{\infty}\frac{x_3^{m'_3}\,dx_3}{(1+x_3)^n}\ \text{and}\ \psi(x_2)=\int_{0}^{x_2}\frac{x_1^{m'_1}\,dx_1}{(1+x_1)^n} \quad \ldots \quad (4.14)$$

Then integrating by parts (the integration being for $1/(1+x_2)^{n-m_2}$ and the differentiation for $\varphi(x_2)\,\psi(x_2).x_2^{m_2}/(1+x_2)^{m_2}$ we should have the expression

$$=(1/n-m_2-1)\{x^{m_2}/(1+x)^{n-1}\}\,\phi(x)\,\psi(x)+(1/n-m_2-1)\int_x^\infty \frac{x_2^{m_2}\,dx_2}{(1+x_2)^{n-1}}[\varphi(x_2)\,\psi'(x_2)+\phi'(x_2)\,\psi(x_2)]$$

$$+(m_2/n-m_2-1)\int_x^\infty \frac{x_2^{m_2-1}\,dx_2}{(1+x_2)^n}\,\varphi(x_2)\,\psi(x_2)$$

$$=\ (1/n-m_2-1)\,F_0(x\,;m_2,n-1)\,F(m'_2,n\,;x)\,F(x\,;m'_1,n)$$
$$+(1/n-m_2-1)\,F(m'_2,n\,;m_2+m'_1,2n-1\,;x)$$
$$-(1/n-m_2-1)\,F(m'_2+m_2,2n-1\,;x\,;m'_1,n)+(m_2/n-m_2-1)\,F(m'_2,n\,;m_2-1,n\,;x\,;m_1',n)$$
$$=\ (1/n-m_2-1)\,F_0(x\,;m_2,n-1)\,F(m'_2,n\,;x)\,F(x\,;m'_1,n)$$
$$+(1/n-m_2-1)\,F(m'_2,n\,;m_2,n-1\,;x\,;\overleftarrow{m'_1},\overleftarrow{n})$$
$$-(1/n-m_2-1)\,F(m'_2,n\,;\overleftarrow{m_2},\overleftarrow{n-1}.\,x\,;m'_1,n)$$
$$+(m_2/n-m_2-1)\,F(m'_2,n\,;m_2-1,n\,;x\,;m'_1,n) \qquad \ldots \quad (4.15)$$

Hence we have

$$F\begin{Bmatrix} m_p,n & m_{p-1},n & .. & m_1,n\\ m_p,n & m_{p-1},n & .. & m_1,n\\ . & . & .. & .\\ x;\ (m_p,n & m_{p-1},n & .. & m_1,n)\\ . & . & .. & .\\ m_p,n & m_{p-1},n & .. & m_1,n \end{Bmatrix}=(1/n-m_p-1)\ F\begin{Bmatrix} \overrightarrow{m_p,n-1} & m_{p-1},n & .. & m_1,n\\ \overrightarrow{m_p,n-1} & m_{p-1},n & .. & m_1,n\\ \to\cdot\to & . & .. & .\\ x;(m_p,n-1 & m_{p-1},n & .. & m_1,n)\\ \to\cdot\to & . & .. & .\\ m_p,n-1 & m_{p-1},n & .. & m_1,n\\ \square & m_{p-1},n & .. & m_1,n \end{Bmatrix}$$

$$-(1/n-m_p-1)\ F\begin{Bmatrix} \square & m_{p-1},n & .. & m_1,n\\ \overleftarrow{m_p,n-1} & m_{p-1},n & .. & m_1,n\\ \leftarrow\cdot\leftarrow & . & .. & .\\ x;(m_p,n-1 & m_{p-1},n & .. & m_1,n)\\ \leftarrow\cdot\leftarrow & . & .. & .\\ m_p,n-1 & m_{p-1},n & .. & m_1,n \end{Bmatrix}+(m_p/n-m_p-1)F\begin{Bmatrix} m_p-1,n & m_{p-1},n & .. & m_1,n\\ m_p-1,n & m_{p-1},n & .. & m_1,n\\ . & . & .. & .\\ x;(m_p-1,n & m_{p-1},n & .. & m_1,n)\\ . & . & .. & .\\ m_p-1,n & m_{p-1},n & .. & m_1,n \end{Bmatrix}$$

