Representations of Partial Linear Spaces of Prime Order

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A thesis submitted to the Indian Statistical Institute
in partial fulfillment of the requirements for the award of

Doctor of Philosophy
in
Mathematics

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Indian Statistical Institute
Bangalore Center
2007

To

Maa & Bapa

Acknowledgements

It is a great pleasure to express my appreciation to all who put efforts towards shaping this thesis.

At the outset I am extremely grateful to my thesis supervisor Prof. N. S. Narasimha Sastry. He suggested this topic for a dissertation and supervised me very closely throughout the time I worked on it. His stimulating suggestions and encouragement helped me all the time of research for and writing of this thesis. He taught me many group theoretic and geometric techniques and shared his ideas with infinite patience. During the last couple of years he has patiently read, reread – and sometimes reread again – my research works as carefully as possible. His observations and comments helped me to move forward with investigations in depth and decorate the fruits of my research.

I acknowledge the financial support provided by *National Board for Higher Mathematics (NBHM), Department of Atomic Energy, Government of India* through a Research Fellowship *DAE Grant 39/3/2000-R&D-II* to pursue my doctoral studies.

I thank Prof. Swadheenananda Pattnayak for developing my interest in Mathematics. His foresight and values paved the way for me to enter into the world of Mathematics.

I am thankful to all the faculty of Statistics and Mathematics Unit at the Indian Statistical Institute, Bangalore Center, particularly to Prof. Gadadhar Misra and Prof. A. Sitaram, for their constant support. I would like to express my sincere thanks to Dr. Shreedhar Inamdar and Dr. Maneesh Thakur for generously sharing their time in providing critical comments and suggestions. My special thanks to Dr. B. Sury for his constant encouragement during the period of my doctoral work. His office door was always open for me to discuss and share craziest ideas about my research.

I want to express my deep gratitude to Dr. Rudra P. Sarkar for his decisive encouragement and a sense of humor about life. I owe a lot to

Dr. Sanjay Parui for his emotional support, for communing his mathematical thoughts and experiences while sharing the same office with me for four years. His assistance concerning the many LATEXnical difficulties I have experienced has helped me a lot.

No words can express my indebtedness to Dr. Manoranjan Mishra and Dr. Bibhuti Bhusan Sahoo for their brotherly advice that helped me to stay in the right path all along.

The burden of completing this thesis was lessened substantially by the support and humor of my friends Subrata, Anupam, Sanjoynath and Probal who made my life in the ISI campus so worthwhile and memorable. However, the latter, I call him 'Bhai', deserves a notable thank, for keeping me update on the day to day hot topics in the institute campus.

I would like to express my gratitude to all the administrative and technical staff members of the institute who have been kind enough to advise and help me in their respective roles. Especially, Ms. Asha Lata and Ms. Mohana Devi deserve a great amount of gratitude not only for their loyal assistance but also for being their great souls and friendship. However, the former sometimes shows her artificial displeasure by saying that I always create disruption in her official work.

Finally, last but not the least, I thank my parents, brother, sister and brother-in-law for extending their unconditional love, patience and support – they allowed me to spend most of the time in doing my research work.

Contents

Preface

Throughout, p denotes a fixed prime number. All groups considered here are finite.

The elements of order p play a significant role in the classification of simple groups. If p divides the order of a group G, then Cayley's theorem says that there exists a subgroup of G of order p. For a positive integer k, let $\mathcal{E}_k^p(G)$ denote the collection of all elementary abelian p-subgroups of order at least p^k in a group G. The graph with vertex set $\mathcal{E}_k^p(G)$ in which two vertices A and B are adjacent if $[A, B] = 1$ is called the *commuting graph on* $\mathcal{E}_k^p(G)$. The determination of the groups G for which the commuting graph on $\mathcal{E}_k^p(G)$ is disconnected for small k plays a crucial role in the classification of simple groups ([**1**], Section 46).

Involutions and their centralizers in a simple group also play a very important role in determining the structure of the group. Many simple groups have been characterized in terms of the centralizer of an involution. The Odd Order Theorem [**30**] of Feit and Thompson says that every non-abelian simple group G possess an involution t. The Brauer and Fowler Theorem [**4**] says that there is only a finite number of simple groups G_0 possessing an involution t_0 with $C_{G_0}(t_0) \simeq C_G(t)$. From the classification of finite simple groups, with a small number of exceptions, G is the unique simple group with such a centralizer. Even in the exceptional cases, at most three simple groups possess the same centralizer. For example: $L_5(2)$, M_{24} and He (Held-Higman-McKay group of order $2^{10}.3^3.5^2.7^3.17$) are the only simple groups possessing an involution with centralizer $L_3(2)/D_8^3$.

Given a set C of involutions in a group G, we can define a point-line geometry $\mathcal{I}_2(G, C)$ whose point set is C and the line set consists of all subsets $\{x, y, xy\}$ of C, where x and y are distinct commuting elements in C. If C is the set of all involutions in G, then $\mathcal{I}_2(G, C)$ is called the *involution geometry* of G (see [**43**], p.111) and denoted by $\mathcal{I}_2(G)$. A *triangular set* [**60**] in G is a G-invariant (under conjugation) subspace of $\mathcal{I}_2(G)$. For any two disjoint triangular sets T_1 and T_2 of G, Shult ([**60**], Theorem 1) proved that $[T_1, T_2]$ is a subgroup of the largest normal subgroup of G of odd order. This result, together with the Odd Order Theorem of Feit and Thompson, imply that every non-abelian simple group contains a unique minimal triangular set ([**60**], Corollary 1). In view of this result of Shult, it seems to be of interest to study the triangular sets in groups.

For a general prime p, we can consider the point-line geometry $\mathcal{I}_p(G,T)$ whose point set T is a collection of subgroups of G of order p. Two distinct points x and y in T are *collinear* if they generate an elementary abelian subgroup of G of order p^2 and each subgroup of order p in it is a member of T. The line containing x and y is the set of $p{+}1$ subgroups of order p in $\langle x, y\rangle$. We could also consider the simplicial complex determined by the partially ordered set of singular subspaces of $\mathcal{I}_p(G,T)$. It seems to be of interest to study the distribution of conjugacy classes of subgroups of order p of the group in terms of this point-line geometry. There are two aspects to this study:

(1) Determining the structure of the point-line geometry or the simplicial complex mentioned above.
(2) Embeddibility of standard point-line geometries in the above mentioned geometry.

There has been a considerable amount of interest in the first problem. The uniqueness of the Monster simple group F_1 is obtained as the automorphism group of the collinearity graph of the geometry $\mathcal{I}_2(F_1,C)$, where C is the set of all conjugates of an involution whose centralizer is of $2\cdot F_2$-type [**34**]. The uniqueness of the Baby Monster group F_2 and the sporadic group F_5 of Harada-Norton is proved by the same method considering C to be the set of all conjugates of an involution whose centralizer is isomorphic to $2\cdot {}^2E_6(2):2$ [**58**] and $(2\cdot \mathrm{HS}):2$ [**59**], respectively. In the latter case, F_5 is determined as the commutator subgroup of the automorphism group of the collinearity graph.

Another important class of examples of $\mathcal{I}_p(G,T)$ are the *root group geometries* ([**16**], p.75), where G is a group of Lie type defined over a field $\mathbb{F}_p$ with p elements and T is the collection of all (long) root subgroups of G. An important motivation for the study of these geometries was the possibility of generating subgroups of the group defined by the substructures of the geometry. Cooperstein [**18**] proved that if G is of type G_2 or ${}^3D_4$, then $\mathcal{I}_p(G,T)$ is a generalized hexagon (see [**67**]).

The question of characterizing buildings of spherical type in terms of point-line geometries with certain properties had been an active topic of research during the 1980's. In the spirit of the characterization of projective spaces in terms of Veblen and Young axioms and the characterization of polar spaces in terms of Buekenhout and Shult axioms, Cohen and Cooperstein [**16**] characterized the root group geometries of type E_6, E_7 and E_8 as some point-line geometries (the so called parapolar spaces) satisfying certain conditions.

The main thrust of this thesis is on the second problem: that is, to recognize the standard point-line geometries like projective spaces, generalized quadrangles, polar spaces, near polygons etc. in groups. An initial motivation for this work was to initiate the search for new point-line geometries like generalized quadrangles etc. with $p+1$ points per line which are embedded in groups.

For a point-line geometry $\mathcal{G} = (P, L)$ with three points per line, the *universal embedding module* $V(\mathcal{G})$ of $\mathcal{G}$ is a $\mathbb{F}_2$-vector space defined as: $V(\mathcal{G}) = \langle v_x : x \in P; v_x + v_y + v_z = 0, \{x, y, z\} \in L\rangle$. In [**39**] and [**40**], the universal embedding modules for the 2-local parabolic geometries $\mathcal{G}(J_4)$, $\mathcal{G}(F_2)$ and $\mathcal{G}(F_1)$ were shown to be trivial. Here $\mathcal{G}(J_4)$ and $\mathcal{G}(F_2)$ are the Petersen type geometries of the fourth Janko group J_4 and the Baby Monster group F_2 respectively, and $\mathcal{G}(F_1)$ is the tilde type geometry of the Monster group F_1 (see [**36**] for definitions). This result played an important role in the proof that none of these three geometries appear as a residue in a flag-transitive tilde or Petersen type geometry of a higher rank [**41**].

The notion of a universal representation group of a geometry was introduced in [**40**] to prove the triviality of $V(\mathcal{G}(F_2))$. The *universal representation group* $R(\mathcal{G})$ of $\mathcal{G}$ has the presentation: $R(\mathcal{G}) = \langle r_x : x \in P, r_x^2 = 1, r_x r_y r_z = 1, \{x, y, z\} \in L\rangle$. In [**38**] and [**42**], Ivanov et al. studied the structure of $R(\mathcal{G})$ when $\mathcal{G}$ is one of the geometries $\mathcal{G}(J_4)$, $\mathcal{G}(F_2)$ and $\mathcal{G}(F_1)$ and proved that $R(\mathcal{G}(F_2)) \simeq 2 \cdot F_2$, $R(\mathcal{G}(F_1)) \simeq F_1$ and $R(\mathcal{G}(J_4)) \simeq J_4$. The calculation of the universal representation group $R(\mathcal{G}(F_1))$ has been used to identify the Y-group Y_{555} with the Bi-Monster (see [**36**], Section 8.6).

In [**37**], Ivanov introduced the concept of a representation in groups of a point-line geometry $\mathcal{G} = (P, L)$ with lines of size $p+1$. His definition of representation is similar to the definition of the root group geometries over $\mathbb{F}_p$ studied by Cooperstein [**18**], and Cohen and Cooperstein [**16**].

A *representation* of $\mathcal{G}$ is a pair (R, ρ) where R is a (possibly non-abelian) group and ρ is a mapping from P to the set of subgroups of R of order p such that R is generated by the image of ρ and for every $l \in L$ and $x \neq y$ in l, $\rho(x)$ and $\rho(y)$ are distinct and the subgroup generated by $\rho(l)$ has order p^2. This definition of representations of geometries led to a new research area in the theory of groups and geometries [**37**]. The knowledge of the representations is crucial for the construction of affine and c-extensions of geometries and non-split extensions of groups and modules (see Sections 2.7 and 2.8 of [**43**]).

In this thesis, we study the representations of finite projective spaces, generalized quadrangles and non-degenerate polar spaces with lines of size $p + 1$ and dense near hexagons with three points per line in the sense of Ivanov [**37**].

In Chapter 1, we review some basic results related to generalized quadrangles, polar spaces, generalized polygons and near polygons that are needed in the subsequent chapters.

In Chapter 2, we determine the triangular sets in the finite irreducible Coxeter groups.

In Chapter 3, we present the notion of representations of partial linear spaces introduced by Ivanov [**37**]. The representation group for a non-abelian representation of a finite partial linear space with lines of size $p+1$ need not be finite (see Example 3.14). We give a sufficient condition on the partial linear space and on the non-abelian representation of it to ensure that the representation group is a finite p-group (Theorem 3.23). We use this result as a basic tool to study non-abelian representations of finite non-degenerate polar spaces with lines of size $p + 1$ and slim dense near hexagons. By definition, every representation of a projective space is abelian and faithful. So the study of the representations of a projective space of dimension m over $\mathbb{F}_p$ in a group G is the same thing as the study of elementary abelian p-subgroups of G of order p^{m+1}. We study elementary abelian p-subgroups of the symmetric group $Sym(I)$ on a finite set I and describe the maximal elementary abelian p-groups of $Sym(I)$, up to conjugacy (Theorems 3.28 and 3.33).

In Chapter 4, we study non-abelian representations of finite non-degenerate polar spaces of rank at least two with $p+1$ points per line. We characterize the finite symplectic polar spaces of rank at least two with $p+1$ points per line, p odd, as the only finite non-degenerate polar spaces with $p+1$ points per line admitting non-abelian representations (Theorems 4.1 and 4.2).

In Chapter 5, we recall some results about $(2,t)$-GQs. We present a proof of the finiteness of t. We study complete arcs of $(2,t)$-GQs in detail. Every representation of a $(2,t)$-GQ is necessarily abelian (Theorem 4.1(i)). However, the representation need not be faithful (Example 3.11). We study the faithful representations of these geometries. They play an important role in the study of non-abelian representations of slim dense near hexagons.

In Chapter 6, we study slim dense near hexagons. We present the classification of these geometries due to Brouwer et al. [**9**]. There are eleven such geometries, up to isomorphism. We denote them by $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathbb{G}_3$, $DH_6(2^2)$, $Q_6^-(2)\otimes Q_6^-(2)$, $DW_6(2)$, $\mathbb{H}_3$, $Q_6^-(2)\times\mathbb{L}_3$, $W_4(2)\times\mathbb{L}_3$ and $Q_4^+(2)\times\mathbb{L}_3$ (see Theorem 6.1). We give a construction for each of them, though we only need to work with their parameters. We give new constructions for $DW_6(2)$ and $\mathbb{H}_3$ (Theorems 6.3 and 6.10). Except $\mathbb{E}_1$ and $\mathbb{E}_2$, they all admit big quads. We study the structure of the slim dense near hexagons having big quads relative to a subspace generated by two of its disjoint big quads.

In Chapter 7, we study non-abelian representations of slim dense near hexagons. We show that $DH_6(2^2)$, $\mathbb{E}_3$ and $\mathbb{G}_3$ do not admit non-abelian representations (Theorem 7.1). If S denotes one of the remaining eight near hexagons, we show that the representation group R for a non-abelian representation of S is of order 2^β, $1+n(S)\leq\beta\leq 1+dimV(S)$, where $dimV(S)$ is the dimension of the universal embedding module of S and $n(S)$ is given as in Theorem 6.1. Further, if $\beta=1+n(S)$, then $R=2_\epsilon^{1+n(S)}$, where $\epsilon=-$ or $+$ according as $S=Q_6^-(2)\otimes Q_6^-(2)$ or not (Theorem 7.2). If S is one of the near hexagons $Q_6^-(2)\otimes Q_6^-(2)$, $DW_6(2)$, $\mathbb{H}_3$, $Q_6^-(2)\times\mathbb{L}_3$, $W_4(2)\times\mathbb{L}_3$ and $Q_4^+(2)\times\mathbb{L}_3$ having big quads, then we show that S admits a non-abelian representation such that the representation group is extraspecial of order $2^{1+n(S)}$ (Theorem 7.3). There is a Fischer space structure on the big quads of a slim dense near hexagon ([**9**], Sections 3 and 4). We use this structure to give a sufficient condition for a representation of S to be abelian (Theorem 7.20) and deduce Theorem 7.1 as a consequence of it. We also use this structure to construct a non-abelian representation of $Q_6^-(2)\otimes Q_6^-(2)$.

CHAPTER 1

Point-Line Geometries

In this chapter we summarize some basic concepts on point-line geometries and introduce the notation that we shall use later in this thesis.

1.1. Graphs

By a *graph* $\mathcal{G} = (X, \approx)$ we mean a set X together with a symmetric, anti-reflexive relation $\approx$, referred to as *adjacency*. The elements of X are called *vertices*. If x and y are adjacent for distinct vertices $x, y \in X$, then the pair $\{x, y\}$ is called an *edge*. If any two distinct vertices are adjacent, then $\mathcal{G}$ is called a *clique*. A *path* from x to y, $x, y \in X$, is a finite sequence of vertices $x = x_0, x_1, \cdots, x_n = y$ where x_{i-1} is adjacent to x_i for $i = 1, \cdots, n$. The number n is called the *length* of such a path. The graph $\mathcal{G}$ is *connected* if there is a path between any two of its vertices. A *geodesic* from x to y is a path from x to y of minimum length. The *distance* between two vertices x and y, denoted by $d(x, y)$, is the length of a geodesic joining x to y, if a geodesic exists, otherwise $d(x, y) = \infty$. The *diameter* of $\mathcal{G}$ is $sup\{d(x, y) : x, y \in X\}$. A sequences of vertices $x_0, x_1, \cdots, x_m$ is a *circuit* of length m if $m \geq 2$; $x_0 = x_m$; $x_0, x_1, \cdots, x_{m-1}$ are pairwise distinct; and $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \cdots, m$. The *girth* of $\mathcal{G}$ is the length of a shortest circuit in $\mathcal{G}$. The graph $\mathcal{G}$ is *bipartite* if the vertex set X can be partitioned into two non-empty subsets X_1 and X_2 such that every edge of $\mathcal{G}$ has one vertex in X_1 and the other vertex in X_2. The *complement graph* of $\mathcal{G}$ is the graph $\mathcal{G}' = (X, \approx')$, whose vertex set is X and two distinct vertices x and y are defined to be adjacent (that is, $x \approx' y$) if and only if they are non-adjacent vertices of $\mathcal{G}$.

1.2. Partial Linear Spaces

A *point-line geometry* is a pair $S = (P, L)$ consisting of a set P and a collection L of subsets of P of size at least 2. The elements of P and L

are called *points* and *lines* of S, respectively. If any two distinct points of S are contained in at most one line, then S is called a *partial linear space*. Two distinct points x and y of S are *collinear*, written as $x \sim y$, if there is a line of S containing them. If x and y are not collinear, we write $x \nsim y$. If each pair of distinct points of S is contained in exactly one line, then S is called a *linear space*. Important examples of linear spaces are the projective spaces and affine spaces (as point-line geometries). For $x \in P$ and $A \subseteq P$, we define

$$x^\perp = \{x\} \cup \{y \in P : x \sim y\} \text{ and } A^\perp = \bigcap_{x \in A} x^\perp.$$

If $P^\perp$ is empty, then S is called a *non-degenerate* point-line geometry.

Let $S = (P, L)$ be a partial linear space. If x and y are two collinear points of S, then we denote by xy the unique line containing x and y. In that case, $\{x, y\}^\perp = xy$. If P is a finite set, then S is called a *finite partial linear space*. A point of S is *thick* if it is contained in at least three lines. A line of S is *thick* if it contains at least three points. If all points and all lines of S are thick, then S itself is called *thick*. If each line of S contains $s + 1$ points, then S is of *order* s. Further, if each point of S is contained in $t + 1$ lines, then S is said to have *parameters* (s, t). If S is of order 2, then S is called a *slim* partial linear space. In that case, if $x, y \in P$ are collinear, then we define $x * y$ by $xy = \{x, y, x * y\}$.

1.2.1. Collinearity and Incidence graph. With each point-line geometry $S = (P, L)$, there is associated a graph $\Gamma(P)$, called the *collinearity graph* of S. The vertices of $\Gamma(P)$ are the points of S, and two distinct vertices are *adjacent* whenever they are collinear in S. For $x, y \in P$, the *distance* $d(x, y)$ between x and y is measured in $\Gamma(P)$. If $\Gamma(P)$ is connected, then S is called a *connected* point-line geometry. For a non-negative integer i, we define

$$\begin{aligned} \Gamma_i(x) &= \{y \in P : d(x, y) = i\}; \\ \Gamma_{\leq i}(x) &= \{y \in P : d(x, y) \leq i\}. \end{aligned}$$

Thus $x^\perp = \{x\} \cup \Gamma_1(x)$ for $x \in P$. For $z \in P$ and $X, Y \subseteq P$, we define $d(z, X) = \inf_{x \in X} d(z, x)$; and $d(X, Y) = \inf_{x \in X, y \in Y} d(x, y)$. The *incidence graph* $\Gamma(S)$ of S has vertex set $P \cup L$, in which two distinct vertices x and y are adjacent if and only if either $x \in P, y \in L$ and $x \in y$; or $x \in L, y \in P$ and $y \in x$. Clearly, $\Gamma(S)$ is a bipartite graph.

1.2.2. Subspaces. Let $S = (P, L)$ be a point-line geometry. A subset of P is a *subspace* of S if each line containing at least two of its points is entirely contained in it. The empty set, singletons and P are all subspaces of S. If S is a partial linear space, then the lines are also subspaces. Clearly, intersection of subspaces is again a subspace. For a subset X of P, the *subspace* $\langle X \rangle$ *generated by* X is the intersection of all subspaces of S containing X. It is well defined as P is a subspace of S containing X. Thus, $\langle X \rangle$ is the smallest subspace of S containing X. If S is a partial linear space and if $x, y \in P$ are collinear, then $xy = \langle x, y \rangle$, where $\langle x, y \rangle$ is short of $\langle \{x, y\} \rangle$. A subspace of S is *singular* if each pair of its points is collinear. Thus, a singular subspace is a subspace which is also a clique in the collinearity graph. A *geometric hyperplane* of S is a subspace of S different from P that meets each line of S non-trivially.

1.2.3. Isomorphisms. Let $S = (P, L)$ and $S' = (P', L')$ be two point-line geometries. A map $\alpha : P \longrightarrow P'$ is an *isomorphism* from S to S' if it is a bijection, $\alpha(x) \sim \alpha(y)$ in S' whenever $x \sim y$ in S and it induces a bijection from L to L'. In that case, S and S' are called *isomorphic* and written as $S \simeq S'$. An isomorphism from S onto itself is called an *automorphism* of S.

1.2.4. Direct product. Let $S_1 = (P_1, L_1)$ and $S_2 = (P_2, L_2)$ be two partial linear spaces. Then, their *direct product* $S_1 \times S_2$ is the partial linear space whose point set is $P_1 \times P_2$ and the line set consists of all subsets of $P_1 \times P_2$ projecting to a single point in P_i and projecting in P_j onto an element of L_j, where $\{i, j\} = \{1, 2\}$.

1.3. Polar Spaces

A *polar space* is a point-line geometry $S = (P, L)$ such that the following 'one or all' axiom holds:

> *For each point-line pair* $(x, l) \in P \times L$ *with* $x \notin l$, x *is collinear with one or all points of* l.

We refer to ([**66**], 7.1, p.102) for the original definition of a polar space. The above equivalent definition of a polar space is due to Buekenhout and Shult [**11**], where there is no restriction on the intersection of two lines. However, a remarkable discovery of Buekenhout and Shult is the following.

Theorem 1.1 ([**11**], Theorem 3, p.161)**.** *A non-degenerate polar space is a partial linear space.*

Rank of a polar space S is the supremum of the lengths m of chains $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m$ of singular subspaces of S.

Let $S = (P, L)$ be a non-degenerate polar space of finite rank n. Each singular subspace of S is isomorphic to a projective space. The *dimension* of a singular subspace of S is the dimension of the associated projective space. Each maximal singular subspace of S has dimension $n-1$. For singular subspaces X, Y of S with $Y \subset X$, the *co-dimension* of Y in X is the dimension of X minus the dimension of Y.

1.3.1. Finite classical polar spaces. We shall use the following notation for finite classical polar spaces of rank $r \geq 2$ over the field $\mathbb{F}_q$ with q elements, where q is a prime power.

- $W_{2r}(q)$: the points of $PG(2r-1, q)$ together with the totally isotropic lines with respect to a symplectic polarity;
- $H_{2r}(q^2)$: the points together with the lines of a non-singular Hermitian variety in $PG(2r-1, q^2)$;
- $H_{2r+1}(q^2)$: the points together with the lines of a non-singular Hermitian variety in $PG(2r, q^2)$;
- $Q^+_{2r}(q)$: the points together with the lines of a non-singular hyperbolic quadric in $PG(2r-1, q)$;
- $Q_{2r+1}(q)$: the points together with the lines of a non-singular quadric in $PG(2r, q)$;
- $Q^-_{2r+2}(q)$: the points together with the lines of a non-singular elliptic quadric in $PG(2r+1, q)$.

The study of polar spaces was initiated by Veldkamp [**68**]. Building on the work of Veldkamp, Tits [**66**] classified polar spaces whose rank is at least three. For polar spaces of possibly infinite rank, see [**44**]. Tits classification implies

Theorem 1.2. *A finite thick non-degenerate polar space of rank $r \geq 3$ is isomorphic to either the symplectic polar space $W_{2r}(q)$; or one of the orthogonal polar spaces $Q^+_{2r}(q)$, $Q_{2r+1}(q)$ and $Q^-_{2r+2}(q)$; or one of the unitary polar spaces $H_{2r}(q^2)$ and $H_{2r+1}(q^2)$.*

Theorem 1.3 ([**62**], Theorem 1, p.330)**.** *The number of points of the finite classical polar spaces are given by the formulae:*

$$\begin{aligned}
|W_{2r}(q)| &= (q^{2r}-1)/(q-1);\\
|Q^+_{2r}(q)| &= (q^{r-1}+1)(q^r-1)/(q-1);\\
|Q_{2r+1}(q)| &= (q^{2r}-1)/(q-1);\\
|Q^-_{2r+2}(q)| &= (q^r-1)(q^{r+1}+1)/(q-1);\\
|H(2r,q^2)| &= (q^{2r}-1)(q^{2r-1}+1)/(q^2-1);\\
|H(2r+1,q^2)| &= (q^{2r+1}+1)(q^{2r}-1)/(q^2-1).
\end{aligned}$$

The following inductive property ([**14**], Section 6.4, p.90) of the classical polar spaces is useful for us.

Lemma 1.4. *Let $S = (P, L)$ be a classical polar spaces of finite rank $r \geq 3$ and let $x, y \in P$ be two non-collinear points of S. Then, $\{x, y\}^\perp$ is a polar space of rank $r - 1$ and is of the same type as S.*

1.4. Generalized Polygons

A *generalized n-gon*, $n \geq 1$, is a partial linear space $S = (P, L)$ satisfying the following:

- The incidence graph $\Gamma(S)$ of S has girth $2n$ and diameter n,
- Any two elements of $P \cup L$ are contained in some circuit in $\Gamma(S)$ of length $2n$.

The concept of a generalized polygon was introduced by Tits [**65**] in his celebrated work on triality. These geometries form spherical buildings of rank two. For a detailed discussion of these structures, we refer to [**67**]. A lot of restrictions are known concerning the integer n and the parameters (s, t) of a finite generalized n-gon.

Lemma 1.5 ([**67**], Corollary 1.5.3, p.19)**.** *Every thick generalized n-gon admits parameters (s, t). Further, if n is odd, then $s = t$.*

Finite generalized n-gons with parameters (s, t) which are not ordinary polygons exist only for $n = 2, 3, 4, 6, 8$ and 12. This was proved by Feit and Higman [**29**], see Kilmoyer and Solomon [**47**] for a different proof. However, their classification is very difficult since, for instance, projective planes and generalized 3-gons are the same.

Lemma 1.6 ([**29**])**.** *Finite thick generalized n-gons exist only for $n = 3, 4, 6$ and 8.*

For $n = 3, 4, 6, 8$, generalized n-gons are called *generalized triangles*, *generalized quadrangles*, *generalized hexagons* and *generalized octagons*, respectively. In the next section we give another definition of generalized quadrangles, it can be seen that both the definitions are equivalent.

Lemma 1.7 ([**67**], Lemma 1.5.4, p.19)**.** *Let $S = (P, L)$ be a finite generalized n-gon, $n \geq 3$, with parameters (s, t). Then,*

$$|P| = (s+1)\left(\frac{(st)^{n/2}-1}{st-1}\right); |L| = (t+1)\left(\frac{(st)^{n/2}-1}{st-1}\right).$$

Lemma 1.8 ([**67**], Theorem 1.7.2, p.24)**.** *Let $S = (P, L)$ be a finite thick generalized n-gon, $n \geq 4$, with parameters (s, t). Then, one of the following holds:*

(i) $n = 4$, $s \leq t^2$ and $t \leq s^2$;
(ii) $n = 6$, $s \leq t^3$ and $t \leq s^3$;
(iii) $n = 8$, $s \leq t^2$ and $t \leq s^2$.

The known finite thick generalized quadrangles to date have parameters $(q-1, q+1), (q+1, q-1), (q, q), (q^2, q^3), (q^3, q^2), (q, q^2)$ and (q^2, q) for a prime power q ([**50**] and [**63**]). All known finite thick generalized hexagons have parameters $(q, q), (q, q^3)$ and (q^3, q) for a prime power q. All known finite thick generalized octagons have parameters $(2^a, 2^{2a})$ and $(2^{2a}, 2^a)$, a being odd.

1.5. Generalized Quadrangles

A *generalized quadrangle* (GQ, for short) is a non-degenerate partial linear space $S = (P, L)$ satisfying the following 'exactly one' axiom:

> *For each point-line pair $(x, l) \in P \times L$ with $x \notin l$, x is collinear with exactly one point of l.*

Let $S = (P, L)$ be a generalized quadrangle. There is a point-line geometry $S^D = (P', L')$ associated with S, whose point set is $P' = L$ and with every point $x \in P$ is associated a line $x' \in L'$ which is the collection of all points of P' containing x. Then, S^D is a generalized quadrangle, called the *dual* of S. If S and S^D are isomorphic, then S is said to be *self-dual*.

The following is an improvement of Lemma 1.5 when $n = 4$.

Lemma 1.9 ([**14**], Theorem 7.1, p.98). *Let $S = (P, L)$ be a generalized quadrangle with at least one thick line and one thick point. Then, S admits parameters (s, t).*

The theory of generalized quadrangles is extremely important in the theory of polar spaces and dense near polygons. Non-degenerate polar spaces of rank 2 are precisely the generalized quadrangles. The quads in a dense near polygon are generalized quadrangles. In this section, we write down those results about generalized quadrangles which we shall use later in this thesis. We refer to [**50**], [**51**] and [**67**] for several examples of finite generalized quadrangles.

If S is a generalized quadrangle with parameters (s, t), then we say that S is a (s, t)-GQ. From Subsection 1.3.1, the finite classical generalized quadrangles are $W_4(q)$, $H_4(q^2)$, $H_5(q^2)$, $Q_4^+(q)$, $Q_5(q)$ and $Q_6^-(q)$. The parameters of these generalized quadrangles are (q, q), (q^2, q), (q^2, q^3), $(q, 1)$, (q, q) and (q, q^2), respectively.

Finite generalized quadrangles are classified only for $s = 2, 3$, see ([**51**], Chapter-6) and ([**63**], 5.1, p.401). Regarding the finite GQs with $s = p$, p a prime, Kantor showed that if a finite thick generalized quadrangle S of order p admits a rank three automorphism group G on the point set of S, then one of the following holds ([**45**], Theorem 1.1):

(*i*) $t = p^2 - p - 1$ and $p^3 \nmid |G|$;
(*ii*) $G \simeq PSp(4, p)$ and $S \simeq W_4(p)$ or $G \simeq P\Gamma U(4, p)$ and $S \simeq Q_4^+(p)$;
(*iii*) $p = 2$, $G = Alt(6)$ and $S \simeq W_4(2)$.

A question posed by Tits that is still open is whether there exists a (s, t)-GQ with $s > 1$ finite and t infinite. It is known that there is no such generalized quadrangle with $s = 2, 3$ and 4. This is due to Cameron [**12**] for $s = 2$, Brouwer [**7**] for $s = 3$, and Cherlin [**15**] for $s = 4$.

1.5.1. Regularity and anti-regularity. Let $S = (P, L)$ be a finite (s, t)-GQ. A *triad* of points of S is a triple of non-collinear points of S. For a triad T of points of S, an element of $T^\perp$ is called a *center* of T. A pair $\{x, y\}$ of distinct points of S is *regular* if $x \sim y$; or if $x \nsim y$ and $|\{x, y\}^{\perp\perp}| = t + 1$. A point x is *regular* if $\{x, y\}$ is regular for each $y \in P \setminus \{x\}$. A pair $\{x, y\}$ of non-collinear points of S is *anti-regular* if $|z^\perp \cap \{x,y\}^\perp| \leq 2$ for each $z \in P \setminus \{x,y\}$. A point x is *anti-regular* if $\{x,y\}$ is anti-regular for each $y \in P \setminus x^\perp$.

Dually, we define a triad of lines, center of a triad of lines, regularity and anti-regularity of a line.

Lemma 1.10 ([**51**], 1.2.4, p.4)**.** *Let $S = (P,L)$ be a finite thick (s,t)-GQ. Then, $s^2 = t$ if and only if each triad of points of S has $s+1$ centers.*

Lemma 1.11 ([**51**], 1.5.2, p.13)**.** *Let $S = (P,L)$ be a finite thick (s,t)-GQ. The following hold:*

- (i) *If S has a regular point x and a regular line l with $x \notin l$, then $s = t$ is even.*
- (ii) *If $s = t$ is odd and if S contains two regular points, then S is not self-dual.*

Lemma 1.12 ([**51**], 3.2.1, p.43)**.** *$Q_5(q)$ is isomorphic to the dual of $W_4(q)$. Further, $Q_5(q)$ (or $W_4(q)$) is self-dual if and only if q is even.*

A triad $T = \{x,y,z\}$ of S is 3-*regular* if $s^2 = t > 1$ and $|T^{\perp\perp}| = s+1$. A point x of S is 3-*regular* if $s^2 = t > 1$ and each triad of S containing x is 3-regular.

Lemma 1.13 ([**51**], 3.3.1, p.51)**.** *The following hold:*

- (i) *In $Q_5(q)$, all lines are regular; all points are regular if an only if q is even; all points are anti-regular if and only if q is odd.*
- (ii) *In $Q_6^-(q)$, all lines are regular and all points are 3-regular.*

Lemma 1.14 ([**51**], 5.2.1, p.77)**.** *A generalized quadrangle with parameters (q,q) is isomorphic to $W_4(q)$ if and only if all its points are regular.*

1.5.2. Ovoids and spreads. Let $S = (P,L)$ be a finite (s,t)-GQ. A *k-arc* (of points) of S is a set of k pair-wise non-collinear points of S. An empty set is a 0-arc or a trivial arc, a 1-arc is just a singleton and a 3-arc is a triad. A k-arc is *complete* if it is not contained in a $(k+1)$-arc. A point x of S is called a *center* of a k-arc of S if x is collinear with every point of it. An *ovoid* of S is a k-arc of S that meets each line of S non-trivially. A *spread* of S is a set of lines partitioning the point set P.

Lemma 1.15 ([**51**], 1.8.1, p.20)**.** *Let $S = (P, L)$ be a finite (s, t)-GQ. Let O be an ovoid and K be a spread of S. Then, $|O| = |K| = 1 + st$.*

Lemma 1.16 ([**51**], 3.4.1, p.55)**.** *The following hold:*

(i) *$Q_5(q)$ has ovoids. It has spreads if and only if q is even.*

(ii) *$Q_6^-(q)$ has spreads but no ovoids.*

The following result appears in ([**20**], Theorem 2.2, p.19) which is proved by induction on $r \geq 3$. However, for the case $r = 2$ which is needed to start the induction, the authors refer to other papers available in the literature. We include a proof of it for the sake of completeness.

Proposition 1.17. *Let $q = p^e$ be odd and $r \geq 2$. Then, $W_{2r}(q)$ is generated by a set $A_r = \{a_i, b_i : 1 \leq i \leq r\}$ consisting of $2r$ points such that, for distinct $u, v \in A_r$, $u \nsim v$ if and only if $\{u, v\} = \{a_i, b_i\}$ for some i.*

Proof. First, assume that $r = 2$. Let $A_2 = \{a_1, a_2, b_1, b_2\}$ be a quadrangle in $W_4(q)$ with $a_1 \nsim b_1$ and $a_2 \nsim b_2$. Consider the parallel lines $m_0 = a_1a_2$ and $m_1 = b_1b_2$. Let $\{m_0, m_1\}^\perp = \{l_0, l_1, \cdots, l_q\}$. Then, $\bigcup_{i=0}^{q} l_i \subset \langle A_2 \rangle$. For a point z, we denote by L_z the set of lines containing z. Let x be a point not in $\bigcup_{i=0}^{q} l_i$. For each line l_i, there is a unique line l_i^x in L_x meeting l_i. This defines a map δ from $\{m_0, m_1\}^\perp$ to L_x. If δ is not one-one then there exists $i \neq j$ such that $l_i^x = l_j^x$. If $l_i^x \cap l_i = \{u\}$ and $l_i^x \cap l_j = \{v\}$, then $l_i^x = uv$. Since u, v both are points in $\langle A_2 \rangle$ it follows that l_i^x is a line in $\langle A_2 \rangle$. So $x \in \langle A_2 \rangle$.

Assume now that δ is one-one. For $k \in \{0, 1\}$, let x be collinear with u_k in the line m_k. Consider the lines l_{i_0} and l_{i_1} in $\{m_0, m_1\}^\perp$, where $l_{i_k} \cap m_k = \{u_k\}$. Let $\{l_{i_0}, l_{i_1}\}^\perp = \{m_0, m_1, \cdots, m_q\}$. Now for each line m_i, there is a unique line m_i^x in L_x meeting m_i. This defines a map σ from $\{l_{i_0}, l_{i_1}\}^\perp$ to L_x. Applying the argument as in the first paragraph, we may assume that σ is one-one. Then, $m_i^x = l_j^x$ for some $i, j \in \{0, 1, \cdots, q\} \setminus \{i_0, i_1\}$. (Note that $l_{i_0}^x = m_0^x$ and $l_{i_1}^x = m_1^x$.) Let $m_i^x \cap m_i = \{v_i\}$ and $m_i^x \cap l_j = \{v_j\}$. Since q is odd, all lines of $W_4(q)$ are anti-regular, by Lemmas 1.12 and 1.13(i). So $m_i \cap l_j = \Phi$. Thus, $v_i \neq v_j$ and $v_iv_j = m_i^x$. Since $\bigcup_{i=0}^{q} m_i \subset \langle A_2 \rangle$, v_i and v_j both are contained in $\langle A_2 \rangle$ and it follows that m_i^x is a line in $\langle A_2 \rangle$. So $x \in \langle A_2 \rangle$.

Now, assume that $r \geq 3$. Let a_r, b_r be two non-collinear points of $W_{2r}(q)$ and set $H = \{a_r, b_r\}^{\perp}$. Then, H is a subspace and $H \simeq W_{2(r-1)}(q)$ (Lemma 1.4). By induction, let H be generated by a set $A_{r-1} = \{a_i, b_i : 1 \leq i \leq r-1\}$ satisfying the required property. We prove that $W_{2r}(q)$ is generated by the set $A_r = A_{r-1} \cup \{a_r, b_r\}$ and this would complete the proof.

We have $H \subset \bigcup_{l \in L_w} l \subset \langle A_r \rangle$ for $w \in \{a_r, b_r\}$. Let $z \in P \setminus (L_{a_r} \cup L_{b_r})$. For each line l in L_w, there is a unique line in L_z meeting l because $z \nsim w$ and this defines a bijection τ_w from L_w onto L_z. Suppose that $z \nsim y$ for some $y \in H$. Let $l \in L_{a_r}$ be such that $y \in l$ and let $m \in L_{b_r}$ be such that $\tau_{a_r}(l) = \tau_{b_r}(m)$. If $\{z_1\} = l \cap \tau_{a_r}(l)$ and $\{z_2\} = m \cap \tau_{b_r}(m)$, then $z_1 \neq z_2$ and $z_1 z_2 = \tau_{a_r}(l)$. Since z_1 and z_2 are in $\langle A_r \rangle$, so also the line $z_1 z_2$. Hence $z \in \langle A_r \rangle$. Suppose that z is collinear with every point of H. Fix two non-collinear points x and y in H. Let c be a point in the line yz different from y and z. Then, $c \nsim x$ because $x \sim z$ and $x \nsim y$. So $c \in \langle A_r \rangle$ by the above argument. Then, yc is contained in $\langle A_r \rangle$, so $z \in \langle A_r \rangle$. □

1.6. Near Polygons

A *near polygon* is a connected partial linear space $S = (P, L)$ such that the following 'near polygon' property holds:

> *For each point-line pair $(x, l) \in P \times L$ with $x \notin l$, there exists a unique point on l nearest to x.*

Let $S = (P, L)$ be a near polygon. If the diameter of S is n, then S is called a *near $2n$-gon*. The sets $\Gamma_{\leq n-1}(x)$, $x \in P$, are called *special* geometric hyperplanes of S. Important examples of near polygons are the generalized n-gons. A near 0-gon is just a point. A near 2-gon is a line. Near $2n$-gons for $n = 2, 3, 4$ are called *near quadrangles*, *near hexagons* and *near octagons*, respectively. Near $2n$-gons exist for each n. In the next section, we give examples of three infinite families of slim near $2n$-gons.

The concept of a near polygon was introduced by Shult and Yanushka [**61**] to study system of lines in an Euclidean space. A structure theory of these geometries was developed by Brouwer and Wilbrink [**10**]. The possible 'line-line' relations in a near polygon are given in the following.

Theorem 1.18 ([**10**], Lemma 1,p.146)**.** *Let l and m be two lines of a near polygon $S = (P, L)$ with at least one thick line. Then, one of the following posiibilities occurs.*

(*i*) *There exists a unique point $x \in l$ and a unique point $y \in m$ such that $d(u, v) = d(u, x)+d(x, y)+d(y, v)$ for all points $u \in l$ and $v \in m$.*
(*ii*) *There exists a positive integer i such that $d(u, m) = d(v, l) = i$ for all points $u \in l$ and $v \in m$.*

The lines l and m satisfying Theorem 1.18(*ii*) are called *parallel lines*.

1.6.1. Quads. Let $S = (P, L)$ be a near polygon. A subspace C of P is *convex* if every geodesic in $\Gamma(P)$ between two points of C is entirely contained in the induced subgraph $\Gamma(C)$ of $\Gamma(P)$. A *quad* of S is a non-degenerate convex subspace of S of diameter two. Thus a quad is a generalized quadrangle.

Theorem 1.19 ([**61**], Proposition 2.5, p.10)**.** *Let $S = (P, L)$ be a near polygon and $x_1, x_2 \in P$ with $d(x_1, x_2) = 2$. If x_1 and x_2 have at least two common neighbors y_1 and y_2 such that at least one of the lines x_iy_j is thick, then x_1 and x_2 are contained in a unique quad. This quad consists of all points of S which have distance at most 2 from each of x_1, x_2, y_1 and y_2.*

The unique quad containing x_1 and x_2 in Theorem 1.19 is denoted by $Q(x_1, x_2)$. An immediate consequence of Theorem 1.19 is the following 'quad-quad' relation.

Corollary 1.20. *Two distinct quads of a near polygon are either disjoint, or meet in a point or a line.*

The possible 'point-quad' relations are given in the following.

Theorem 1.21 ([**61**], Proposition 2.6, p.12)**.** *Let $S = (P, L)$ be a near polygon. Let $x \in P$ and Q be a quad of S. Then, either*

(*i*) *there is a unique point $y \in Q$ closest to x (depending on x) and $d(x, z) = d(x, y) + d(y, z)$ for all $z \in Q$; or*
(*ii*) *the points in Q closest to x form an ovoid $\mathcal{O}_x$ of Q.*

In the first case, this means that Q is *gated* with respect to x, in the sense of [**28**]. The point-quad pair (x, Q) in Theorem 1.21 is called *classical* in the first case and *ovoidal* in the second case. A quad Q is *classical* if (x, Q) is classical for each $x \in P$. If (x, Q) is not classical for at least one $x \in P$, then Q is called *ovoidal*.

1.6.2. Dense near polygons. A near polygon is said to be *dense* if each of its line is thick and each pair of points at distance two from each other have at least two common neighbours. By Theorem 1.19, any such pair is contained in a unique quad of S. All dense near $2n$-gons with parameters (s, t) are classified for $s = 2$ and $n = 1, 2, 3, 4$ (see [**26**]). A dense near polygon is *classical* if each quad of it is classical.

Lemma 1.22 ([**10**], Theorem 2, p.151)**.** *Let $S = (P, L)$ be a dense near polygon. Let $x, y \in P$ with $d(x, y) = i$ and $x = x_0, x_1, \cdots, x_i = y$ be a geodesic between x and y. Then, there exists a geodesic $y = y_0, y_1, \cdots, y_i = x$ such that $d(x_j, y_j) = i$ for $0 \leq j \leq i$.*

As an immediate consequence of Lemma 1.22, we have

Corollary 1.23. *Let $S = (P, L)$ be a dense near polygon. Let $x, y \in P$ with $d(x, y) = i \geq 2$. Then, for every line $l \in L$ containing x, there exists a line m containing y such that l and m are parallel lines and $d(l, m) = i$.*

Lemma 1.24 ([**10**], Lemma 19, p.152)**.** *Let $S = (P, L)$ be a finite dense near polygon. Then, the number of lines containing a point of S is independent of the point.*

Lemma 1.25 ([**10**], Corollary to Theorem 3, p.156)**.** *Let $S = (P, L)$ be a dense near $2n$-gon. Then, the induced subgraph of $\Gamma(P)$ on $\Gamma_n(x)$ is connected for each $x \in P$.*

The following proposition is an improvement of Lemma 1.25.

Proposition 1.26. *Let $S = (P, L)$ be a dense near $2n$-gon and H be a geometric hyperplane of S. Set $H' = P \setminus H$. Then, the subgraph $\Gamma(H')$ of $\Gamma(P)$ is connected.*

Proof. Let $x, y \in H'$ and $d(x, y) = k$ in $\Gamma(P)$. We use induction on k. For any geodesic $x = x_0, x_1, \cdots, x_k = y$ from x to y in $\Gamma(P)$, we may assume that the intermediate points x_i $(1 \leq i \leq k-1)$ are in H. For if $x_i \notin H$ for some i $(1 \leq i \leq k-1)$, then we can connect x and x_i (respectively, x_i and y) by a path in $\Gamma(H')$ by induction.

Now, fix a geodesic $x = x_0, x_1, \cdots, x_k = y$ from x to y in $\Gamma(P)$. There exists a geodesic $y = y_0, y_1, \cdots, y_k = x$ from y to x in $\Gamma(P)$

such that $d(x_i, y_i) = k, 0 \leq i \leq k$ (Lemma 1.22). Let a be a point in the line x_0x_1 different from x_0 and x_1. Since $d(y_0, x_0) = k$ and $d(y_0, x_1) = k - 1$, $d(y_0, a) = k$ in $\Gamma(P)$. Similarly, $d(a, y_1) = k$ in $\Gamma(P)$. So there exists a point b different from y_0 and y_1 in the line y_0y_1 such that $d(a, b) = k - 1$ in $\Gamma(P)$. By our assumption, $x_1, y_1 \in H$. So $a, b \in H'$ because, $x_0 \notin H, y_0 \notin H$ and H is a geometric hyperplane of S. By induction, a and b are connected by a path $a = a_1, \cdots, a_m = b$ in $\Gamma(H')$. Then, $x, a = a_1, \cdots, a_m = b, y$ is a path from x to y in $\Gamma(H')$. This completes the proof. □

Proposition 1.26 holds for a generalized polygon also, except in a few cases, see [**8**].

1.6.3. Near polygons from dual polar spaces. Let $S = (P, L)$ be a polar space of rank $n \geq 2$. Consider the point-line geometry $DS = (P', L')$ constructed as follows:

- P' is the collection of all maximal singular subspaces of S;
- A line of DS is the collection of all maximal singular subspaces of S containing a specific singular subspace of S of co-dimension 1.

Then, DS is a partial linear space, called the *dual polar space of rank* n associated with S. These geometries are characterized in terms of points and lines by Cameron [**13**]. Dual polar spaces of rank n are near $2n$-gons.

Lemma 1.27 ([**13**], Theorem 1,p.75)**.** *The dual polar spaces of rank* n *are the classical dense near* $2n$*-gons.*

Let S and DS be as above. For $a \in P$, define $a' = \{X \in P' : a \in X\}$. Let A be a subset of P and set $A' = \bigcup_{a \in A} a'$. Then, the collinearity graph $\Gamma(P')$ of DS induces a graph structure on A'. An edge in $\Gamma(A')$ is a pair of elements of A' sharing a subspace of S of co-dimension 1. Distance between two points of A' is the same in $\Gamma(A')$ as well as in $\Gamma(P')$. If A' is a subspace of DS, then A' is a near polygon which may not have quads ([**9**], p.352).

1.7. Slim Dense Near Polygons

Let $S = (P, L)$ be a slim dense near $2n$-gon, $n \geq 1$. If $n = 1$, then $S \simeq \mathbb{L}_3$, a line of size 3. If $n = 2$, then S is a $(2, t)$-GQ. In that case, P is finite and $t = 1, 2$ or 4. Further, for each value of t there exists a unique $(2,t)$-GQ, up to isomorphism (see [**14**], Theorem 7.3, p.99). Thus, S is isomorphic to the classical generalized quadrangles $Q_4^+(2) \simeq \mathbb{L}_3 \times \mathbb{L}_3$, $W_4(2) \simeq Q_5(2)$ and $Q_6^-(2)$, respectively, for $t = 1, 2$ and 4. In Chapter 5, we study these generalized quadrangles in detail. If $n = 3$, then S is called a *slim dense near hexagon.*

The dual polar spaces $DW_{2n}(2)$ and $DH_{2n}(2^2)$ of rank n are slim dense near $2n$-gons (Lemma 1.27). In the rest of this section, we describe three infinite families of slim dense near $2n$-gons $\mathbb{H}_n, \mathbb{G}_n$ and $\mathbb{I}_n$, and three 'exceptional' slim dense near hexagons $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_3$. We refer to [**26**] for more on slim dense near polygons.

1.7.1. The near polygon $\mathbb{H}_n$, $n \geq 1$. Let X be a set of size $2n+2$, $n \geq 1$. Let $\mathbb{H}_n = (P, L)$ be the partial linear space, where:

- P is the set of all partitions of X into $n+1$ sets of size 2;
- Lines are the collections of partitions sharing $n-1$ common 2-subsets.

Then, $\mathbb{H}_n$ is a dense near $2n$-gon with parameters $(s,t) = (2, \frac{n(n+1)}{2})$ (see [**9**], p.355). Clearly, $\mathbb{H}_1 \simeq \mathbb{L}_3$ and $\mathbb{H}_2 \simeq W_4(2)$. Each quad of $\mathbb{H}_n, n \geq 3$, is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3$ or $W_4(2)$.

1.7.2. The near polygon $\mathbb{G}_n$, $n \geq 1$. Let $\mathbb{F}_4^{2n}$ denote the $2n$-dimensional vector space over $\mathbb{F}_4$, let $\{e_0, \cdots, e_{2n-1}\}$ be a basis of it and $(-,-)$ be the non-singular hermitian form on it defined by $(x,y) = x_0y_0^2 + x_1y_1^2 + \cdots + x_{2n-1}y_{2n-1}^2$, where $x = \sum x_ie_i$ and $y = \sum y_ie_i$. Let H denote the corresponding hermitian variety in $PG(2n-1, 2^2)$. The *support* S_α of a point $\alpha = \mathbb{F}_4x$ of $PG(2n-1, 2^2)$ is the set of all $i \in \{0, 1, \cdots, 2n-1\}$ for which $(x, e_i) \neq 0$ and its cardinality is called the *weight* of α. A point of $PG(2n-1, 2^2)$ belongs to H if and only if its weight is even. A subspace π of H is said to be *good* if it is generated by a set $G_\pi \subseteq H$ of points whose supports are pair-wise disjoint. If π is good, then G_π is uniquely determined. If G_π contains k_{2i} points of weight $2i$, $i \in \mathbb{N} \setminus \{0\}$, then π is said to be of *type* $(2^{k_2}, 4^{k_4}, \cdots)$. Let Y (respectively, Y') denote the set of all good subspaces of dimension $n-1$ (respectively, $n-2$). Then, every element of Y has type (2^n). The type of an element of Y' is either (2^{n-1}) or $(2^{n-1}, 4^1)$. The subspace (Y, Y') of $DH_{2n}(2^2)$ is a slim dense near $2n$-gon [**24**], denoted by $\mathbb{G}_n$. It has parameters $(s,t) = (2, \frac{3n^2-n-2}{2})$. The number of points of $\mathbb{G}_n$ equals $\frac{3^n.(2n)!}{2^n.n!}$. Clearly, $\mathbb{G}_1 \simeq \mathbb{L}_3$ and $\mathbb{G}_2 \simeq Q_6^-(2)$. If $n \geq 3$, then all the three types of quads occur in $\mathbb{G}_n$.

1.7.3. The near polygon $\mathbb{I}_n$, $n \geq 2$. Consider a non-singular quadric $Q(2n,2)$, $n \geq 2$, in $PG(2n,2)$ and a hyperplane Π of $PG(2n,2)$ intersecting $Q(2n,2)$ in a non-singular hyperbolic quadric $Q^+(2n-1,2)$. Let $\mathbb{I}_n$ be the subspace of the orthogonal dual polar space $DQ_{2n+1}(2)$ whose points are the maximal subspaces of $Q(2n,2)$ which are not contained in $Q^+(2n-1,2)$. Then, $\mathbb{I}_n$ is a dense near $2n$-gon ([**9**], p.352). It has parameters $(s,t) = (2, 2^n - 3)$. Each quad of $\mathbb{I}_n$, $n \geq 3$ is isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3$ or to $W_4(2)$. For $n = 2$, $\mathbb{I}_3 \simeq Q_4^+(2)$ and for $n = 3$, $\mathbb{I}_3 \simeq \mathbb{H}_3$.

1.7.4. The near hexagon $\mathbb{E}_1$. Let $\mathbb{F}_3^{12}$ denote the 12-dimensional vector space over $\mathbb{F}_3$ and let A denote the following matrix over $\mathbb{F}_3$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Let C be the 6-dimensional subspace of $\mathbb{F}_3^{12}$ generated by the rows of A. The subspace C is called the *extended ternary Golay code* (see [**17**], p.85). A near hexagon $\mathbb{E}_1$ can be constructed from C as follows:

- The points of $\mathbb{E}_1$ are the 729 cosets of C in $\mathbb{F}_3^{12}$;
- The lines are the triples of the form $\{v+C, v+e_i+C, v-e_i+C\}$, $v \in \mathbb{F}_3^{12}$, $i \in \{1, 2, \cdots, 12\}$, where e_i is the element of $\mathbb{F}_3^{12}$ whose i-th co-ordinate equals 1 and all other co-ordinates equal 0.

Thus, two cosets are collinear whenever they contain vectors that differ in only one position, and the line containing them is the line of the affine space $\mathbb{F}_3^{12}/C$ containing them. The near hexagon $\mathbb{E}_1$ is dense and has parameters $(s,t) = (2,11)$ ([**61**], p.30). All quads of $\mathbb{E}_1$ are isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3$. Up to isomorphism, there is only one slim dense near hexagon on 729 points [**5**].

Here is another description of $\mathbb{E}_1$ [**27**]. Consider again the matrix A above. The columns of A define a set K of twelve points in $PG(5,3)$. Any set of five different points generate a four dimensional subspace containing precisely six points of K (see [**21**]). This gives a model for the Witt design $S(5,6,12)$. Consider the point-line geometry $T_5^*(K)$ obtained as follows: Embed $PG(5,3)$ as a hyperplane Π_∞ in a 6-dimensional projective space Π. The points of $T_5^*(K)$ are the points of $\Pi \setminus \Pi_\infty$ and the lines of $T_5^*(K)$ are the lines of Π intersecting Π_∞ in a point of K. It is shown in [**27**] that $\mathbb{E}_1$ is isomorphic to $T_5^*(K)$.

1.7.5. The near hexagon $\mathbb{E}_2$. First, recall that a Steiner system $S(24, 8, 5)$ is a set X of size 24 and a family B of 759 subsets of X of size eight each such that any 5-subset of X is in a unique member of B (see [**17**], p.276). It is known this is unique up to isomorphism. Taking B as the set of points and the collection of triples of members of B which partition X as the set of lines, we get a dense near hexagon with parameters $(s, t) = (2, 14)$ ([**61**], p.40). We denote this near hexagon by $\mathbb{E}_2$. All quads of $\mathbb{E}_2$ are isomorphic to $W_4(2)$. There is only one slim dense near hexagon on 759 points, up to isomorphism [**6**].

1.7.6. The near hexagon $\mathbb{E}_3$. A non-empty set X of points of a partial linear space S is called a *hyperoval* if every line of S intersects X in zero or two points. The unitary polar space $H_6(2^2)$ has two isomorphism classes of hyperovals [**49**]. The hyperovals of one class contain 126 points and the hyperovals of the other class contain 162 points. Let X be a hyperoval of $H_6(2^2)$ of size 126. Each maximal singular subspace of $H_6(2^2)$ has zero or six points in common with X. The near hexagon $\mathbb{E}_3$ is a subspace of the dual polar space $DH_6(2^2)$ consisting of the points (that is, maximal singular subspaces of $H_6(2^2)$) intersecting X in six points (see [**26**], p.159). $\mathbb{E}_3$ is dense, has 567 points and parameters $(s, t) = (2, 14)$. Each quad of $\mathbb{E}_3$ is isomorphic to $W_4(2)$ or to $Q_6^-(2)$.

CHAPTER 2

Triangular Sets

Let G be a finite group and $\mathcal{I}(G)$ be the *involution geometry* of G (see [**43**], p.111). That is, $\mathcal{I}(G)$ is a partial linear space whose point set is the set $I_2(G)$ of all involutions in G and the line set consists of all triples $\{a, b, ab\}$, where a and b are distinct commuting involution in G. Here we do not assume that G is generated by $I_2(G)$. A *triangular set* in G is a G-invariant subspace (under conjugation) of $\mathcal{I}(G)$. Clearly, the empty set and $I_2(G)$ are triangular sets in G, called the *trivial* triangular sets. If T_1 and T_2 are two triangular sets in G, then $T_1 \cap T_2$ is a triangular set in G. If T is a triangular set in G and H is a subgroup of G, then $T \cap H$ is a triangular set in H.

In this chapter, we determine the non-trivial triangular sets in the finite irreducible Coxeter groups $W(A_n)$, $W(B_n)$, $W(D_n)$, $W(I_n)$, $W(H_3)$, $W(H_4)$, $W(F_4)$ and $W(E_6)$, except for $W(E_7)$ and $W(E_8)$.

The following fundamental lemma is useful for us.

Lemma 2.1 ([**48**], Proposition 4.6.1, p.132)**.** *Let G be a finite group and N be a normal subgroup of G of index 2. For $x \in N$,*

(i) $x^G = x^N$ if and only if $C_N(x) \neq C_G(x)$.

(ii) If $C_N(x) = C_G(x)$, then x^G splits into exactly two conjugacy classes in N, both of equal cardinality.

Proof. (i) If $x^G = x^N$, then $C_N(x) \neq C_G(x)$, otherwise, $[G : C_G(x)] = |x^G| = |x^N| = [N : C_N(x)]$ which is not possible. Conversely, let $[x, t] = 1$ for some $t \notin N$. Let $y, z \in x^G$ and $a, b \in G$ be such that $y = axa^{-1}$ and $x = bzb^{-1}$. Then, we can write:

$$\begin{aligned} y &= axa^{-1} = (ab)z(ab)^{-1}, \text{ and} \\ y &= axa^{-1} = (at)x(at)^{-1} = (atb)z(atb)^{-1}. \end{aligned}$$

Since N is normal in G and $t \notin N$, $(ab)^{-1}(atb) = b^{-1}tb \notin N$. So $(ab)N \neq (atb)N$. Since N is of index 2 in G, either $(ab)N = N$ or $(atb)N = N$. So $ab \in N$ or $atb \in N$. In either case, y and z are conjugate in N.

(ii) This follows from $|x^N| = [N : C_N(x)] = \frac{1}{2}[G : N][N : C_N(x)] = \frac{1}{2}[G : C_G(x)] = \frac{1}{2}|x^G|$. □

2.1. Triangular Sets in $W(A_n), n \geq 1$

The group $W(A_n)$ is isomorphic to $Sym(n+1)$ where $Sym(n)$ denotes the symmetric group defined on a set of n elements. An involution in $Sym(n)$ is *even* (respectively, *odd*) if it is an even (respectively, odd) permutation. If $n = 1$ or 2, then $W(A_n)$ has only one conjugacy class of involutions, so it has no non-trivial triangular set.

Proposition 2.2. *For $n \geq 3$, the set T consisting of all even involutions of $W(A_n)$ is the only non-trivial triangular set in $W(A_n)$.*

Proof. The product of two distinct commuting even involutions is an even involution. So T is a subspace of $I_2(W(A_n))$. Since T is $W(A_n)$-invariant, it is a triangular set in $W(A_n)$.

Let T' be a triangular set in $W(A_n)$. First, suppose that each element of T' is an even involution. We show that $T = T'$. For this it is enough to show that T' contains an element which is a product of two disjoint transpositions. Let $x \in T'$. We write x as a product of pair-wise disjoint transpositions

$$x = (a_1, b_1) \cdots (a_r, b_r) \tag{2.1.1}$$

for some $r \geq 2$ even. If $r = 2$, then we are done. Assume that $r \geq 4$. Let $y \in W(A_n)$ be the involution defined as

$$y = (a_1, a_2)(b_1, b_2)(a_3, b_3) \cdots (a_r, b_r). \tag{2.1.2}$$

Since x and y are conjugate in $W(A_n)$, $y \in T'$. Since $[x, y] = 1$, $xy = (a_1, b_2)(a_2, b_1) \in T'$.

Now, suppose that T' contains an odd involution. We show that $T' = I_2(W(A_n))$. For this it is enough to show that T' contains a transposition. Let $x \in T'$ be an odd involution. We write x as in (2.1.1) with $r \geq 1$ odd. If $r = 1$, then we are done. Assume that $r \geq 3$. Defining y as in (2.1.2), the above argument yields $T \subset T'$. Let $z \in W(A_n)$ be the involution $z = (a_1, b_1) \cdots (a_{r-1}, b_{r-1})$. Since $z \in T$ and $[x, z] = 1$, it follows that $xz = (a_r, b_r) \in T'$. □

We denote by $Alt(n)$ the alternating group defined on n symbols.

Lemma 2.3. *For $x \in I_2(Alt(n)), n \geq 4$, $x^{Sym(n)} = x^{Alt(n)}$.*

Proof. We write x as in (2.1.1) for some $r \geq 2$ even. Let $t = (a_1, b_1)$. Since $t \in C_{Sym(n)}(x)$ and $t \notin C_{Alt(n)}(x)$, Lemma 2.1(i) completes the proof. □

Proposition 2.4. *$Alt(n), n \geq 4$, has no non-trivial triangular set.*

Proof. Let T be a non-empty triangular set in $Alt(n)$. We show that $T = I_2(Alt(n))$. By Lemma 2.3, it is enough to show that T contains an element which is a product of two disjoint transpositions. For $x \in T$, we write x as in (2.1.1) for some $r \geq 2$ even and take y as in (2.1.2). Then, by Lemma 2.3, x and y are conjugate in $Alt(n)$. Since $[x, y] = 1$, $xy = (a_1, b_2)(a_2, b_1) \in T$. □

2.2. Triangular Sets in $W(B_n), n \geq 2$

The group $W(B_n)$ may be considered as the group of signed permutations on n symbols $1, \cdots, n$ (see [**35**], p.5). Define the i^{th} *sign change* to be the element of $W(B_n)$ sending i to $-i$ and fixing all other j. The set of such elements generate a group of order 2^n isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Then, $W(B_n)$ is the semidirect product of this group by $Sym(n)$.

For our purpose, we define $W(B_n)$ as follows: Let $Sym(2n)$, $n \geq 2$, be the symmetric group defined on the $2n$ symbols $\pm 1, \cdots, \pm n$. Then:

$$W(B_n) = \{x \in Sym(2n) : x(i) + x(-i) = 0, \text{ for all } i \in \{1, \cdots n\}\}.$$

Each element $x \in W(B_n)$ is of the form

$$x = \begin{pmatrix} 1 & \cdots & n & -n & \cdots & -1 \\ \epsilon_1 y(1) & \cdots & \epsilon_n y(n) & -\epsilon_n y(n) & \cdots & -\epsilon_1 y(1) \end{pmatrix}, \tag{2.2.1}$$

for unique $(\epsilon_1, \cdots, \epsilon_n) \in C_2^n$ and unique $y \in Sym(n)$, where $C_2 = \{1, -1\}$ is the multiplicative cyclic group of order 2 and $Sym(n)$ is the symmetric group on the symbols $1, \cdots, n$.

Let $Sym(n)$ be the symmetric group defined on $I_1, \cdots, I_n$, where $I_i = \{i, -i\}$. Then, the map

$$\phi : W(B_n) \longrightarrow Sym(n) \tag{2.2.2}$$

defined by $\phi(x)(I_i) = I_j$, where $x \in W(B_n)$ and $x(I_i) = \{x(i), x(-i)\} = \{j, -j\} = I_j$, is a surjective homomorphism with $ker(\phi) \simeq C_2^n$. So, $W(B_n)$ is an extension of C_2^n by $Sym(n)$. Since C_2^n has a complement in $W(B_n)$ isomorphic to $Sym(n)$, $W(B_n)$ is a semi-direct product of C_2^n by $Sym(n)$ and consequently, $|W(B_n)| = 2^n n!$.

Remark 2.5. *If* $Sym(2n)$ *is the symmetric group defined on the* $2n$ *symbols* $1, \cdots, 2n$, *then* $W(B_n)$ *can be defined as*

$$W(B_n) = \{x \in Sym(2n) : x(i) + x(2n - i + 1) = 2n + 1, i \in \{1 \cdots 2n\}\}.$$

An element in $W(B_n)$ which is a product of two l-cycles in $Sym(2n)$ of the form $(a_1, \cdots, a_l)(-a_1, \cdots, -a_l)$ is called a *positive l-cycle*; and which is a $2l$ cycle in $Sym(2n)$ of the form $(a_1, \cdots, a_l, -a_1, \cdots, -a_l)$ is called a *negative l-cycle*;

Note that the order of a positive l-cycle is l where as that of a negative l-cycle is $2l$. A positive 2-cycle $(a, b)(-a, -b)$ is called a *positive transposition*. A negative 1-cycle $(a, -a)$ is called a *negative transposition*. Every positive (respectively, negative) cycle in $W(B_n)$ is a product of positive (respectively, negative) transpositions. Positive transpositions of the form $t_i = (i, i + 1)(-i, -(i + 1)), 1 \leq i \leq n - 1$, generate a subgroup of $W(B_n)$ isomorphic to $Sym(n)$ and the negative transpositions generate a normal subgroup of $W(B_n)$ isomorphic to C_2^n.

Lemma 2.6 ([**48**], Proposition 7.2.1, p.201)**.** *Every element of* $W(B_n)$ *can be uniquely expressed as a product of disjoint positive and negative cycles.*

Corollary 2.7. *An element of* $W(B_n)$ *is an involution if and only if it is a product of disjoint positive and negative transpositions.*

An element x of $W(B_n)$ is said to be of *type* (p_x, n_x) if it is a product of p_x number of positive cycles and n_x number of negative cycles. The integers p_x and n_x are called the *positive* and the *negative parts* of x, respectively.

Lemma 2.8 ([**48**], Theorem 7.2.5, p.202)**.** *Any two involutions in* $W(B_n)$ *of the same type are conjugate in* $W(B_n)$.

We next determine the triangular sets in $W(B_n)$. Consider the following subsets of $I_2(W(B_n))$:

$T_1 = \{(1, -1) \cdots (n, -n)\}$;
$T_2 = \{x : p_x = 0, n_x \text{ is even}\}$;
$T_3 = \{x : p_x = 0\}$;
$T_4 = \{x : p_x \text{ is even }, n_x \text{ is even}\}$;
$T_5 = \{x : n_x \text{ is even}\}$;
$T_6 = \{x : p_x \text{ is even}\}$;
$T_7 = \{x : \text{both } p_x \text{ and } n_x \text{ have the same parity}\}$.

Proposition 2.9. *$T_i, 1 \leq i \leq 7$, is a triangular set in $W(B_n)$.*

Proof. Clearly, T_1, T_2 and T_3 are triangular sets in $W(B_n)$. Now, T_5 is a triangular set in $W(B_n)$ because, it is the intersection of $W(B_n)$ with the non-trivial triangular set (consisting of all even involutions) in $Sym(2n)$. Since $T_5 \cap T_6 = T_4$, we only need to show that T_6 and T_7 are triangular sets in $W(B_n)$.

It is clear that T_6 and T_7 are normal subsets of $W(B_n)$. Consider the map ϕ defined in (2.2.2). For every involution $z \in W(B_n)$, p_z is even if and only if $\phi(z)$ is an even involution in $Sym(n)$. Let $x, y \in T_6, x \neq y$, with $[x, y] = 1$. Since p_x and p_y are even, $\phi(xy) = \phi(x)\phi(y)$ is an even involution in $Sym(n)$. So p_{xy} is even and T_6 is a subspace of $\mathcal{I}(W(B_n))$. A similar argument, together with the fact that $x \in W(B_n)$ is an even involution in $Sym(2n)$ if and only if n_x is even, shows that T_7 is a subspace of $\mathcal{I}(W(B_n))$. So, T_6 and T_7 are triangular sets in $W(B_n)$. ☐

Lemma 2.10. *Let T be a non-empty triangular set in $W(B_n)$ such that $p_y = 0$ for all $y \in T$. Then, $T = T_1, T_2$ or T_3.*

Proof. We may assume that $T \neq T_1$. Let $x = (a_1, -a_1) \cdots (a_r, -a_r) \in T$ be such that $T_1 \neq \{x\}$. For $b \notin \{\pm a_1, \cdots, \pm a_r\}$, let $y = (a_1, -a_1) \cdots (a_{r-1}, -a_{r-1})(b, -b)$. Since x and y are of the same type, they are conjugate in $W(B_n)$. Since $[x, y] = 1$, $xy = (a_r, -a_r)(b, -b) \in T$. This implies that $T_2 \subseteq T$. If $T \neq T_2$, then n_z is odd for some $z \in T$. Then, it follows that T contains a negative transposition and so $T = T_3$. ☐

Notation 2.11. *For a positive cycle $x = (a_1, \cdots, a_l)(-a_1, \cdots, -a_l)$ in $W(B_n)$, we write $x = x_1\overline{x}_1$, where $x_1 = (a_1, \cdots, a_l), \overline{x}_1 = (-a_1, \cdots, -a_l)$. If $x \in W(B_n)$ is of type (p_x, n_x), then we write x as*

$$x = x_1\overline{x}_1 \cdots x_{p_x}\overline{x}_{p_x}x'_1 \cdots x'_{n_x}, \tag{2.2.3}$$

where x'_i are negative cycles.

Lemma 2.12. *Let T be a triangular set in $W(B_n)$ such that $p_x \neq 0$ for some $x \in T$. Then, $T_2 \subseteq T$. Further, if $p_x \geq 2$, then $T_4 \subseteq T$.*

Proof. We first prove that $T_2 \subseteq T$. For this it is enough to show that T contains an element which is a product of two negative transpositions. We write x as in (2.2.3), where $x_i\overline{x}_i = (a_i, b_i)(-a_i, -b_i)$. Let y be the involution in $W(B_n)$ defined as

$$y = y_1\overline{y}_1x_2\overline{x}_2 \cdots x_{p_x}\overline{x}_{p_x}x'_1 \cdots x'_{n_x}, \tag{2.2.4}$$

where $y_1\overline{y}_1 = (a_1, -b_1)(-a_1, b_1)$. Then, x and y are conjugate in $W(B_n)$ and $[x, y] = 1$. So $y \in T$, $xy = (a_1, -a_1)(b_1, -b_1) \in T$ and $T_2 \subset T$.

Now, assume that $p_x \geq 2$. Let z be the involution in $W(B_n)$ defined as

$$z = z_1\overline{z}_1 z_2\overline{z}_2 x_3\overline{x}_3 \cdots x_{p_x}\overline{x}_{p_x} x'_1 \cdots x'_{n_x}, \tag{2.2.5}$$

where $z_1\overline{z}_1 = (a_1, a_2)(-a_1, -a_2)$ and $z_2\overline{z}_2 = (b_1, b_2)(-b_1, -b_2)$. Since x and z are conjugate in $W(B_n)$, $z \in T$. Now, $[x, z] = 1$ implies that

$$xz = (a_1, b_2)(-a_1, -b_2)(a_2, b_1)(-a_2, -b_1) \in T.$$

This, together with $T_2 \subseteq T$, implies that $T_4 \subseteq T$. □

Theorem 2.13. $T_i, 1 \leq i \leq 7$, *are the only non-trivial triangular sets in* $W(B_n)$.

Proof. Let T be a non-empty triangular set in $W(B_n)$. By Lemma 2.10, we assume that $p_x \neq 0$ for some $x \in T$. Then, $T_2 \subseteq T$ by Lemma 2.12.

First, assume that p_y is even for all $y \in T$. Then, $p_x \geq 2$ and so $T_4 \subseteq T$ by Lemma 2.12. If $T \neq T_4$, then n_y is odd for some $y \in T$. Since $T_2 \subseteq T$, it follows that T contains all negative transpositions and so, $T = T_6$.

Now, we may assume that p_x is odd. By Lemmas 2.10 and 2.12, we may assume $(p_x, n_x) = (1, 0)$ or (1,1) according as n_x is even or not. If $(p_x, n_x) = (1, 0)$, then it follows that T contains all positive transpositions. So $T_5 \subseteq T$ since $T_2 \subset T$. If $T_5 \neq T$, then T contains an element whose negative part is odd. Then, it follows that T contains all negative transpositions also and so $T = I_2(W(B_n))$.

If $(p_x, n_x) = (1, 1)$, then it follows that T contains all involutions whose positive and negative parts are either even or odd. So $T_7 \subseteq T$. If $T \neq T_7$, then there exists $y \in T$ such that either p_y odd and n_y even or p_y even and n_y odd. Then, T contains all positive transpositions in the first case and all negative transpositions in the latter case. In both cases, it follows that T contains all positive and negative transpositions and so, $T = I_2(W(B_n))$. □

2.3. Triangular Sets in $W(D_n), n \geq 4$

For $n \geq 4$, the group $W(D_n)$ is defined to be the subgroup of $W(B_n)$ given by

$$W(D_n) = \{x \in W(B_n) : |\{i : x(i) < 0, 1 \leq i \leq n\}| \text{ is even}\},$$

or equivalently,

$$W(D_n) = \{x \in W(B_n) : x(1) \cdots x(n) > 0\}.$$

Each element $x \in W(D_n)$ is of the form (2.2.1) with $\epsilon_1 \cdots \epsilon_n = 1$. Thus, $W(D_n)$ is a normal subgroup of $W(B_n)$ of index 2. Observe that if $x \in W(B_n)$ is an involution, then $x \in W(D_n)$ if and only if n_x is even.