$$+(1/n-m_p-1)\,F_0(x\,;m_p,n-1)[F\{(m_{p-1},n;m_{p-2},n;..m_j,n);x\}\,F\{x;(m_{j-1},n;m_{j-2},n;..m_1,n)\}$$
$$-F\{(m_{p-1},n\,;m_{p-2},n\,;..m_{j+1},n\,;m_{j-1},n)\,;x\}\ F\{x\,;(m_j,n\,;m_{j-2},n\,;..m_1,n)\}\pm..] \qquad (4.16)$$

$$=\ (2/n-m_p-1)\sum_{s=p-1}^{j}(-1)^{p-s}\,F\{m_{p-1},n;..m_{s+1},n;m_{s-1},n;..m_j,n;\ x;m_{j-1},n;..m_1,n\}R_1$$

$$+(2/n-m_p-1)\sum_{s'=j-1}^{1}-(1)^{p-1-s'}\,F\{m_{p-1},n;..m_j,n;x;m_{j-1},n;..m_{s'+1},n\,;m_{s'-1},n;..m_1,n\}R_2$$
$$+(m_p/n-m_p-1)\ F\{m_p-1,n;m_{p-1},n;..m_j,n;x;\ m_{j-1},n;..m_1,n\}$$
$$+(1/n-m_p-1)\,F_0(x;m_p,n-1)\,\Sigma\pm F\{(m'_{p-1},n\,;m'_{p-2},n\,;..m'_{j+1},n\,m'_j,n);x\}R_3 \quad .. \quad (4.17)$$

where $\qquad R_1=F(m_s+m_p,2n-1\,;x),\ R_2=F(x;m_s'+m_p,2n-1)$

and $\qquad R_3=F\{x\,;(m'_{j-1},n\,;m'_{j-2},n\,;..m'_1,n)\}$

and where (*i*) $m'_{p-1}>m'_{p-2}>..>m'_{j+1}>m'_j$ (*ii*) $m'_{j-1}>m'_{j-2}>..m'_1$ and (*iii*) $m'_{p-1},m'_{p-2},..m'_j$ is any selection subject to (*i*), of $p-1-(j-1)$, that is, $p-j$ out of $m_{p-1},m_{p-2}\ldots m_1(m_{p-1}>\ldots>m_1)$ and (*iv*) $m'_{j-1},\ldots m'_1$ are those of $m_{p-1},\ldots,m_1$ that are left out after the selection (*iii*). We take the $+ve$ or $-ve$ sign in this summation, acording as the corresponding term in the formal determinantal expansion is $+ve$ or $-ve$.

By successive reduction this comes out as

$$2 \sum_{s=p-1}^{j} (-1)^{p-1-s} F(m_{p-1},n ; \ldots m_{s+1},n ; m_{s-1},n ; \ldots m_j,n ; x ; m_{j-1},n , \ldots m_1,n) \sum_{r=1}^{m_p-m_{p-1}} R$$

$$+2 \sum_{s'=j-1}^{1} (-1)^{p-1-s'} F(m_{p-1},n ; \ldots m_j,n ; x; m_{j-1},n ; \ldots m_{s'+1},n ; m_{s'-1},n ; \ldots m_1,n) \sum_{r=1}^{m_p-m_{p-1}} R_1$$

$$+\Sigma \pm F((m'_{p-1},n ; m'_{p-2},n ; \ldots m'_j,n); x) \; F(x ; (m'_{j-1},n ; \ldots m'_1 n)) \sum_{r=1}^{m_p-m_{p-1}} R_2 \quad (4\cdot18)$$

where $R = ({}_{m_p}P_{r-1}/{}_{n-m_p-2+r}P_r) \; F(m_s+m_p-r+1, \; 2n-1 \; ; \; x),$