Lemma 2.14 ([**48**], Theorem 8.2.1, p.222)**.** *Let $x \in W(D_n)$ be an involution.*

(i) *If $n_x \neq 0$ or $4p_x \neq 2n$, then $x^{W(B_n)} = x^{W(D_n)}$.*
(ii) *If $n_x = 0$ and $4p_x = 2n$, then $x^{W(B_n)}$ splits into a union of two conjugacy classes in $W(D_n)$.*

In particular, if n is odd, then $x^{W(B_n)} = x^{W(D_n)}$.

We next determine the triangular sets in $W(D_n)$. Consider the following subsets of $I_2(W(D_n))$:

$P_1 = \{(1, -1) \cdots (n, -n), n \text{ even}\}$;
$P_2 = \{x : p_x = 0\}$;
$P_3 = \{x : p_x \text{ is even}\}$.

Proposition 2.15. *P_1, P_2 and P_3 are triangular sets in $W(D_n)$.*

Proof. This follows because, $P_1 = T_1 \cap W(D_n)$ if n is even, $P_2 = T_2 \cap W(D_n)$ and $P_3 = T_4 \cap W(D_n)$ for the triangular sets T_1, T_2 and T_4 in $W(B_n)$ as defined in the previous section. □

Lemma 2.16. *Let T be a non-empty triangular set in $W(D_n)$ such that $p_y = 0$ for every $y \in T$. Then, $T = P_2$ if n is odd, and $T = P_1$ or P_2 if n is even.*

Proof. If n is even, we assume that $T \neq P_1$. Then, using Lemma 2.14, the proposition follows from the proof of Lemma 2.10. □

Lemma 2.17. *Assume that $n \geq 5$. Let T be a non-trivial triangular set in $W(D_n)$ such that $p_x \geq 1$ for some $x \in T$. Then, $T = P_3$.*

Proof. It is enough to show that $P_3 \subseteq T$. We write x as in (2.2.3) with n_x even, where $x_i\overline{x}_i = (a_i, b_i)(-a_i, -b_i)$. First, assume that $n_x \neq 0$ or $4p_x \neq 2n$ (this is the case if n is odd). So $x^{W(B_n)} = x^{W(D_n)}$, by Lemma 2.14. Taking y as in (2.2.4) and applying the argument as in the proof of Lemma 2.12, it follows that $P_2 \subset T$. Then, $p_x \geq 2$, otherwise T would contain all positive transpositions also and so, $T = I_2(W(D_n))$, a contradiction to the non-triviality of T. Now, taking z as in (2.2.5) and applying again the argument as in the proof of Lemma 2.12, we get $xz = (a_1, b_2)(-a_1, -b_2)(a_2, b_1)(-a_2, -b_1) \in T$. Now, it follows that T contains all involutions in $W(D_n)$ which are product of two disjoint positive transpositions. This, together with $P_2 \subset T$, implies that $P_3 \subseteq T$.

Now, assume that $n_x = 0$ and $4p_x = 2n$. Then $n \geq 6$ is even. Again, we take z as in (2.2.5). Let $w = (a_2, b_1)(-a_2, -b_1) \in W(D_n)$. Then, $wxw^{-1} = z$. So x and z are conjugate in $W(D_n)$ and $z \in T$. The above argument yields that T contains all involutions in $W(D_n)$ which are product of two disjoint positive transpositions. Now, taking

$$x = (a_1, b_1)(-a_1, -b_1)(a_2, b_2)(-a_2, -b_2) \in T,$$
$$y = (a_1, -b_1)(-a_1, b_1)(a_2, b_2)(-a_2, -b_2) \in T,$$

the above argument again yields $P_2 \subset T$ and so, $P_3 \subseteq T$. □

Now, let $n = 4$. We denote by C^1 and C^2 the two conjugacy classes of involutions in $W(D_4)$ containing elements which are product of two disjoint positive transpositions. Consider the following subsets of $I_2(D_4)$:

$$P_4 = C_1 \cup \{(1, -1) \cdots (4, -4)\}\};$$
$$P_5 = C_2 \cup \{(1, -1) \cdots (4, -4)\}.$$

Let e denote the identity element in $W(D_4)$. For $i \in \{4, 5\}$, it can be seen that $P_i \cup \{e\}$ is an elementary abelian 2-subgroup of $W(D_4)$ of order 8. So P_4 and P_5 are triangular sets in $W(D_4)$.

Lemma 2.18. *Let T be a non-trivial triangular set in $W(D_4)$. Then, T is one of sets P_1, P_2, P_3, P_4 and P_5.*

Proof. If $T \notin \{P_1, P_2, P_3, P_4, P_5\}$, then $p_x = 1$ for some $x \in T$. We write $x = (a, b)(-a, -b)x'_1 x'_{n_x}$, where x'_i are negative transpositions with $n_x = 0$ or 2. Take $y = (a, -b)(-a, b)x'_1 x'_{n_x}$. Then, x and y are conjugate in $C(D_4)$ and $[x, y] = 1$. So $y \in T$ and $xy = (a, -a)(b, -b) \in T$. Now, it follows that T contains all involutions which are product of two negative transpositions. This implies that T contains all positive transpositions also and so $T = I_2(W(D_4))$, a contradiction to the non-triviality of T. □

We now summarize the above results.

Theorem 2.19. *Let T be a non-trivial triangular set in $W(D_n)$. Then, one of the following hold:*

(*i*) $n \geq 6$ *is even and* $T = P_1, P_2$ *or* P_3.
(*ii*) $n \geq 5$ *is odd and* $T = P_2$ *or* P_3.
(*iii*) $n = 4$ *and* $T = P_1, P_2, P_3, P_4$ *or* P_5.

2.4. Triangular Sets in the Exceptional Groups

The group $W(I_n)$, $n \geq 3$, is the dihedral group of order $2n$. It has the presentation $W(I_n) = \langle x, y : x^n = y^2 = 1, x^y = x^{-1} \rangle$. An element of $W(I_n)$ is of the form $x^i y^j$ for some $i \in \{0, 1, \cdots, n-1\}$ and $j \in \{0, 1\}$. Each element of the form $x^i y$ is an involution in $W(I_n)$.

If n is odd, then $W(I_n)$ has only one conjugacy class of involutions (see [**1**], 45.2, p.242). In that case, $W(I_n)$ has no non-trivial triangular set. If $n = 2m$ is even, then $Z(W(I_n)) = \{1, x^m\}$. In that case, $W(I_n)$ has three conjugacy classes A_1, A_2 and A_3 of involutions, where

$A_1 = \{x^m\}$;
$A_2 = \{x^i y : i \text{ is odd}\}$;
$A_3 = \{x^i y : i \text{ is even}\}$.

Proposition 2.20. *Let T be a non-trivial triangular set in $W(I_n)$. Then, $n = 2m$ is even and one of the following hold:*

(*i*) m *is odd and* $T = A_1, A_2$ *or* A_3. *In particular, T has no lines.*
(*ii*) m *is even and* $T = A_1, A_1 \cup A_2$ *or* $A_1 \cup A_3$.

Proof. Let m be odd. Then, no two distinct elements in A_2 (as well as in A_3) commute. For $a = xy \in A_2$ and $b = x^{m+1}y \in A_3$, $[a, b] = 1$ and $ab = x^m$. Since T is a non-trivial triangular set, it follows that $T = A_1, A_2$ or A_3.

Let m be even. Then, no element in A_2 commute with any element in A_3. For $a = xy, b = x^{m+1}y$ in A_2, $[a, b] = 1$ and $ab = x^m$. Again, for $c = y, d = x^m y$ in A_3, $[c, d] = 1$ and $cd = x^m$. Since T is a non-trivial triangular set in $W(I_n)$, it follows that $T = A_1, A_1 \cup A_2$ or $A_1 \cup A_3$. □

There is no convenient means of representing the group elements for the other exceptional groups. We calculate the triangular sets in the finite irreducible Coxeter groups $W(E_6), W(H_3), W(H_4)$ and $W(F_4)$ with the help of the computational discrete algebra package GAP [**64**]. (The other two exceptional groups of type E_7 and E_8 are very large in size. GAP does not support the calculation in these two cases as

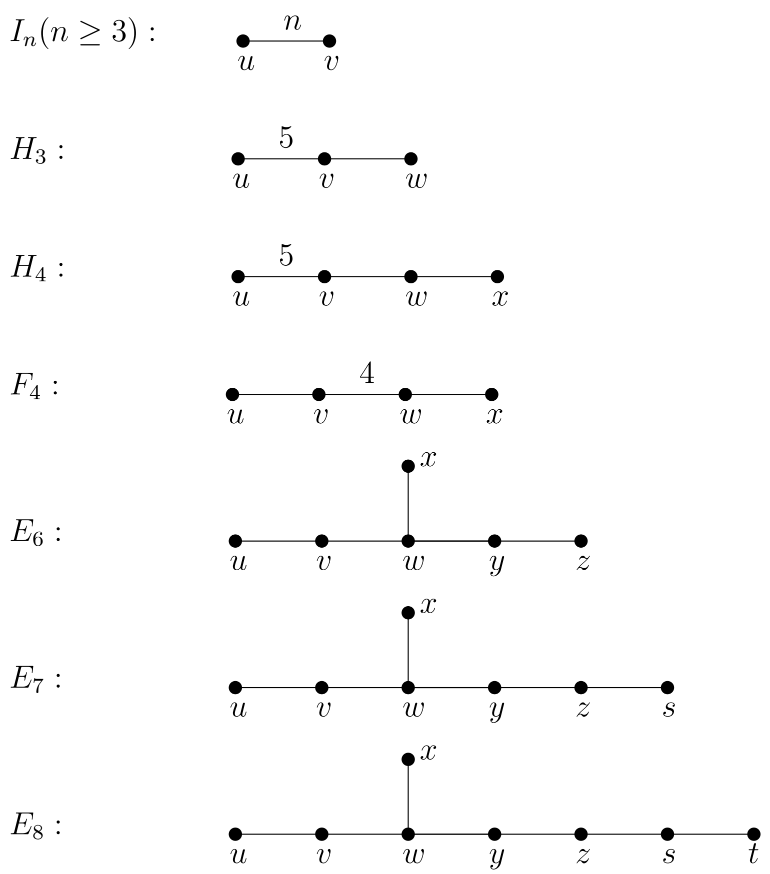

FIGURE 1

it limits program's workspace to 128 MB of memory. So we have not determined the triangular sets in these two groups.) In each of these groups there exists a unique element of maximum length, we denote it by α. This element α is an involution contained in the center of the group, except in the case $W(E_6)$.

The Coxeter graphs of these groups are given in Figure 1. In Table 1, we give a representative for each conjugacy class of involutions in terms of the generators of the groups as given in their Coxeter graphs. We denote the conjugacy classes of involutions by $C_1, C_2, \cdots$ and give the size of each of them. In the last column of the table we make a list of the non-trivial triangular sets as a union of the conjugacy classes.

In order to determine the triangular sets with the help of GAP in each of these exceptional groups we use the following steps:

Table 1. Triangular sets

group	conjugacy class	representative	size	non-trivial triangular sets
$W(H_3)$	C_1	α	1	$C_1,\ C_3$
	C_2	u	15	
	C_3	uw	15	
$W(H_4)$	C_1	α	1	$C_1,\ C_1 \cup C_4$
	C_2	u	60	
	C_3	αu	60	
	C_4	uw	450	
$W(F_4)$	C_1	α	1	$C_1, C_1 \cup C_6,$ $C_1 \cup C_6 \cup C_7,$ $C_1 \cup C_3 \cup C_4 \cup C_6,$ $C_1 \cup C_2 \cup C_5 \cup C_6$
	C_2	u	12	
	C_3	wxw	12	
	C_4	αwxw	12	
	C_5	αu	12	
	C_6	$uwxvwvu(wv)^2wxw$	18	
	C_7	uw	72	
$W(E_6)$	C_1	u	36	$C_2 \cup C_3$
	C_2	α	45	
	C_3	ux	270	
	C_4	uxz	540	

Step 1. Construct the group, say G, as a finitely presented group in terms of the generators and relations obtained from its Coxeter graph.

Step 2. Get a set 'reps' of representatives consisting of exactly one element from each conjugacy class of involutions in G using the command:
> reps := Filtered(List(ConjugacyClasses(G),Representative),
> $i-$ >Order(i) = 2);

Step 3. For each representative r in 'reps', get the corresponding conjugacy class in G using the command
> ConjugacyClass(G, r);

Step 4. Consider a set C which is a union of conjugacy classes of involutions in G and check whether C is a triangular set or not using the following command:
> ForAll(Combinations(C,2), $p-$ > $p[1] * p[2]$ in C or
> $p[1] * p[2]$ <> $p[2] * p[1]$);

If it returns 'true', then C is a triangular set otherwise not. We perform this test for all possible cases of C, that is, for all possible unions of conjuagcy classes of involutions in G.

CHAPTER 3

Representations of Partial Linear Spaces

The notion of a representation of a finite partial linear space of prime order was introduced by Ivanov [**37**] in his investigations of Petersen and Tilde geometries (motivated in large measure by questions about the finite simple groups Monster and Baby Monster). In this chapter, we give a sufficient condition (Theorem 3.23) on the partial linear space and on the non-abelian representation of it to ensure that the representation group is a finite p-group. This result is crucial when we study non-abelian representations of finite non-degenerate polar spaces in Chapter 4 and that of slim dense near hexagons in Chapter 7. Contents of this chapter (except the last section) appear in ([**55**], Sections 1 and 2).

3.1. Basic Definitions

Throughout this chapter, p denotes a fixed prime number.

A finite p-group is *elementary abelian* if it is abelian and each of its non-trivial elements is of order p. An elementary abelian p-group G of order p^k can be considered as a k-dimensional vector space over the field $\mathbb{F}_p$ with p elements and hence is determined by its order. Further, the automorphism group of G is isomorphic to $GL_k(p)$, the group of all $k \times k$ invertible matrices with entries from $\mathbb{F}_p$.

3.1.1. Hall-commutator formula. For elements x, y in a group, we write $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. We repeatedly use the following Hall's commutator formula ([**32**], Theorem 2.1, p.18), mostly without mention.

Lemma 3.1. *For elements x, y, z in a group,*

(*i*) $[xy, z] = [x, z]^y[y, z]$;
(*ii*) $[x, yz] = [x, z][x, y]^z$.

3.1.2. Extraspecial p-groups. A finite p-group G is *extraspecial* if the Frattini subgroup $\Phi(G)$, the commutator subgroup $G' = [G, G]$ and the center $Z(G)$ of G coincide and have order p.

Let D_8 and Q_8, respectively, denote the *dihedral* and *quaternion groups* of order 8. For an odd prime p, we set

$$E = \langle x, y, z : x^p = y^p = z^p = 1, [x, z] = [y, z] = 1, [x, y] = z\rangle;$$
$$F = \langle x, y : x^{p^2} = y^p = 1, x^y = x^{1+p}\rangle.$$

A non-abelian p-group of order p^3 is extraspecial and is isomorphic to one of the groups D_8, Q_8, E or F ([**32**], Theorem 5.1, p.203).

Let G be an extraspecial p-group. Then, the quotient group $G/Z(G)$ is elementary abelian. Set $Z = Z(G) = \langle z\rangle$ and $V = G/Z$. We consider V as a vector space over $\mathbb{F}_p$. The map $f : V \times V \longrightarrow \mathbb{F}_p$ defined by

$$f(xZ, yZ) = i, \tag{3.1.1}$$

where $[x, y] = z^i$ $(0 \le i \le p-1)$, is a non-degenerate symplectic bilinear form on V ([**22**], Theorem 20.4, p.78). Write V as an orthogonal direct sum of m hyperbolic planes K_i, $1 \le i \le m$, in V. Let H_i be the inverse image of K_i in G under the natural surjective homomorphism from G to $G/Z = V$. Then, H_i is generated by two elements x_i and y_i such that $[x_i, y_i] = z$. Hence, H_i is an extraspecial p-group of order p^3 and $[H_i, H_j] = 1$. This implies that G is a central product of its subgroups $H_1, \cdots, H_m$ and G is of order p^{1+2r}. These facts are used to prove the following classification of extraspecial p-groups, see ([**22**], Theorem 20.5, p.79).

Lemma 3.2. *Let G be an extraspecial p-group of order p^{1+2m}. Then, exactly one of the following four cases arises:*

(*i*) *$p \neq 2$, exponent of G is p, and G is a central product of m copies of E;*
(*ii*) *$p \neq 2$, exponent of G is p^2, and G is a central product of $m-1$ copies of E and a copy of F;*
(*iii*) *$p = 2$, exponent of G is 4, and G is a central product of m copies of D_8;*
(*iv*) *$p = 2$, exponent of G is 4, and G is a central product of $m-1$ copies of D_8 and a copy of Q_8.*

In particular, if p is odd, then an extraspecial p-group is uniquely determined by its order and exponent. In cases (i), (ii) and (iii), G

possesses a maximal abelian subgroup of order p^{m+1} *and exponent* p*; and in case* (iv)*, each maximal abelian subgroup of* G *is isomorphic to* $C_2^{m-1} \times C_4$.

We denote by p_+^{1+2m} (respectively, p_-^{1+2m}) an extraspecial p-group of order p^{1+2m} if (i) and (iii) (respectively, (ii) and (iv)) hold in Lemma 3.2. Note that p_+^{1+2} is isomorphic to the group of 3×3 upper triangular matrices with entries from $\mathbb{F}_p$ and 1 on the main diagonal. For more on extraspecial p-groups, see ([**32**], Chapter 5, Section 5, p.203) and ([**33**], Section 3, p.127 and Appendix 1, p.141).

Notation 3.3. *For a group* G*,* $G^* = G \setminus \{1\}$ *and* $I_p(G)$ *denotes the set of all elements of order* p *in* G*.*

3.1.3. Embeddings and Representations. Let $S = (P, L)$ be a connected partial linear space of order p.

Definition 3.4. *An embedding of* S *is a mapping* ρ *from* P *into the point set of a projective space* $\mathbb{P}(V)$ *associated with a vector space* V *over* $\mathbb{F}_p$ *such that the following hold:*

(i) *V is generated by* $Im(\rho)$*;*

(ii) *For each line* $\{x_0, x_1, \cdots, x_p\}$ *of* S*, the set* $\{\rho(x_0), \rho(x_1), \cdots, \rho(x_p)\}$ *is a line of* $\mathbb{P}(V)$*.*

This is well-defined for a prime power also. In the definition of an embedding, we did not require ρ to be one-one. In [**53**], Ronan studied the construction of embeddings using presheaves and geometric hyperplanes. When $p = 2$, he proved that if S admits at least one embedding, then there exists an embedding ρ_0 of S such that any other embedding of S is a composition of ρ_0 and a linear mapping. Such an embedding ρ_0 is unique up to an isomorphism of the associated projective spaces and is called the *universal embedding* of S. The $\mathbb{F}_2$-vector space $V(S)$ generated by the image of ρ_0 is called the *universal embedding module* of S. As an abstract group with additive group operation, $V(S)$ has the presentation:

$$\begin{aligned} V(S) \ = \ \langle v_x : x \in P;\ 2v_x = 0;\ v_x + v_y = v_y + v_x \text{ for } x, y \in P; \\ v_x + v_y + v_{x*y} = 0 \text{ if } x \sim y \rangle \end{aligned}$$

and ρ_0 is defined by $\rho_0(x) = \langle v_x \rangle$ for $x \in P$. We refer to ([**46**], Theorem 1, p.266) for a sufficient condition on the point-line geometry for the existence of its universal embeddings in general case.

Definition 3.5 (Ivanov [**37**], p.305)**.** *A representation of S is a mapping ψ from the point set P of S into the set of subgroups of order p of a group R such that the following hold:*

(*i*) *R is generated by the subgroups $\psi(x), x \in P$;*

(*ii*) *For each line $l \in L$, the subgroups $\psi(x)$, $x \in l$, are pairwise distinct and generate an elementary abelian p-subgroup of R of order p^2.*

We write (R, ψ) to mean that ψ is a representation of S with *representation group* R and say that (R, ψ) is a representation of S. For each $x \in P$, we fix a generator r_x of $\psi(x)$. We denote by R_ψ the union of the subgroups $\langle r_x \rangle$ for $x \in P$ and by R_ψ^* the set $R_\psi \setminus \{1\}$. Note that R_ψ^* is a subset of $I_p(R)$. The representation (R, ψ) is *faithful* if ψ is injective. A representation (R, ψ) of S is *abelian* or *non-abelian* according as R is abelian or not. (Note that, in [**37**], 'non-abelian representation' means 'the representation group is not necessarily abelian'.)

Lemma 3.6 ([**51**], 4.2.4, p.68)**.** *Let $S = (P, L)$ be a finite thick (p, t)-GQ. Let (R, ψ) be a faithful abelian representation of S and $H_x = \langle r_y : y \in x^\perp \rangle$ for $x \in P$. Then, H_x is a subgroup of index 2 in R for each $x \in P$.*

A representation (R_1, ψ_1) of S is a *cover* of a representation (R_2, ψ_2) of S if there exist an automorphism β of S and a group homomorphism $\varphi : R_1 \longrightarrow R_2$ such that $\psi_2(\beta(x)) = \varphi(\psi_1(x))$ for every $x \in P$. Further, if φ is an isomorphism, then the two representations (R_1, ψ_1) and (R_2, ψ_2) are said to be *equivalent*. Given a representation (R, ψ) of S, there is a *universal representation* (R_U, ψ_U) covering (R, ψ) such that if (R_1, ψ_1) is a representation of S covering (R, ψ), then (R_U, ψ_U) is a cover of (R_1, ψ_1) (see [**37**], p.306).

In general, the universal representation (R_U, ψ_U) depends on the particular choice of (R, ψ). However, when $p = 2$, there is a unique universal representation $(R(S), \psi_S)$ which is the cover of every other representation of S (see [**37**], p.306). The *universal representation group* $R(S)$ of S has the presentation:

$$R(S) = \langle r_x : x \in P, r_x^2 = 1, r_x r_y r_z = 1 \text{ if } \{x, y, z\} \in L \rangle.$$

The only difference in the definition of $V(S)$ and that of $R(S)$ is that in the latter the generating elements r_x need not commute. In fact,

Lemma 3.7 ([**38**], p.525)**.** $V(S) = R(S)/[R(S), R(S)]$.

Lemma 3.8 ([**37**], Lemma 3.5, p.310). *Let $S = (P, L)$ be a connected slim partial linear space and (R, ψ) be a representation of S. Assume that, for every point $x \in P$, there are two subsets $A(x)$ and $B(x)$ of P satisfying the following:*

(*i*) *if $y \in A(x)$, then $[r_x, r_y] = 1$;*
(*ii*) *the subgraph $\Gamma(B(x))$ of $\Gamma(P)$, in which two points of $B(x)$ are adjacent if they are collinear and the line containing them contains a point of $A(x)$, is connected;*
(*iii*) *if $z \in B(x)$, then $x \in B(z)$; and*
(*iv*) *the graph with vertex set P, in which x is adjacent to the points in $B(x)$, is connected.*

Then, the subgroup $\langle [r_x, r_z] : x \in P, z \in B(x)\rangle$ of R is of order at most 2 and contained in $Z(R)$. In particular, if $P = A(x) \cup B(x)$ for every $x \in P$, then R' is of order at most 2.

The following is a sufficient condition for the universal representation group to be infinite.

Lemma 3.9 ([**37**], Lemma 3.6, p.310). *Let $S = (P, L)$ be a connected slim partial linear space. If S contains a geometric hyperplane H such that the induced subgraph $\Gamma(P \setminus H)$ of $\Gamma(P)$ has at least two connected components, then the universal representation group of S is infinite.*

3.2. Examples

We now indicate various possibilities for a representation of a partial linear space of prime order and the corresponding representation group.

Embeddings of partial linear spaces of order p in projective spaces over $\mathbb{F}_p$ are examples of abelian representations. The representation group is the underlying vector space considered as an abelian group.

Example 3.10. *Let Ω be a set of size 24 on which the Mathieu group M_{24} acts 5-fold transitively preserving a family of 8-element subsets called octads which form a Steiner system $\mathcal{S}$ of type $S(5, 8, 24)$. For $a, b \in \Omega$, the set-wise stabilizer of $\{a, b\}$ in $Aut(\mathcal{S}) \simeq M_{24}$ is isomorphic to $Aut(M_{22})$. Consider the point-line system of the rank 3 Petersen type geometry $\mathcal{G}(M_{22})$ and the rank 3 tilde type geometry $\mathcal{G}(M_{24})$ which are constructed as follows.*

The points of $\mathcal{G}(M_{22})$ are all the 2-element subsets of $\Omega_0 = \Omega \setminus \{a, b\}$. Three points $\{c, d\}, \{e, f\}$ and $\{g, h\}$ form a line whenever

$\{a, b, c, d, , e, f, g, h\}$ *is an octad. The points of* $\mathcal{G}(M_{24})$ *are all the sextets of* $\mathcal{S}$. *Two sextets* $S = \{S_1, \cdots, S_6\}$ *and* $T = \{T_1, \cdots, T_6\}$ *are collinear whenever* $|S_i \cap T_j|$ *is even for all* $1 \leq i, j \leq 6$. *If* O *is an octad and* X, Y *are two 4-subsets in* O *such that* $|X \cap Y| = 2$, *then the sextets defined by* X *and* Y *are collinear; moreover any pair of collinear sextets appears in this way. If* S *and* T *are two distinct collinear sextets then the third sextet on the line containing* S *and* T *is defined by any 4-set which is a symmetric difference of* S_i *and* T_j, *where* $S_i \cap T_j$ *is non-empty.*

*The universal representation groups of these two geometries are abelian ([**38**], Lemmas 3.1 and 3.2, p.528, 529). In particular, they don't admit non-abelian representations.*

Example 3.10 shows that not all partial linear spaces admit non-abelian representations. In Chapter 4, we prove that this is the case for every finite non-degenerate polar space which is not of symplectic type of odd prime order. The following example shows that representation of a partial linear space need not be faithful.

Example 3.11. *Let* $S = (P, L)$ *be a* $(2, 1)$-*GQ and let* P_1, P_2, P_3 *be three triads of* S *partitioning* P. *Let* $R = \{1, r_1, r_2, r_3\}$ *be the Klein four group. Define a map* ψ *from* P *to the set of subgroups of* R *of order two by* $\psi(x) = \langle r_i \rangle$ *if* $x \in P_i$. *Then,* (R, ψ) *is an abelian representation of* S *which is not faithful.*

Example 3.12. *Let* G *be a finite group generated by its elements of order* p. *Consider the partial linear space* $\Delta_p = (P, L)$, *whose points set* P *is the collection of all subgroups of* G *of order* p. *A line of* S *is the set of* $p + 1$ *subgroups each of order* p *in an elementary abelian* p-*subgroup of* G *of order* p^2. *Then,* (G, I_d), *where* I_d *is the identity map, is a representation of* Δ_p *and* G *is a representation group of* Δ_p. *When* $p = 2$, *the universal representation group of* $\Delta_2(Alt(7))$ *is* $3.Alt(7)$ *([**37**], Lemma 3.7, p.311) and that of* $\Delta_2(M_{22})$ *is* $3.M_{22}$ *([**38**], Proposition 4.4, p.531). The universal representation group of* $\Delta_2(U_4(3))$ *is* $3^2.U_4(3)$ *[**52**].*

Example 3.13. *Let* G *be a finite simple group of Lie type defined over* $\mathbb{F}_p$. *Let* $\mathcal{G} = (P, L)$ *be the root group geometry of* G. *That is, the point set* P *is the collection of all (long) root subgroups of* G. *Two distinct root subgroups* $x, y \in P$ *are collinear if they generate an elementary abelian subgroup of* G *of order* p^2 *and each subgroup of order*

*p in it is a member of P. Then, the line xy is the set of $p+1$ subgroups of order p in $\langle x, y\rangle$. The identity map defines a representation of $\mathcal{G}$ in G and so G is a representation group of $\mathcal{G}$. Note that if G is of type E_6, E_7 or E_8, then $\mathcal{G}$ is a parapolar space ([**16**], p.75); if it is of type G_2 or ${}^3D_4$, then $\mathcal{G}$ is a generalized hexagon with parameters (p, p) and (p, p^3) respectively (see [**18**], p.322 and 328 for p odd; and [**19**], Lemma 2.2, p.2 for $p = 2$); if it is type F_4 or ${}^2E_6$, then $\mathcal{G}$ is a metasymplectic space ([**18**], Section 4); and if it is of type ${}^2F_4$, then $\mathcal{G}$ is a generalized octagon with parameters (2,8) (see [**57**]).*

The following example shows that the universal representation group of a finite partial linear space could be infinite.

Example 3.14. *Let $S = (P, L)$ be a generalized hexagon with parameters $(2, 2)$. Then, S is isomorphic to either $H(2)$ (the one admitting an embedding in $O_7(2)$) or its dual $H(2)^*$ (see [**63**], Theorem 4, p.402). For each $x \in P$, $H(x) = \{y \in P : d(x, y) < 3\}$ is a geometric hyperplane of S. The subgraph of $\Gamma(P)$ induced on the complement of $H(x)$ in P is connected if $S \simeq H(2)$ and has two connected components if $S \simeq H(2)^*$ (see [**31**], Section 3). By Lemma 3.9, the universal representation group of $H(2)^*$ is infinite.*

We refer to [**37**] (also see [**38**]) for more examples of non-abelian representations of partial linear spaces of order p. The representation theory of a geometry $\mathcal{G} = (P, L)$ in a group R tries to understand the structure of R in terms of the commutative relations on a set of generators of R. Questions regarding the commutativity of R, finiteness of R, structure of R etc. in terms of the relations defined by $\mathcal{G}$ seems to be of interest for further investigations. Theorem 3.23 is an example of the kind of results we have in mind. It is interesting to observe that in all the examples of non-abelian representations of various geometries, the representation map is always injective on the set of points.

3.3. A Sufficient Condition

In this section, we give a sufficient condition on the partial linear space and on the non-abelian representation of it (Theorem 3.23) in order that the non-abelian representation group be a finite p-group.

Let $S = (P, L)$ be a connected partial linear space. We assume that with each $x \in P$ is associated a geometric hyperplane $H(x)$ of S containing x such that the following conditions on S hold:

($C1$) If $y \in H(x)$, then $x \in H(y)$.
($C2$) The subgraph $\Gamma(H'(x))$ of $\Gamma(P)$ induced on the complement $H'(x)$ of $H(x)$ in P is connected.
($C3$) If $y \in H'(x)$ then there exist lines l_1 and l_2 with $x \in l_1$ and $y \in l_2$ such that for each $w \in l_1$, $H(w)$ intersects l_2 at exactly one point. Further, this correspondence is a bijection from l_1 onto l_2.
($C4$) The graph $\Sigma(P)$ with vertex set P, in which two points x and y are adjacent if $y \in H'(x)$, is connected.

We give two examples of partial linear spaces in which the above four conditions hold.

Example 3.15. *Let $S = (P, L)$ be a polar space of rank $r \geq 2$. Then, S is connected. For each $x \in P$, we associate the geometric hyperplane $H(x) = x^{\perp}$ of S. Then, $(C1), \cdots, (C4)$ hold.*

Example 3.16. *Let $S = (P, L)$ be a dense near $2n$-gon, $n \geq 2$, with thick lines. By the definition of a near polygon, S is connected. For each $x \in P$, we associate the special geometric hyperplane $H(x) = \Gamma_{\leq n-1}(x)$ of S. Clearly $(C1)$ holds. By Lemma 1.25, $(C2)$ holds. Now, by Corollary 1.23, if $d(x, y) = n$, $x, y \in P$ and l_1 is any line containing x, then there exists a line l_2 containing y such that $(C3)$ holds. This also implies that if $u \sim v$, $u, v \in P$, then there exists $w \in P$ such that $d(u, w) = d(v, w) = n$. So u, w, v is a path in $\Sigma(P)$. Then, connectedness of $\Sigma(P)$ follows from that of $\Gamma(P)$. Thus $C(4)$ holds.*

Remark 3.17. *If $S = (P, L)$ is a generalized $2n$-gon and $H(x), x \in P$, is as in Example 3.16, then $(C2)$ need not hold (see Example 3.14).*

For the rest of this section, we assume that each line of S contains $p + 1$ points.

Let (R, ψ) be a representation of S. For $x, y \in P$, we define $u_{xy} = [r_x, r_y]$. We assume that:

$$u_{xy} = 1 \text{ whenever } x \in P \text{ and } y \in H(x).$$

Proposition 3.18. *Assume that $(C1)$ and $(C2)$ hold in S. Then, the following hold:*

(i) If $u_{vw} = 1$ for $v, w \in P$ with $v \in H'(w)$, then $r_w \in Z(R)$.
(ii) If $a \in P$ and $r_a \in Z(R)$, then $r_c \in Z(R)$ for every $c \sim a$.

Proof. (i) Let $y \in H'(w)$, $y \sim v$ and $vy \cap H(w) = \{x\}$. Then, $u_{wy} = 1$ because $x \notin \{v, y\}$ and $u_{wx} = u_{vw} = 1$. Now, connectedness of $\Gamma(H'(w))$ implies that $u_{wz} = 1$ for every $z \in H'(w)$. Since $u_{wz} = 1$ for $z \in H(w)$ also, $r_w \in Z(R)$.

(ii) By definition, $H(a) \subsetneq P$. Let $b \in H'(a)$. By $(C1)$, $a \in H'(b)$. By (i), $r_b \in Z(R)$ because $u_{ab} = 1$. Now, $ac \cap H(b)$ is a singleton. Since each line contains at least 3 points, there exists a point z in $ac \cap H'(b)$ different from a. Now, $b \in H'(z)$ by $(C1)$ and $u_{bz} = 1$. So, $r_z \in Z(R)$ by (i) again. So the subgroup generated by $\psi(ac)$ is contained in $Z(R)$ and $r_c \in Z(R)$. □

Corollary 3.19. *Assume that* $(C1)$ *and* $(C2)$ *hold in* S. *If* R *is non-abelian, then the following hold:*

(i) $u_{xy} \neq 1$ *whenever* $x, y \in P$ *and* $y \in H'(x)$.
(ii) $R_\psi \cap Z(R) = \{1\}$.
(iii) *If* $x \sim y$, *then* $y \in H(x)$.
(iv) *If* $H(x) \neq H(y)$ *for each pair of non-collinear points* x *and* y, *then* ψ *is faithful.*

Proof. (i) follows from Proposition 3.18 and the connectedness of $\Gamma(P)$. (ii) and (iii) follow from (i). We now prove (iv). Suppose that $\langle r_x \rangle = \langle r_y \rangle$ for distinct x, y in P. Then, $x \nsim y$ by Definition 3.5(ii). By (i), $u \in H(x)$ if and only if $u \in H(y)$. So $H(x) = H(y)$, a contradiction. □

Proposition 3.20. *Assume that* $(C3)$ *holds in* S. *Then, for* $x, y \in P$, $[u_{xy}, r_x] = [u_{xy}, r_y] = 1$. *If* $u_{xy} \neq 1$, *then* u_{xy} *is of order* p *and* $\langle r_x, r_y \rangle = p_+^{1+2}$.

Proof. Let $x \in P$, $y \in H'(x)$ and l_1, l_2 be lines as in $(C3)$. Let x, a, u be three pairwise distinct points in l_1 and y, b, v be points in l_2 such that $y \in H(a)$, $b \in H(x)$ and $v \in H(u)$. By $(C3)$, y, b, v are pairwise distinct. Write $r_x = r_a^i r_u^j$, $r_y = r_v^k r_b^m$ for some i, j, k, m, $(1 \leq i, j, k, m \leq p-1)$. Now,

$$u_{xy} = [r_a^i r_u^j, r_y] = [r_u^j, r_y] = [r_u^j, r_v^k r_b^m] = [r_u^j, r_b^m] = [r_x r_a^{-i}, r_b^m] = [r_a^{-i}, r_b^m].$$

Since $[r_a^{-i}, r_b^m] = [r_b^m, r_a^i]^{r_a^{-i}}$,

$$\begin{aligned} u_{xy} &= [r_b^m, r_a^i]^{r_a^{-i}} = [r_y r_v^{-k}, r_a^i]^{r_a^{-i}} = [r_v^{-k}, r_a^i]^{r_a^{-i}} = [r_v^{-k}, r_u^{-j} r_x]^{r_a^{-i}} \\ &= [r_v^{-k}, r_x]^{r_a^{-i}} = [r_b^m r_y^{-1}, r_x]^{r_a^{-i}} = [r_y^{-1}, r_x]^{r_a^{-i}} = [r_y^{-1}, r_x]. \end{aligned}$$

So $u_{xy}r_y^{-1} = r_x^{-1}r_y^{-1}r_x = r_y^{-1}[r_y^{-1}, r_x] = r_y^{-1}u_{xy}$. Thus $[u_{xy}, r_y] = 1$. Similarly, $u_{yx} = [r_x^{-1}, r_y]$. This, together with $[r_y, r_x^{-1}] = [r_x^{-1}, r_y]^{-1} = u_{yx}^{-1} = u_{xy}$ implies that $[u_{xy}, r_x] = 1$. Now, $[r_x^i, r_y] = [r_x, r_y]^i = u_{xy}^i$ for all $i \geq 0$. So $u_{xy}^p = 1$ and $\langle r_x, r_y\rangle = p_+^{1+2}$. □

Proposition 3.21. *Assume that $(C1), \cdots, (C4)$ hold in S. Then, $R' \leq Z(R)$ and $|R'| \leq p$.*

Proof. For $x, y \in P$, let $U_{xy} = \langle u_{xy}\rangle$. Let a, b be adjacent in $\Gamma(H'(x))$ and $ab \cap H(x) = \{c\}$. Now $r_b = r_a^i r_c^j$ for some i, j, $1 \leq i, j \leq p-1$. Since $[r_x, r_c] = 1$, we have

$$u_{xb} = [r_x, r_b] = [r_x, r_a^i r_c^j] = [r_x, r_a^i] = [r_x, r_a]^i = u_{xa}^i.$$

So $U_{xb} = U_{xa}$. This, together with $(C2)$, implies that U_{xy} is independent of the choice of y in $H'(x)$. Since $u_{xy} = u_{yx}^{-1}$, we have $U_{xy} = U_{yx}$. So, if $x, y \in P$ with $y \in H'(x)$, then $U_{xy} = U_{yx}$. Now, by $(C4)$, U_{xy} is independent of the edge $\{x, y\}$ in $\Sigma(P)$. We denote this common subgroup by U.

We now show that $U \leq Z(R)$. Let $x \in P$ and $y \in H'(x)$. We show that $[u_{xy}, r_z] = 1$ for each $z \in P$. We may assume that $z \in H'(x) \cup H'(y)$. In this case it is clear from Proposition 3.20 because $U_{xy} = U_{xz}$ if $z \in H'(x)$. Similarly, if $z \in H'(y)$.

Now, since $R = \langle r_x : x \in P\rangle$, $u_{xy} \in Z(R)$ and $u_{xy} = 1$ if $y \in H(x)$, it follows that $R' = \langle u_{xy} : x \in P, y \in H'(x)\rangle = U$ and is of order at most p (Proposition 3.20). □

Proposition 3.22. *Assume that $(C1), \cdots\cdot, (C4)$ hold in S. If R is non-abelian, then exponent of R is p or 4 according as p is odd or $p = 2$. In particular, if P is finite, then R is finite and $\Phi(R) = R'$.*

Proof. Let $r = r_1 r_2 \cdots r_n \in R$, $r_i \in R_\psi$. We use induction on n. Let $r = hr_n$, where $h = r_1 r_2 \cdots r_{n-1}$. Since $R' \subseteq Z(R)$, $r_n^i h = hr_n^i[r_n^i, h] = hr_n^i[r_n, h]^i$. So $r^{i+1} = h^{i+1}r_n^{i+1}[r_n, h]^{1+2+\cdots+i}$ for all $i \geq 0$. Now, the result follows because by induction $h^p = 1$ if p is odd and $h^4 = 1$ if $p = 2$. Note that if $p = 2$, exponent of R can not be 2 as R is non-abelian.

Now, if P is finite then R/R' and so R are finite and $\Phi(R) = R'\langle r^p : r \in R\rangle = R'$. For $p = 2$, the last equality holds because $r^2 \in R'$ for every $r \in R$. □

We now summarize the above results of this section.

Theorem 3.23. *Let $S = (P, L)$ be a connected partial linear space of prime order p. Suppose that for each $x \in P$, there is associated a geometric hyperplane $H(x)$ containing x such that $(C1), \cdots, (C4)$ hold. Let (R, ψ) be a non-abelian representation of S such that $[\psi(x), \psi(y)] = 1$ for all $x, y \in P$ with $y \in H'(x)$. Then, the following hold:*

(*i*) *$[\psi(x), \psi(y)] \neq 1$ and $\langle \psi(x), \psi(y) \rangle = p_+^{1+2}$ for $x, y \in P$ with $y \in H'(x)$;*
(*ii*) *$|R'| = p$, $R' \subseteq Z(R)$, R is a p-group, and exponent of R is p or 4 according as p is odd or $p = 2$;*
(*iii*) *$R_\psi \cap Z(R) = \{1\}$;*
(*iv*) *ψ is faithful if $H(x) \neq H(y)$ whenever $x \nsim y$;*
(*v*) *R is finite with $R' = \Phi(R)$ if P is finite.*

Remark 3.24. *For $p = 2$, Theorem 3.23(ii) is a consequence of Lemma 3.8 without the assumption of $(C3)$. Our proof of Proposition 3.21 is similar to that of ([**38**], Lemma 2.2, p.526).*

Corollary 3.25. *Let S and (R, ψ) be as in Theorem 3.23. If P is finite, then (R, ψ) is the cover of a representation (R_1, ψ_1) of S where R_1 is extraspecial, or $p = 2$ and $Z(R_1)$ is cyclic of order 4.*

Proof. If $Z(R)$ is elementary abelian (this is the case if p is odd), write $Z(R) = R'T$, $R' \cap T = \{1\}$ for some subgroup T of $Z(R)$. Let $R_1 = R/T$. Then, R_1 is extraspecial. Define ψ_1 from P to R_1 by $\psi_1(x) = \langle r_x T \rangle$, $x \in P$. Since $r_x \notin Z(R)$, $\langle r_x T \rangle$ is a subgroup of R_1 of order p for each $x \in P$. Then, (R_1, ψ_1) is a non-abelian representation of S and (R, ψ) is a cover of (R_1, ψ_1).

If $Z(R)$ is not elementary abelian, then $p = 2$. Write $Z(R) = \langle a \rangle K$, $\langle a \rangle \cap K = \{1\}$ where $K \leq Z(R)$ and a is of order 4. Since $r^2 \in R'$ for every $r \in R$, it follows that $R' = \langle a^2 \rangle$. Now taking $R_1 = R/K$, the above argument completes the proof. □

3.4. Maximal Elementary Abelian Subgroups of $Sym(I)$

Every representation of a projective space (as a point-line geometry) is necessarily faithful, by condition (*ii*) of Definition 3.5. The representation group of a finite projective space of dimension m over $\mathbb{F}_p$, is an elementary abelian p-group of order p^{m+1}. Thus, the study of representation of projective spaces of dimension n over $\mathbb{F}_p$ in a group G is the same thing as finding elementary abelian p-subgroups of G of order p^{n+1}. In this section, we study in detail the maximal elementary abelian p-subgroups, up to conjugacy, of the symmetric group $Sym(I)$ defined on a finite set I.

3.4.1. Conjugacy classes. An element of $Sym(I)$ is *regular* if it is the identity element or if it has no fixed points and is the product of disjoint cycles of the same length. A subgroup H of $Sym(I)$ is *regular* if the action of H on I is 'sharply transitive' - that is, H acts transitively on I and no non-trivial element of H fixes any element of I. If M is an elementary abelian p-subgroup of $Sym(I)$, then the non-trivial cycles in the cyclic decomposition of each element of M^* are of length p. So if M is regular also, then every element of M is regular. The converse need not be true.

Notation 3.26. *For a bijective map $\alpha : I \longrightarrow J$, we denote by α^* the isomorphism from $Sym(I)$ to $Sym(J)$ induced by α - that is, $\alpha^* : Sym(I) \longrightarrow Sym(J)$ is the isomorphism defined by $\alpha^*(x)(j) = \alpha x \alpha^{-1}(j)$ for $x \in Sym(I)$ and $j \in J$.*

Proposition 3.27. *Let M be an elementary abelian p-subgroup of $Sym(I)$ acting transitively on I. Then, the following hold:*

(a) *M is regular.*
(b) *For distinct $i, j \in I$, there is a unique $x \in M^*$ with $x(i) = j$. In particular, M is maximal subject to being elementary abelian.*
(c) $|I| = |M|$.
(d) *The permutation group M acting on I with the natural action is isomorphic to the permutation group M acting on itself by left translation.*

Proof. Since M acts transitively on I, for every $x \in M^*$, x fixes no element of I, otherwise, the set $K_x = \{j \in I : x(j) = j\}$ would be a proper M-invariant subset of I. This proves (a). If $x(i) = j = y(i)$ for some $x, y \in M^*$, then $y^{-1}x(i) = i$. So $y = x$ by (a). This proves (b). For $i \in I$, the set $I(i) = \{x(i) : x \in M\}$ is M-invariant and $|I(i)| = |M|$ by (b). Now, the transitive action of M on I implies that $I(i) = I$ and so $|I| = |M|$. This proves (c). We now prove (d). Let $f : M \longrightarrow Sym(I)$ and $h : M \longrightarrow Sym(M)$ be the embeddings where f is the inclusion map and h is defined by $h_x(y) = xy$, $x, y \in M$. Fix $i \in I$. Define $g : M \longrightarrow I$ by $g(x) = x(i)$, $x \in M$. Then, g is a bijection. Let $j \in I$. There exists $y \in M$ such that $y(i) = j$. Now $g^*(h_x)(j) = gh_xg^{-1}(y(i)) = gh_x(y) = g(xy) = xy(i) = x(j)$. So $g^*(h_x) = x = f(x)$ for every $x \in M$. □

Theorem 3.28. *Let M be a maximal elementary abelian p-subgroup of $Sym(I)$. Then, M is maximal subject to being abelian if and only if at most one element of I is fixed by every element of M.*

Proof. We need to prove the 'if' part. Let x be an element of $Sym(I)$ centralizing M. We prove that $x \in M$. Let $I = I_1 \cup \cdots \cup I_r$ be the orbit decomposition of I under M and M_i be the restriction of M to I_i. We may assume that $|I_i| \geq p$ for each i. Since M is maximal elementary abelian, M_i is a subgroup of M and the action of M_i on I_i is transitive.

We first show that each I_i is x-invariant. Suppose that $x(i) = j$ for some $i \in I_t$ and $j \in I_s$ with $t \neq s$. Let $m \in M_t^*$ and $m(i) = l$. Then, $l \in I_t$ and is different from i because M_t acts regularly on I_t by Proposition 3.27(a). Now, $m(j) = j$ since M_t acts trivially on I_s. So $x(l) = xm(i) = mx(i) = m(j) = j$, a contradiction to that $i \neq l$ and $x(i) = j$. Thus, each I_i is x-invariant.

Now, we show that x is of order p. Then, the maximality of M being elementary abelian would complete the proof. Let $(i_1 i_2 \cdots)$ be a nontrivial cycle in the cyclic decomposition of x. Then, $i_1, i_2 \in I_t$ for some t, since the M-orbits are x-invariant. There exists $m \in M_t^*$ such that $m(i_1) = i_2$ because M_t acts transitively on I_t. Let $c = (i_1 i_2 i_3' \cdots i_p')$ be the cycle in the cyclic decomposition of m containing i_1 and i_2. Now, $c' = (x(i_1)x(i_2)x(i_3') \cdots x(i_p')) = (i_2 x(i_2)x(i_3') \cdots x(i_p'))$ is a nontrivial cycle in the cyclic decomposition of $xmx^{-1} = m$. Since i_2 is common to both c and c', we must have $c = c'$ and $x(i_2) = i_3', x(i_3') = i_4', \cdots, x(i_{p-1}') = i_p', x(i_p') = i_1$. So, $(i_1 i_2 i_3' \cdots i_p')$ is the cycle in the cyclic decomposition of x containing i_1 and i_2 which has length p. Thus, x has order p. $\square$

The following is a partial converse to Proposition 3.27.

Proposition 3.29. *Let M be a maximal elementary abelian p-subgroup of $Sym(I)$ such that $|M| = |I|$. Then, M acts transitively on I except when $|I| = 4$.*

Proof. Let $I_1, \cdots, I_r$ be the M-orbits and M_i be the restriction of M to I_i. By maximality of M, M_i is a subgroup of M acting transitively on I_i, $M_i \cap M_j = \{1\}$ for $i \neq j$ and $M = M_1 \cdots M_r$. By Proposition 3.27(c), $|M_i| = |I_i| = p^{k_i}$ for some integer $k_i \geq 0$. If some $k_i = 0$, then the subset J of I consisting of elements which are left fixed by each element of M is non-empty and p divides $|J|$. But this is not possible, because $0 < |J| < p$ by the maximality of M. Thus, each $k_i \geq 1$. Let $|M| = p^n$ for some positive integer n. Now, $|M| = |I|$ implies that $p^n = p^{k_1} + \cdots + p^{k_r}$. Further, $M = M_1 \cdots M_r$ implies that $p^n = p^{k_1} \cdots p^{k_r}$. So either $r = 1$ and $k_1 = n$ or $p = n = r = 2$ and $k_1 = k_2 = 1$. Thus, M acts transitively on I if $|I| \neq 4$. □

When $|I| = 4$, Proposition 3.29 need not hold. To see this, we take M to be a subgroup of $Sym(I)$ generated by two disjoint transpositions. Then, M is a maximal elementary abelian 2-subgroup of $Sym(I)$ of order 4 whose action is not transitive on I.

Proposition 3.30. *If $|I| = p^n$ for some integer $n \geq 1$, then there exists an elementary abelian p-subgroup M of $Sym(I)$ acting transitively on I.*

Proof. We use induction on n. If $n = 1$, then we take M to be a cyclic subgroup of $Sym(I)$ generated by a cycle of length p. Assume the result for $n-1$. Let $I = I_1 \cup \cdots \cup I_p$ be a partition of I into subsets of size p^{n-1}. By induction, let H_j be an elementary abelian p-subgroup of $Sym(I_j)$ of order p^{n-1} acting transitively on I_j. Let $\{x_{js} : 1 \leq s \leq n-1\}$ be a basis for H_j (considering H_j as a vector space over $\mathbb{F}_p$) and let H be the elementary abelian p-subgroup of $Sym(I)$ of order p^{n-1} with basis $\{y_s : 1 \leq s \leq n-1\}$, where $y_s = \prod_{j=1}^{p} x_{js}$. Fix $i_j \in I_j$, $j = 1, \cdots, p$, and let $a \in Sym(I)$ be defined by $a = \prod_{x \in H} (x(i_1), \cdots, x(i_p))$. Then, $a \notin H$. Note that $(x(i_1), \cdots, x(i_p))$ is a p-cycle for every $x \in H$ and for distinct $x, y \in H$, the cycles $(x(i_1), \cdots, x(i_p))$ and $(y(i_1), \cdots, y(i_p))$ are disjoint. So a has order p.

Set $M = \langle H, a \rangle$. We show that M is an elementary abelian p-subgroup of $Sym(I)$ acting transitively on I. Let $k \in I$. Then, $k \in I_l$ for some l. There exists $x \in H$ such that $x(i_l) = k$. Now, for each t $(1 \leq t \leq n-1)$, $ay_t(k) = ay_t x(i_l) = y_t x(i_{l+1}) = y_t a x(i_l) = y_t a(k)$. Thus a commutes with every element of H and so M is elementary abelian.

Now, we show that M acts transitively on I. Let $k, l \in I$ be distinct. If k, l are in the same I_t for some t, then some element of H would take k to l. We assume that $k \in I_s$ and $l \in I_t$ for some $s \neq t$. Since H acts transitively on each I_j, there exists $x, y \in H$ such that $x(k) = i_s$ and $y(i_t) = l$. Also, observe that there exists $z \in \langle a \rangle$ such that $z(i_s) = i_t$. Now, $yzx(k) = l$ and so M acts transitively on I. □

Proposition 3.31. *Let M_1 and M_2 be elementary abelian p-subgroups of $Sym(I)$ acting transitively on I. Then, M_1 and M_2 are conjugate in $Sym(I)$.*

Proof. Fix $s_1, s_2 \in I$. Define α_i from M_i to I by $\alpha_i(a) = a(s_i)$, $a \in M_i$. Then, α_i is a bijection by Proposition 3.27(b). Let g from M_1 to M_2 be a group isomorphism and set $\tau = \alpha_2 g \alpha_1^{-1}$. Then, τ is a bijection of I and so $\tau \in Sym(I)$.

We show that $\tau M_1 \tau^{-1} = M_2$. Let $x \in M_1$, $t \in I$ and $z \in M_2$ be such that $z(s_2) = t$. Now, for $y \in M_2$, a routine calculation shows that $(\tau x \tau^{-1})y(t) = yzg(x)(s_2) = y(\tau x \tau^{-1})(t)$. Thus, $\tau x \tau^{-1}$ commutes with every element of M_2. By Proposition 3.27(b), M_2 is a maximal elementary abelian p-subgroup of $Sym(I)$. So, $\tau x \tau^{-1} \in M_2$ and $\tau M_1 \tau^{-1} = M_2$. □

Let p be a prime and n be a positive integer and let $l = n - pm$, where m is the largest integer such that $pm \leq n$. By a *p-partition* of n we mean an expression of the form $n = p^{r_1} + \cdots + p^{r_k}$, where $r_1 \geq \cdots \geq r_k$, $r_{k-l} \geq 1$ and $r_j = 0$ for $j \geq k - l + 1$. Every maximal elementary abelian p-subgroup of $Sym(I)$ gives rise to a p-partition of $|I|$ via the orbit decomposition of I. We prove that the converse also holds. Let $|I| = p^{r_1} + \cdots + p^{r_k}$ be a p-partition of $|I|$. We make a partition of I into $J_1 \cup J_2 \cup \cdots \cup J_k$ with $|J_i| = p^{r_i}$. Let K_t be an elementary abelian p-subgroup of $Sym(J_t)$ of order p^{r_t} acting transitively on J_t (Proposition 3.30).

Proposition 3.32. *Let $K = K_1 \cdots K_k$. Then, K is a maximal elementary abelian p-subgroup of $Sym(I)$.*

Proof. Clearly, K is an elementary abelian p-subgroup of $Sym(I)$. We show that if $x \in Sym(I)$ centralizes K and is of order p, then $x \in K$. Let $i \in J_t$ with $|J_t| \neq 1$. Let $m \in K_t^*$ and $m(i) = j \in J_t$. Since m is regular on I_t, $i \neq j$ and so $x(i) \neq x(j)$. Now, $mx(i) = mxm^{-1}(j) = x(j)$. So, $x(i) \in J_t$ because m acts trivially on J_s if $s \neq t$. Thus, J_t is x-invariant. Let x_t be the restriction of x to J_t. Then, x_t commute with every element of K_t. By Proposition 3.27(b), K_t is a maximal elementary abelian p-subgroup of $Sym(J_t)$. So $x_t \in K_t$.

Now, let J be the set of all elements of I which are left fixed by elements of K. Then, J is x-invariant and $|J| < p$ (by the definition of a p-partition). Since x has order p, it follows that x acts trivially on J. Thus, $x \in K$. □

Theorem 3.33. *There is a bijection between the set of conjugacy classes of maximal elementary abelian p-subgroups of $Sym(I)$ and the set of p-partitions of $|I|$.*

Proof. We need only to prove that if M and H are maximal elementary abelian p-subgroups of $Sym(I)$ giving rise to the same p-partition $|I| = p^{r_1} + \cdots + p^{r_k}$ of $|I|$ (via the orbit decomposition of I), then they are conjugate in $Sym(I)$. Let $I = I_1 \cup I_2 \cup \cdots \cup I_k$ and $I = J_1 \cup J_2 \cup \cdots \cup J_k$ be the orbit decompositions of I with respect to M and H respectively, where $|I_i| = p^{r_i} = |J_i|$. Let M_i and H_i be the restrictions of M and H to I_i and J_i respectively.

Fix $i_l \in I_l$ and $j_l \in J_l$ $(1 \leq l \leq k)$. Let x_l be the map from M_l to I_l taking m to $m(i_l)$, $m \in M_l$ and let y_l be the map from H_l to J_l which takes h to $h(j_l)$, $h \in H_l$. Let g_l from M_l to H_l be a group isomorphism. Then, $z_l = y_l g_l x_l^{-1}$ is a bijection from I_l to J_l. Let z be the bijection of I onto itself which coincides with z_l on I_l.

We show that $zMz^{-1} = H$. It is enough to show that zaz^{-1} centralizes H for each $a \in M$. Let $b \in H$ and $s \in I$. Then, $s \in J_l$ for some l. There exists $c \in H_l$ such that $c(j_l) = s$. Now, a simple calculation shows that $(zaz^{-1})b(s) = bcg_l(a_l)(j_l) = b(zaz^{-1})(s)$, where a_l is the restriction of a to I_l. Thus, zaz^{-1} centralizes H for every $a \in M$. □

3.4.2. Structure of the normalizers. Let M be an elementary abelian p-subgroup of $Sym(I)$ of order p^n. We write $N = N_{Sym(I)}(M)$ and $C = C_{Sym(I)}(M)$. The map ϕ from N to $Aut(M)$ taking x to ϕ_x, where $\phi_x(m) = xmx^{-1}$ for $m \in M$, defines a homomorphism with $Ker(\phi) = C$. Since $Aut(M)$ is isomorphic to $GL_n(p)$, it follows that N/C is isomorphic to a subgroup of $GL_n(p)$. We next prove that if M acts transitively on I, then these two groups are isomorphic.

Proposition 3.34. *Assume that M acts transitively on I. Then, N/C is isomorphic to $GL_n(p)$.*

Proof. We need only to show that the map ϕ defined above is surjective. Let $h \in Aut(M)$. Fix $i \in I$. The map g from M to I defined by $g(m) = m(i)$, $m \in M$, is a bijection by Proposition 3.27(b). Then, $x = ghg^{-1}$ is a bijection of I onto itself and so $x \in Sym(I)$. We show that $x \in N$ and $\phi_x = h$. Let $m \in M$, $t \in I$ and $y \in M$ be such that $y(i) = t$. Now, for every $z \in M$, $(xmx^{-1})z(t) = zh(m)y(i) = z(xmx^{-1})(t)$. So xmx^{-1} centralizes M. By Proposition 3.27(b), M

is maximal subject to being elementary abelian and so $xmx^{-1} \in M$. This proves that $x \in N$. By a similar calculation as above, $\phi_x(m)(t) = h(m)(t)$ for all $m \in M$ and $t \in I$. Thus, $\phi_x = h$ and N/C is isomorphic to $Aut(M)$. □

We note that Proposition 3.34 is not true if the action of M on I is not transitive even if it is maximal subject to being elementary abelian.

In the next two propositions, we assume M to be a maximal elementary abelian p-subgroup of $Sym(I)$. Let $I = I_1 \cup \cdots \cup I_r$ be the M-orbit decomposition of I and M_i be the restriction of M to I_i. Set $J = \{I_1, \cdots, I_r\}$ and $N_i = N_{Sym(I_i)}(M_i)$ for $i = 1, \cdots, r$.

Proposition 3.35. *Assume that* $|I_i| = |I_j|$ *for all* i, j. *Then,* N *is an extension of* $N_1 \cdots N_r$ *by* $Sym(J)$.

Proof. The map ψ from N to $Sym(J)$ taking $g \in N$ to ψ_g, where $\psi_g(I_i) = g(I_i)$, defines a homomorphism with $Ker(\psi) = N_1 \cdots N_r$. We show that ψ is surjective. Since $Sym(J)$ is generated by transpositions, we need only to show that for every transposition (I_iI_j), there exists $x \in N$ such that ψ_x takes I_i to I_j. For $k \in \{i, j\}$, fix $a_k \in I_k$ and define a bijection f_k from M_k to I_k by $f_k(x) = x(a_k)$. Let β from M_i to M_j be a group isomorphism. Then, $z = f_j\beta f_i^{-1}$ is a bijection from I_i to I_j and it induces an element $x \in Sym(I)$ of order two defined by $x = \prod_{l \in I_i}(l\ z(l))$. Now, observe that $x(I_i) = I_j$, $xM_ix^{-1} = M_j$, $xM_jx^{-1} = M_i$ and $xM_kx^{-1} = M_k$ for all $k \notin \{i, j\}$. Thus, $x \in N$ and $\psi_x(I_i) = I_j$. □

Now, assume that not all orbits are of the same sizes. We make a partition of I as $I = J_1 \cup \cdots \cup J_k$, where J_l are formed by taking the union of all orbits of the same sizes. Let K_i be the restriction of M to J_i and set $H_i = N_{Sym(J_i)}(K_i)$.

Proposition 3.36. N *is isomorphic to* $H_1 \times \cdots \times H_k$.

Proof. Note that each J_i is x-invariant for $x \in N$. So, the map from N to $H_1 \times \cdots \times H_k$ taking x to $(x_1, \cdots, x_k)$, where x_i is the restriction of x to J_i, defines an isomorphism. □

CHAPTER 4

Representations of Polar Spaces

Throughout, p denotes a fixed prime number. In this chapter, we study non-abelian representations of finite non-degenerate polar spaces of order p and prove the following ([**55**], Sections 3,4 and 5).

Theorem 4.1. *Let $S = (P, L)$ be a finite non-degenerate polar space of rank $r \geq 2$ and of order p. If S admits a non-abelian representation (R, ψ), then:*

(*i*) *p is odd;*
(*ii*) $R = p_+^{1+2r}$*;*
(*iii*) *S is isomorphic to $W_{2r}(p)$.*

Theorem 4.2. *$W_{2r}(p)$, $r \geq 2$, admits a non-abelian representation. Further, any two such representations are equivalent.*

4.1. Non-abelian Representation Group

Let $S = (P, L)$ be a non-degenerate polar space of finite rank $r \geq 2$ and of order p. Let (R, ψ) be a representation of S. By Definition 3.5(*ii*), $[r_x, r_y] = 1$ for every $x, y \in P$ with $y \in x^{\perp}$. By Example 3.15, all the results of Section 3.3 hold for S and (R, ψ).

First we consider the case when $p = 2$.

Proposition 4.3. *Let $S = (P, L)$ be a $(2, t)$-GQ and (R, ψ) be a representation of S. Then, R is an elementary abelian 2-group.*

Proof. It is enough to show that R is abelian. Let x, y be two non-collinear points of S. Let T be a $(2, 1)$-subGQ of S. Such a T exists because each line has three points. Let $\{x, y\}^{\perp} = \{a, b\}$ in T. For $u \sim v$, we define $u * v \in P$ by $uv = \{u, v, u * v\}$. In T, since $[r_b, r_y] = [r_b, r_x] = 1$ and $r_{(a*x)*(b*y)} = r_{(a*y)*(b*x)}$, it follows that $r_x r_y = r_y r_x$. □

Proposition 4.4. *Let $S = (P, L)$ be a non-degenerate polar space of finite rank $r \geq 2$ and of order two. Then, every representation of S is abelian.*

Proof. Let (R, ψ) be a representation of S. By Lemma 1.4, there exists a chain of subspaces $Q_0 = P \supsetneq Q_1 \supsetneq Q_2 \supsetneq \cdots \supsetneq Q_{r-2}$ such that Q_i is a polar space of rank $r - i$. Thus Q_{r-2} is a $(2, t)$-GQ. By Proposition 4.3, $\langle \psi(Q_{r-2}\rangle$ is abelian. Now, Theorem 3.23(i) completes the proof. □

Now on, let (R, ψ) be a non-abelian representation of S.

By Proposition 4.4, p is an odd prime. Note that if $r \geq 3$, then finiteness of P and that of the rank r of S are equivalent. However, if S is a generalized quadrangle of order s, $s > 1$, then finiteness of P is not known except when $s = 2, 3, 4$ (see Section 1.5). The rest of this section is devoted to prove that if P is finite, then R is extraspecial.

Lemma 4.5. *ψ is faithful and $[r_x, r_y] \neq 1$ if $x \nsim y$.*

Proof. This follows from Theorem 3.23(iv). □

Notation 4.6. *Given a line l of S and two distinct points a and b on it, we write:*

$$\psi(l) = \{\langle r_a\rangle, \langle r_b\rangle, \langle r_a r_b\rangle, \langle r_a^2 r_b\rangle, \cdots, \langle r_a^{p-1} r_b\rangle\}.$$

Let $x, y \in P$, $x \nsim y$ and $u, v \in \{x, y\}^\perp$, $u \nsim v$. Then, $[r_x, r_y] \neq 1$ and $[r_u, r_v] \neq 1$. Let $l_0 = xu$, $l_1 = vy$, $m_0 = xv$ and $m_1 = uy$. Consider the lines l_0 and l_1. By 'one or all' axiom, each point of l_0 is collinear with exactly one point of l_1 and vice-versa. Let $l_0 = \{x, u, x_1, x_2, \cdots, x_{p-1}\}$ and $\langle r_{x_i}\rangle = \langle r_x^i r_u\rangle$ for $1 \leq i \leq p-1$. Let $x_i \sim v_i$ in l_1. Then, $l_1 = \{v, y, v_1, v_2, \cdots, v_{p-1}\}$. Replacing the generator r_v by r_v^j for some j $(2 \leq j \leq p-1)$, if necessary, we may assume that $\langle r_{v_1}\rangle = \langle r_v r_y\rangle$. So $[r_x r_u, r_v r_y] = 1$. Then, $[r_x^i r_u, r_v^i r_y] = 1$ for all $i \geq 0$ because $R' \subseteq Z(R)$. By Lemma 4.5, $[r_x^i r_u, r_v^j r_y] \neq 1$ if $i \neq j$. So $\langle r_{v_i}\rangle = \langle r_v^i r_y\rangle$. Let m_{i+1} be the line such that $\psi(m_{i+1}) = \langle r_x^i r_u, r_v^i r_y\rangle$, $1 \leq i \leq p-1$.

Let $z \in m_i \backslash (l_0 \cup l_1)$ and $w \in m_j \backslash (l_0 \cup l_1)$ for $i \neq j$, $0 \leq i, j \leq p$. If $i = 0$, then $\langle r_z\rangle = \langle r_x^{k_1} r_v\rangle$ and if $i > 0$ then $\langle r_z\rangle = \langle (r_x^{i-1} r_u)^{k_1} (r_v^{i-1} r_y)\rangle$ for some k_1, $1 \leq k_1 \leq p-1$. Similarly, $\langle r_w\rangle = \langle r_x^{k_2} r_v\rangle$ or $\langle (r_x^{j-1} r_u)^{k_2} (r_v^{j-1} r_y)\rangle$ for some k_2, $1 \leq k_2 \leq p-1$, according as $j = 0$ or $j > 0$. Now, from $R' \subseteq Z(R)$, the identity $[r_x, r_y] = [r_v, r_u]$ (a consequence of $[r_x r_u, r_v r_y] = 1$) and the fact that each point of m_i is collinear with exactly one point of m_j for $i \neq j$ (a consequence of 'one or all' axiom), the following lemma is straight forward.

Lemma 4.7. *$z \sim w$ if and only if $k_1 + k_2 = p$.*

Proposition 4.8. *If $a, d \in R_\psi$, then $ad[a,d]^{(p-1)/2} \in R_\psi$.*

Proof. Let $a, d \in R_\psi^*$. Let $x_1, x_2 \in P$ be such that $\langle r_{x_1}\rangle = \langle a\rangle$ and $\langle r_{x_2}\rangle = \langle d\rangle$. We may assume that $x_1 \nsim x_2$. Then, $[a,d] \neq 1$ by Lemma 4.5. We show that $\langle ad[a,d]^{(p-1)/2}\rangle$ is the image of some element of P. Let $y_1, y_2 \in \{x_1, x_2\}^\perp$ be such that $y_1 \nsim y_2$, $\langle r_{y_1}\rangle = \langle b\rangle$ and $\langle r_{y_2}\rangle = \langle c\rangle$. Consider the lines $l_0 = x_1y_1$ and $l_1 = x_2y_2$. Let $z_1 \in l_0$ be such that $\langle r_{z_1}\rangle = \langle ab\rangle$ and let $z_1 \sim z_2 \in l_1$. Replacing the generator c by c^j for some j, if necessary, we may assume that $\langle r_{z_2}\rangle = \langle cd\rangle$. Let $m_0 = x_1y_2$ and $m_1 = z_1z_2$. Let $u \in m_0$ be such that $\langle r_u\rangle = \langle a^{(p-1)/2}c\rangle$. Then, $x_1 \neq u \neq y_2$. Let $u \sim v$ in m_1. By Lemma 4.7, $\langle r_v\rangle = \langle (ab)^{(p+1)/2}(cd)\rangle$. If $y_1 \sim w$ in the line uv, then $\langle r_w\rangle = \langle (a^{(p-1)/2}c)^k(ab)^{(p+1)/2}(cd)\rangle$ for some k $(1 \leq k \leq p-1)$. Now $[b, (a^{(p-1)/2}c)^k(ab)^{(p+1)/2}(cd)] = 1$. So, $[b,c]^{k+1} = 1$ and $k+1 = p$. The subgroup $\langle b^{(p-1)/2}(a^{(p-1)/2}c)^{p-1}(ab)^{(p+1)/2}(cd)\rangle$ is the image of some point of y_1w. But

$$b^{(p-1)/2}(a^{(p-1)/2}c)^{p-1}(ab)^{(p+1)/2}(cd) = ad[b,c]^{(p+1)/2} = ad[a,d]^{(p-1)/2}.$$

In the last equality we have used $[a,d] = [b,c]^{-1}$, a consequence of $[ab, cd] = 1$. Thus, $ad[a,d]^{(p-1)/2} \in R_\psi$. □

Proposition 4.9. *The set R_ψ is a complete set of coset representatives of R' in R.*

Proof. Let $r_1R' = r_2R'$ for some r_1,$r_2 \in R_\psi$. Since $R' \subseteq Z(R)$, r_1 and r_2 are both trivial or are both nontrivial (Theorem 3.23(iii)). Assume that the latter holds and that $r_1 = r_2w$ for some $w \in R'$. Let $x_1, x_2 \in P$ be such that $\langle r_{x_1}\rangle = \langle r_1\rangle$ and $\langle r_{x_2}\rangle = \langle r_2\rangle$. Since $[r_1, r_2] = 1$, either $x_1 = x_2$ or $x_1 \sim x_2$ (Lemma 4.5). If $x_1 \sim x_2$ then $w \neq 1$ by Definition 3.5(ii) and $\langle w\rangle$ would be the image of some point in the line x_1x_2, a contradiction to Theorem 3.23(iii). So $x_1 = x_2$ and $r_1 = r_2^i$ for some i $(1 \leq i \leq p-1)$. Then, $r_2^{i-1} = w \in R' \subseteq Z(R)$. Now, Theorem 3.23($iii$) implies that $i = 1$ and so $w = 1$ and $r_1 = r_2$.

Now, let $sR' \in R/R'$. Write $s = r_1r_2\cdots r_k$, $r_i \in R_\psi$. Let $R' = \langle z\rangle$. Since $R' \subseteq Z(R)$, there is some integer j such that $r_1r_2\cdots r_kz^j$ is an element, say r, of R_ψ by Proposition 4.8. Then, $sR' = rR'$, completing the proof of the proposition. □

Proposition 4.10. *Assume that P is finite. Then, $|R| = p(1 + (p-1)|P|)$ and $R = p_+^{1+2m}$ for some $m \geq 1$.*

Proof. Since $|R'| = p$ (Theorem 3.23(ii)), the first assertion follows from Proposition 4.9. Also, $R' = Z(R)$ because $R_\psi \cap Z(R) = \{1\}$ and $R' \subseteq Z(R)$. Now, Theorem 3.23(v) completes the proof. □

Corollary 4.11. *If S is a finite classical polar space of rank $r \geq 2$ admitting a non-abelian representation, then S is isomorphic to $W_{2m}(p)$ or $Q_{2m+1}(p)$.*

Proof. By Proposition 4.10, $|P| = (p^{2m}-1)/(p-1)$ for some $m > 0$. So the corollary follows from the number of points of the classical polar spaces (see Theorem 1.3). □

By proposition 4.9, S admits an abelian faithful representation with representation group R/R'. Considering R/R' as a vector space over $\mathbb{F}_p$, it has dimension $2m$. Since $Q_{2m+1}(p)$ does not possess an abelian $2m$-dimensional faithful representation, it follows that the only possibility is $S \simeq W_{2m}(p)$. In the next section, we prove this fact giving a geometrical argument involving triads of points of a generalized quadrangle.

4.2. Proof of Theorem 4.1

Recall that a triad of lines of a (s,t)-GQ is a triple T of pair-wise disjoint lines and a line in $T^\perp$ is called a center of T.