$R_1 = ({}_{m_p}P_r/{}_{n-m_p-2+r}P_r) \; F(x \; ; \; m_{s'}+m_p-r+1, 2n-1)$

and $R_2 = ({}_{m_p}P_{r-1}/{}_{n-m_p-2+r}P_r) \; F(x; m_p-r+1, n-1)$

If on the other hand we had tried to increase m_1 successively by one to m_2 then we would have had.

$F(m_p,n ; \ldots m_j,n ; x ; m_{j-1},n ; \ldots m_1,n)$

$$= (-1)^p \sum_{s=p}^{j} (-1)^{p-s} F(m_p,n ; \ldots m_{s+1},n ; m_{s-1},n ; \ldots m_j,n ; x ; m_{j-1},n ; \ldots m_1,n) \sum_{r=1}^{m_2-m_1} R_6$$

$$+(-1)^p \sum_{s'=j-1}^{2} (-1)^{p-s'} F(m_p,n ; \ldots m_j,n; x ; m_{j-1},n ; \ldots m_{s'+1},n ; m_{s'-1},n ; \ldots m_2,n) \sum_{r=1}^{m_2-m_1} R_7$$

$$+\Sigma \pm F(m'_p,n ; m'_{p-1},n ; \ldots m'_{j-1},n); x) \; F(x ; (m'_j,n ; \ldots m'_2 n)) \sum_{r=1}^{m_2-m_1} R_8$$

where $R_6 = ({}_{n-m_1-r}P_{r-1}/{}_{m_1+r}P_r) \; F(m_1+r-m_s, \; 2n-1; \; x),$

$R_7 = ({}_{n-m_1-r}P_{r-1}/{}_{m_1+r}P_r) \; F(m_1+r+m_{s'}, 2n-1 \; ; \; x)$

and $R_8 = ({}_{n-m_1-r}P_{r-1}/{}_{m_1+r}P_r) \; F_0(x \; ; \; m_1+r, n-1)$ .. (4.19)

and where $m'_p, \; m'_{p-1}, \ldots, m'_{j-1}$ is any selection (in proper order) of $(m_p, m_{p-1}, \ldots, m_2)$ and $(m'_j \ldots m'_2)$ is the complementary set.

In actual statistical problems we have $m_i = m+i-1$ $(i=p, \; p-1, \ldots 1)$ and hence $m_p - m_{p-1} = 1$, and in (4.18) and (4.19) there would be only one term each in the 'r' summations although the numbers of terms in the 's' summations would be the same (for any 'r', which latter is only one here). We shall consider in particular the case of $p=3$ for which

$F(m+2, n \; ; m+1, n \; ; x \; ; m,n) = (2 \; n-m-3)[F(m+1, n \; ; x)F(2m+2, 2n-1 \; ; x)$

$\quad -F(x \; ; m, n)F(x \; ; 2m+3, n)] + (1/n-m-3)F_0(x \; ; m+p-2, n-1)$

$\quad \times [(F(m+1, n \; ; x)F(x \; ; m, n) - F(m, n \; ; \; x) \; F(x \; ; m+1, n)]$ .. (4.2)

or $\quad = -(2/m+1) \; [F(m+2, n \; ; x) \; F(2m+1, 2n-1 \; ; x) - F(x \; ; m+1, n) \; F(x \; ; 2m+2, 2n-1)]$

$\quad + (1/m+1) \; F_0(x \; ; m+1, n-1) \; [F(m+2, n \; ; \; x) \; F(m+1, n \; ; x) - F(\;)F(\;)]$ .. (4.21)

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Paper received: 30 November, 1943.