Proposition 4.12. *Let $S = (P, L)$ be a (p,t)-GQ. If S admits a triad of lines with at least three centers, then every representation of S is abelian.*

Proof. Let $\{l_1, l_2, l_3\}$ be a triad of lines in S with centers m_1, m_2, m_3. Let $\{x_{ij}\} = l_i \cap m_j, 1 \leq i,j \leq 3$. Consider the lines l_1 and l_2. Replacing $r_{x_{11}}$ by $r_{x_{11}}^k$ for some k, if necessary, we may assume that the point a of l_1 with $\langle r_a\rangle = \langle r_{x_{11}} r_{x_{12}}\rangle$ is collinear with the point b with $\langle r_b\rangle = \langle r_{x_{21}} r_{x_{22}}\rangle$. So $[r_{x_{11}} r_{x_{12}}, r_{x_{21}} r_{x_{22}}] = 1$. Then, $[r_{x_{11}}^i r_{x_{12}}, r_{x_{21}}^i r_{x_{22}}] = 1$ for $0 \leq i \leq p-1$. Let $\langle r_{x_{13}}\rangle = \langle r_{x_{11}}^i r_{x_{12}}\rangle$ and $\langle r_{x_{23}}\rangle = \langle r_{x_{21}}^j r_{x_{22}}\rangle$ for some i, j, $1 \leq i,j \leq p-1$. If $i \neq j$ then R is abelian (Theorem 3.23(i)). So we assume that $i = j$. Let $\langle r_{x_{31}}\rangle = \langle r_{x_{11}}^k r_{x_{21}}\rangle$ and $\langle r_{x_{33}}\rangle = \langle (r_{x_{11}}^i r_{x_{12}})^n (r_{x_{21}}^i r_{x_{22}})\rangle$ for some k, n, $1 \leq k, n \leq p-1$. If $n \neq p-k$, then R is abelian by Lemma 4.7. So, we assume that $\langle r_{x_{33}}\rangle = \langle (r_{x_{11}}^i r_{x_{12}})^{p-k} (r_{x_{21}}^i r_{x_{22}})\rangle$. By a similar argument, we assume that $\langle r_{x_{32}}\rangle = \langle r_{x_{21}}^{p-k} r_{x_{22}}\rangle$. Now, Lemma 4.7 implies that R is abelian because $x_{32} \sim x_{33}$ and $p-k \neq p-(p-k)$. □

Corollary 4.13. *If S admits a non-abelian representation, then every line of S is anti-regular and no line of S is regular.*

Proposition 4.14. *Let $S = (P, L)$ be a finite (p, t)-GQ. If S admits a non-abelian representation (R, ψ), then $t = p$ and $R = p_+^{1+4}$.*

Proof. We have $|P| = (p + 1)(pt + 1)$ (Lemma 1.7(i)). So $|R| = p^2(t(p^2 - 1) + p)$ (Proposition 4.10). By Corollary 4.13, $t \geq 2$. So, $p^2(t(p^2 - 1) + p) \geq p^4$. Now, $|R| = p^{2m+1}$ for some integer $m \geq 1$. Thus,

$$t = p(p^{2(m-2)} + p^{2(m-3)} + \cdots + p^2 + 1).$$

Since $t \leq p^2$ (Lemma 1.8(i)), $m = 2$, $t = p$ and $R = p_+^{1+4}$. □

In $Q_5(p)$ all lines are regular (Lemma 1.13(i)). So every representation of $Q_5(p)$ is abelian. On the other hand, since p is odd, $W_4(p)$ is not self-dual and is isomorphic to the dual of $Q_5(p)$ (Lemma 1.12). Since p is odd, no point of $Q_5(p)$ is regular (Lemma 1.11(i)). So no line of $W_4(p)$ is regular. Again, all points of $Q_5(p)$ are anti-regular (Lemma 1.13(i)), so all lines of $W_4(p)$ are anti-regular. We prove

Proposition 4.15. *Let $S = (P, L)$ be a (p, p)-GQ. If S admits a non-abelian representation, then S is isomorphic to $W_4(p)$.*

Proof. Since $W_4(p)$ is characterized by the regularity of all of its points (Lemma 1.14), it is enough to show that if $x, y \in P$ and $x \nsim y$ then $\{x, y\}^{\perp\perp}$ contains $\{a, b\}^{\perp}$ for distinct $a, b \in \{x, y\}^{\perp}$. Let (R, ψ) be a non-abelian representation of S. Let $z \in \{a, b\}^{\perp}$ and $w \in \{x, y\}^{\perp}$. We claim that $z \sim w$. Write $H = C_R(r_a) \cap C_R(r_b)$. Then,

$$|H| = \frac{|C_R(r_a)||C_R(r_b)|}{|C_R(r_a)C_R(r_b)|} = \frac{p^4p^4}{p^5} = p^3.$$

Let $K = \langle r_x, r_y \rangle$. By Theorem 3.23($i$), $|K| = p^3$. So $K = H$ because $K \leq H$. Then, $[r_w, r_z] = 1$ because $[r_w, K] = 1$. So $z \sim w$ again by Theorem 3.23(i). □

Proof of Theorem 4.1. By Proposition 4.4, p is an odd prime. By Lemma 1.4 and Proposition 4.15, S is isomorphic to $W_{2r}(p)$. Proposition 4.10 implies that $R = p_+^{1+2r}$. This completes the proof. □

4.3. Proof of Theorem 4.2

Proposition 1.17 suggests that $W_{2r}(p)$ may admit a non-abelian representation with representation group p_+^{1+2r}. We prove Theorem 4.2 in Propositions 4.17 and 4.18 below. In view of Proposition 4.8, we first prove the following.

Proposition 4.16. *Let $G = p_+^{1+2r}$. There exists a set T of coset representatives of $Z(G)$ in G such that $t_1t_2[t_1,t_2]^{(p-1)/2} \in T$ when $t_1, t_2 \in T$. Further, T is unique up to conjugacy in G.*

Proof. Let $Z = Z(G) = \langle z \rangle$ and $V = G/Z$. We write V as an orthogonal direct sum of r hyperbolic planes K_i $(1 \leq i \leq r)$ in V with respect to the non-degenerate symplectic bilinear form f defined in (3.1.1). Let H_i be the inverse image of K_i in G. Then, H_i is generated by two elements x_{i_1} and x_{i_2} such that $[x_{i_1}, x_{i_2}] = z$. Let $A_j = \langle x_{i_j}, 1 \leq i \leq r \rangle$, $j = 1, 2$. Then, A_j is an elementary abelian p-subgroup of G of order p^r, $A_j \cap Z = \{1\}$ and $A_1Z \cap A_2Z = Z$. Set

$$T = \{xy[x,y]^{\frac{p-1}{2}} : x \in A_1, y \in A_2\}.$$

We show that T has the required property. Let $\alpha = xy[x,y]^{\frac{p-1}{2}}$, $\beta = uv[u,v]^{\frac{p-1}{2}}$ be elements of T where $x, u \in A_1$ and $y, v \in A_2$. If $\alpha Z = \beta Z$, then $u^{-1}xZ = y^{-1}vZ$ and is equal to Z because $A_1Z \cap A_2Z = Z$. So $x = u$ and $y = v$ because $A_j \cap Z = \{1\}$. Thus $\alpha Z = \beta Z$ if and only if $x = u, y = v$. So, $|T| = p^{2r}$ and T is a complete set of coset representatives. Since $G' = Z$, a routine calculation shows that $\alpha\beta[\alpha,\beta]^{(p-1)/2} = (xu)(yv)[xu,yv]^{(p-1)/2} \in T$. Thus, T has the stated property.

Now we prove the uniqueness part. In fact, we show that the group of inner automorphisms of G acts regularly on the set $\mathcal{X}$ of all sets of coset representatives of Z in G, each of which is closed under the binary operation $(t_1, t_2) \mapsto t_1t_2[t_1,t_2]^{(p-1)/2}$.

Fix an ordered basis $\{v_1Z, \cdots, v_{2r}Z\}$ for V. Each $T \in \mathcal{X}$ is determined by the sequence $(x_1, \cdots, x_{2r})$, where $T \cap v_iZ = \{x_i\}$. In fact, if $aZ = x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} Z \in V$, where $i_1 < \cdots < i_n$ and $1 \leq j_k \leq p-1$, then $aZ \cap T = \{x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} z^m\}$, where

$$z^m = [x_{i_1}^{j_1}, x_{i_2}^{j_2}]^{(p-1)/2}[x_{i_1}^{j_1}x_{i_2}^{j_2}, x_{i_3}^{j_3}]^{(p-1)/2} \cdots [x_{i_1}^{j_1} \cdots x_{i_{n-1}}^{j_{n-1}}, x_{i_n}^{j_n}]^{(p-1)/2}.$$

Thus, $|\mathcal{X}| \leq p^{2r}$. Further, for $T \in \mathcal{X}$ and $g \in G$, $g^{-1}Tg = T$ implies $g \in Z$. To see this, let $t \in T$ and $g^{-1}tg = t' \in T$. Then, $tZ = g^{-1}tgZ =$

$t'Z$. Since T contains exactly one element from each coset, it follows that $t = t'$ and $g \in C_G(t)$. Thus, $g \in C_G(T) = Z$. Since $|G : Z| = p^{2r}$, $|\mathcal{X}| = p^{2r}$ and G acts transitively on $\mathcal{X}$. □

Proposition 4.17. *$W_{2r}(p)$, $r \geq 2$, admits a non-abelian representation and the representation group is p_+^{1+2r}.*

Proof. Let $G = p_+^{1+2r}$ and T be as in Proposition 4.16. Consider the partial linear space $S = (P, L)$, where $P = \{\langle x \rangle : 1 \neq x \in T\}$ and a line is of the form $\{\langle x \rangle, \langle y \rangle, \langle xy \rangle, \cdots, \langle x^{p-1}y \rangle\}$ for distinct $\langle x \rangle$, $\langle y \rangle$ in P with $[x, y] = 1$. Note that $x^i y \in T$ for each i and $|P| = (p^{2r}-1)/(p-1)$. We show that S is a polar space of rank r.

Since $T \cap Z(G) = \{1\}$, S is non-degenerate. Let $\langle x \rangle \in P$, $l \in L$ and $\langle x \rangle \notin l$. Then, $\langle x \rangle$ is collinear with one or all points of l because $C_G(x)$ intersects nontrivially with the subgroup H of G generated by the points of l. Note that H is a subgroup of order p^2 and disjoint from $Z(G)$. Rank of S is r because singular subspaces in S correspond to elementary abelian subgroups of G which intersect $Z(G)$ trivially and p^r is the maximum of the orders of such subgroups of G. Thus S is a polar space of rank r.

Clearly G is a representation group of S. So, S is isomorphic to $W_{2r}(p)$ (Theorem 4.1(iii)). □

Proposition 4.18. *Any two representations of $W_{2r}(p)$, $r \geq 2$, are equivalent.*

Proof. Let (R_1, ψ_1) and (R_2, ψ_2) be two representations of $W_{2r}(p)$. By Theorem 4.1(ii), we may assume that $R_1 = R_2 = R$. Then, R_{ψ_i} is a set of coset representatives of $Z(R)$ in R (Proposition 4.9). Let φ be an automorphism of R such that $\varphi(R_{\psi_1}) = R_{\psi_2}$ (Proposition 4.16). Define a map $\beta : P \longrightarrow P$ by $\beta = \psi_2^{-1}\varphi\psi_1$. Now, Lemma 4.5 implies that β is an automorphism of $W_{2r}(p)$. Then, (R, ψ_1) and (R, ψ_2) are equivalent with respect to φ and β. □

CHAPTER 5

Representations of $(2,t)$-GQs

In Section 5.1, we recall some results about $(2,t)$-GQs. We present a proof of the finiteness of t. In Section 5.2, we study k-arcs of a $(2,t)$-GQ in detail. Every representation of a $(2,t)$-GQ is abelian (Theorem 4.1(i)). However, the representation need not be faithful (Example 3.11). In Section 5.3, we study faithful representations of these geometries. Most of the results are well-known, but a detailed study is helpful in studying non-abelian representations of slim dense near hexagons in Chapter 7.

5.1. $(2,t)$-GQs

We first write a brief summary of the results known for $(2,t)$-GQs.

Lemma 5.1 ([**14**], Theorem 7.3, p.99)**.** *Let* $S=(P,L)$ *be a* $(2,t)$*-GQ. Then,* t *is finite and* $t=1,2$ *or* 4.

We present a proof of Lemma 5.1 later in this section.

Lemma 5.2. *There exists a unique* $(2,t)$*-GQ, up to isomorphism, for each value of* $t\in\{1,2,4\}$.

For the uniqueness of the $(2,2)$-GQ, see ([**51**], 5.2.3, p.78) and for that of the $(2,4)$-GQ, see ([**51**], 5.3.2(i), p.90). Thus, S is isomorphic to the classical generalized quadrangles $Q_4^+(2)$, $W_4(2)\simeq Q_5(2)$ and $Q_6^-(2)$, respectively, for $t=1,2$ and 4 (see Subsection 1.3.1 for notation).

A model for a $(2,2)$**-GQ**: Let $\Omega=\{1,2,3,4,5,6\}$. A *factor* of Ω is a set of three pair-wise disjoint 2-subsets of Ω. Let $\mathcal{E}$ be the set of all 2-subsets of Ω and $\mathcal{F}$ be the set of all factors of Ω. Then, $|\mathcal{E}|=|\mathcal{F}|=15$ and the pair $(\mathcal{E},\mathcal{F})$ is a $(2,2)$-GQ ([**51**], 6.1.1, p.122).

There exists a unique projective plane of order four. We refer to [**3**] for a connection between the $(2,2)$-GQ and the projective plane of order four.

A model for a (2,4)-GQ: Let Ω, $\mathcal{E}$ and $\mathcal{F}$ be as in the above model of a (2,2)-GQ. Let $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Take

$$\begin{aligned} P &= \mathcal{E} \cup \Omega \cup \Omega'; \\ L &= \mathcal{F} \cup \{\{i, \{i,j\}, j'\} : 1 \leq i \neq j \leq 6\}. \end{aligned}$$

Then, $|P| = 27$, $|L| = 45$ and the pair (P, L) is a $(2,4)$-GQ ([**51**], 6.1.3, p.123).

Let $S = (P, L)$ be a $(2,t)$-GQ. Since each line contains exactly three points, any two non-collinear points of S are contained in a subspace of S which is a $(2,1)$-subGQ: For $x, y \in P$ with $x \nsim y$, consider $a, b \in \{x,y\}^\perp$ with $a \neq b$. Let $w = (a * x) * (b * y)$. By 'exactly one' axiom, $w \sim b * x$ and $w \sim a * y$. Since $b * x \sim a * y$ also, $\{w, b * x, a * y\}$ is a line.

Proposition 5.3. *Let $S = (P, L)$ be a $(2,t)$-GQ and K be a $(2,1)$-subGQ of S. Then, for $a \in P \setminus K$, the subspace $\langle K \cup \{a\}\rangle$ of S is a $(2,2)$-subGQ.*

Proof. Fix $x \in K$ such that $a \sim x$ and let $l_1 = xa = \{a, x, b\}$. There are two triads of K, say $T_1 = \{x, y, z\}$ and $T_2 = \{x, u, v\}$, containing x. If a is collinear with two points of T_i, then it is a center of T_i and b is a center of T_j, where $\{i, j\} = \{1, 2\}$. We may assume that $a \in T_1^\perp$ and $b \in T_2^\perp$. We get 4 new lines in $\langle K \cup \{a\}\rangle$; namely, $l_2 = \{a, y, c\}$, $l_3 = \{a, z, d\}$, $l_4 = \{b, u, e\}$, $l_5 = \{b, v, g\}$. Consider the set $R = K \cup \{a, b, c, d, e, g\}$ consisting of 15 points, where $K = \{x, y, z, u, v, w, p, q, r\}$ with six lines $n_1 = \{y, v, p\}$, $n_2 = \{y, u, q\}$, $n_3 = \{z, u, r\}$, $n_4 = \{z, v, w\}$, $n_5 = \{x, p, r\}$, $n_6 = \{x, q, w\}$. By 'exactly one' axiom, together with the fact that if three points are pair-wise collinear then they form a line, we get 4 more lines in R; namely, $l_6 = \{q, g, d\}$, $l_7 = \{w, c, e\}$, $l_8 = \{p, e, d\}$, $l_9 = \{r, c, g\}$. Observe that R together with these 15 distinct lines is a subspace of S which is a $(2,2)$-subGQ. Since $R \subseteq \langle K \cup \{a\}\rangle$, $R = \langle K \cup \{a\}\rangle$ by the definition of $\langle K \cup \{a\}\rangle$. $\square$

Proposition 5.4. *Let $S = (P, L)$ be a $(2,2)$-GQ and let $T = \{x, y, z\}$ be a triad of points of S. Then, the following hold:*

(*i*) $|T^\perp| = 1$ *or 3.*

(*ii*) $|T^\perp| = 1$ *if and only if T is contained in a unique $(2,1)$-subGQ of S.*

(*iii*) $|T^\perp| = 3$ *if and only if T is a complete triad.*

Proof. (ii) Let $\{x,y\}^\perp = \{a,b,c\}$ and let $T^\perp = \{a\}$. Then, $\langle T \cup \{b\}\rangle$ is the unique $(2,1)$-subGQ of S containing T because, it contains the point c also. Conversely, if Q is the $(2,1)$-subGQ of S containing T, then Q contains two points of $\{x,y\}^\perp$ and the other point of $\{x,y\}^\perp$ is in $T^\perp$.

(i) Clearly $|T^\perp| \leq 3$. Consider $a \in \{x,y\}^\perp$. If $z \sim a$, then $T^\perp$ is non-empty. If $z \nsim a$ then $\langle T \cup \{a\}\rangle$ is a $(2,1)$-subGQ of S. Then, (ii) implies that $|T^\perp| = 1$. In particular, $T^\perp$ is non-empty. Now assume that $|T^\perp| \geq 2$ and let $a, b \in T^\perp$. Let Q be the $(2,1)$-subGQ generated by $\{a,b,x,y\}$. Let c be the unique point of Q such that $\{a,b,c\}$ is a triad of Q. By (ii), let $\{a,b,c\}^\perp = \{\alpha\}$. Since $\alpha \in \{a,b\}^\perp = \{x,y,z\}$ and $\alpha \notin \{x,y\}$, it follows that $z = \alpha$. Then, both x and y are collinear with $z * c$ in the line zc. Thus $T^\perp = \{a,b,z*c\}$ and $|T^\perp| = 3$.

(iii) T is a complete triad if and only if $|x^\perp \cup y^\perp \cup z^\perp| = 15$. The latter holds if and only if $|T^\perp| = |x^\perp \cap y^\perp \cap z^\perp| = 3$. □

Proposition 5.5. *Let $S = (P,L)$ be a $(2,2)$-GQ. Then, every incomplete triad of point of S is contained in a unique 5-arc. In particular, S has an ovoid.*

Proof. Let T be an incomplete triad of points of S. Then, T is contained in a $(2,1)$-subGQ of S (Proposition 5.4). Let $R = T \cup \{y \in P : y \sim x \text{ for some } x \in T\}$. Then, $|R| = 13$. Let $P \setminus R = \{a,b\}$ and $O = T \cup \{a,b\}$. Then, O is a 5-arc because, if $a \sim b$, then each point of T must be collinear with $a * b$ in the line ab and this implies that there are three more lines (different from ab) containing $a * b$, which is not possible because S is a $(2,2)$-GQ. □

Corollary 5.6. *Every 4-arc of a $(2,2)$-GQ is incomplete.*

Proof. Follows from the proof of Proposition 5.5. □

We now prove Lemma 5.1.

Proof of Lemma 5.1. Let K be a fixed $(2,1)$-subGQ of S and let $T = \{a,b,c\}$ be a fixed triad of points of K. Let $\{Q_i\}$ be the collection of distinct $(2,2)$-subGQs containing K (Proposition 5.3) and let $O_i = T \cup \{x_i, y_i\}$ be the ovoid of Q_i containing T (Proposition 5.5). Let $\{p_i\} = T^\perp$ in Q_i (Proposition 5.4(ii)). Then, $p_i \nsim x_i$ and $p_i \nsim y_i$ since Q_i is a $(2,2)$-GQ. For $i \neq j$, we prove that $p_i \in \{x_j, y_j\}^\perp$ in S and each of x_i, y_i is collinear with exactly one of x_j and y_j. Considering the $(2,1)$-subGQ of Q_j containing the incomplete triad $\{a,b,x_j\}$ (Proposition

5.4), it follows that $p_i \sim x_j$ because $p_i \sim a$ and $p_i \sim b$. The same argument implies $p_i \sim y_j$. Let R be the $(2,1)$-subGQ of Q_j containing the incomplete triad $\{a, x_j, y_j\}$. Since $x_i \nsim a$, it follows from R that x_i is not collinear with both x_j and y_j. Assume that $x_i \nsim x_j$. Now $x_i \sim a * p_i$ in the line $p_i a$ because $x_i \nsim a$ and $x_i \nsim p_i$. Since $p_i \in \{a, x_j, y_j\}^\perp$, it follows from the $(2,2)$-subGQ generated by $R \cup \{p_i\}$ that $x_i \sim y_j$.

If $t = 3$, then $P = Q_1 \cup Q_2$. Consider the line $p_1 x_2$. We have $p_1 * x_2 \in Q_1$ or $p_1 * x_2 \in Q_2$. This implies that $p_1 x_2$ is a line in Q_1 or in Q_2 accordingly. But this is not possible since $p_1 \notin Q_2$ and $x_2 \notin Q_1$.

Assume that $t \geq 5$. Then, there exists at least four $(2,2)$-subGQs of S containing K, say Q_1, Q_2, Q_3, Q_4. We may assume that $x_2 \sim x_3$ between the two pairs $\{x_2, y_2\}$ and $\{x_3, y_3\}$. Then, $l = \{p_1, x_2, x_3\}$ is a line since $p_1 \in \{x_2, x_3\}^\perp$. Consider the point-line pair (p_4, l). We have $p_4 \in O_1^\perp$. So $p_4 \nsim p_1$ as $p_1 \sim a$, $p_4 \nsim x_2$ and $p_4 \nsim x_3$ as each of x_2, x_3 is collinear with one of x_1, y_1, a contradiction to the 'exactly one' axiom. Thus $t = 1, 2$ or 4. This completes the proof. $\square$

We conclude this section with the following description of the subspaces of a $(2,t)$-GQ.

Proposition 5.7. *let $S = (P, L)$ be a $(2,t)$-GQ and let K be a subspace of S. Then, one of the following hold:*

- (a) *K is a k-arc for some $k \geq 0$.*
- (b) *K consists of k lines, all through some point $x \in K$, $1 \leq k \leq t+1$.*
- (c) *K is a $(2, t_1)$-GQ, $1 \leq t_1 \leq t$.*

Proof. Assume that K is not of type (a) or (b). We prove that K is of type (c). Consider a point x of K and suppose that x is contained in $t_1 + 1$ lines of K for some $t_1 \leq t$. We show that $t_1 + 1$ is the number of lines through every point of K. Since the set of lines of K is non-empty, $t_1 \geq 0$. Let y be a point of K such that $x \nsim y$. Such a point y exists because K is not of type (b). If y is contained in $t_2 + 1$ lines of K, then by 'exactly one' axiom $t_1 = t_2$. Hence $t_1 + 1$ is the number of lines of K containing any point not collinear with at least one of the points x or y.

Now, consider a point z of K which is collinear with both x and y. We claim that $t_1 \geq 1$. Suppose that $t_1 = 0$. Let l be a line of K different from the lines xz and yz. Then, $x \notin l$ and $y \notin l$ as $t_1 = 0$. By 'exactly one' axiom, there exists a line of K containing x and intersecting l. So

xz and l intersect at z or $x * z$. The same argument shows that yz and l intersect at z or $y * z$. Hence $z \in l$. Thus every line of K contain z and K is of type (b), a contradiction. So $t_1 \geq 1$. Let l be a line of K containing x and $l \neq xz$. Then, $z \notin l$. There is a point $a \in l$ such that $a \nsim y$ and $a \nsim z$. Then, the number of lines of K containing z equals the number of lines of K containing a and that equals the number of lines of K containing y and hence equals $t_1 + 1$. Thus each point of K is contained in $t_1 + 1$ lines of K. This completes the proof. □

5.2. Complete Arcs

We have studied the complete arcs of a $(2, 2)$-GQ in the previous section. In a $(2, 1)$-GQ, the complete arcs are precisely the triads. Here we describe the complete arcs of a $(2, 4)$-GQ.

Let $S = (P, L)$ be a $(2, 4)$-GQ. Then, S has no ovoid (Lemma 1.16(ii)) and $k \leq 6$ for any k-arc of S by ([**51**], 2.7.1, p.34). Thus, any 6-arc of S is complete.

Proposition 5.8. *Every triad of S is contained in a unique $(2, 1)$-subGQ of S.*

Proof. Uniqueness of the $(2, 1)$-GQ containing T is clear. Let $T = \{a, b, c\}$ be triad of S. Let Q_1 be a $(2, 1)$-subGQ of S containing a and b. We may assume that $c \notin Q_1$. Let Q_2 be the $(2, 2)$-subGQ generated by $Q_1 \cup \{c\}$ (Proposition 5.3). We assume again that there is no $(2, 1)$-subGQ of Q_2 containing T. So T is a complete triad in Q_2 (Proposition 5.4). Let $\{x, y\} = \{a, b\}^\perp$ in Q_1 and z be the point in Q_1 such that $\{x, y, z\}$ is triad. Since T is a complete triad in Q_2, $c \in \{x, y, z\}^\perp$. Let $w \in \{a, b\}^\perp \setminus Q_2$. Since $\{a, b, z\}$ is a triad in Q_1, it follows from Q_1 that $w \sim z$ and so $w \nsim c$. Now T is contained in the $(2, 1)$-subGQ generated by $T \cup \{w\}$. □

Corollary 5.9. *Any 4-arc of S is contained in a unique $(2, 2)$-subGQ of S. In particular, any 4-arc has at most two centers.*

Proof. This follows from Propositions 5.8 and 5.3. □

Proposition 5.10. *Every triad of S has exactly three centers.*

Proof. Let $T = \{a, b, c\}$ be a triad of S and let Q be the $(2, 1)$-subGQ containing T. Then, it follows from Q that $x \sim c$ for every $x \in \{a, b\}^\perp \setminus Q$. □

We note that Proposition 5.10 is a particular case of Lemma 1.10. The latter is proved by a sophisticated counting method. Here, our proof is mainly based on the fact that each line contains exactly three points.

Proposition 5.11. *Let O be a 5-arc of S. If O is contained in a $(2,2)$-subGQ of S, then O is complete.*

Proof. Let Q be a $(2,2)$-subGQ of S containing O. Clearly O is complete in Q being an ovoid of it. There are three centers in S of any triad $T \subset O$ and Q contains exactly one of them (Proposition 5.4). So the $(2,1)$-subGQ of S containing T, in fact, is contained in Q. Let B be the set of points of S which are collinear with some point of O. We show that $B = P$ and this will prove the completeness of O in S. Fix two points $a, b \in O$. Let l and m be the two lines of S containing a which are not the lines of Q and let b be collinear with the points x of l and y of m. Then, $c \sim x$ and $c \sim y$ for $c \in O \setminus \{a,b\}$ because the $(2,1)$-subGQ containing the triad $T = \{a,b,c\}$ is contained in Q. For $c \in O \setminus \{a\}$, let B_c be the $(2,1)$-subGQ generated by $l \cup m \cup \{c\}$ and let $A_c = \{z \in B_c : z \sim c \text{ and } z \nsim a\}$. Then, A_c contains exactly two points which are not points of Q. Also $A_c \cap A_d$ is empty for $c, d \in O \setminus \{a\}$ with $c \neq d$. So B contains 27 points which is the disjoint union of $B = Q \cup (l \cup m) \setminus \{a\} \bigcup_{c \in O \setminus \{a\}} A_c$. Thus $B = P$. $\square$

Proposition 5.12. *Let O be a 5-arc of S. Then, O is contained in a $(2,2)$-subGQ of S if and only if $|O^\perp| = 2$. In particular, O is complete if $|O^\perp| = 2$.*

Proof. Let $O = \{a,b,c,d,e\}$. Let Q be a $(2,2)$-subGQ of S containing O and let $\{x,y\} = \{a,b\}^\perp \setminus Q$. We show that $O^\perp = \{x,y\}$. Let $z \in \{x,y\}$ and $w \in \{c,d,e\}$. Considering the $(2,1)$-subGQ of Q containing the triad $\{a,b,w\}$, it follows that $z \sim w$. Thus $O^\perp = \{x,y\}$ since $|O^\perp| \leq 2$ by Corollary 5.9.

Now let $O^\perp = \{x,y\}$. Let Q be a $(2,2)$-subGQ of S containing the 4-arc $\{a,b,c,d\}$ (Corollary 5.9). Then, $x, y \notin Q$. We show that $e \in Q$. Assume that $e \notin Q$ and let w be the point in Q such that $T \cup \{w\}$ is a 5-arc of Q. Considering the $(2,1)$-subGQ of Q containing the triad $\{a,b,w\}$, it follows that $x \sim w, y \sim w$. Also $e \sim w$ because $T \cup \{w\}$ is a complete 5-arc (Proposition 5.11). This implies that $\{x,e,w\}$ and $\{y,e,w\}$ are two lines containing e and w, a contradiction. Thus $e \in Q$. $\square$

Corollary 5.13. *Any 4-arc of S has exactly two centers and is contained in a unique complete 5-arc.*

Proof. This follows from Corollaries 5.6 and 5.9 and Proposition 5.12. □

The following result is a converse to Proposition 5.11.

Proposition 5.14. *Let O be a 5-arc of S. If O is complete, then it is contained in a $(2,2)$-subGQ of S.*

Proof. Let $T \subset O$ be a 4-arc and $\{x\} = O \setminus T$. Let Q be the $(2,2)$-subGQ of S containing T (Corollary 5.9). We show that $x \in Q$. Suppose that $x \notin Q$. Let z be the point of Q such that $T_1 = T \cup \{z\}$ is a complete 5-arc of Q. Let B be the set of points of S which are collinear with some point of T. Then, $|B| = (4 \times 11) - (6 \times 5) + (4 \times 3) - 2 = 24$ and $P \setminus B = \{z, a, b\}$, where $a, b \notin Q$ and $x \in \{a, b\}$. By Proposition 5.11, T_1 is a complete 5-arc of S. So $z \sim a, z \sim b$. Also $a \not\sim b$, otherwise $\{z, a, b\}$ would be a line and $w \sim a$ or $w \sim b$ for $w \in T$, which is not possible. This implies that $T \cup \{a, b\}$ is a 6-arc of S containing O, a contradiction to the completeness of O. □

There is an immediate corollary of the proof of Proposition 5.14.

Corollary 5.15. *S has 6-arcs. Any 4-arc of S is contained in a unique 6-arc.*

Proposition 5.16. *Every incomplete 5-arc of S has exactly one center.*

Proof. Let $O = \{a, b, c, d, e\}$ be an incomplete 5-arc of S. Let $\{x, y\} = \{a, b, c, d\}^{\perp}$ (Corollary 5.13). Note that both x and y can not be in $O^{\perp}$ (Proposition 5.12). We show that either $x \in O^{\perp}$ or $y \in O^{\perp}$. Suppose that that $x \not\sim e$ and $y \not\sim e$. Let Q be the $(2,1)$-subGQ of S containing the triad $\{a, b, e\}$. Since $x \sim a, x \sim b$ and $x \not\sim e$, it follows from Q that $x \in Q$. The same argument implies $y \in Q$. Thus $\{x, y, e\}$ is a triad of Q. Since $c \sim x, c \sim y$ and $c \notin Q$, it follows from Q that $c \sim e$, a contradiction. □

We summarize the above results in the following:

Theorem 5.17. *Let $S = (P, L)$ be a $(2,4)$-GQ and let O be a k-arc of S. Then, the following hold.*

(a) *If* $k=3$, *then* O *is contained in a unique* $(2,1)$*-subGQ of* S *and* $|O^{\perp}|=3$.

(b) *If* $k=4$, *then* O *is contained in a unique complete* 5*-arc and in a unique complete* 6*-arc of* S, *and* $|O^{\perp}|=2$.

(c) *If* $k=5$, *then* O *is complete if and only if it is contained in a* $(2,2)$*-subGQ of* S *if and only if* $|O^{\perp}|=2$. *Further,* O *is incomplete if and only if* $|O^{\perp}|=1$.

(d) S *has* 6*-arcs and any such arc is complete.*

5.3. Representations

The contents of this section appear in ([**56**], Section 3). Throughout this section, let $S=(P,L)$ be a $(2,t)$-GQ. For $t'=1,2$, a $(2,t')$-subGQ of S and a point outside it generate a $(2,2t')$-subGQ in S. Further, the minimum number of points generating S is 4 if $t=1$, 5 if $t=2$ and 6 if $t=4$.

For the rest of this section, let (R,ψ) be a faithful representation of S. By Proposition 4.3, R is an elementary abelian 2-group. Recall that $R_{\psi}^{*}=\{r_x : x\in P\}\subseteq I_2(R)$ and $R_{\psi}=R_{\psi}^{*}\cup\{1\}$.

Proposition 5.18. *The following hold:*

(i) $|R|=2^4$ *if* $t=1$*;*

(ii) $|R|=2^4$ *or* 2^5 *if* $t=2$, *and both possibilities occur;*

(iii) $|R|=2^6$ *if* $t=4$.

Proof. The $\mathbb{F}_2$-dimension of R is at most k, since S is generated by k points where $(t,k)\in\{(1,4),(2,5),(4,6)\}$. So $|R|\leq 2^k$.

(i) If $t=1$, then $|R|\geq 2^4$ because $|P|=9$ and ψ is faithful. So $|R|=2^4$.

(ii) If $t=2$, then $|R|\geq 2^4$ because S contains a $(2,1)$-subGQ. The rest follows from the fact that S has a symplectic embedding in a $\mathbb{F}_2$-vector space of dimension 4 as well as an orthogonal embedding in a $\mathbb{F}_2$-vector space of dimension 5 (see Section 5.1).

To prove (iii), we need Proposition 5.19 below which is a partial converse to the fact that if $x\sim y$, $x,y\in P$, then $r_xr_y\in R_{\psi}^{*}$. □

Proposition 5.19. *Assume that* $(t,|R|)\neq(2,2^4)$. *If* $r_xr_y\in R_{\psi}^{*}$ *for distinct* $x,y\in P$, *then* $x\sim y$.

Proof. Let $z\in P$ be such that $r_z=r_xr_y$. If $x\nsim y$, then $T=\{x,y,z\}$ is a triad of S because ψ is faithful. There is no (2,1)-subGQ of S containing T because the subgroup of R generated by the image of such a GQ is of order 2^4 (Proposition 5.18(i)). Every triad of a $(2,4)$-GQ is contained in a unique (2,1)-subGQ (Proposition 5.8). So $t = 2$ and T is a complete triad. Let Q be a (2,1)-subGQ of S containing x and y. Then, $z \notin Q$ and $P = \langle Q, z\rangle$. Since $r_z \in \langle\psi(Q)\rangle$, $|R| = |\langle\psi(Q)\rangle| = 2^4$, a contradiction to $(t, |R|) \neq (2, 16)$. □

Remark 5.20. *If* $(t, |R|) = (2, 2^4)$*, then Proposition 5.19 is not true because in this case* $R^* = R^*_\psi$*, so* $r_x r_y \in R^*_\psi$ *for non-collinear points* x *and* y*.*

Proof of Proposition 5.18(iii). If $t = 4$, then there are 16 points of S not collinear with a given point x. By Proposition 5.19, $|R^* \setminus R_\psi| \geq 16$. Thus, $|R| > 2^5$ and so $|R| = 2^6$. □

Corollary 5.21. *Let* $t = 4$ *and* Q *be a (2,2)-subGQ of* S*. Then,* $\langle\psi(Q)\rangle$ *is of order* 2^5*.*

Proof. This follows from Proposition 5.18(iii) and the fact that $P = \langle Q, x\rangle$ for $x \in P \setminus Q$ (see Proposition 5.7). □

Proposition 5.22. *If* $t = 2$*, then* $|R| = 2^4$ *if and only if* $r_a r_b r_c = 1$ *for every complete triad* $\{a, b, c\}$ *of points of* S*.*

Proof. Let $T = \{a, b, c\}$ be a complete triad of points of S and Q be a (2,1)-subGQ of S containing a and b. Then, $c \notin Q$ and $P = \langle Q, c\rangle$.

If $r_a r_b r_c = 1$, then $r_c \in \langle\psi(Q)\rangle$ and $|R| = |\langle\psi(Q)\rangle| = 2^4$. Now, assume that $|R| = 2^4$. Let $\{x, y\} = \{a, b\}^\perp$ in Q. Then, $x, y \in T^\perp$, since T is a complete triad. Let z be the point in Q such that $\{x, y, z\}$ is a triad in Q. Then, $c \sim z$ and $r_z = (r_a r_x)(r_b r_y)$. Since $H = \langle r_y : y \in x^\perp\rangle$ is a maximal subgroup of R (Lemma 3.6), $|H| = 2^3$. So $r_c = r_a r_b$ or $r_a r_b r_x$, since ψ is faithful. If the latter holds, then $r_{c*z} = r_y$. But this is not possible because, ψ is faithful and $y \neq c * z$. Hence $r_c = r_a r_b$. □

Corollary 5.23. *Assume that* $(t, |R|) = (2, 2^4)$*. Let* $a, b, c \in P$ *be such that* $r_a r_b r_c = 1$*. Then,* $T = \{a, b, c\}$ *is a line or a complete triad.*

Proof. Assume that T is not a line. Then, T is a triad, since ψ is faithful. We show that T is complete. Suppose that T is not complete. Let $\{a, b, d\}$ be the complete triad of S containing a and b. Then, $r_a r_b r_d = 1$ (Proposition 5.22) and $c \neq d$. So $r_c = r_d$, contradicting that ψ is faithful. □

Lemma 5.24. *If S contains a triad $T = \{a,b,c\}$ such that $r_ar_br_c \in R^*_\psi$, then $(t,|R|) = (2,2^4)$. In particular, T is incomplete.*

Proof. Let $x \in P$ be such that $r_x = r_ar_br_c$. Since ψ is faithful, $x \notin T$. Let Q_1 be the (2,1)-subGQ of S containing T. If $x \in Q_1$, then $\langle\psi(Q_1)\rangle = \langle r_a, r_b, r_c, r_x\rangle$ would be of order 2^3, contradicting Proposition 5.18(i). So $x \notin Q_1$ and $t \neq 1$. Let Q_2 be the (2,2)-subGQ of S generated by Q_1 and x. Then, $|\langle\psi(Q_2)\rangle| = 2^4$. By Corollary 5.21, $t \neq 4$. So $t = 2$, $P = Q_2$ and $|R| = 2^4$. Now, Proposition 5.22 implies that T is incomplete. □

Lemma 5.25. *Let $a,b \in P$ with $a \nsim b$. Set $A = \{r_ar_x : x \nsim a\}$ and $B = \{r_br_x : x \nsim b\}$. Then, $|A \cap B| = t+2$.*

Proof. It is enough to prove that $r_ar_x = r_br_y$ for $r_ar_x \in A, r_br_y \in B$ if and only if either $x = b$ and $y = a$ holds or there exists a point c such that $\{c,a,y\}$ and $\{c,b,x\}$ are lines. We need to prove the 'only if' part. Since ψ is faithful, $x \neq b$ if and only if $y \neq a$. Assume that $x \neq b$ and $y \neq a$. For this, we show that $y \sim a$ and $x \sim b$. Then, $r_{a*y} = r_ar_y = r_br_x = r_{b*x}$. Since ψ is faithful, it would then follow that $a * y = b * x$ and this would be our choice of c.

First, assume that $(t,|R|) \neq (2,2^4)$. Since $a \nsim b$, $r_ar_b \notin R_\psi$ by Proposition 5.19. Since $r_xr_y = r_ar_b$, Proposition 5.19 again implies that $x \nsim y$. Now, $r_ar_br_y = r_x \in R_\psi$. By Lemma 5.24, $\{a,b,y\}$ is not a triad. This implies that $y \sim a$. By a similar argument, $x \sim b$.

Now, assume that $(t,|R|) = (2,2^4)$. Suppose that $x \nsim b$. Then, $T = \{a,b,x\}$ is a triad of S. By Proposition 5.24, T is incomplete. Let Q be the $(2,1)$-subGQ in S containing T and let $\{c,d\} = \{a,b\}^\perp$ in Q. Then, $r_x = r_ar_br_cr_d = r_xr_yr_cr_d$. So $r_yr_cr_d = 1$. By Corollary 5.23, $\{c,d,y\}$ is a complete triad. Since $b \in \{c,d\}^\perp$, it follows that $b \in \{c,d,y\}^\perp$, a contradiction to that $b \nsim y$. So $x \sim b$. A similar argument shows that $y \sim a$. This completes the proof. □

Proposition 5.26. *Let $K = R \setminus R_\psi$. Each element of K is of the form r_yr_z for some $y \nsim z$ in P, except when $(t,|R|) = (2,2^5)$. In this case, exactly one element, say α, of K can not be expressed in this way. Moreover, $\alpha = r_ur_vr_w$ for every complete triad $\{u,v,w\}$ of S.*

Proof. Since K is empty when $(t,|R|) = (2,2^4)$, we assume that $(t,|R|) = (1,2^4)$, $(2,2^5)$ or $(4,2^6)$. Fix $a,b \in P$ with $a \nsim b$. Then, $r_ar_b \in K$ (Proposition 5.19). Let A and B be as in Lemma 5.25, and set

$$C = \{r_a r_b r_x : \{a, b, x\} \text{ is a triad which is incomplete if } t = 2\}.$$

By proposition 5.19, $A \subseteq K$ and $B \subseteq K$ and by Lemma 5.24, $C \subseteq K$. Each element of C corresponds to a triad which is contained in a (2,1)-subGQ of S. Let $r_a r_b r_x \in C$ and Q be the (2,1)-subGQ of S containing the triad $\{a, b, x\}$. If $\{a, b\}^\perp = \{p, q\}$ in Q, then $r_{a*p} r_{b*q} = r_x$ implies that $r_a r_b r_x = r_p r_q$. Thus, every element of C can be expressed in the required form.

By Proposition 5.19, $A \cap C$ and $B \cap C$ are empty. By Lemma 5.25, $|A \cap B| = t + 2$. Then, an easy count shows that

$$|A \cup B \cup C| = \begin{cases} 10t - 4 & \text{if } t = 1 \text{ or } 4 \\ 10t - 5 & \text{if } t = 2 \end{cases}.$$

So $K = A \cup B \cup C$ if $t = 1$ or 4, and $K \setminus (A \cup B \cup C)$ is a singleton if $t = 2$. This proves the proposition for $t = 1, 4$ and tells that if $(t, |R|) = (2, 2^5)$, then at most one element of K can not be written in the desired form.

Now, let $(t, |R|) = (2, 2^5)$ and $T = \{u, v, w\}$ be a complete triad of S. By Lemma 5.24, $\alpha = r_u r_v r_w \in K$. Suppose that $\alpha = r_x r_y$ for some $x, y \in P$. Then, $x \nsim y$ by Lemma 5.24 and $\{x, y\} \cap T = \Phi$ by Proposition 5.19. Suppose that $x \in T^\perp$ and Q be the $(2, 1)$-subGQ of S generated by $\{x, u, v, y\}$. Since $w \notin Q$ and $r_w = r_u r_v r_x r_y$, it follows that $|R| = 2^4$, a contradiction. So, $x \notin T^\perp$. Similarly, $y \notin T^\perp$. Thus, each of x and y is collinear with exactly one point of T. Let $x \sim u$. Then, $y \nsim x * u$, since $x * u \in T^\perp$ and $\alpha = r_x r_y$. Let U be the (2,1)-subGQ of S generated by $\{u, x, y, v\}$. Note that $y \sim u$ in U. Let z be the unique point in U such that $\{u, v, z\}$ is a triad of U. Then, $r_z = r_x r_y r_u r_v = r_w$. Since $w \neq z$ (in fact, $w \notin U$), this is a contradiction to the faithfulness of ψ. Thus, α can not be expressed as $r_x r_y$ for any x, y in P. This, together with the last sentence of the previous paragraph, implies that α is independent of the complete triad T of S. This completes the proof. □

CHAPTER 6

Slim Dense Near Hexagons

In Section 6.1, we present the classification of slim dense near hexagons due to Brouwer, et al. [**9**] and give a construction for each of them. In Section 6.2, we give new constructions for the near hexagons $DH_6(2^2)$ and $DW_6(2)$. In Section 6.3, we describe the structure of a slim dense near hexagon having big quads with respect to a subspace of it generated by two of its big quads.

6.1. Classification Result

Let $S = (P, L)$ be a dense near hexagon with parameters $(2, t)$. Let $t_2 + 1 = |\Gamma_1(x) \cap \Gamma_1(y)|$ for $x, y \in P$ with $d(x, y) = 2$. Note that t_2 depends on the points x and y and $t > t_2$. We say that a quad of S is of *type* $(2, t_2)$ if it is a $(2, t_2)$-GQ. A quad of S is *big* if it has distance at most one from each point of S. Thus, a big quad of S is classical and vice-versa. The following result is due to Brouwer, et al. [**9**].

Theorem 6.1 ([**9**], Theorem 1.1, p.349)**.** *Let $S = (P, L)$ be a dense near hexagon with parameters $(2, t)$. Then, P is finite and S is isomorphic to one of the eleven near hexagons with parameters as given below:*

	$\|P\|$	t	t_2	$dimV(S)$	$n(S)$	a_1	a_2	a_4
$\mathbb{E}_2$	759	14	2	23	22	0	35	0
$\mathbb{E}_1$	729	11	1	24	24	66	0	0
$DH_6(2^2)$	891	20	$4^\star$	22	20	0	0	21
$\mathbb{E}_3$	567	14	$2, 4^\star$	21	20	0	15	6
$\mathbb{G}_3$	405	11	$1, 2, 4^\star$	20	20	9	9	3
$Q_6^-(2) \otimes Q_6^-(2)$	243	8	$1, 4^\star$	18	18	16	0	2
$Q_6^-(2) \times \mathbb{L}_3$	81	5	$1, 4^\star$	12	12	5	0	1
$DW_6(2)$	135	6	$2^\star$	15	8	0	7	0
$\mathbb{H}_3$	105	5	$1, 2^\star$	14	8	3	4	0
$W_4(2) \times \mathbb{L}_3$	45	3	$1, 2^\star$	10	8	3	1	0
$Q_4^+(2) \times \mathbb{L}_3$	27	2	$1^\star$	8	8	3	0	0

Here, $\mathbb{L}_3$ denotes a line of size 3, $n(S)$ is the $\mathbb{F}_2$-rank of the matrix $A_3 : P \times P \longrightarrow \{0, 1\}$ defined by $A_3(x, y) = 1$ if $d(x, y) = 3$ and zero otherwise. For each row, the t_2 column lists all possible values of t_2. Further, we add a '$*$' to this entry if and only if the corresponding quad is big. The number of quads of type $(2, r)$, $r = 1, 2, 4$, containing a point of S is indicated by a_r.

We refer to [**26**] for other classification results about slim dense near polygons. It is proved that there are 24 slim dense near octagons, up to isomorphism. The classification is mainly based on the fact that all slim dense near octagons have a *big hex* - that is, a convex sub near hexagon with distance at most 1 from every point of the near octagon.

The near hexagons $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathbb{G}_3$ and $\mathbb{H}_3$ are described in Section 1.7. The near hexagons $DH_6(2^2)$ and $DW_6(2)$ are the unitary and symplectic dual polar spaces of rank three, respectively. All slim dense near hexagons having big quads are subspaces of $DH_6(2^2)$ (see [**9**], p.353). A direct product of near polygons is a near polygon ([**10**], Theorem 1, p.146). In that case, the number of points in a line is independent of that line. The near hexagons $Q_6^-(2)\times\mathbb{L}_3$, $W_4(2)\times\mathbb{L}_3$ and $Q_4^+(2)\times\mathbb{L}_3$ are direct products of a $(2, t)$-GQ for $t = 4, 2, 1$, respectively, with $\mathbb{L}_3$.

We now describe $Q_6^-(2) \otimes Q_6^-(2)$. The following description of this near hexagon is taken from [**23**]. Let $S = (P, L)$ be a (2,4)-GQ, $T = \{l_1, \cdots, l_9\} \subset L$ be a spread of S and l be an arbitrary line in T. Let $\phi_j : P \longrightarrow l_j$ be the map taking each $x \in P$ to the unique point of l_j nearest to x in S. Let $\mathcal{G} = \mathcal{G}(S, T, l)$ be the graph with vertex set $l \times T \times T$. Two distinct vertices (x, l_i, l_j) and (y, l_m, l_n) are *adjacent* whenever at least one of the following two conditions hold:

(1) $j = n$ and $\phi_i(x)$ and $\phi_m(y)$ are collinear points in S;
(2) $i = m$ and $\phi_j(x)$ and $\phi_n(y)$ are collinear points in S.

Note that if $i = m$ and $j = n$, then both (1) and (2) are satisfied. Any two adjacent vertices of $\mathcal{G}$ are contained in a unique maximal clique of size three. The points and the lines of $Q_6^-(2) \otimes Q_6^-(2)$ are the vertices and the maximal cliques of $\mathcal{G}$, respectively.

We present another description of the near hexagon $\mathbb{H}_3$ taken from [**25**].

Example 6.2 ([**25**], p.51)**.** *Let $S = (P, L)$ be the $(2, 2)$-GQ. A partial linear space $S' = (P', L')$ can be constructed from S as follows.*

Set $P' = \{(x, y) \in P \times P : x = y \text{ or } x \sim y\}$. With each $l \in L$, we associate four types of elements of L' as follows. Let $l = \{x, y, z\}$ and $T = \{k, m, n\}$ be an incomplete triad of lines of S such that $T^\perp = \{l\}$. We may assume that $k \cap l = \{x\}$, $m \cap l = \{y\}$ and $n \cap l = \{z\}$. Let x' be an arbitrary point of $k \setminus \{x\}$. Let y' (respectively, z') be the unique point of $m \setminus \{y\}$ (respectively, $n \setminus \{z\}$) not collinear with x'. The four types of elements of L' are the following:

(I) $\{(x, x), (y, y), (z, z)\}$,
(II) $\{(x, x), (x, y), (x, z)\}$,
(III) $\{(x, y), (y, z), (z, x)\}$.
(IV) $\{(x, x'), (y, y'), (z, z')\}$.

*The elements of L' of type (II) and (III) are up to a permutation of the points of l. Then, S' is a slim dense near hexagon isomorphic to $\mathbb{H}_3$ ([**25**], p.51). It can be seen that $\{x', y', z'\}$ is a complete triad of points of S.*

6.2. New Constructions for $\mathbb{H}_3$ and $DW_6(2)$

In this section, we give new constructions for $DW_6(2)$ and $\mathbb{H}_3$ from two copies of a (2,2)-GQ. We first construct $\mathbb{H}_3$ and then construct $DW_6(2)$ in which $\mathbb{H}_3$ appears as a geometric hyperplane. We use these constructions of $DW_6(2)$ and $\mathbb{H}_3$ in the proof of the existence of their non-abelian representations in Chapter 7. The contents of this section appear in [**54**].

6.2.1. Construction of $\mathbb{H}_3$. Let $S = (P, L)$ and $S' = (P', L')$ be two (2,2)-GQs. Fix an isomorphism from S to S' and denote the image of $x \in P \cup L$ by x'. We define a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ as follows. Take the point set to be

$$\mathcal{P} = \{(x, y') \in P \times P' : y' \in x'^\perp\},$$

and the lines to be subsets of P of the form

$$\{(x, u'), (y, v'), (z, w')\},$$

where $T = \{x, y, z\}$ is a line or a complete triad of points of S and $T'^\perp = x'^\perp \cap y'^\perp \cap z'^\perp = \{u', v', w'\}$ in S'.

Theorem 6.3. *$\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a slim dense near hexagon with parameters $(s, t, t_2) = (2, 5, \{1, 2\})$.*

We develop some structure before we prove the theorem. Let $\alpha = (x, u')$ and $\beta = (y, v')$ be two distinct points of $\mathcal{S}$. By the construction of the lines of $\mathcal{S}$, $\alpha \sim \beta$ if and only if $x \neq y$, $u' \neq v'$, $u' \in y'^{\perp}$ and $v' \in x'^{\perp}$. Let α and β be distinct non-collinear points of $\mathcal{S}$. Then, one of the following possibilities occur:

(A1) $x = y, u' \neq v'$;
(A2) $x \neq y, u' = v'$;
(A3) $x \neq y, u' \neq v'$, $u' \notin y'^{\perp}$ and $v' \notin x'^{\perp}$;
(A4) $x \neq y, u' \neq v'$ and either $u' \in y'^{\perp}$ and $v' \notin x'^{\perp}$ or $u' \notin y'^{\perp}$ and $v' \in x'^{\perp}$.

Lemma 6.4. *Assume that* $(A1)$ *or* $(A2)$ *holds. Then,* $|\{\alpha, \beta\}^{\perp}| \geq 2$.

Proof. Assume that $(A1)$ holds. Then, $x' \in \{u', v'\}^{\perp}$. If $u' \sim v'$, we may assume that $v' \neq x'$. So $x'v' = u'v'$. Then, $(v, u' * v')$ and $(x * v, u' * v')$ are in $\{\alpha, \beta\}^{\perp}$. If $u' \nsim v'$, let $\{u', v', w'\}$ be the complete triad of S' containing u' and v'. Then, (a, w') and (b, w') are in $\{\alpha, \beta\}^{\perp}$, where $\{a', b'\} = \{u', v'\}^{\perp} \setminus \{x'\}$.

Now assume that $(A2)$ holds. Then, $u \in \{x, y\}^{\perp}$ in S. If $x \sim y$, we assume that $u \neq y$. Then, $(x*y, y')$ and $(x*y, y'*u')$ are in $\{\alpha, \beta\}^{\perp}$. If $x \nsim y$, let $\{x, y, w\}$ be the complete triad of S containing x and y. Let $\{x, y\}^{\perp} \setminus \{u\} = \{a, b\}$ in S. Then, (w, a') and (w, b') are in $\{\alpha, \beta\}^{\perp}$. $\square$

Lemma 6.5. *Assume that* $(A3)$ *holds. Then,* $|\{\alpha, \beta\}^{\perp}| \geq 3$.

Proof. If $x \sim y$ and $u' \sim v'$, then $\{x', y', u', v'\}$ defines a quadrangle in S'. Then, (u, x'), (v, y') and $(u * v, x' * y')$ are in $\{\alpha, \beta\}^{\perp}$.

If $x \sim y$ and $u' \nsim v'$, let $T' = \{u'*x', v'*y', z'\}$ be the complete triad of S' containing $u' * x'$ and $v' * y'$. Then, $u', v' \in T'^{\perp}$ and $x' * y' \notin T'$. Now, $x' * y' \sim z'$, because $x' * y' \nsim u' * x'$, $x' * y' \nsim v' * y'$ and T' is a complete triad. Then, $(u*x, x'), (v*y, y')$ and $(z, x'*y')$ are in $\{\alpha, \beta\}^{\perp}$.

By a similar argument, if $x \nsim y$ and $u' \sim v'$ then $(u, u'*x'), (v, v'*y')$ and $(u * v, z')$ are in $\{\alpha, \beta\}^{\perp}$, where $\{u' * x', v' * y', z'\}$ is the complete triad of S' containing $u' * x'$ and $v' * y'$.

Now, assume that $x \nsim y$ and $u' \nsim v'$. If $u' = x'$ and $v' = y'$, then $(a, a'), (b, b')$ and (c, c') are in $\{\alpha, \beta\}^{\perp}$, where $\{u', v'\}^{\perp} = \{a', b', c'\}$ in S'. We may assume that $v' \neq y'$. Then, the complete triads $\{x', y'\}^{\perp}$ and $\{u', v'\}^{\perp}$ of S' intersect at $w' = v' * y'$. This fact is independent of whether $u' = x'$ or not. Let $\{x', y'\}^{\perp} = \{a', b', w'\}$ and $\{u', v'\}^{\perp} =$

$\{p', q', w'\}$ in S'. Since $a' \not\sim w'$, $b' \not\sim w'$ and $\{p', q', w'\}$ is a complete triad of S', each of a' and b' is collinear with exactly one of p' and q'. Similarly, each of p' and q' is collinear with exactly one of a' and b'. So, we may assume that $a' \sim p'$ and $b' \sim q'$. Then, $(p, a'), (q, b')$ and (w, w') are contained in $\{\alpha, \beta\}^{\perp}$. □

Lemma 6.6. *Assume that* $(A4)$ *holds. Then,* $d(\alpha, \beta) = 3$.

Proof. We may assume that $u' \in y'^{\perp}$ and $v' \notin x'^{\perp}$. Suppose that $d(\alpha, \beta) = 2$ and $(z, w') \in \{\alpha, \beta\}^{\perp}$. Then, $z \notin \{x, y\}$, $w' \notin \{u', v'\}$, $u', v' \in z'^{\perp}$ and $w' \in \{x', y'\}^{\perp}$. Let $T' = \{x', y', z'\}$. Then, T' is either a line or a complete triad of S', because x, y and z are pair-wise distinct and $u', w' \in T'^{\perp}$ with $u' \neq w'$. Since $v' \in \{y', z'\}^{\perp}$, it follows that $v' \in T'^{\perp}$ and $v' \in x'^{\perp}$, a contradiction to our assumption.

So $d(\alpha, \beta) \neq 2$. Now, choose $w' \in \{x', y'\}^{\perp}$ with $w' \neq u'$. Then, $\alpha \sim (y, w')$ and $d((y, w'), \beta) = 2$ by Lemma 6.4. Hence $d(\alpha, \beta) = 3$. □

As a consequence of the above results, we have

Corollary 6.7. *The diameter of* $\mathcal{S}$ *is 3.*

We next prove that the near-polygon property (NP) is satisfied in $\mathcal{S}$. Let $\mathtt{L} = \{\alpha, \beta, \gamma\}$ be a line and θ be a point of $\mathcal{S}$. Note that any two points of $\mathcal{S}$ have only one common neighbour. So if θ has ditance 1 from two points of $\mathtt{L}$, then it is itself a point of $\mathtt{L}$. Let $\alpha = (x, u'), \beta = (y, v'), \gamma = (z, w')$ and $\theta = (p, q')$. Then, $T = \{x, y, z\}$ is either a line or a complete triad of points of S and $T'^{\perp} = \{u', v', w'\}$.

Proposition 6.8. *If* θ *has distance 2 from two points of* $\mathtt{L}$*, then it is collinear with the third point of* $\mathtt{L}$.

Proof. Let $d(\theta, \alpha) = d(\theta, \beta) = 2$. We prove $\theta \sim \gamma$ by showing that $p \neq z, q' \neq w'$, $q' \in z'^{\perp}$ and $w' \in p'^{\perp}$.

If $p = x$ and $q' \neq v'$ (respectively, $p \neq x$ and $q' = v'$), then $d(\theta, \beta) = 2$ (respectively, $d(\theta, \alpha) = 2$) yields $v' \notin x'^{\perp}$, a contradiction. So $p = x$ if and only if $q' = v'$. Similarly, $p = y$ if and only if $q' = u'$. Thus, if $p \in \{x, y\}$, then $p \neq z, q' \neq w'$, $q' \in z'^{\perp}$ and $w' \in p'^{\perp}$.

If $p \notin \{x, y\}$, then the above argument implies that $p \neq z$ and $q' \neq w'$. Also, $d(\theta, \alpha) = d(\theta, \beta) = 2$ yields $x', y' \notin q'^{\perp}$ and $u', v' \notin p'^{\perp}$. This implies that $q' \in z'^{\perp}$ and $w' \in p'^{\perp}$. □

Proposition 6.9. *If* θ *has distance 3 from two points of* $\mathtt{L}$*, then it has distance 2 from the third point of* $\mathtt{L}$.

Proof. Let $d(\theta,\alpha) = d(\theta,\beta) = 3$. We prove $d(\theta,\gamma) = 2$. By Lemma 6.4, we may assume that $p \neq z$ and $q' \neq w'$. This, together with $d(\theta,\alpha) = d(\theta,\beta) = 3$, implies that $p' \notin T'$, $q' \notin T'^{\perp}$. We show that $q' \notin z'^{\perp}$ and $w' \notin p'^{\perp}$. This would complete the proof.

Suppose that $q' \in z'^{\perp}$. Since $q' \notin T'^{\perp}$, $q' \notin x'^{\perp}$ and $q' \notin y'^{\perp}$. Then, $d(\theta,\alpha) = d(\theta,\beta) = 3$ yields $u', v' \in p'^{\perp}$. This implies that $p' \in \{u',v'\}^{\perp} = T'$, a contradiction. A similar argument shows that if $w' \in p'^{\perp}$, then $q' \in T'^{\perp}$, a contradiction. □

Proof of Theorem 6.3. Propositions 6.8 and 6.9, together with Corollary 6.7, imply that $\mathcal{S}$ is a near hexagon. By Lemmas 6.4 and 6.5, $\mathcal{S}$ is dense. Since $|\mathcal{P}| = 105$, Theorem 6.1 completes the proof. □

Thus, quads of $\mathcal{S}$ are $(2,1)$ or $(2,2)$-GQs. In fact, it can be shown that equality holds in Lemmas 6.4 and 6.5.

6.2.2. Construction of $DW_6(2)$. Let $S = (P, L)$, $S' = (P', L')$ and $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be as in the construction of $\mathbb{H}_3$ in Subsection 6.2.1. We define a partial linear space $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ as follows. Take the point set to be

$$\mathbb{P} = \mathcal{P} \cup P \cup P'$$

and the line set to be

$$\mathbb{L} = \mathcal{L} \cup \mathbb{L}_1,$$

where $\mathbb{L}_1$ consists of subsets of $\mathbb{P}$ of the form $\{x, (x,u'), u'\}$ for every point $(x,u') \in \mathcal{P}$.

Theorem 6.10. $\mathbb{S} = (\mathbb{P}, \mathbb{L})$ *is a slim dense near hexagon with parameters* $(s,t,t_2) = (2,6,2)$.

An immediate consequence of Theorems 6.3 and 6.10 is the following.

Corollary 6.11. $\mathbb{H}_3$ *is a geometric hyperplane of* $DW_6(2)$.

By the construction of the lines of $\mathbb{S}$, no two points of P, as well as of P', are collinear in $\mathbb{S}$. Further, if $x \in P$ and $u' \in P'$, then $x \sim u'$ if and only if $(x,u') \in \mathcal{P}$, or equivalently, $u' \in x'^{\perp}$ in S'. Let α and β be two distinct non-collinear points of $\mathbb{S}$ with $\alpha \in P \cup P'$. Then, one of the following possibilities occur:

- $(B1)$ $\alpha = x$ and $\beta = y$ for some $x, y \in P$ with $x \neq y$;
- $(B2)$ $\alpha = u'$ and $\beta = v'$ for some $u', v' \in P'$ with $u' \neq v'$;
- $(B3)$ $\alpha = x \in P$ and $\beta = u' \in P'$ with $u' \notin x'^{\perp}$;

($B4$) $\alpha = x \in P$ and $\beta = (y, v') \in \mathcal{P}$ with $x \neq y$ and $v' \in x'^{\perp}$ in S';
($B5$) $\alpha = u' \in P'$ and $\beta = (y, v') \in \mathcal{P}$ with $u' \neq v'$ and $y \in u^{\perp}$ in S;
($B6$) $\alpha = x \in P$ and $\beta = (y, v') \in \mathcal{P}$ with $x \neq y$ and $v' \notin x'^{\perp}$ in S';
($B7$) $\alpha = u' \in P'$ and $\beta = (y, v') \in \mathcal{P}$ with $u' \neq v'$ and $y \notin u^{\perp}$ in S.

Lemma 6.12. *Assume that* ($B1$) *or* ($B2$) *holds. Then,* $|\{\alpha, \beta\}^{\perp}| \geq 3$ *in* $\mathbb{S}$.

Proof. If ($B1$) holds, then $w' \in \{x, y\}^{\perp}$ in $\mathbb{S}$ for each $w' \in \{x', y'\}^{\perp}$ in S'. So $|\{\alpha, \beta\}^{\perp}| \geq 3$. Similarly, if ($B2$) holds then $|\{\alpha, \beta\}^{\perp}| \geq 3$. □

Lemma 6.13. *Assume that* ($B3$) *holds. Then,* $d(\alpha, \beta) = 3$.

Proof. Clearly $d(\alpha, \beta) \geq 3$ since $u' \notin x'^{\perp}$. Let $v' \in \{u', x'\}^{\perp}$ in S'. Then, x, v', v, u' is a path of length 3 in $\Gamma(\mathbb{P})$. So $d(\alpha, \beta) = 3$. □

Lemma 6.14. *Assume that* ($B4$) *or* ($B5$) *holds. Then,* $|\{\alpha, \beta\}^{\perp}| \geq 3$.

Proof. Assume that ($B4$) holds. If $x \sim y$ in S, then $v' \in x'y'$ in S'. If $v' = x'$, then $v', (x, y')$ and $(x, v' * y')$ are in $\{\alpha, \beta\}^{\perp}$. We may assume that $v' \neq x'$. Then, $v', (x, x')$ and $(x, v' * x')$ are in $\{\alpha, \beta\}^{\perp}$. If $x \nsim y$ in S, let $\{x', y'\}^{\perp} = \{u', v', w'\}$ in S'. Then, $v', (x, u')$ and (x, w') are in $\{\alpha, \beta\}^{\perp}$. A similar argument applies if ($B5$) holds. □

Lemma 6.15. *Assume that* ($B6$) *or* ($B7$) *holds. Then,* $d(\alpha, \beta) = 3$.

Proof. Assume that ($B6$) holds. Suppose that $\theta \in \{\alpha, \beta\}^{\perp}$. Then, $\theta \neq v'$, since $v' \notin x'^{\perp}$ in S'. So $\theta = (x, w')$ for some $w' \in x'^{\perp}$. Then, $\theta \sim \beta$ implies that $v' \in x'^{\perp}$ in S', a contradiction. So $d(\alpha, \beta) \neq 2$. Now, $y \sim \beta$ and $d(\alpha, y) = 2$ (Lemma 6.12). So $d(\alpha, \beta) = 3$. A similar argument can be applied if ($B7$) holds. □

Note that the embedding of $\mathcal{S}$ into $\mathbb{S}$ is isometric. As a consequence of the above results of this subsection, together with Corollary 6.7, we have

Corollary 6.16. *The diameter of* $\mathbb{S}$ *is 3.*

Next we prove that property (NP) is satisfied in $\mathbb{S}$.

Proposition 6.17. *Let* L *be a line of* $\mathbb{S}$ *of type* $(\mathbb{L}_1)$ *and* α *be a point of* $\mathbb{S}$ *not contained in* L*. Then,* α *is nearest to exactly one point of* L*.*

Proof. Let $\mathtt{L} = \{x, \beta, u'\}$ where $\beta = (x, u') \in \mathcal{P}$. Let $\alpha = v' \in P'$. Then, $v' \neq u'$ and $d(\alpha, u') = 2$ (Lemma 6.12). Now $d(\alpha, \beta) = 2$ or 3 according as $x \in v^{\perp}$ in S or not. In the first case, $\alpha \sim x$, and in the latter case, $d(\alpha, x) = 3$ (Lemma 6.12). A similar argument holds if $\alpha \in P$.

Let $\alpha = (y, v') \in \mathcal{P}$. If $x = y$, then $u' \neq v'$ and $x \in \{u, v\}^{\perp}$ in S. So $\alpha \sim x$ and $d(\alpha, \beta) = d(\alpha, u') = 2$ (Lemmas 6.4 and 6.14). Similarly, if $u' = v'$ then $\alpha \sim u'$ and $d(\alpha, \beta) = d(\alpha, x) = 2$. Assume that $x \neq y$ and $u' \neq v'$. If $\alpha \sim \beta$, then $u' \in y'^{\perp}$ and $v' \in x'^{\perp}$ in S'. So $d(\alpha, x) = d(\alpha, u') = 2$ (Lemma 6.14). If $d(\alpha, \beta) = 2$, then $u' \notin y'^{\perp}$ and $v' \notin x'^{\perp}$ in S'. By Lemma 6.15, $d(\alpha, x) = d(\alpha, u') = 3$. If $d(\alpha, \beta) = 3$, then either $u' \in y'^{\perp}$ and $v' \notin x'^{\perp}$, or $u' \notin y'^{\perp}$ and $v' \in x'^{\perp}$ in S'. Then, $d(\alpha, x) = 3$ and $d(\alpha, u') = 2$ in the first case, and $d(\alpha, x) = 2$ and $d(\alpha, u') = 3$ in the latter. □

Now, let $\mathtt{L} = \{\beta, \theta, \gamma\} \in \mathcal{L}$ be a line of $\mathbb{S}$ and $\alpha \in P \cup P'$. We take $\beta = (x, u'), \theta = (y, v')$ and $\gamma = (z, w')$.

Proposition 6.18. *If* α *has distance 2 from two points of* L*, then it is collinear with the third point of* L*.*

Proof. Let $\alpha = q' \in P'$ and $d(\alpha, \beta) = d(\alpha, \theta) = 2$. Then, $q' \notin \{u', v'\}$ and $x, y \in q^{\perp}$ in S. Thus, $q' \in \{x', y'\}^{\perp} = \{u', v', w'\}$ in S'. So $q' = w'$ and $\alpha \sim \gamma$. A similar argument holds if $\alpha \in P$. □

Proposition 6.19. *If* α *has distance 3 from two points of* L*, then it has distance 2 from the third point of* L*.*

Proof. Let $\alpha = q' \in P'$ and $d(\alpha, \beta) = d(\alpha, \theta) = 3$. Then, $q' \notin \{u', v'\}$ and $x, y \notin q^{\perp}$ in S. So $q' \neq w'$ and $q' \in z'^{\perp}$ in S'. The latter follows from the fact that $\{x, y, z\}$ is a line or a complete triad of points of S. Thus, $d(\alpha, \gamma) = 2$. A similar argument holds if $\alpha \in P$. □

Proof of Theorem 6.10. By the results of this subsection, together with Theorem 6.3, $\mathbb{S}$ is a slim dense near hexagon. Since $|\mathcal{P}| = 135$, Theorem 6.1 completes the proof. □

6.3. Structural Properties

Let $S = (P, L)$ be a slim dense near hexagon. Since a (2,4)-GQ admits no ovoids, every quad of S of type $(2, 4)$ is big by Theorem 1.21. The following lemma says that all big quads of S are of the same type.

Lemma 6.20 ([**9**], p.359)**.** *Let Q be a quad of S of type $(2, t_2)$. Then, $|P| \geq |Q|(1 + 2(t - t_2))$. Equality holds if and only if Q is big. In particular, if a quad of S of type $(2, t_2)$ is big then so are all quads of S of that type.*

Notation 6.21. *For a big quad Q of S and a point $x \in P \setminus Q$, we denote by x_Q the unique point of Q collinear with x.*

Lemma 6.22 ([**10**], Lemma 5, p.148)**.** *Let Q be a big quad of S and $\{a, b, c\}$ be a line of S disjoint from Q. Then, $\{a_Q, b_Q, c_Q\}$ is a line of Q.*

Lemma 6.23 ([**9**], Proposition 4.3, p.354)**.** *Let Q_1 and Q_2 be two disjoint big quads of S. Let τ be the map from Q_1 to Q_2 defined by $\tau(x) = x_{Q_2}$, $x \in Q_1$. Then,*

(i) τ is an isomorphism from Q_1 to Q_2.
(ii) The set $Q_1 * Q_2 = \{x * x_{Q_2} : x \in Q_1\}$ is a big quad of S.

Further, $Y = Q_1 \cup Q_2 \cup Q_1 * Q_2$ is a subspace of S isomorphic to the near hexagon $Q_1 \times \mathbb{L}_3$, a direct product of Q_1 with $\mathbb{L}_3$.

A *Fischer space* ([**2**], p.92) is a partial linear space satisfying the following:

(i) Each line contains exactly three points.
(ii) The subspace generated by any two intersecting lines is isomorphic to the dual affine plane of order two, or the affine plane of order three.

Lemma 6.24 ([**9**], Corollary 4.4, p.354)**.** *Let B be the collection of all big quads of S and L_B be the collection of subsets $\{Q_1, Q_2, Q_1 * Q_2\}$ of B, where Q_1 and Q_2 are disjoint. Then, (B, L_B) is a Fischer space, called the Fischer space on big quads of S.*

Let $S = (P, L)$ be a slim dense near hexagon having big quads. Fix two disjoint big quads Q_1 and Q_2 of S. Let $Q_3 = Q_1 * Q_2$ and $Y = Q_1 \cup Q_2 \cup Q_3$. By Lemma 6.23, Y is a subspace of S isomorphic to the near hexagon $Q_4^+(2) \times \mathbb{L}_3$, $W_4(2) \times \mathbb{L}_3$ or $Q_6^-(2) \times \mathbb{L}_3$ according as Q_1 and Q_2 are of type (2,1), (2,2) or (2,4). Now, fix a big quad Q of S disjoint from Y. Let $\{i, j, k\} = \{1, 2, 3\}$. We use the following notation:

* For $x \in P \setminus Y$, we define $x^j = x_{Q_j}$ and, for $x \in Q_i$, we define $z_x^j = x_{Q_j}$. Thus, for $x \in Q_i$, $\{x, z_x^j, z_x^k\}$ is a line of Y meeting each of Q_i, Q_j and Q_k.
* For a line $l = \{a, b, c\}$ of S, we set $l_Q = \{a_Q, b_Q, c_Q\}$ if $l \cap Q$ is empty, and $l^j = \{a^j, b^j, c^j\}$ if $l \cap Q_j$ is empty.
* We denote by τ_j the isomorphism from Q to Q_j defined by $\tau_j(x) = x^j$, $x \in Q$ and by τ_{ij} the isomorphism from Q_i to Q_j defined by $\tau_{ij}(x) = z_x^j$, $x \in Q_i$ (see Lemma 6.23(i)).
* For $x \in P \setminus (Y \cup Q)$, we denote by x_Q^i the point $(x_Q)^i$ in Q_i. For a line l disjoint from both Y and Q, we denote by l_Q^i the line $(l_Q)^i$ in Q_i.

Lemma 6.25. *Let $x \in P \setminus Y$. Then:*

(i) *$d(z_{x^i}^j, x^j) = 1$ and $d(x^i, x^j) = 2$.*
(ii) *$\{x^i, z_{x^j}^i, z_{x^k}^i\}$ is a line in Q_i.*

Proof. (i) Since $x \in \Gamma_1(x^i) \cap \Gamma_1(x^j)$, $d(x^i, x^j) = 2$. Further, $d(x^i, x^j) = d(x^i, z_{x^i}^j) + d(z_{x^i}^j, x^j)$. So $d(z_{x^i}^j, x^j) = 1$.

(ii) By (i), $x^i \sim z_{x^j}^i$ and $x^i \sim z_{x^k}^i$. We show that $z_{x^j}^i \sim z_{x^k}^i$. The quad $Q(x^j, x^k)$ of Y is of type (2,1) and $\{x^j, x^k\}^\perp = \{z_{x^j}^k, z_{x^k}^j\}$ in Y. Now, from the parallel lines $\{x^j, z_{x^j}^i, z_{x^j}^k\}$ and $\{x^k, z_{x^k}^i, z_{x^k}^j\}$ in $Q(x^j, x^k)$, it follows that $z_{x^j}^i \sim z_{x^k}^i$. □

Lemma 6.26. *Let $l = \{a, b, c\}$ be a line of S intersecting Y at $\{c\}$.*

(i) *If $c \in Q_i \cup Q_j$, then $d(a^i, b^j) = 2$.*
(ii) *If $c \in Q_k$, then $d(a^i, b^j) = 1$. In fact, $a^i = z_{b^j}^i$.*

Proof. (i) Let c be in, say, Q_i. Since $a^i = b^i = c$, $d(a^i, b^j) = d(b^i, b^j) = 2$ by Lemma 6.25(i).

(ii) We have $a^k = b^k = c$. Since l is disjoint from Q_i, $l^i = \{a^i, b^i, c^i = z_c^i\}$ is a line of Q_i. By Lemma 6.25(ii), $\{b^i, z_{b^j}^i, z_{b^k}^i = z_c^i\}$ is also a line of Q_i. Since these lines share two points, they are the same and so, $a^i = z_{b^j}^i$ and $d(a^i, b^j) = 1$. □

Lemma 6.27. *Let l be a line of S disjoint from Y and $x, y \in l$ with $x \neq y$.*

(*i*) *If* $l^j = x^j z^j_{x^i}$ *in* Q_j, *then* $(y^i, y^j) = (z^i_{x^j}, x^j * z^j_{x^i})$ *or* $(x^i * z^i_{x^j}, z^j_{x^i})$. *In other words, the following three statements are equivalent:*

$$l^i = x^i z^i_{x^j} \text{ in } Q_i;\ \ l^j = x^j z^j_{x^i} \text{ in } Q_j;\ \text{ and } \tau_{ij}(l^i) = l^j.$$

(*ii*) $d(x^i, y^j) \leq 2$ *if and only if* $l^i = x^i z^i_{x^j}$ *in* Q_i.

Proof. (*i*) If $l^j = x^j z^j_{x^i}$, then $y^j \in \{z^j_{x^i}, x^j * z^j_{x^i}\}$. Assume that $y^j = x^j * z^j_{x^i}$. Since $\tau_{ji}(x^j z^j_{x^i}) = x^i z^i_{x^j}$, $z^i_{y^j} = x^i * z^i_{x^j}$ and so $y^i \sim x^i * z_{x^j}$ (Lemma 6.25(*i*)). Since $y^i \sim x^i$ also, y^i is a point in the line $x^i z^i_{x^j}$. Now, $d(y^i, y^j) = 2$ implies that $y^i = z^i_{x^j}$.

If $y^j = z^j_{x^i}$, then applying the above argument to $(x * y)^j = x^j * z^j_{x^i}$, we get $(x * y)^i = z^i_{x^j}$ and so, $y^i = x^i * z^i_{x^j}$.

(*ii*) If $l^i = x^i z^i_{x^j}$ in Q_i, then $\tau_{ij}(l^i) = l^j$ by (*i*) and it follows that $d(x^i, y^j) \leq 2$. Now, let $l^i \neq x^i z^i_{x^j}$ in Q_i. By (*i*), $l^j \neq x^j z^j_{x^i}$ in Q_j. So $y^j \nsim z^j_{x^i}$, and $d(x^i, y^j) = d(x^i, z^j_{x^i}) + d(z^j_{x^i}, y^j) = 1 + 2 = 3$. ☐

Lemma 6.28. *For* $x, y \in Q$ *with* $x \nsim y$, $\{d(z^j_{x^i}, z^i_{y^j}), d(z^i_{x^j}, z^j_{y^i})\} = \{2, 3\}$.

Proof. By Lemma 6.23, there exist $w \in \{x, y\}^\perp$ in Q such that $x^i w^i = x^i z^i_{x^j}$. By Proposition 6.27(*i*), $(w^i, w^j) = (z^i_{x^j}, x^j * z^j_{x^i})$ or $(x^i * z^i_{x^j}, z^j_{x^i})$. Assume that $(w^i, w^j) = (z^i_{x^j}, x^j * z^j_{x^i})$. Then, $d(z^i_{x^j}, z^j_{y^i}) = d(w^i, z^j_{y^i}) = d(w^i, z^j_{w^i}) + d(z^j_{w^i}, z^j_{y^i}) = 2$. Now, $y^j \sim w^j$ and $y^j \nsim x^j$ in Q_j implies that $x^i \nsim z^i_{y^j}$. So $d(x^i, z^i_{y^j}) = 2$ and $d(z^j_{x^i}, z^i_{y^j}) = d(z^j_{x^i}, x^i) + d(x^i, z^i_{y^j}) = 3$. A similar argument holds if $(w^i, w^j) = (x^i * z^i_{x^j}, z^j_{x^i})$. ☐

Lemma 6.29. *For every* $x \in Q$, *there exists a unique line* l *in* Q *containing* x *such that* $\tau_{ij}(l^i) = l^j$. *In particular,* $l^i = \{x^i, z^i_{x^j}, z^i_{x^k}\}$.

Proof. Since τ_i is an isomorphism from Q to Q_i, there exists a line l of Q containing x such that $l^i = x^i z^i_{x^j}$. By Lemma 6.27(*i*), $\tau_{ij}(l^i) = l^j$. The line l in Q through x such that $\tau_{ij}(l^i) = l^j$ is unique because, for any other line $\bar{l}$ of Q containing x, $\tau_{ij}(\bar{l}^i)$ and $\bar{l}^j$ are two disjoint lines in Q_j containing $z^j_{x^i}$ and x^j, respectively. Now, $l^i = x^i z^i_{x^j} = \{x^i, z^i_{x^j}, z^i_{x^k}\}$ (see Lemma 6.25(*ii*)). ☐

Notation 6.30. *For* $x \in Q$, *we denote by* ζ_x *the unique line* l *in* Q *containing* x *as in Lemma 6.29 and we write* $T_Q = \{\zeta_x : x \in Q\}$.

Corollary 6.31. T_Q *is a spread of* Q.

Proof. This follows because, $\zeta_x = \zeta_y$ for $x \in Q$ and $y \in \zeta_x$, by Lemma 6.29. □

Let $l = \{a, b, c\}$ be a line of Q. First, let $l \in T_Q$. Set $T^l = l^i \cup l^j \cup l^k$ and $T^l_{jk} = l^i \cup \tau_{ij}(l^i) \cup \tau_{ik}(l^i)$. The set T^l_{jk} is a quad of Y of type (2,1) whose lines are the rows and the columns of the matrix

$$T^l_{jk} = \begin{bmatrix} a^i & z^j_{a^i} & z^k_{a^i} \\ b^i & z^j_{b^i} & z^k_{b^i} \\ c^i & z^j_{c^i} & z^k_{c^i} \end{bmatrix}. \tag{6.3.1}$$

Since $l \in T_Q$, Lemma 6.29 implies that the subsets T^l and T^l_{jk} of P coincide. So T^l is a quad of Y of type $(2,1)$ whose lines are the rows and columns of one of the matrices

$$\begin{bmatrix} a^i & c^j & b^k \\ b^i & a^j & c^k \\ c^i & b^j & a^k \end{bmatrix}; \text{ or } \begin{bmatrix} a^i & b^j & c^k \\ b^i & c^j & a^k \\ c^i & a^j & b^k \end{bmatrix}. \tag{6.3.2}$$

Note that if $b^k \sim a^i$, then the line containing them is $\{a^i, c^j, b^k\}$.

Now, let $l \notin T_Q$. Then, $\tau_{ij}(l^i)$ and l^j are disjoint lines in Q_j. The set $T^l_i = l^i \cup \tau_{ji}(l^j)\tau_{ki}(l^k)$ form a $(2,1)$-GQ in Q_i. We can write

$$T^l_i = \begin{bmatrix} a^i & b^i & c^i \\ z^i_{a^j} & z^i_{b^j} & z^i_{c^j} \\ z^i_{a^k} & z^i_{b^k} & z^i_{c^k} \end{bmatrix}; T^l_j = \begin{bmatrix} z^j_{a^i} & z^j_{b^i} & z^j_{c^i} \\ a^j & b^j & c^j \\ z^j_{a^k} & z^j_{b^k} & z^j_{c^k} \end{bmatrix}; \text{ and} \tag{6.3.3}$$

$$T^l_k = \begin{bmatrix} z^k_{a^i} & z^k_{b^i} & z^k_{c^i} \\ z^k_{a^j} & z^k_{b^j} & z^k_{c^j} \\ a^k & b^k & c^k \end{bmatrix}.$$

Each row as well as each column in T^l_i (respectively, T^l_j, T^l_k) is a line of Q_i (respectively, Q_j, Q_k). Further, the (m,n)-th entries from T^l_i, T^l_j and T^l_k form a line of Y.

As a consequence of the above, we have

Corollary 6.32. *Let l be a line of Q. For distinct $a, b \in l$, $d(a^i, b^j) \leq 2$ or $d(a^i, b^j) = 3$ according as $l \in T_Q$ or not.*

CHAPTER 7

Representations of Slim Dense Near Hexagons

In this chapter, we study non-abelian representations of slim dense near hexagons. We prove the following.

Theorem 7.1. *Let $S = (P, L)$ be one of the slim dense near hexagons $DH_6(2^2), \mathbb{E}_3$ and $\mathbb{G}_3$. Then, every representation of S is abelian.*

Theorem 7.2. *Let $S = (P, L)$ be a slim dense near hexagon different from $DH_6(2^2), \mathbb{E}_3$ and $\mathbb{G}_3$. Let (R, ψ) be a non-abelian representation of S. Then,*

(*i*) *R is of order 2^β, where $1 + n(S) \leq \beta \leq 1 + dimV(S)$.*
(*ii*) *If $\beta = 1 + n(S)$, then $R = 2_\epsilon^{1+n(S)}$ with $\epsilon = -$ or $+$ according as S is equal to $Q_6^-(2) \otimes Q_6^-(2)$ or not.*

Theorem 7.3. *Let $S = (P, L)$ be one of the slim dense near hexagons $Q_6^-(2) \otimes Q_6^-(2)$, $Q_6^-(2) \times \mathbb{L}_3$, $DW_6(2)$, $\mathbb{H}_3$, $W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$. Then, S admits a non-abelian representation in a group of order $2^{1+n(S)}$.*

Theorems 7.1 and 7.3 deals with the question of existence of non-abelian representations of a slim dense near hexagon having big quads. The only slim dense near hexagons not admitting big quads are $\mathbb{E}_1$ and $\mathbb{E}_2$. For these two geometries it is not known to us whether they admit non-abelian representations.

7.1. Initial Results

Let $S = (P, L)$ be a slim dense near hexagon and (R, ψ) be a non-abelian representation of S. For $x \in P$ and $y \in \Gamma_{\leq 2}(x)$, $[r_x, r_y] = 1$: if $d(x, y) = 2$, we apply Proposition 4.3 to the restriction of ψ to the quad $Q(x, y)$. From Example 3.16 and Theorem 3.23 applied to S,

Proposition 7.4. *The following hold:*

(i) *For* $x, y \in P$, $[r_x, r_y] \neq 1$ *if and only if* $d(x, y) = 3$. *In that case,* $\langle r_x, r_y \rangle \simeq 2_+^{1+2}$, *a dihedral group of order 8.*

(ii) R *is a finite* 2*-group of exponent* 4, $|R'| = 2$ *and* $R' = \Phi(R) \subseteq Z(R)$.

(iii) $r_x \notin Z(R)$ *for each* $x \in P$ *and* ψ *is faithful.*

We repeatedly use Proposition 7.4(i) and (iii), mostly without mention. Corollary 3.25 implies

Corollary 7.5. (R, ψ) *is the cover of a representation* (R_1, ψ_1) *of* S*, where* R_1 *is extraspecial or* $Z(R_1)$ *is cyclic of order* 4.

Throughout this chapter, we write $R' = \langle \mu \rangle$.

Since R' is of order two, Lemma 3.7 implies

Corollary 7.6. $|R| \leq 2^{1+dimV(S)}$.

Proposition 7.7. $R = EZ(R)$*, where* E *is an extraspecial 2-subgroup of* R *and* $E \cap Z(R) = Z(E)$.

Proof. We consider $V = R/R'$ as a vector space over $\mathbb{F}_2$. The map $f : V \times V \longrightarrow \mathbb{F}_2$ taking (xR', yR') to 0 or 1 accordingly $[x, y] = 1$ or μ, is a symplectic bilinear form on V (see (3.1.1)). This is non-degenerate if and only if $R' = Z(R)$. Let W be a complement in V of the radical of f and E be its inverse image in R. Then, E is extraspecial and the proposition follows. □

As a consequence of Proposition 7.7, we have

Corollary 7.8. *Let* M *be an abelian subgroup of* R *of order* 2^m *intersecting* $Z(R)$ *trivially. Then,* $|R| \geq 2^{2m+1}$. *Further, equality holds if and only if* R *is extraspecial and* M *is a maximal abelian subgroup of* R *intersecting* $Z(R)$ *trivially.*

Lemma 7.9. *The natural homomorphism from* R *to* R/R' *is one-one on* R_ψ^*.

Proof. Suppose that $r_u R' = r_v R'$ for some $u, v \in P, u \neq v$. Then, $r_u = r_v r$ for $r \in R'$. Since $[r_u, r_v] = 1$, $d(u, v) \leq 2$. Now, $d(u, v) \neq 1$, otherwise $r = r_{u*v} \in Z(R)$, contradicting Proposition 7.4(iii). Let $w \in P \setminus Q(u, v)$ be such that $w \sim v$. Then, $[r_w, r_u] = [r_w, r_v r] = [r_w, r_v] = 1$. But $d(w, u) = 3$, a contradiction to Proposition 7.4(i). □

The following lemma is useful for us.

Lemma 7.10. *Let $x \in P$ and $Y \subseteq \Gamma_3(x)$. Then, $[r_x, \prod_{y \in Y} r_y] = 1$ if and only if $|Y|$ is even.*

Proof. Since $R' \subseteq Z(R)$, $[r_x, \prod_{y \in Y} r_y]$ is well-defined (though $\prod_{y \in Y} r_y$ depends on the order of multiplication) and $[r_x, \prod_{y \in Y} r_y] = \prod_{y \in Y} [r_x, r_y]$. Let $y, z \in \Gamma_3(x)$ be distinct. The subgraph of $\Gamma(P)$ induced on $\Gamma_3(x)$ is connected (Lemma 1.25). Let $y = y_0, y_1, \cdots, y_k = z$ be a path in $\Gamma_3(x)$. Then, $r_y r_z = \Pi r_{y_i * y_{i+1}}$ $(0 \leq i \leq k-1)$. Since $d(x, y_i * y_{i+1}) = 2$, $[r_x, r_y][r_x, r_z] = [r_x, r_y r_z] = 1$. Now, Theorem 7.4($i$) completes the proof. □

Notation 7.11. *For a quad Q of S, we denote by M_Q the subgroup of R generated by $\psi(Q)$. Note that M_Q is elementary abelian by Proposition 4.3.*

Proposition 7.12. *Let Q be a quad of S and $M_Q \cap Z(R) \neq \{1\}$. Then, Q is of type (2,2); $|M| = 2^5$; and $M_Q \cap Z(R) = \{1, r_a r_b r_c\}$, where $\{a, b, c\}$ is any complete triad of points of S.*

Proof. Let $1 \neq m \in M_Q \cap Z(R)$. Then, $m \neq r_x$ for each $x \in P$ (Proposition 7.4(iii)). If Q is of type (2,1) or (2,4), then by Proposition 5.26, $m = r_y r_z$ for some $y, z \in Q, y \nsim z$. Choose $w \in P \setminus Q$ with $w \sim y$. Then, $[r_w, r_z] = [r_w, r_y r_z] = [r_w, m] = 1$. But $d(w, z) = 3$, a contradiction to Proposition 7.4(i). So Q is a (2,2)-GQ.

Now, $|M_Q| \neq 2^4$ otherwise, $M_Q^* = \{r_x : x \in Q\}$ and $m = r_x \in Z(R)$ for some $x \in Q$, contradicting Proposition 7.4(iii). So $|M_Q| = 2^5$. Now, either $m = r_u r_v$ for some $u, v \in Q, u \nsim v$ or $m = r_a r_b r_c$ for every complete triad $\{a, b, c\}$ of Q (Proposition 5.26). The above argument in the first paragraph implies that the first possibility does not occur. □

Proposition 7.13. *Let Q be a quad of S of type $(2,2)$. Then, Q is ovoidal if and only if $|M_Q| = 2^5$ and $M_Q \cap Z(R) = \{1\}$.*

Proof. First, assume that Q is ovoidal. Let $z \in P \setminus Q$ be such that the pair (z, Q) is ovoidal. Let $\mathcal{O}_z = \{x_1, \cdots, x_5\}$ be the ovoid in Q defined in Theorem 1.21(ii). Let $\{x_1, x_2, y\}$ be the complete triad of Q containing x_1 and x_2. If $|M_Q| = 2^4$, then $d(y, z) = 3$ and $r_{x_1} r_{x_2} r_y = 1$ (Proposition 5.22). But $[r_z, r_y] = [r_z, r_{x_1} r_{x_2} r_y] = 1$, a contradiction to Proposition 7.4(i). So $|M_Q| = 2^5$. If $1 \neq m \in M_Q \cap Z(R)$, then $m = r_a r_b r_c$ for any complete triad $\{a, b, c\}$ of Q (Proposition 7.12).

In particular, $m = r_{x_1}r_{x_2}r_y$. Since $m \in Z(R)$, applying the above argument we get a contradiction. So $M_Q \cap Z(R) = \{1\}$.

Now, assume that $|M_Q| = 2^5$ and $M_Q \cap Z(R) = \{1\}$. Suppose that Q is classical. Let $\{a, b, c\}$ be a complete triad of Q. Then, $r_ar_br_c \neq 1$, by Proposition 5.22. Since (x, Q) is classical for each $x \in P \setminus Q$, either each of a, b, c is at distance 2 from x or exactly two of them are at distance 3 from x. In either case, $[r_x, r_ar_br_c] = 1$ (see Lemma 7.10). So $1 \neq r_ar_br_c \in M_Q \cap Z(R)$, a contradiction to that $M_Q \cap Z(R) = \{1\}$. □

Proposition 7.14. *Let Q and Q' be two disjoint big quads of S of type $(2, t_2)$. Then, $M_Q \cap M_{Q'} = \{1\}$ in the following cases:*

(i) $t_2 \in \{1, 4\}$,

(ii) $t_2 = 2$*; and* $|M_Q| = 2^4$ *or* $|M_{Q'}| = 2^4$.

Further, if $t_2 = 2$ and $|M_Q| = 2^5 = |M_{Q'}|$, then $M_Q \cap M_{Q'}$ is contained in $Z(R)$ and is of order at most 2.

Proof. Replacing $Z(R)$ by $M_{Q'}$ and choosing w in Q', the argument in the first paragraph of the proof of Proposition 7.12 implies (i).

Let $t_2 = 2$. If $|M_Q| = 2^4$, then $M_Q^* = \{r_x : x \in Q\}$. If $M_Q \cap M_{Q'} \neq \{1\}$, then $r_x \in M_{Q'}$ for some $x \in Q$. Then, $[r_x, r_z] = 1$ for every $z \in Q'$, since $M_{Q'}$ is abelian. In particular, $[r_x, r_z] = 1$ for every $z \in Q'$ with $d(x, z) = 3$, a contradiction to Proposition 7.4(i). This proves (ii).

Now, let $|M_Q| = 2^5 = |M_{Q'}|$. Let $1 \neq m \in M_Q \cap M_{Q'}$. By the second paragraph of the proof of Proposition 7.12 with $Z(R)$ replaced by $M_{Q'}$, it follows that $m = r_ar_br_c$ for every complete triad $\{a, b, c\}$ of Q. Since Q is classical, $M_Q \cap Z(R) \neq \{1\}$ (Proposition 7.13); and so, $M_Q \cap Z(R) = \{1, r_ar_br_c\} = \{1, m\}$ (Proposition 7.12). Thus, $M_Q \cap M_{Q'} \subseteq M_Q \cap Z(R) \subseteq Z(R)$. Since $|M_Q \cap Z(R)| = 2$, we get that $M_Q \cap M_{Q'}$ is of order at most 2. □

7.2. Proof of Theorem 7.2

In this section, we prove Theorem 7.2, except for the near hexagon $Q_6^-(2) \otimes Q_6^-(2)$. We prove Theorem 7.2 for $Q_6^-(2) \otimes Q_6^-(2)$ in Section 7.5. Throughout this section, we assume that $S = (P, L)$ is a slim dense near hexagon different from $Q_6^-(2) \otimes Q_6^-(2)$, $DH_6(2^2)$, $\mathbb{E}_3$ and $\mathbb{G}_3$. By Proposition 7.4(ii), R is a finite 2-group. So $|R| = 2^\beta$ for some β. By Corollary 7.6, $\beta \leq 1 + dimV(S)$. We find an elementary abelian 2-subgroup of R of order 2^ξ, $\xi = \frac{n(S)}{2}$, intersecting $Z(R)$ trivially. Then, by Corollary 7.8, $\beta \geq 1 + 2\xi = 1 + n(S)$ and $R = 2_+^{1+2\xi}$ if equality holds. This would complete the proof.

7.2.1. The near hexagons $Q_6^-(2) \times \mathbb{L}_3, DW_6(2), \mathbb{H}_3, W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$. Let S be one of the five near hexagons mentioned above. Let Q be a big quad of S. Set $M = M_Q$. If Q is of type $(2,1)$ or $(2,4)$, then $M \cap Z(R) = \{1\}$ (Proposition 7.12). Also, $|M| = 2^4$ or 2^6 according as Q is of type (2,1) or (2,4) (Proposition 5.18). If Q is of type (2,2), then $|M| = 2^4$ or 2^5. Also, if $|M| = 2^5$, then $|M \cap Z(R)| = 2$, since Q is classical (Propositions 7.12 and 7.13). Thus, R has an elementary abelian 2-subgroup of order 2^ξ intersecting $Z(R)$ trivially.

7.2.2. The near hexagons $\mathbb{E}_1$ and $\mathbb{E}_2$. Let $S = (P, L)$ be one of the near hexagons $\mathbb{E}_1$ and $\mathbb{E}_2$. Fix $a \in P$ and $b \in \Gamma_3(a)$. Let $l_1, \cdots, l_{t+1}$ be the lines containing a; x_i be the point in l_i with $d(b, x_i) = 2$; and set $A = \{x_i : 1 \leq i \leq t+1\}$. For a subset X of A, we set $T_X = \{r_x : x \in X\}$; $M_X = \langle T_X \rangle$; and $M = \langle r_b \rangle M_X$. Then, M_X and M are elementary abelian 2-subgroups of R.

Proposition 7.15. *Let X be a subset of A such that*

(*i*) $M_X \cap Z(R) = \{1\}$,
(ii) T_X *is linearly independent.*

Then, $|M| = 2^{|X|+1}$ *and* $M \cap Z(R) = \{1\}$. *In particular,* $|R| \geq 2^{2|X|+3}$.

Proof. By (*ii*), $2^{|X|} \leq |M| \leq 2^{|X|+1}$. If $|M| = 2^{|X|}$, then r_b can be expressed as a product of some of the elements r_x, $x \in X$. Since $[r_a, r_x] = 1$ for each $x \in X$, it follows that $[r_a, r_b] = 1$, a contradiction to Proposition 7.4(*i*). So $|M| = 2^{|X|+1}$.

Suppose that $M \cap Z(R) \neq \{1\}$ and $1 \neq z \in M \cap Z(R)$. Let $z = \prod_{y \in X \cup \{b\}} r_y^{i_y}$, $i_y \in \{0, 1\}$. Since $z \in Z(R)$, $i_b = 0$ by the previous argument. Then, it follows that $z \in M_X$, a contradiction to (*i*). So $M \cap Z(R) = \{1\}$.

By Corollary 7.8, $|R| \geq 2^{2(|X|+1)+1} = 2^{2|X|+3}$. □

A subset X of A is *good* if (*i*) and (*ii*) of Proposition 7.15 hold. The next Lemma gives a necessary condition for a subset of A to be good.

Lemma 7.16. *Let X be a subset of A which is not good, $\alpha \in M_X \cap Z(R)$ (possibly $\alpha = 1$) and*

$$\alpha = \prod_{x_k \in X} r_{x_k}^{i_k} \tag{7.2.1}$$

where $i_k \in \{0,1\}$. Set $B = \{k : x_k \in X\}$, $B' = \{k \in B : i_k = 1\}$ and $A_{i,j} = \{k \in B' : x_k \in Q(x_i, x_j)\}$ for $1 \le i \ne j \le t+1$. Assume that B' is non-empty when $\alpha = 1$. Then:

(*i*) $|B'| \ge 3$,

(*ii*) $|B'|$ *is even if and only if* $|A_{i,j}|$ *is even.*

Proof. (*i*) $|B'| \ge 2$ because $r_{x_k} \notin Z(R)$ for each k (Proposition 7.4(*iii*)). If $|B'| = 2$, then $r_x r_y = \alpha$ for some pair of distinct $x, y \in X$. Since ψ is faithful and r_x, r_y are involutions, $\alpha \ne 1$. For the quad $Q = Q(x,y)$ of S, $1 \ne \alpha \in M_Q \cap Z(R)$. By Proposition 7.12, Q is a $(2,2)$-GQ and $r_a r_b r_c = \alpha$ for each complete triad $\{a,b,c\}$ of Q. In particular, if $\{x,y,w\}$ is the complete triad of Q containing x and y, then $r_x r_y r_w = \alpha$. It follows that $r_w = 1$, a contradiction. So $|B'| \ge 3$.

(*ii*) Let $w \in Q(x_i, x_j)$ and $w \nsim a$. For each $m \in B'_{i,j} = B' \setminus A_{i,j}$, $d(w, x_m) = 3$ because $x_m \sim a$. Now, $[r_w, \prod_{m \in B'_{i,j}} r_{x_m}] = [r_w, \prod_{m \in B'} r_{x_m}] = [r_w, \alpha] = 1$. So $|B'_{i,j}|$ is even by Lemma 7.10. This implies (*ii*). □

In what follows, for any subset X of A which is not good, B' is defined relative to an expression as in (7.2.1) for an arbitrary but fixed element of $M_X \cap Z(R)$.

We now prove Theorem 7.2 for $\mathbb{E}_1$ and $\mathbb{E}_2$. By Proposition 7.15, it is enough to find good subsets of A of size $(2\xi - 2)/2$. In the following we use the notation of Lemma 7.16.

First, consider the case $\mathbb{E}_1$. Let $X = \{x_i : 1 \le i \le 11\}$. Then, X is a good subset of A. Otherwise, for some $i, j \in B'$ with $i \ne j$ (see Lemma 7.16(*i*)), $A_{i,j} = \{i,j\}$ and $A_{i,12} = \{i\}$ and, by Lemma 7.16(*ii*), $|B'|$ would be both even and odd.

Now, consider the case $\mathbb{E}_2$. There are 7 quads of S containing the point $x_1 \in A$. This partitions the 14 points ($\ne x_1$) of A, say

$$\{x_2,x_3\} \cup \{x_4,x_5\} \cup \{x_6,x_7\} \cup \{x_8,x_9\} \cup \{x_{10},x_{11}\} \cup \{x_{12},x_{13}\} \cup \{x_{14},x_{15}\}.$$

Consider the quad $Q(x_{10}, x_{12})$. We may assume that $Q(x_{10}, x_{12}) \cap A = \{x_{10}, x_{12}, x_{15}\}$. We show that

$$X = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{10}, x_{12}, x_{14}\}$$

is a good subset of A.

Assume otherwise. Let $C_1 = \{8, 10, 12, 14\}$ and $C_2 = B \setminus C_1$. For $k \in C_1$, $Q(x_1, x_k) \cap A = \{x_1, x_k, x_{k+1}\}$. So $A_{1,k} \subseteq \{k\}$. By Lemma 7.16(ii), either $C_1 \subseteq B'$ or $C_1 \cap B'$ is empty. Now, $C_1 \nsubseteq B'$ because, otherwise, $A_{1,14} = \{14\}$ and $A_{10,12} = \{10, 12\}$ and, by Lemma 7.16(ii), $|B'|$ would be both odd and even.

So $C_1 \cap B'$ is empty. Then, $B' \subseteq C_2$. Since $A_{1,8}$ is empty, $|B'|$ is even. Choose $j \in B'$ (see Lemma 7.16(i)). Note that there exists $k \in \{8, \cdots, 15\}$ such that $Q(x_j, x_k) \cap \{x_i : i \in C_2\} = \{x_j\}$. Then, $A_{j,k} = \{j\}$ and $|B'|$ is odd also, a contradiction. So, X is good and $|X| = 10$. This completes the proof.

7.3. Proof of Theorem 7.1

Let $S = (P, L)$ be a slim dense near hexagon having big quads of type $(2, 4)$ and Y be a proper subspace of S isomorphic to the near hexagon $Q_6^-(2) \times \mathbb{L}_3$. Big quads of Y are also of type (2,4). There are three pair-wise disjoint big quads of Y and any two of them generate Y. Fix two disjoint big quads Q_1 and Q_2 in Y. Let (R, ψ) be a non-abelian representation of S. Set $M = \langle \psi(Y) \rangle$ and $M_i = M_{Q_i}$ for $i \in \{1, 2\}$. Then, M_i is elementary abelian of order 2^6 (Proposition 5.18(iii)); $M_i \cap Z(R) = \{1\}$ (Proposition 7.12); $M_1 \cap M_2 = \{1\}$ (Proposition 7.14); and $M = 2_+^{1+12}$ with $M = M_1 M_2 R'$ (Theorem 7.2 for $Q_6^-(2) \times \mathbb{L}_3$). Let $N = C_R(M)$. Then, R is a central product of M and N, we write $R = M \circ N$. This can be seen considering the orthogonal decomposition of R/R' with respect to the bilinear form defined in (3.1.1) which is non-degenerate if and only if $R' = Z(R)$. Since M is a central product of six copies of D_8, the centralizer in M_2 of a maximal subgroup of M_1 is of order two and vise-versa.

Let $\{i, j\} = \{1, 2\}$. In the following, we use the notation of Section 6.3.

Proposition 7.17. *For each $x \in P \setminus Y$, r_x has a unique decomposition as $r_x = m_1^x m_2^x n_x$, where $m_j^x = r_{z^j_{x^i}} \in M_j$ and $n_x \in N$ is an involution not in $Z(R)$. In particular, $r_x \notin M$.*

Proof. We can write $r_x = m_1^x m_2^x n_x$ for some $m_1^x \in M_1$, $m_2^x \in M_2$ and $n_x \in N$. Set $H_j = \langle r_w : w \in Q_j \cap x^{j\perp} \rangle \leq M_j$. Then, H_j is a maximal subgroup of M_j (Lemma 3.6). Since $d(x, w) \leq 2$ for each

$w \in Q_j \cap x^{j\perp}$, $r_x \in C_R(H_1) \cap C_R(H_2)$. For all $h \in H_j$,

$$[m_i^x, h] = [m_1^x m_2^x n_x, h] = [r_x, h] = 1.$$

So $m_i^x \in C_{M_i}(H_j)$. If $m_i^x = 1$, then $r_x = m_j^x n_x$ commutes with every element of M_j. In particular, $[r_x, r_y] = 1$ for every $y \in Q_j \cap \Gamma_3(x)$, a contradiction to Theorem 7.4(i). So, $m_i^x \neq 1$. Further, since $C_{M_i}(H_j) = \langle r_{z^i_{x^j}} \rangle$ is of order two, it follows that $m_i^x = r_{z^i_{x^j}}$. Now, $[m_1^x, m_2^x] = 1$, since $d(z^2_{x^1}, z^1_{x^2}) = 2$ (Proposition 6.25(i)). Since $r_x^2 = 1$, $n_x^2 = 1$.

We show that $n_x \neq 1$ and $n_x \notin Z(R)$. The quad $Q = Q(x^1, x^2)$ is of type (2,2) or (2,4) because, x^1 and x^2 have at least three common neighbours $x, z^2_{x^1}$ and $z^1_{x^2}$. Let U be the $(2,2)$-GQ in Q generated by $\{x^1, x^2, x, z^2_{x^1}, z^1_{x^2}\}$. If Q is of type (2,4), then $\langle \psi(U) \rangle$ is of order 2^5 (Corollary 5.21). If Q is of type (2,2), then $U = Q$ and it is ovoidal because it is not a big quad. So $\langle \psi(U) \rangle$ is of order 2^5 (Propositions 7.13). In either case, $r_a r_b r_c \neq 1$ for every complete triad $\{a, b, c\}$ of U (Proposition 5.22). In particular, $n_x = r_x r_{z^2_{x^1}} r_{z^1_{x^2}} \neq 1$ for the complete triad $\{x, z^2_{x^1}, z^1_{x^2}\}$ of U. Now, applying Proposition 7.12 (respectively, Proposition 7.13) when Q is of type (2,4) (respectively, of type (2,2)), we conclude that $n_x \notin Z(R)$. The argument clearly yields the uniqueness of the decomposition. □

Proposition 7.18. *Let Q be a big quad of S disjoint from Y. Then:*

(i) *For $x, y \in Q$, $[n_x, n_y] = 1$ if and only if $x = y$ or $x \sim y$;*

(ii) *For each $x \in Q$, there is a unique line $l_x = \{x, y, x * y\}$ in Q containing x such that $n_{x*y} = n_x n_y$. For any other line $l = \{x, z, x * z\}$ in Q containing x, $n_{x*z} = n_x n_z \mu$.*

Proof. (i) Let $x \sim y$. By Lemma 6.27(ii), $[m_2^x, m_1^y] = [m_1^x, m_2^y] = 1$ or μ . Then, $[n_x, n_y] = [m_1^x m_2^x n_x, m_1^y m_2^y n_y] = [r_x, r_y] = 1$.

Now, let $x \nsim y$. Lemma 6.28 implies that $\{[m_1^x, m_2^y], [m_2^x, m_1^y]\} = \{1, \mu\}$. Since $[r_x, r_y] = 1$, it follows that $[n_x, n_y] = \mu$.

(ii) Let $x \in Q$ and l_x be the line in Q containing x whose projection on Q_j is the line $x^j z^j_{x^i}$. This is possible by Lemma 6.23. For $u, v \in l_x$, $d(z^i_{u^j}, z^j_{v^i}) \leq 2$ by Lemma 6.27(ii). So $[m_i^u, m_j^v] = 1$. Then, $r_{u*v} = (m_1^u m_1^v)(m_2^u m_2^v)(n_u n_v)$. So $n_{u*v} = n_u n_v$. Let l be a line ($\neq l_x$) in Q containing x. For $y \neq w$ in l, $[m_2^y, m_1^w] = \mu$, because $d(z^2_{y^1}, z^1_{w^2}) = 3$ (Lemma 6.27). Then,

$$r_{y*w} = r_y r_w == (m_1^y m_1^w)(m_2^y m_2^w) n_y n_w \mu = m_1^{y*w} m_2^{y*w} n_y n_w \mu.$$

So, $n_{y*w} = n_y n_w \mu$. □

Proposition 7.19. *Let Q be a big quad of S disjoint from Y. Define δ from Q to $I_2(N)$ by $\delta(x) = n_x$. Then:*

(*i*) $[\delta(x), \delta(y)] = 1$ *if and only if* $x = y$ *or* $x \sim y$.
(*ii*) δ *is one-one.*
(*iii*) *There exists a spread T of Q such that for $x, y \in Q$ with $x \sim y$,*

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in T \\ \delta(x)\delta(y)\mu & \text{if } xy \notin T \end{cases}.$$

Proof. (*i*) and (*iii*) follows from Proposition 7.18. We prove (*ii*). Let $\delta(x) = \delta(y)$ for $x, y \in Q$. By (*i*), $x = y$ or $x \sim y$. If $x \sim y$, then $r_{x*y} = r_x r_y = (m_1^x m_1^y)(m_2^x m_2^y)\alpha \in M$, where $\alpha = [m_2^x, m_1^y] \in R'$. But this is not possible as $x * y \notin Y$ (Proposition 7.17). So $x = y$. □

Proposition 7.20. *Let $S = (P, L)$ be a slim dense near hexagon having big quads of type $(2, 4)$. Suppose that the Fischer space on big quads of S contains a subspace H isomorphic to the dual affine plane of order 2. Then, every representation of S is abelian.*

Proof. Let $H = \{Q_1, Q_2, Q_3, Q, T_1, T_2\}$ with the four lines $\{Q_1, Q_2, Q_3\}$, $\{Q_1, Q, T_1\}$, $\{T_1, T_2, Q_3\}$ and $\{Q, T_2, Q_2\}$. Then, $Y = Q_1 \cup Q_2 \cup Q_3$ is isomorphic to $Q_6^-(2) \times \mathbb{L}_3$ and Q is a big quad of S disjoint from Y.

Suppose that (R, ψ) is a non-abelian representation of S and let M and N be as in the beginning of this section. For each $x \in P$, we can write $r_x = r_{z^1_{x^2}} r_{z^2_{x^1}} n_x$, where $n_x \in N \setminus Z(R)$ is an involution (Proposition 7.17). Let $l = \{a, b, c\}$ be a line of S intersecting T_1 at $\{a\}$, T_2 at $\{b\}$ and Q_3 at $\{c\}$. We show that $n_a = n_b, n_a = n_{a_Q}$ and $n_b = n_{b_Q}$. Since $a_Q \neq b_Q$, $n_{a_Q} = n_{b_Q}$ would contradict Lemma 7.19(*ii*), thus completing the proof.

For $m \in \{1, 2\}$, l is disjoint from Q_m, so $l^m = \{a^m, b^m, c^m = z_c^m\}$ is a line of Q_m. By Lemma 6.26(*ii*), $(a^1, b^1) = (z^1_{b^2}, z^1_{a^2})$ and $(a^2, b^2) = (z^2_{b^1}, z^2_{a^1})$. So $r_a = r_{z^1_{a^2}} r_{z^2_{a^1}} n_a = r_{b^1} r_{b^2} n_a$ by Lemma 7.17. Similarly, $r_b = r_{a^1} r_{a^2} n_b$. Now, $r_a r_b = (r_{b^1} r_{b^2})(r_{a^1} r_{a^2}) n_a n_b = (r_{b^1} r_{a^1})(r_{b^2} r_{a^2}) n_a n_b = r_{c^1} r_{c^2} n_a n_b$. The second equality holds, since $d(a^1, b^2) = 1$ (Lemma 6.26(*ii*)). Since $c^1 = z_c^1, c^2 = z_c^2$ and $\{c, z_c^1, z_c^2\}$ is a line of Y, we get $r_a r_b = r_c n_a n_b$. But $r_a r_b = r_c$ by the definition of a representation. So $n_a = n_b$.

Now, consider the line $l_a = \{a, a_Q, a^1 = a_Q^1\}$ intersecting T_1 at $\{a\}$, Q at $\{a_Q\}$ and Q_1 at $\{a^1 = a_Q^1\}$. We have $r_a r_{a_Q} = r_{a^1}$. Since l_a is disjoint from Q_2, $l_a^2 = \{a^2, a_Q^2, z^2_{a^1} = z^2_{a_Q^1}\}$ is a line of Q_2. Now,

$r_a r_{a_Q} = r_{z^1_{a^2}} r_{z^2_{a^1}} r_{z^1_{a^2_Q}} r_{z^2_{a^1_Q}} n_a n_{a_Q}$. By Lemma 6.26($i$), $d(a^1, a^2_Q) = 2$ and so, $[r_{z^2_{a^1}}, r_{z^1_{a^2_Q}}] = 1$. Since $a^1 = a^1_Q$, we get $r_a r_{a_Q} = r_{z^1_{a^2}} r_{z^1_{a^2_Q}} n_a n_{a_Q}$. Since the line l^2_a is disjoint from Q_1, its projection on Q_1 is the line $\{a^1 = a^1_Q, z^1_{a^2}, z^1_{a^2_Q}\}$. So $r_a r_{a_Q} = r_{a^1} n_a n_{a_Q}$. Thus, $n_a = n_{a_Q}$. Similarly, considering the line $l_b = \{b, b_Q, b^2 = b^2_Q\}$ disjoint from Q_1, the above argument yields that $n_b = n_{b_Q}$. This completes the proof. □

Proof of Theorem 7.1**.** Let $S = (P, L)$ be one of the near hexagons $\mathbb{E}_3$ and $\mathbb{G}_3$. Let Δ_S be the graph on big quads of S, two distinct big quads being *adjacent* when they have non-empty intersection. If $S = \mathbb{G}_3$, then $\Delta_{\mathbb{G}_3}$ is the 3-coclique extension of the (2,2)-GQ, and if $S = \mathbb{E}_3$, then $\Delta_{\mathbb{E}_3}$ is locally the collinearity graph of the (2,4)-GQ (see [**9**], p.361). In either case, it follows that for two adjacent vertices V_1 and V_2 of Δ_S, there exists a vertex V of Δ_S which is adjacent neither to V_1 nor to V_2. Consider the Fischer space $\mathcal{F}$ on big quads of S as a slim partial linear space. Then, the subspace H of $\mathcal{F}$ generated by the two intersecting lines $\{V, V_1, V * V_1\}$ and $\{V, V_2, V * V_2\}$ is isomorphic to the dual affine plane of order 2. So, by Proposition 7.20, every representation of S is abelian. Since S is a subspace of the near hexagon $DH_6(2^2)$, Proposition 7.4(i) implies that every representation of $DH_6(2^2)$ is abelian. This completes the proof. □

7.4. Proof of Theorem 7.3

In this section, we construct non-abelian representations for each of the near hexagons in Theorem 7.3, except for $Q^-_6(2) \otimes Q^-_6(2)$. In the latter case, we construct a non-abelian representation in Section 7.5.

7.4.1. $Q^-_6(2) \times \mathbb{L}_3$**,** $W_4(2) \times \mathbb{L}_3$ **and** $Q^+_4(2) \times \mathbb{L}_3$**.** Let $R = 2^{1+2k}_+$, $k \in \{4, 6\}$, $R' = \{1, \mu\}$ and $V = R/R'$. We consider V as a vector space over $\mathbb{F}_2$ and write V as an orthogonal direct sum of k hyperbolic planes K_i ($1 \leq i \leq k$) in V with respect to the non-degenerate symplectic bilinear form defined in (3.1.1). Let H_i be the inverse image of K_i in R. Then, H_i is generated by two elements x_i and x^1_i such that $[x_i, x^1_i] = \mu$. Let $M = \langle x_i : 1 \leq i \leq k\rangle$ and $M^1 = \langle x^1_i : 1 \leq i \leq k\rangle$. Then, M and M^1 are elementary abelian 2-subgroups of R of order 2^k each. Further, M, M^1 and $Z(R)$ pairwise intersect trivially and $R = MM^1Z(R)$.

Let $F = (Q, B)$ be a $(2, t)$-GQ in M with $M = \langle Q\rangle$. Then, $(k, t) = (4, 1), (4, 2)$ or (6,4). Note that if $(k, t) = (4, 2)$, then F is of symplectic type. For each $m \in Q$, the subgroup $H_m = \langle z \in Q : z \in m^{\perp} \rangle$ of M is of index 2 in M (Lemma 3.6). The centralizer of H_m in M^1 is a subgroup $\langle \kappa_m^1 \rangle$ of M^1 of order 2. The map $m \mapsto \kappa_m^1$ from Q to M^1 is one-one. So, $\kappa_a^1 \kappa_b^1 \kappa_c^1 = 1$ for every line $\{a, b, c\}$ in B. Let $M^2 = \langle m\kappa_m^1 : m \in Q \rangle$. Then, M^2 is an elementary abelian 2-subgroup of R. Since $M \cap M^1$ is trivial, it follows that M^2 is of order 2^k. We set

$$Q^1 = \{\kappa_m^1 \in M^1 : m \in Q\},$$
$$Q^2 = \{m\kappa_m^1 \in M^2 : m \in Q\},$$
$$B^1 = \{\{\kappa_a^1, \kappa_b^1, \kappa_c^1\} : \{a, b, c\} \in B\},$$
$$B^2 = \{\{a\kappa_a^1, b\kappa_b^1, c\kappa_c^1\} : \{a, b, c\} \in B\}.$$

Then, $M^1 = \langle Q^1 \rangle$; $M^2 = \langle Q^2 \rangle$; and $F^1 = (Q^1, B^1)$ and $F^2 = (Q^2, B^2)$ are $(2, t)$-GQs in M^1 and M^2, respectively. Now, take

$$\mathsf{Q} = Q \cup Q^1 \cup Q^2,$$
$$\mathsf{B} = B \cup B^1 \cup B^2 \cup \{\{m, m\kappa_m^1, \kappa_m^1\} : m \in Q\}.$$

Then, $\mathsf{S} = (\mathsf{Q}, \mathsf{B})$ is a partial linear space, isomorphic to $Q_4^+(2) \times \mathbb{L}_3$ if $(k, t) = (4, 1)$; $W_4(2) \times \mathbb{L}_3$ if $(k, t) = (4, 2)$; and $Q_6^-(2) \times \mathbb{L}_3$ if $(k, t) = (6, 4)$. Note that F, F^1 and F^2 are the only big quads in the last two cases. Thus we get non-abelian representations for $Q_6^-(2) \times \mathbb{L}_3, W_4(2) \times \mathbb{L}_3$ and $Q_4^+(2) \times \mathbb{L}_3$.

7.4.2. $\mathbb{H}_3$ and $DW_6(2)$. Let $R = 2_+^{1+8}$. Let F and F^1 be the symplectic (2,2)-GQs, the case $(k, t) = (4, 2)$ in Subsection 7.4.1. The map $\sigma : Q \longrightarrow Q^1$ taking $m \mapsto \kappa_m^1, m \in Q$, defines an isomorphism from F to F^1. We set

$$\mathcal{Q} = \{mn^1 : m \in Q, n^1 \in Q^1, [m, n^1] = 1\}$$

and define collinearity in $\mathcal{Q}$ as follows. For distinct $m_1n_1^1, m_2n_2^1 \in \mathcal{Q}$ with $m_1, m_2 \in Q$ and $n_1^1, n_2^1 \in Q^1$, we say that $m_1n_1^1 \sim m_2n_2^1$ if and only if $[m_1, n_2^1] = [m_2, n_1^1] = 1$ and $(m_1m_2)(n_1^1n_2^1) \in \mathcal{Q}$. The second condition implies that $m_1 \neq m_2$ and $n_1^1 \neq n_2^1$. The line containing $m_1n_1^1$ and $m_2n_2^1$ is $\{m_1n_1^1, m_2n_2^1, (m_1m_2)(n_1^1n_2^1)\}$. [Note that if $m_1n_1^1$ and $m_2n_2^1$ are distinct points of $\mathcal{Q}$ with $m_1, m_2 \in Q$ and $n_1^1, n_2^1 \in Q^1$, then the following hold in $\Gamma(\mathcal{Q})$:

(1) $d(m_1n_1^1, m_2n_2^1) = 1$ if and only if $m_1 \neq m_2, n_1^1 \neq n_2^1$ and $[m_1, n_2^1] = [m_2, n_1^1] = 1$.
(2) $d(m_1n_1^1, m_2n_2^1) = 2$ if and only if one of the following occur:
  (i) $m_1 = m_2, n_1^1 \neq n_2^1$;
  (ii) $m_1 \neq m_2, n_1^1 = n_2^1$;

(iii) $m_1 \neq m_2, n_1^1 \neq n_2^1$ and $[m_1, n_2^1] = [m_2, n_1^1] = \mu$.

(3) $d(m_1n_1^1, m_2n_2^1) = 3$ if and only if $m_1 \neq m_2, n_1^1 \neq n_2^1$ and $\{[m_1, n_2^1], [m_2, n_1^1]\} = \{1, \mu\}$.]

Let $\mathcal{B}$ be the set of all such lines in $\mathcal{Q}$. Set

$$\mathbb{Q} = Q \cup Q^1 \cup \mathcal{Q} \text{ and } \mathbb{B} = \mathcal{B} \cup \mathcal{B}^1,$$

where $\mathcal{B}^1 = \{\{m, mn^1, n^1\} : mn^1 \in \mathcal{Q}\}$. We show that $\mathcal{F} = (\mathcal{Q}, \mathcal{B}) \simeq \mathbb{H}_3$ and $\mathbb{F} = (\mathbb{Q}, \mathbb{B}) \simeq DW_6(2)$, thus giving non-abelian representation for $\mathbb{H}_3$ and $DW_6(2)$.

We first recall the constructions of $\mathbb{H}_3$ and $DW_6(2)$ given in Section 6.2. Let $S = (P, L)$ and $S^1 = (P^1, L^1)$ be two (2,2)-GQs and let $\pi : x \mapsto x^1$, $x \in P, x^1 \in P^1$, denote an isomorphism from S to S^1. Let

$\mathcal{P} = \{(x, y^1) \in P \times P^1 : y^1 \in x^{1\perp}\}$;

$\mathcal{L} = \{\{(x, u^1), (y, v^1), (z, w^1)\} : \{x, y, z\}$ is a line or a complete triad of points of S and $\{x^1, y^1, z^1\}^\perp = \{u^1, v^1, w^1\}$ in $S^1\}$;

$\mathbb{P} = \mathcal{P} \cup P \cup P^1$;

$\mathbb{L} = \mathcal{L} \cup \mathcal{L}^1$, where $\mathcal{L}^1 = \{\{x, (x, u^1), u^1\} : (x, u^1) \in \mathcal{P}\}$.

Then, $\mathcal{S} = (\mathcal{P}, \mathcal{L}) \simeq \mathbb{H}_3$ and $\mathbb{S} = (\mathbb{P}, \mathbb{L}) \simeq DW_6(2)$ (see Theorems 6.3 and 6.10).

Now, let $\alpha : P \longrightarrow Q$ be an isomorphism from S to F and $\beta : P^1 \longrightarrow Q^1$ be the isomorphism from F^1 to Q^1 such that the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & P^1 \\ \alpha \downarrow & & \downarrow \beta \\ Q & \xrightarrow[\sigma]{} & Q^1. \end{array}$$

Thus, $\beta(u^1) = \sigma\alpha\pi^{-1}(u^1)$, $u^1 \in P^1$. We show that, if $x \in P$ and $u^1 \in P^1$, then $(x, u^1) \in \mathcal{P}$ if and only if $\alpha(x)\beta(u^1) \in \mathcal{Q}$. First, assume that $(x, u^1) \in \mathcal{P}$ and $u \in P$ be such that $\pi(u) = u^1$. Since $(x, u^1) \in \mathcal{P}$, $x \in u^\perp$ and $\alpha(x) \in \alpha(u)^\perp$. This implies that $[\alpha(x), \sigma(\alpha(u))] = 1$, since $\kappa^1_{\alpha(u)} = \sigma(\alpha(u))$. But $[\alpha(x), \sigma(\alpha(u))] = [\alpha(x), \sigma\alpha\pi^{-1}(u^1)] = [\alpha(x), \beta(u^1)]$. So $\alpha(x)\beta(u^1) \in \mathcal{Q}$. Reversing the argument we conclude that $(x, u^1) \in \mathcal{P}$ when $\alpha(x)\beta(u^1) \in \mathcal{Q}$.

Let the map $\rho : \mathbb{P} \longrightarrow \mathbb{Q}$ be equal to α on P, β on P^1 and $\rho((x, u^1)) = \alpha(x)\beta(u^1)$ for $(x, u^1) \in \mathcal{P}$. Then, ρ induces a bijection from $\mathcal{L}$ to $\mathcal{B}$ and from $\mathcal{L}^1$ to $\mathcal{B}^1$. For the injectivity on $\mathcal{L}$, we use the fact that if $\{u, v, w\}$ is either a line or a complete triad in Q or Q^1, then

$uvw = 1$ (see Proposition 5.22). So $\mathbb{S} \simeq \mathbb{F}$. Further, the restriction of ρ to $\mathcal{P}$ is an isomorphism from $\mathcal{S}$ to $\mathcal{F}$.

7.5. Proof of Theorems 7.2 and 7.3 for $Q_6^-(2) \otimes Q_6^-(2)$

Let $S = (P, L)$ be the near hexagon $Q_6^-(2) \otimes Q_6^-(2)$. We refer to ([**9**], p.363) for the following description of the Fischer space on the set of the 18 big quads of S. This set partitions into two families $\mathcal{F}_1$ and $\mathcal{F}_2$ of size 9 each such that each $\mathcal{F}_i$ defines a partition of the point set P of S. Let $\mathcal{U}_i$, $i = 1, 2$, be the linear space whose points are the big quads of $\mathcal{F}_i$. If Q_1 and Q_2 are two distinct points of $\mathcal{U}_i$, then the line containing them is $\{Q_1, Q_2, Q_3\}$, where $Q_3 = Q_1 * Q_2$ (Lemma 6.23(ii)). Then, $\mathcal{U}_i$ is an affine plane of order 3.

Consider the affine plane $\mathcal{U}_1$. Fix an affine line $\{Q_1, Q_2, Q_3\}$ in $\mathcal{U}_1$. Then, $Y = Q_1 \cup Q_2 \cup Q_3$ is isomorphic to $Q_6^-(2) \times \mathbb{L}_3$. Fix an affine point Q in $\mathcal{U}_1$ such that $Q \cap Y$ is empty.

Proof of Theorem 7.2. Let the subgroups M and N of R be as in the beginning of Section 7.3. Then, $|N| \leq 2^7$ because, $|R| \leq 2^{1+dimV(S)} = 2^{19}$ (Corollary 7.6). We show that $N = 2_-^{1+6}$. This would prove Theorem 7.2 for $Q_6^-(2) \otimes Q_6^-(2)$.

Let $\{a_1, a_2, b_1, b_2\}$ be a quadrangle in Q, where $a_1 \nsim a_2$ and $b_1 \nsim b_2$. Let the map δ be as in Corollary 7.19. Then, the subgroup $\langle \delta(a_1), \delta(a_2), \delta(b_1), \delta(b_2) \rangle$ of R is isomorphic to a central product $H = \langle \delta(a_1), \delta(a_2) \rangle \circ \langle \delta(b_1), \delta(b_2) \rangle$. We can write N as a central product $N = H \circ K$, where $K = C_N(H)$. Then, $|K| \leq 2^3$. There are three more common neighbours, say w_1, w_2, w_3, of a_1 and a_2 in Q different from b_1 and b_2. We can write

$$\delta(w_m) = \delta(a_1)^{i_1} \delta(a_2)^{i_2} \delta(b_1)^{j_1} \delta(b_2)^{j_2} k_m$$

for some $k_m \in K$, where $i_1, i_2, j_1, j_2 \in \{0, 1\}$. Now, by Corollary 7.19(i), $[\delta(w_m), \delta(a_r)] = 1$ and $[\delta(w_m), \delta(b_r)] \neq 1$ for $r = 1, 2$. This implies that $i_1 = i_2 = 0$ and $j_1 = j_2 = 1$; that is, $\delta(w_m) = \delta(b_1)\delta(b_2)k_m$. In particular, k_m is of order 4. Since $[\delta(w_m), \delta(w_n] \neq 1$ for $m \neq n$ (Corollary 7.19(i)), it follows that $[k_m, k_n] \neq 1$. Thus, K is non-abelian and is of order 8; and k_1, k_2 and k_3 are three pair-wise distinct elements of order 4 in K. So, K is isomorphic to Q_8 and $N = 2_-^{1+6}$. This completes the proof. □

The rest of this section is devoted to prove Theorem 7.3 for $Q_6^-(2) \otimes Q_6^-(2)$. Taking $\{i, j, k\} = \{1, 2, 3\}$, we make use of the notation and the results of Section 6.3.

Let $l = \{a, b, c\}$ be a line of S not contained in Y. If l meets Y at some point c, say, and is disjoint from Q, then exactly one of the lines aa_Q and bb_Q meet Y. If l meets Q at some point and is disjoint from Y, then l corresponds to the affine line of $\mathcal{U}_1$ containing Q and parallel to $\{Q_1, Q_2, Q_3\}$. Further, if $x \in l \setminus (l \cap Q)$, then the line xx^i is disjoint from Q. Now, let l be disjoint from both Y and Q. Then l is contained in a point of $\mathcal{U}_1$ different from Q and Q_i, $i \in \{1, 2, 3\}$; or it corresponds to the affine line of $\mathcal{U}_1$ not containing Q and parallel to $\{Q_1, Q_2, Q_3\}$. So, the lines aa_Q, bb_Q and cc_Q either meet Y or all have empty intersection with Y. In the first case, if $xx_Q \cap Y = \{x_Y\}$ for $x \in l$ and $l_Y = \{x_Y : x \in l\}$, then l_Y is a line of Q_i for some $i \in \{1, 2, 3\}$; or $|l_Y \cap Q_i| = 1$ for each $i \in \{1, 2, 3\}$ (and l_Y need not be a line in this case).

For the rest of the section, recall the definition of the spread T_Q of Q from Corollary 6.31.

Lemma 7.21. *Let $l = \{a, b, c\}$ be a line of S disjoint from $Y \cup Q$. Assume that, for each $x \in l$, the line xx_Q meets Y at a point, say x_Y. Let $m, n \in \{1, 2, 3\}$ with $m \neq n$. Then, the following hold:*

- *(i) If l is contained in a point of $\mathcal{U}_1$, then $d(a^m, b^n) \leq 2$ or $d(a^m, b^n) = 3$ according as $l_Q \in T_Q$ or not.*
- *(ii) If l corresponds to the affine line of $\mathcal{U}_1$ not containing Q and parallel to $\{Q_1, Q_2, Q_3\}$, then $l_Q \notin T_Q$ and $d(a^m, b^n) = 3$.*

Proof. (i) Let $l_Y = \{a_Y, b_Y, c_Y\}$. Then, l_Y is a line of Q_i, Q_j or Q_k, say Q_i. (If K is the affine point of $\mathcal{U}_1$ containing l, then $Q_i = K * Q$.) Let $x \in l$. Then $x^i = x_Q^i = x_Y \in Q_i$, so $l^i = l_Q^i$. The line $l_x = \{x, x_Q, x_Y\}$ is disjoint from Q_j and Q_k. So $l_x^j = \{x^j, x_Q^j, z_{x_Y}^j = z_{x_Q^i}^j\}$ and $l_x^k = \{x^k, x_Q^k, z_{x_Y}^k = z_{x_Q^i}^k\}$ are lines of Q_j and Q_k, respectively.

If $l_Q \in T_Q$, then $l_Q^j = \{x_Q^j, z_{x_Q^i}^j, z_{x_Q^k}^j\}$ by Lemma 6.29. Since $|l_Q^j \cap l_x^j| \geq 2$, we get $l_Q^j = l_x^j$. Thus $x^j \in l_Q^j$ for each $x \in l$ and so, $l^j = l_Q^j$. Similarly, $l^k = l_Q^k$. Now, Corollary 6.32 completes the proof of (i) in this case.

If $l_Q \notin T_Q$, then consider (6.3.3) for the line l_Q and the lines l_x^j and l_x^k above. Then, l_x^j and l_x^k are the lines corresponding to the x-column in $T_j^{l_Q}$ and $T_k^{l_Q}$, respectively. So $z^j_{x^k_Q} = x^j$ and $z^k_{x^j_Q} = x^k$ and (i) in this case follows from Corollary 6.32.

(ii) Here l_Y meets each of Q_i, Q_j and Q_k. We may assume that $a_Y \in Q_i$, $b_Y \in Q_j$ and $c_Y \in Q_k$. Then $a^i = a^i_Q = a_Y$, $b^j = b^j_Q = b_Y$ and $c^k = c^k_Q = c_Y$. Suppose that $l_Q \in T_Q$. Since $\tau_{ik}(l^i_Q) = l^k_Q$ (Lemma 6.29), we may assume that $b^k_Q \sim a^i_Q$ (see (6.3.2)). Then, $z^j_{a^i_Q} = c^j_Q$. The line $l_a = \{a, a_Q, a_Y = a^i_Q\}$ is disjoint from Q_j. So $l^j_a = \{a^j, a^j_Q, z^j_{a^i_Q} = c^j_Q\}$ is a line in Q_j. But $l^j_Q = \{a^j_Q, b^j = b^j_Q, c^j_Q\}$ is a line in Q_j, and so $a^j = b^j$, a contradiction to the fact that $\{a^j, b^j, c^j\}$ is a line in Q_j. So $l_Q \notin T_Q$. Since $a^i = a^i_Q, b^j = b^j_Q, c^k = c^k_Q$, (6.3.3) applied to the line l_Q together with Corollary 6.32 implies (ii). □

Lemma 7.22. *Let x be a point in $P \setminus (Y \cup Q)$ such that the line xx_Q is disjoint from Y. Let $\zeta_{x_Q} = \{x_Q, a_x, b_x\} \in T_Q$ and $xx_Q = \{x, x_Q, y\}$. Then $\{(x^1, x^2, x^3), (y^1, y^2, y^3)\} = \{(a^1_x, a^2_x, a^3_x), (b^1_x, b^2_x, b^3_x)\}$.*

Proof. Let $l = xx_Q$. If $x^i \in \zeta^i_{x_Q}$, then $\zeta^i_{x_Q} = l^i$. By the definition of ζ_{x_Q} and Lemma 6.29, $\tau_{ij}(\zeta^i_{x_Q}) = \zeta^j_{x_Q}$. So $z^j_{x^i} \in \zeta^j_{x_Q}$. Since $x^j \sim z^j_{x^i}$ and $x^j \sim x^j_Q$ in the line $\zeta^j_{x_Q}$, it follows that $x^j \in \zeta^j_{x_Q}$. So $l^i = x^i z^i_{x^j}$. Then, $\tau_{ij}(l^i) = l^j$ (Lemma 6.27(i)). So $l^j = \zeta^j_{x_Q}$ and the result follows (see (6.3.2)). Thus, it is enough we show that $x^i \in \zeta^i_{x_Q}$.

Suppose that $x^i \notin \zeta^i_{x_Q}$. Let $\bar{l} = \{x, x^i, w\}$ be the line xx^i of S. Then, $\bar{l}$ is disjoint from Q. Consider the line $\bar{l}_Q = \{x_Q, (x^i)_Q, w_Q\}$ of Q. Since $(x^i)_Q \notin \zeta_{x_Q}$, $\bar{l}_Q \neq \zeta_{x_Q}$ and $\zeta_{x_Q} \cap \bar{l}_Q = \{x_Q\}$. The line ww_Q meets either Q_j or Q_k, say Q_k. Since $\bar{l}$ is disjoint from both Q_j and Q_k, $\bar{l}^j = \{x^j, z^j_{x^i}, w^j\}$ and $\bar{l}^k = \{x^k, z^k_{x^i}, w^k = w^k_Q\}$ are lines of Q_j and Q_k, respectively. Applying Lemma 6.26(ii) to $\bar{l}$, we get $w^j \sim x^k$ and $w^k \sim x^j$.

Now, $d(x^k, x_Q) = d(x^k, x) + d(x, x_Q) = 2$ and $d(x^k, w_Q) = d(x^k, w^k) + d(w^k, w_Q) = 2$. So, $d(x^k, (x^i)_Q) = 1$. Again, $d(x^j, x_Q) = d(x^j, x) + d(x, x_Q) = 2$ and $d(x^j, w_Q) = d(x^j, w^k) + d(w^k, w_Q) = 2$ (since $w^k \sim x^j$). So, $d(x^j, (x^i)_Q) = 1$. Let $c = (x^i)_Q$. Then, $c^j = x^j$ and $c^k = x^k$. Now, $\bar{l}^k_Q = c^k w^k_Q = c^k w^k = c^k z^k_{x^j} = c^k z^k_{c^j}$. Applying Lemma 6.27$(i)$ to $\bar{l}_Q$, we get $\tau_{kj}(\bar{l}^k_Q) = \bar{l}^j_Q$. So $\bar{l}_Q \in T_Q$ (see Lemma 6.29). But $\zeta_{x_Q} \in T_Q$ and $\zeta_{x_Q} \cap \bar{l}_Q = \{x_Q\}$. This leads to a contradiction to the fact that T_Q is a spread of Q (Corollary 6.31). So $x^i \in \zeta^i_{x_Q}$. □

In view of Proposition 7.19, we now prove

Lemma 7.23. *Let $N = 2_-^{1+6}$ with $N' = \{1, \mu\}$ and let $I_2(N)$ be the set of involutions in N. There exists a map δ from Q to $I_2(N)$ satisfying the following:*

(*i*) *δ is one-one.*
(*ii*) *For $x, y \in Q$, $[\delta(x), \delta(y)] = 1$ if and only if either $x = y$ or $x \sim y$.*
(*iii*) *If $x, y \in Q$ and $x \sim y$, then*

$$\delta(x * y) = \begin{cases} \delta(x)\delta(y) & \text{if } xy \in T_Q \\ \delta(x)\delta(y)\mu & \text{if } xy \notin T_Q \end{cases}.$$

Proof. First, recall the model for Q given in Section 5.1. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Let $\mathcal{E}$ be the set of all 2-subsets of Ω and $\mathcal{F}$ be the set of all factors of Ω. Then, the point set of Q is $\mathcal{E} \cup \Omega \cup \Omega'$ and the line set is $\mathcal{F} \cup \{\{i, \{i, j\}, j'\} : 1 \le i \ne j \le 6\}$.

We may assume that the spread T_Q of Q consists of the following lines:

$$l_1 = \{\{1,2\},\{3,4\},\{5,6\}\};\ l_2 = \{\{1,4\},1,4'\};\ l_3 = \{\{2,6\},2,6'\};$$
$$l_4 = \{\{1,6\},\{2,4\},\{3,5\}\};\ l_5 = \{\{1,5\},1',5\};\ l_6 = \{\{2,3\},2',3\};$$
$$l_7 = \{\{1,3\},\{2,5\},\{4,6\}\};\ l_8 = \{\{3,6\},3',6\};\ l_9 = \{\{4,5\},4,5'\}.$$

We write N as a central product $N = \langle x_1, y_1 \rangle \circ \langle x_2, y_2 \rangle \circ Q_8$, where x_i, y_i are involutions, $\langle x_i, y_i \rangle$ is isomorphic to the dihedral group D_8 of order 8, and Q_8 is the quaternion group of order 8. Let $Q_8 = \{1, \mu, i, j, k, i^3, j^3, k^3\}$, where $i^2 = j^2 = k^2 = \mu; ij = k$; and $ji = k^3 = k\mu$. We define $\delta : Q \longrightarrow I_2(N)$ as follows:

$\delta(l_1) = \{x_1, x_2, x_1x_2\};$
$\delta(l_2) = \{x_1y_1y_2i, x_2y_2j, x_1x_2y_1k\mu\};$
$\delta(l_3) = \{x_1y_1i\mu, x_1x_2y_2k, x_2y_1y_2j\mu\};$
$\delta(l_4) = \{y_1, y_1y_2, y_2\};$
$\delta(l_5) = \{x_1x_2y_1i, x_2y_2k\mu, x_1y_1y_2j\};$
$\delta(l_6) = \{x_2y_1y_2i\mu, x_1x_2y_2j\mu, x_1y_1k\};$
$\delta(l_7) = \{x_1x_2y_1y_2\mu, x_2y_1\mu, x_1y_2\mu\};$
$\delta(l_8) = \{x_1x_2y_2i\mu, x_1y_1j\mu, x_2y_1y_2k\};$
$\delta(l_9) = \{x_2y_2i, x_1x_2y_1j, x_1y_1y_2k\mu\}.$

Here, if $l_i = \{a, b, c\}$, then $\delta(l_i)$ denotes $\{\delta(a), \delta(b), \delta(c)\}$ preserving the order. It can be verified that δ satisfies the conditions $(i), (ii)$ and (iii) of the lemma. □

Consider the map $\delta : Q \longrightarrow I_2(N)$ in Lemma 7.23. We now extend δ to $P \setminus Y$. For $x \in P \setminus (Y \cup Q)$, let $\zeta_{xQ} = \{x_Q, a_x, b_x\} \in T_Q$. If the line xx_Q intersects Y, then we define $\delta(x) = \delta(x_Q)$. If xx_Q is disjoint from Y, let $(b_x^1, b_x^2, b_x^3) = (x^1, x^2, x^3)$ (see Lemma 7.22). In that case, we define $\delta(x) = \delta(a_x)$. That is; for $x \in P \setminus (Y \cup Q)$,

$$\delta(x) = \begin{cases} \delta(x_Q) & \text{if } xx_Q \text{ intersects } Y \\ \delta(a_x) & \text{if } xx_Q \cap Y \text{ is empty and } (x^1, x^2, x^3) = (b_x^1, b_x^2, b_x^3) \end{cases}.$$

We now construct a non-abelian representation of S. Let $R = 2_-^{1+18}$ with $R' = \{1, \mu\}$. We write R as a central product $R = M \circ N$, where $M = 2_+^{1+12}$ and N_-^{1+6}. Let (M, λ) be a non-abelian representation of Y (see Subsection 7.4.1). Define a map $\beta : P \longrightarrow R$ as follows:

$$\beta(x) = \begin{cases} \lambda(x) & \text{if } x \in Y \\ \lambda(z_{x^2}^1)\lambda(z_{x^1}^2)\delta(x) & \text{if } x \in P \setminus Y \end{cases}.$$

For $x \in P \setminus Y$, Lemma 6.25(i) implies that $d(z_{x^1}^2, z_{x^2}^1) = 2$. So $[\lambda(z_{x^2}^1), \lambda(z_{x^1}^2)] = 1$ and $\beta(x)$ is an involution.

Proposition 7.24. (R, β) *is a non-abelian representation of* S.

Proof. Only condition (ii) of Definition 3.5 needs to be verified. Let $l = \{u, v, w\}$ be a line of S. We assume that l is not contained in Y and that $l \cap Y = \{w\}$ if l intersects Y. We show that $\beta(u)\beta(v) = \beta(w)$. We have

$$\beta(u)\beta(v) = \lambda(z_{u^2}^1)\lambda(z_{v^2}^1)\lambda(z_{u^1}^2)\lambda(z_{v^1}^2)\delta(u)\delta(v)r', \tag{7.5.1}$$

where $r' = [\lambda(z_{u^1}^2), \lambda(z_{v^2}^1)] \in R'$.

Case (I) *l intersects Y at w:* In this case, Lemma 6.26 yields that $r' = 1$. If $w \in Q_1$, then $u^1 = v^1 = w$ and $\beta(u)\beta(v) = \lambda(z_{u^2}^1)\lambda(z_{v^2}^1)\delta(u)\delta(v) = \lambda(w)\delta(u)\delta(v)$. The last equality holds because $\{z_{u^2}^1, z_{v^2}^1, w\}$ is a line of Q_1. Similarly, $\beta(u)\beta(v) = \lambda(w)\delta(u)\delta(v)$ if $w \in Q_2$. If $w \in Q_3$, then $\{z_{u^2}^1, z_{v^2}^1, z_w^1\}$ and $\{z_{u^1}^2, z_{v^1}^2, z_w^2\}$ are lines of Q_1 and Q_2 respectively. So, $\beta(u)\beta(v) = \lambda(z_w^1)\lambda(z_w^2)\delta(u)\delta(v) = \lambda(w)\delta(u)\delta(v)$. The last equality holds because $\{z_w^1, z_w^2, w\}$ is a line of Y. Since $\beta(w) = \lambda(w)$, we get $\beta(u)\beta(v) = \beta(w)\delta(u)\delta(v)$. Thus, we need to prove that $\delta(u) = \delta(v)$.

If l intersects Q, say $l \cap Q = \{v\}$, then $u_Q = v$ and so, $\delta(u) = \delta(v)$. Let $l \cap Q$ be empty. Exactly one of the lines uu_Q and vv_Q, say uu_Q, meets Y. So $\delta(u) = \delta(u_Q)$. Let $l_{v_Q} = \{v_Q, a_v, b_v\}$. By Lemma 7.22, we assume that $(v^1, v^2, v^3) = (b_v^1, b_v^2, b_v^3)$. Then $\delta(v) = \delta(a_v)$. Since

$w \in \{v^1, v^2, v^3\}$, it follows that $b_v \sim w$. So $w_Q = b_v$ and $u_Q = a_v$. Thus, $\delta(u) = \delta(u_Q) = \delta(a_v) = \delta(v)$.

Case (II) *l disjoint from Y:* Since $\{z^1_{u^2}, z^1_{v^2}, z^1_{w^2}\}$ and $\{z^2_{u^1}, z^2_{v^1}, z^2_{w^1}\}$ are lines of Q_1 and Q_2 respectively, $\beta(u)\beta(v) = \lambda(z^1_{w^2})\lambda(z^2_{w^1})\delta(u)\delta(v)r'$. To complete the proof, we show that either $r' = 1$ and $\delta(u)\delta(v) = \delta(w)$ or $r' = \mu$ and $\delta(u)\delta(v) = \delta(w)\mu$. This follows from Corollary 6.32 and Lemma 7.23(*iii*) if $l \subset Q$.

Assume that l intersects Q at a point, say w. Let $\zeta_w = \{w, a, b\} \in T_Q$. Applying Lemma 7.22, we get $\zeta^j_w = l^j$ in Q_j and $\{\delta(u), \delta(v)\} = \{\delta(a), \delta(b)\}$. This, together with $\zeta_w \in T_Q$, yields that $\delta(u)\delta(v) = \delta(w)$ (Lemma 7.23(*iii*)) and $r' = 1$ (Corollary 6.32).

Now, assume that $l \cap Q$ is empty. If the lines uu_Q, vv_Q and ww_Q meet Y, then Lemmas 7.21 and 7.23(*iii*) complete the proof. So, we may assume that none of uu_Q, vv_Q and ww_Q meet Y. First, let $l_Q \in T_Q$. Then $l_Q = \zeta_{u_Q} = \zeta_{v_Q} = \zeta_{w_Q}$. Applying Lemma 7.22 to the lines xx_Q, $x \in l$, it follows that $l^j_Q = l^j$ in Q_j and $(\delta(u), \delta(v), \delta(w)) = (\delta(w_Q), \delta(u_Q), \delta(v_Q))$ or $(\delta(v_Q), \delta(w_Q), \delta(u_Q))$. Then, it follows that $\delta(u)\delta(v) = \delta(w)$ (Lemma 7.23(*iii*)) and $r' = 1$ (Corollary 6.32).

Now, let $l_Q \notin T_Q$. For $x \in l$, let $\zeta_{x_Q} = \{x_Q, a_x, b_x\}$. We may assume, by Lemma 7.22, that $(x^1, x^2, x^3) = (a^1_x, a^2_x, a^3_x)$. So, $\delta(x) = \delta(b_x)$. For distinct $x, y \in l$, $a^i_x = x^i \sim y^i = a^i_y$ in Q_i. Thus, $l_a = \{a_u, a_v, a_w\}$ and $l_b = \{b_u, b_v, b_w\}$ are lines of Q. Since $l_b \notin T_Q$, $\delta(u)\delta(v) = \delta(b_u)\delta(b_v) = \delta(b_w)\mu = \delta(w)\mu$. Again, $l_a \notin T_Q$ implies that $d(u^1, v^2) = d(a^1_u, a^2_v) = 3$ (Corollary 6.32) and so, $r' = \mu$. This completes the proof. □

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